The Exact Smile of some Local Stochastic Volatility Models

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Abstract

We introduce a class of local stochastic volatility models. Within our framework, we obtain an expression for both (i) the price of any European option and (ii) the induced implied volatility smile. To illustrate our method, we perform specific computations for a CEV-like model.

Keywords: CEV, local volatility, stochastic volatility, implied volatility.

1 Introduction

A *local volatility* model is a stochastic volatility model in which the volatility σ_t of an asset X is a function of the present level of X. That is, $\sigma_t = \sigma(X_t)$. Among local stochastic volatility models, perhaps the most well-known is the constant elasticity of variance (CEV) model of Cox (1975). An extension of the CEV model to defaultable assets (the Jump-to-Default CEV or JDCEV model) is derived in Carr and Linetsky (2006) . One advantage of these two local stochastic volatility models is that they allow for closed-form pricing formulas for European options written as infinite series of special functions.

In this paper, we introduce a class of local stochastic volatility models which, like the CEV and JDCEV models, allow for European option prices to be written down in closed form as an infinite series. Additionally, we derive an expression for the *exact* implied volatility surface induced by our class of models. Previous studies of the implied volatility surface induced by local volatility models focused on heat-kernal expansions to derive *asymptotic approximations* of the volatility smile (see e.g., Gatheral, Hsu, Laurence, Ouyang, and Wang (2010); Henry-Labordere (2005) and references therein). It is worth mentioning that Dupire (1994) solves the inverse problem of finding a formula for the local volatility function the produces a given observed implied volatility surface exactly.

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The rest of this paper proceeds as follows: in section 2 we present our model and assumptions. In section 3 we derive a formula for the price of a European option in our modeling framework. In section 4 we provide an formula for the implied volatility smile induced by our model. As an example of our framework, in section 5 we perform explicit pricing and implied volatility computations for a CEV-like model. Numerical results are provided at the conclusion of the text. An appendix with some mathematical background is also provided. Concluding remarks can be found in section 6.

2 Model and Assumptions

We assume a frictionless market, no arbitrage and take an equivalent martingale measure $\mathbb{P}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$. The filtration $\{\mathcal{F}_t, t \geq 0\}$ represents the history of the market. All processes defined below live on this space. For simplicity we assume zero interest rates and no dividends so that all assets are martingales. We consider a diffusion X with lifetime ζ whose dynamics are given by

$$dX_t = \left(a^2 + \varepsilon\, \eta(\log X_t)\right)^{1/2} X_t dW_t, \tag{1}$$

were, $a > 0$, $\varepsilon \geq 0$, the function $\eta : \mathbb{R} \to \mathbb{R}^+$ is $C_0^\infty(\mathbb{R})$ and W is a Brownian motion [1]. Note that X has *local* stochastic volatility $\sigma(X_t) = \left(a^2 + \varepsilon\, \eta(\log X_t)\right)^{1/2}$. Obviously, if $\eta = 0$ then X is a geometric Brownian motion. This will be key for our implied volatility analysis in section 4. Observe that both zero and infinity are natural boundaries according to Feller's boundary classification for one-dimensional diffusions (see Borodin and Salminen (2002) pp. 14-15). That is, both zero and infinity are unattainable.

In what follows it will be convenient to introduce $Y = \log X$. A simple application of Itô's formula shows that Y satisfies

$$dY_t = -\frac{1}{2}\left(a^2 + \varepsilon\, \eta(Y_t)\right) dt + \left(a^2 + \varepsilon\, \eta(Y_t)\right)^{1/2} dW_t.$$

3 Option Pricing

We wish to find the time-zero value $u^\varepsilon(t, y)$ of a European-style option with payoff $h(Y_t)$ at time $t > 0$. Using risk-neutral pricing we have

$$u^\varepsilon(t, y) = \mathbb{E}_y\, h(Y_t),$$

[1] The notation $C_0^\infty(\mathbb{R})$ indicates the space of infinitely differentiable functions with compact support.

where the notation $\mathbb{E}_y$ indicates expectation starting from $y = \log X_0$. The function $u^\varepsilon(t,y)$ satisfies the Kolmogorov backward equation

$$(-\partial_t + \mathcal{A}^\varepsilon)\, u^\varepsilon = 0, \qquad\qquad u^\varepsilon(0,y) = h(y). \tag{2}$$

where $\mathcal{A}^\varepsilon$ is the generator of the process Y. The domain of $\mathcal{A}^\varepsilon$ is defined as the set of f for which the limit $\lim_{t\to 0} \frac{1}{t}\left(\mathbb{E}_y f(Y_t) - f(y)\right)$ exists in the strong sense. For any $f \in C_0^2(\mathbb{R})$ the generator $\mathcal{A}^\varepsilon$ has the explicit representation

$$\mathcal{A}^\varepsilon = \mathcal{A}_0 + \varepsilon\, \eta\, \mathcal{A}_1, \qquad \mathcal{A}_0 = \frac{1}{2}a^2\left(\partial^2 - \partial\right), \qquad \mathcal{A}_1 = \frac{1}{2}\left(\partial^2 - \partial\right), \qquad \mathrm{dom}(\mathcal{A}_i) = C_0^2(\mathbb{R}).$$

Remark 1. The operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are normal operators in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dy)$ and satisfy the following (improper) eigenvalue equations (neither $\mathcal{A}_0$ nor $\mathcal{A}_1$ have any proper eigenvalues)

$$\mathcal{A}_0\psi_\lambda = \phi_\lambda\psi_\lambda, \qquad \psi_\lambda = \frac{1}{\sqrt{2\pi}}e^{i\lambda y}, \qquad \phi_\lambda = \frac{1}{2}a^2\left(-\lambda^2 - i\lambda\right),$$
$$\mathcal{A}_1\psi_\lambda = \chi_\lambda\psi_\lambda, \qquad \psi_\lambda = \frac{1}{\sqrt{2\pi}}e^{i\lambda y}, \qquad \chi_\lambda = \frac{1}{2}\left(-\lambda^2 - i\lambda\right).$$

Note that the eigenfunctions satisfy $(\psi_\lambda, \psi_\mu) = \int \overline{\psi_\lambda}(y)\psi_\mu(y)dy = \delta(\lambda - \mu)$. Note also that Borel-measurable functions of normal operators (e.g., $g(\mathcal{A}_0)$) are well-defined, as explained in Appendix A.

We seek a solution to Cauchy problem (2) of the form

$$u^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n\, u_n. \tag{3}$$

We will justify this expansion in Theorem 2. Inserting the expansion (3) into Cauchy problem (2) and collecting terms of like powers of ε we obtain

$$\mathcal{O}(1): \qquad (-\partial_t + \mathcal{A}_0)u_0 = 0, \qquad u_0(0,y) = h(y),$$
$$\mathcal{O}(\varepsilon^n): \qquad (-\partial_t + \mathcal{A}_0)u_n = -\eta\mathcal{A}_1 u_{n-1}, \qquad u_n(0,y) = 0.$$

The solution to the above equations is

$$\mathcal{O}(1): \qquad u_0(t,y) = e^{t\mathcal{A}_0}h(y),$$
$$\mathcal{O}(\varepsilon^n): \qquad u_n(t,y) = \int_0^t ds\, e^{(t-s)\mathcal{A}_0}\eta(y)\mathcal{A}_1 u_{n-1}(s,y).$$

Using the equation (11) from appendix A we obtain

$$\mathcal{O}(1): \qquad u_0(t,y) = \int_{\mathbb{R}} d\lambda\, e^{t\phi_\lambda}(\psi_\lambda, h)\psi_\lambda(y),$$
$$\mathcal{O}(\varepsilon^n): \qquad u_n(t,y) = \int_0^t \int_{\mathbb{R}} ds\, d\mu\, e^{(t-s)\phi_\mu}\left(\psi_\mu, \eta\mathcal{A}_1 u_{n-1}(s,\cdot)\right)\psi_\mu(y),$$

After a bit of algebra, we find an explicit representation for $u_n(t,y)$

$$u_n(t,y) = \underbrace{\int \cdots \int}_{n+1} \left(\prod_{k=0}^{n} d\lambda_k \right) \left(\sum_{k=0}^{n} \frac{e^{t\phi_{\lambda_k}}}{\prod_{j\neq k}^{n} (\phi_{\lambda_k} - \phi_{\lambda_j})} \right) \left(\prod_{k=0}^{n-1} \left(\psi_{\lambda_{k+1}}, \eta \mathcal{A}_1 \psi_{\lambda_k}\right) \right) (\psi_{\lambda_0}, h) \; \psi_{\lambda_n}. \tag{4}$$

We have now obtained a formal expansion for the price of a European option. The following theorem provides conditions under which the expansion is guaranteed to be valid.

Theorem 2 (Option Price)**.** *Suppose* $\varepsilon \leq \frac{a^2}{\|\eta\|}$, *where* $\|\eta\| = \sqrt{(\eta,\eta)}$. *Then the option price* $u^\varepsilon(t,y)$ *is given by* (3) - (4).

Proof. See Appendix B. ☐

4 Implied Volatility

In this section we fix (t,y) and a call option payoff $h(y) = (e^y - e^k)^+$. Note that

$$(\psi_\lambda, h) = \frac{-e^{k - ik\lambda}}{\sqrt{2\pi}\,(i\lambda + \lambda^2)}, \qquad \qquad \mathrm{Im}(\lambda) < -1.$$

The following definitions will be useful:

Definition 3. The *Black-Scholes Price* $u^{BS} : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$u^{BS}(\sigma) := \int d\lambda \, e^{t\phi_\lambda^{BS}(\sigma)} (\psi_\lambda, h) \psi_\lambda, \qquad \qquad \phi_\lambda^{BS}(\sigma) = \frac{1}{2}\sigma^2(-\lambda^2 - i\lambda).$$

Definition 4. The *Implied Volatility* is defined implicitly as the unique number $\sigma^\varepsilon \in \mathbb{R}^+$ such that

$$u^{BS}(\sigma^\varepsilon) = u^\varepsilon, \tag{5}$$

where u^ε is as given in Theorem 2.

Remark 5. Note that $u_0 = u^{BS}(a)$. As shown in Lorig (2012), when u^ε can be expanded as a power series whose first term corresponds to u^{BS}, one can obtain the exact implied volatility surface corresponding to u^ε.

Remark 6. For $0 < t < \infty$ the existence and uniqueness of the implied volatility σ^ε can be deduced by using the general arbitrage bounds for call prices and the monotonicity of u^{BS}.

Remark 7. Note that u^{BS} is an invertible analytic function that satisfies $\partial_\rho u^{BS}(\sigma) > 0$ for all $\sigma > 0$. By the Lagrange inversion theorem, the inverse $[u^{BS}]^{-1}$ of such a function is also analytic.

Clearly, u^ε is an analytic function of ε (we derived its power series expansion). It is a useful fact that the composition of two analytic functions is also analytic (see Brown and Churchill (1996), section 24, p. 74). Thus, in light of Remark 7, we deduce that $\sigma^\varepsilon = [u^{BS}]^{-1}(u^\varepsilon)$ is an analytic function and therefore has a power series expansion in ε. We write this expansion as follows

$$\sigma^\varepsilon = \sigma_0 + \delta^\varepsilon, \qquad \delta^\varepsilon = \sum_{k=1}^{\infty} \varepsilon^k \sigma_k. \tag{6}$$

Taylor expanding u^{BS} about the point σ_0 we have

$$\begin{aligned}
u^{BS}(\sigma^\varepsilon) &= u^{BS}(\sigma_0 + \delta^\varepsilon) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (\delta^\varepsilon \partial_\sigma)^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \varepsilon^k \sigma_k \right)^n \partial_\sigma^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{k=1}^{\infty} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \varepsilon^k \right] \partial_\sigma^n u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(\sigma_0) \\
&= u^{BS}(\sigma_0) + \sum_{k=1}^{\infty} \varepsilon^k \left[\sigma_k \partial_\sigma + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n \right] u^{BS}(\sigma_0).
\end{aligned} \tag{7}$$

Now, we insert expansions (3) and (7) into (5) and collect terms of like order in ε

$$\begin{aligned}
\mathcal{O}(1): &\qquad u_0 = u^{BS}(\sigma_0), \\
\mathcal{O}(\varepsilon^k): &\qquad u_k = \sigma_k \partial_\sigma u^{BS}(\sigma_0) + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{BS}(\sigma_0), \qquad k \geq 1.
\end{aligned}$$

Solving the above equations for $\{\sigma_k\}_{k=0}^{\infty}$ we find

$$\begin{aligned}
\mathcal{O}(1): &\qquad \sigma_0 = a, \\
\mathcal{O}(\varepsilon^k): &\qquad \sigma_k = \frac{1}{\partial_\sigma u^{BS}(\sigma_0)} \left(u_k - \sum_{n=2}^{\infty} \frac{1}{n!} \left(\sum_{j_1+\cdots+j_n=k} \prod_{i=1}^{n} \sigma_{j_i} \right) \partial_\sigma^n u^{BS}(\sigma_0) \right), \qquad k \geq 1.
\end{aligned} \tag{8}$$

Remark 8. The right hand side of (8) involves only σ_j for $j \leq k-1$. Thus, the $\{\sigma_k\}_{k=1}^{\infty}$ can be found recursively.

Remark 9. Note that $\partial_\sigma^n u^{BS}(\sigma)$ is easily computed using

$$\partial_\sigma^n u^{BS}(\sigma) = \int d\lambda \, \left(\partial_\sigma^n e^{t\phi_\lambda^{BS}(\sigma)} \right) (\psi_\lambda, h) \psi_\lambda.$$

Explicitly, up to $\mathcal{O}(\varepsilon^4)$ we have

$$\begin{aligned}
\mathcal{O}(\varepsilon): \qquad & \sigma_1 = \frac{u_1}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^2): \qquad & \sigma_2 = \frac{u_2 - \frac{1}{2!}\sigma_1^2 \partial_\sigma^2 u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^3): \qquad & \sigma_3 = \frac{u_3 - (\sigma_2 \sigma_1 \partial_\sigma^2 + \frac{1}{3!}\sigma_1^3 \partial_\sigma^3) u_0}{\partial_\sigma u_0}, \\
\mathcal{O}(\varepsilon^4): \qquad & \sigma_4 = \frac{u_4 - (\sigma_3 \sigma_1 \partial_\sigma^2 + \frac{1}{2}\sigma_2^2 \partial_\sigma^2 + \frac{1}{2}\sigma_2 \sigma_1^2 \partial_\sigma^3 + \frac{1}{24}\sigma_1^4 \partial_\sigma^4) u_0}{\partial_\sigma u_0}.
\end{aligned}$$

We summarize our implied volatility result in the following theorem:

Theorem 10 (Implied Volatility)**.** *The implied volatility σ^ε defined in* (5) *is given explicitly by* (6) *where $\sigma_0 = \sigma$ and $\{\sigma_k\}_{k=1}^\infty$ are given by* (8)*.*

Remark 11. Everything we have done so far is <u>exact</u>. The accuracy of the implied volatility expansion (6) is limited only by the number of terms one wishes to compute.

5 CEV-like Example

In the constant elasticity of variance (CEV) model of Cox (1975) the dynamics of X are assumed to be of the form $dX_t = \sqrt{\varepsilon}\, X_t^{\beta/2} X_t dW_t$. A key feature of the CEV model is that, when $\beta < 0$, volatility $\sigma(x) = \sqrt{\varepsilon}\, x^{\beta/2}$ *increases* as $x \searrow 0$, which (i) is consistent with the leverage effect and (ii) results in a negative implied volatility skew. However, values of $\beta < 0$ also cause the volatility to drop unrealistically close to zero as x increases. If we choose $\eta(y) = e_\beta(y) := e^{\beta y}$ in (1) then the dynamics of X become

$$dX_t = (a^2 + \varepsilon X_t^\beta)^{1/2} X_t dW_t,$$

Note that the local volatility function $\sigma(x) = (a^2 + \varepsilon\, x^\beta)^{1/2}$ behaves like $\sigma(x) \sim \sqrt{\varepsilon}\, x^{\beta/2}$ as $x \searrow 0$ and behaves like a constant $\sigma(x) \sim a$ as $x \nearrow \infty$. See figure 1 for a comparison of our local volatility function and the CEV local volatility function.

Remark 12. Because $e^{\beta y}$ is unbounded as $y \to -\infty$ (recall $\beta < 0$), the function $e_\beta \notin C_0^\infty(\mathbb{R})$. However, we can modify the domain of u^ε to be $\mathbb{R}^+ \times \mathbb{R}_0$ where $\mathbb{R}_0 := (y_0, \infty)$. The operators $\mathcal{A}_0$ and $\mathcal{A}_1$ would then be defined on $L^2(\mathbb{R}_0, dy)$ and the domain of these operators would include a absorbing boundary condition at y_0 (signifying default of X the first time X reaches the level e^{y_0}). Note that $\|e_\beta\|_0 := (\int_{y_0}^\infty |e_\beta|^2 dy)^{1/2} = e^{\beta y_0}/\sqrt{-2\beta}$. In the analysis that follows, it will simplifyy computations considerably if we continue to work on $L^2(\mathbb{R}, dy)$ as working on $L^2(\mathbb{R}_0, dy)$ would require modifying the eigenfunctions ψ_λ from complex exponentials

$\exp(i\lambda y)$ to cosines $\cos(\lambda y)$. However, the simplification comes at a cost; in light of the conditions of theorem (2) our results may not be valid for values of $y < -\frac{1}{\beta}\log\frac{a^2\sqrt{-2\beta}}{\varepsilon}$.

We wish to find a simplified expression for u_n (4) for the case $\eta = e_\beta$. Noting that

$$(\psi_\mu, e_\beta \mathcal{A}_1 \psi_\lambda) = \chi_\lambda \, \delta(\lambda - \mu - i\beta),$$

we see that the $n+1$-fold integral (4) collapses into a single integral

$$\begin{aligned} u_n &= \int_{\mathbb{R}} d\lambda \left(\sum_{k=0}^{n} \frac{e^{t\phi_{\lambda-ik\beta}}}{\prod_{j\neq k}^{n}(\phi_{\lambda-ik\beta} - \phi_{\lambda-ij\beta})} \right) \left(\prod_{k=0}^{n-1} \chi_{\lambda-ik\beta} \right) (\psi_\lambda, h)\, \psi_{\lambda-in\beta} \\ &= e_{n\beta} \int_{\mathbb{R}} d\lambda \left(\sum_{k=0}^{n} \frac{e^{t\phi_{\lambda-ik\beta}}}{\prod_{j\neq k}^{n}(\phi_{\lambda-ik\beta} - \phi_{\lambda-ij\beta})} \right) \left(\prod_{k=0}^{n-1} \chi_{\lambda-ik\beta} \right) (\psi_\lambda, h)\, \psi_\lambda. \end{aligned} \tag{9}$$

Remark 13. Although we have written the option price as an infinite series (3), from a practical standpoint, one may only compute $u^\varepsilon \approx \sum_{n=0}^{N} \varepsilon^n u_n$. For any finite N we may pass the sum through the integral appearing in (9). Thus, for the purposes of computation, the best way express the approximate option price is

$$u^\varepsilon \approx \int_{\mathbb{R}} d\lambda \, (\psi_\lambda, h)\, \psi_\lambda \sum_{n=0}^{N} \varepsilon^n \, e_{n\beta} \left(\sum_{k=0}^{n} \frac{e^{t\phi_{\lambda-ik\beta}}}{\prod_{j\neq k}^{n}(\phi_{\lambda-ik\beta} - \phi_{\lambda-ij\beta})} \right) \left(\prod_{k=0}^{n-1} \chi_{\lambda-ik\beta} \right).$$

Note, to obtain the approximate value of u^ε, *only a single integration is required.* This makes our pricing formula as efficient as other models in which option prices are expressed as a Fourier-type integral (e.g. Lévy processes, Heston model, etc.).

Some Numerical Results

Define the transition density $p^\varepsilon(t, y; y_0)$ and the $\mathcal{O}(\varepsilon^n)$ approximation of the transition density $p^{(n)}(t, y; y_0)$

$$p^\varepsilon(t, y; y_0) = \mathbb{E}_{y_0} \, \delta_y(Y_t), \qquad\qquad p^{(n)}(t, y; y_0) = \sum_{k=0}^{n} \varepsilon^k p_k(t, y; y_0).$$

In figure 2 we plot the approximate transition density $p^{(n)}$. Next, define the $\mathcal{O}(\varepsilon^n)$ approximation of the implied volatility

$$\sigma^{(n)} := \sum_{k=0}^{n} \varepsilon^k \sigma_k,$$

where the σ_k are given by (8). In figure 3 we provide a numerical example illustrating convergence of $\sigma^{(n)}$ to σ^ε. We compute σ^ε by calculating u^ε first using Theorem 2 and then by inverting the Black-Scholes formula numerically. We plot implied volatility as a function of the log-moneyness to maturity ratio, LMMR $:= (k - y)/t$. Convergence is fastest for values of k near y and slows as k moves away from y.

6 Conclusion

In this paper we introduce a class of local stochastic volatility models. Within our modeling framework, we obtain a formula (written as an infinite series) for the price of any European option. Additionally, we obtain an explicit expression for the implied volatility smile induced by our class of models. As an example of our framework, we introduce a CEV-like model, which corrects one possible short-coming of the CEV model; namely, our choice of volatility does not drop to zero as the value of the underlying increases. In the CEV-like framework, we show that option prices can be computed with the same level of efficiency as other models in which option prices are computed as Fourier-type integrals.

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A Spectral theory of normal operators in a Hilbert space

In this appendix we summarize the theory of normal operators acting on a Hilbert space. A detailed exposition on this topic (including proofs) can be found in Reed and Simon (1980) and Rudin (1973).

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$. A *linear operator* is a pair $(\mathrm{dom}(\mathcal{A}),\mathcal{A})$ where $\mathrm{dom}(\mathcal{A})$ is a linear subset of $\mathcal{H}$ and $\mathcal{A}$ is a linear map $\mathcal{A}:\mathrm{dom}(\mathcal{A})\to\mathcal{H}$. The *adjoint* of an operator $\mathcal{A}$ is an operator $\mathcal{A}^*$ such that $(\mathcal{A}f,g)=(f,\mathcal{A}^*g), \forall\, f\in\mathrm{dom}(\mathcal{A}), g\in\mathrm{dom}(\mathcal{A}^*)$, where

$$\mathrm{dom}(\mathcal{A}^*):=\{g\in\mathcal{H}:\exists\, h\in\mathcal{H} \text{ such that } (\mathcal{A}f,g)=(f,h)\ \forall\, f\in\mathrm{dom}(\mathcal{A})\}.$$

An operator $(\mathrm{dom}(\mathcal{A}),\mathcal{A})$ is said to be *self-adjoint* in $\mathcal{H}$ if

$$\mathrm{dom}(\mathcal{A})=\mathrm{dom}(\mathcal{A}^*), \qquad (\mathcal{A}f,g)=(f,\mathcal{A}g) \qquad \forall\, f,g\in\mathrm{dom}(\mathcal{A}).$$

Throughout this appendix, for any self-adjoint operator $\mathcal{A}$, we will assume that $\mathrm{dom}(\mathcal{A})$ is a dense subset of $\mathcal{H}$. A densely defined self-adjoint operator is closed (see Rudin (1973), Theorem 13.9). An operator $(\mathrm{dom}(\mathcal{A}),\mathcal{A})$ is said to be *normal* in $\mathcal{H}$ if it is closed, densely defined and commutes with its adjoint: $\mathcal{A}^*\mathcal{A}=\mathcal{A}\mathcal{A}^*$. Clearly, every self-adjoint operator is a normal operator.

Given a linear operator $\mathcal{A}$, the *resolvent set* $\rho(\mathcal{A})$ is defined as the set of $\lambda\in\mathbb{C}$ such that the mapping $(\mathcal{A}-\mathrm{Id}\,\lambda)$ is one-to-one and $R_\lambda:=(\mathcal{A}-\mathrm{Id}\,\lambda)^{-1}$ is continuous with $\mathrm{dom}(R_\lambda)=\mathcal{H}$. The operator $R_\lambda:\mathcal{H}\to\mathcal{H}$ is called the *resolvent*. The *spectrum* $\sigma(\mathcal{A})$ of an operator $\mathcal{A}$ is defined as $\sigma(\mathcal{A}):=\mathbb{C}\setminus\rho(\mathcal{A})$. We say that $\lambda\in\sigma(\mathcal{A})$ is an *eigenvalue* of $\mathcal{A}$ if there exists $\psi\in\mathrm{dom}(\mathcal{A})$ such that the *eigenvalue equation* is satisfied

$$\mathcal{A}\,\psi=\lambda\,\psi. \tag{10}$$

A function ψ that solves (10) is called an *eigenfunction* of $\mathcal{A}$ corresponding to λ. The *multiplicity* of an eigenvalue λ is the number of linearly independent eigenfunctions for which equation (10) is satisfied. The spectrum of an operator $\mathcal{A}$ can be decomposed into two disjoint sets called the *discrete* and *essential* [2] spectra: $\sigma(\mathcal{A})=\sigma_d(\mathcal{A})\cup\sigma_e(\mathcal{A})$. For a normal operator $\mathcal{A}$, a number $\lambda\in\mathbb{C}$ belongs to $\sigma_d(\mathcal{A})$ if and only if λ is an isolated point of $\sigma(\mathcal{A})$ and λ is an eigenvalue of finite multiplicity (see Rudin (1973), Theorem 12.29).

A *projection-valued measure* on the measure space $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ is a family of bounded linear operators $\{E(B), B\in\mathcal{B}(\mathbb{C})\}$ in $\mathcal{H}$ that satisfies:

1. $E(\emptyset)=0$ and $E(\mathbb{C})=\mathrm{Id}$.

2. $E(B)$ is an orthogonal projection. That is, $E^2(B)=E(B)$ and $E(B)$ is self-adjoint: $E^*(B)=E(B)$.

[2] The essential spectrum may be further decomposed into the *continuous* spectrum and the *residual* spectrum. It can be shown that the residual spectrum of an ordinary differential operator is empty (see Roach (1982), page 184).

3. $E(A \cap B) = E(A)E(B)$.

4. If $B = \bigcup_{i=1}^{\infty} B_i$ and $B_i \cap B_j = \emptyset$ for $i \neq j$ then $E(B) = \lim_{n\to\infty} \sum_{i=1}^{n} E(B_j)$, where the limit is in the strong operator topology.

5. For every in $f, g \in \mathcal{H}$ the set function $\mu_{f,g}(B) := (f, E(B)g)$ is a complex measure on $\mathcal{B}(\mathbb{C})$.

Theorem 14 (Spectral Representation Theorem)**.** *There is a one-to-one correspondence between normal operators $\mathcal{A}$ and projection-valued measures E on $\mathcal{H}$, the correspondence being given by*

$$\mathcal{A} = \int_{\sigma(\mathcal{A})} \lambda \, E(d\lambda).$$

If $g(\cdot)$ is a Borel function on $\mathbb{C}$ then

$$g(\mathcal{A}) = \int_{\sigma(\mathcal{A})} g(\lambda) E(d\lambda), \qquad \operatorname{dom}(g(\mathcal{A})) = \{f \in \mathcal{H} : \int_{\sigma(\mathcal{A})} |g(\lambda)|^2 \mu_{f,f}(d\lambda) < \infty\}. \tag{11}$$

Proof. See Rudin (1973) Theorems 12.21 and 13.33. ☐

As a practical matter, if $\mathcal{A}$ is an differential operator acting on a Hilbert space $L^2(\mathbb{R}, dy)$, then the operators defined by (11) can be constructed by solving the *proper* and *improper* [3] eigenvalue problems

$$\begin{array}{llll} \text{proper:} & \mathcal{A}\,\psi_n = \phi_n\,\psi_n, & \phi_n \in \sigma_d(\mathcal{A}), & \psi_n \in \mathcal{H}, \\ \text{improper:} & \mathcal{A}\,\psi_\lambda = \phi_\lambda\,\psi_\lambda, & \phi_\lambda \in \sigma_e(\mathcal{A}), & \psi_\lambda \notin \mathcal{H}. \end{array}$$

For the improper eigenvalue problem one extends the domain of $\mathcal{A}$ to include functions all functions ψ for which $\mathcal{A}f$ makes sense and for which the following boundedness conditions are satisfied

$$\lim_{y\to\pm\infty} |\psi(y)|^2 < \infty.$$

After normalizing, the proper and improper eigenfunctions $\mathcal{A}$ satisfy the following orthogonality relations

$$(\psi_n, \psi_m) = \delta_{n,m}, \qquad (\psi_\lambda, \psi_{\lambda'}) = \delta(\lambda - \lambda'), \qquad (\psi_n, \psi_\lambda) = 0.$$

The operator $g(\mathcal{A})$ in (11) is constructed as follows (see Hanson and Yakovlev (2002), section 5.3.2)

$$g(\mathcal{A})f = \sum_{\lambda \in \sigma_d(\mathcal{A})} g(\phi_\lambda)\,(\psi_\lambda, f)\,\psi_\lambda + \int_{\sigma_e(\mathcal{A})} g(\phi_\lambda)\,(\psi_\lambda, f)\,\psi_\lambda d\lambda.$$

It is not always easy to evaluate divergent integrals of the form $(\psi_\lambda, \psi_{\lambda'})$ and verify that they are in fact delta functions $\delta(\lambda - \lambda')$. A method for directly obtaining properly normalised improper eigenfunctions can be found on page 238 of Friedman (1956).

[3]The term "improper" is used because the improper eigenvalues $\lambda \notin \sigma_d(\mathcal{A})$ and the improper eigenfunctions $\psi_\lambda \notin \mathcal{H}$ since $(\psi_\lambda, \psi_\lambda) = \infty$.

B Proof of Theorem 2

Our strategy is to show that $\mathcal{A}^\varepsilon = \mathcal{A}_0 + \varepsilon\,\eta\,\mathcal{A}_1$ generates a semigroup $\mathcal{P}_t^\varepsilon = \exp(t\mathcal{A}^\varepsilon)$. This will guarantee that $u^\varepsilon = \mathcal{P}_t^\varepsilon h(y)$ is an analytic function of ε, which in turn, justifies the use of expansion (3). Throughout this section we will work on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, dy)$. We let $\mathrm{dom}(\mathcal{A}_i) = C_0^\infty(\mathbb{R})$ and we note that $C_0^\infty(\mathbb{R})$ is a dense subset of $\mathcal{H}$. Our analysis begins with a Theorem from Chernoff (1972):

Theorem 15. *Let $\mathcal{A}$ be the generator of a C_0 contraction semigroup $\mathcal{P}_t^0 = \exp(t\mathcal{A})$ on a Banach space. Let $\varepsilon\,\mathcal{B}$ be a dissipative operator with a densely defined adjoint. Assume that the inequality*

$$\|\varepsilon\,\mathcal{B}u\| \le a\,\|u\| + b\,\|\mathcal{A}u\|\,, \qquad\qquad \forall\, u \in \mathrm{dom}(A)$$

holds for some $a \ge 0$ and $b \le 1$ (i.e., the operator $\varepsilon\,\mathcal{B}$ is $\mathcal{A}$-bounded with bound $b \le 1$). Then the closure of $\mathcal{A}^\varepsilon := \mathcal{A} + \varepsilon\,\mathcal{B}$ generates a C_0 contraction semigroup $\mathcal{P}_t^\varepsilon = \exp(t\mathcal{A}^\varepsilon)$.

Remark 16. The operator $\mathcal{A}_0$ is the generator of a C_0 contraction semigroup $\mathcal{P}_t^0 = \exp(t\mathcal{A}_0)$ on $\mathcal{H}$.

To show that $\varepsilon\,\eta\,\mathcal{A}_1$ is dissipative, the following Theorem will be useful:

Theorem 17. *Let $\mathcal{A}$ be a linear operator with domain* $\mathrm{dom}(\mathcal{A}) = C_0^\infty(\mathbb{R})$. *Then $\mathcal{A}$ satisfies the positive maximum principle if and only if*

$$\mathcal{A} = \frac{1}{2}a^2(y)\partial^2 + b(y)\partial + \int \nu(y,dz)\left(e^{z\partial} - 1 - \mathbb{I}_{\{z<R\}}z\partial\right) - c(y), \tag{12}$$

for some $a(x) \ge 0$, $b(x) \in \mathbb{R}$, $c(x) \ge 0$, $R \in [0,\infty]$ and $\nu(y,dz)$ satisfying

$$\int_{\mathbb{R}} \nu(y,dz)\left(1 \wedge z^2\right) < \infty.$$

Operators of the form (12) *are called* Lévy-type operators*.*

Proof. See Theorem 2.12 of Hoh (1998). □

Remark 18. An operator that satisfies the positive maximum principle is dissipative (see Ethier and Kurtz (1986), Lemma 4.2.1 on page 165).

Remark 19. The operator $\varepsilon\,\eta\,\mathcal{A}_1$ is clearly of the form (12). Therefore, $\varepsilon\,\eta\,\mathcal{A}_1$ is dissipative.

Remark 20. The adjoint of $\varepsilon\eta\mathcal{A}_1$, given by $(\varepsilon\eta\mathcal{A}_1)^* = \varepsilon\mathcal{A}_1^*\eta$, has domain $\mathrm{dom}(\varepsilon\mathcal{A}_1^*\eta) = C_0^\infty(\mathbb{R})$, and is therefore densely defined in $\mathcal{H}$.

Since $\mathcal{A}_0$ generates a C_0 semigroup and $\varepsilon\,\eta\,\mathcal{A}_1$ is dissipative and has a densely defined adjoint, we have only to show that $\varepsilon\,\eta\,\mathcal{A}_1$ is $\mathcal{A}_0$-bounded with $b \le 1$.

Proposition 21. *Suppose* $\eta \in C_0^\infty(\mathbb{R})$ *and* $\varepsilon \le \frac{a^2}{\|\eta\|}$ *(which is the condition given in Theorem 2). Then* $\varepsilon\, \eta\, \mathcal{A}_1$ *is* $\mathcal{A}_0$*-bounded with* $a = 0$ *and* $b \le 1$.

Proof. Clearly, for any $u \in \mathrm{dom}(\mathcal{A}_0)$ we have

$$\|\varepsilon\, \eta\, \mathcal{A}_1\, u\| \le \varepsilon\, \|\eta\| \cdot \|\mathcal{A}_1 u\| = \frac{\varepsilon}{a^2}\, \|\eta\| \cdot \|\mathcal{A}_0 u\| \le \|\mathcal{A}_0 u\|\, .$$

□

The proof of Theorem 2 is complete.

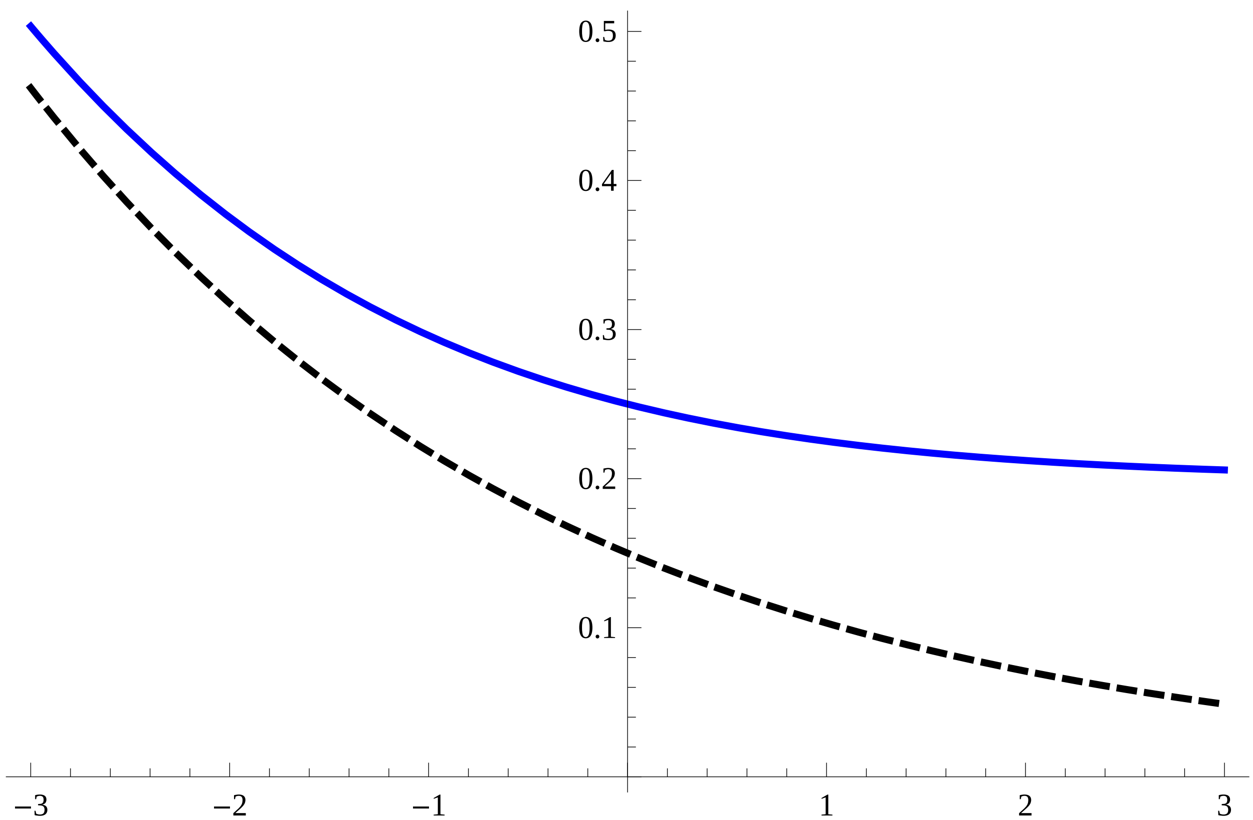

Figure 1: A comparison of the CEV volatility $\sigma(e^y) = \sqrt{\varepsilon} e^{\beta y/2}$ (dashed black) and our "shifted" version of the CEV volatility $\sigma(e^y) = (a^2 + \varepsilon e^{\beta y})^{1/2}$ (solid blue). Notice that the CEV volatility drops to zero as $y \to \infty$ whereas our shifted version stays above a. The following parameters are used in this plot: $a = 0.20$, $\sqrt{\varepsilon} = 0.15$, $\beta = -0.75$.

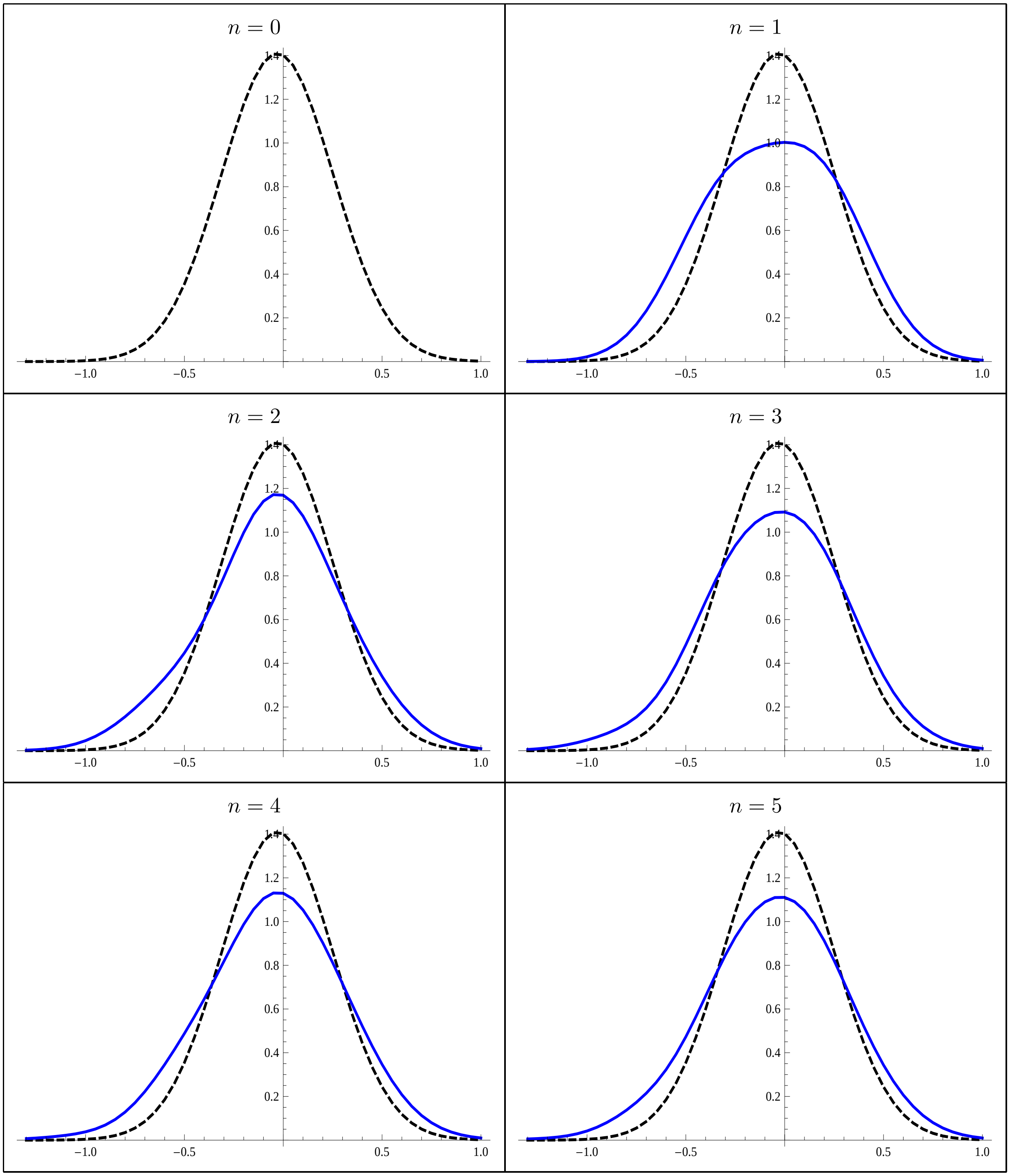

Figure 2: A plot of the approximate transition density $p^{(n)}(t, y; 0)$ (solid blue) for different values of n. For comparison, we also plot $p^{(0)}$ (dashed black). Note that the density of Y_t has a fat tail to the left, which is expected since $\sigma(e^y)$ increase as $y \to -\infty$. The following parameters are used in these plots: $a = 0.20$, $\sqrt{\varepsilon} = 0.15$, $\beta = -0.75$, $t = 2.0$.

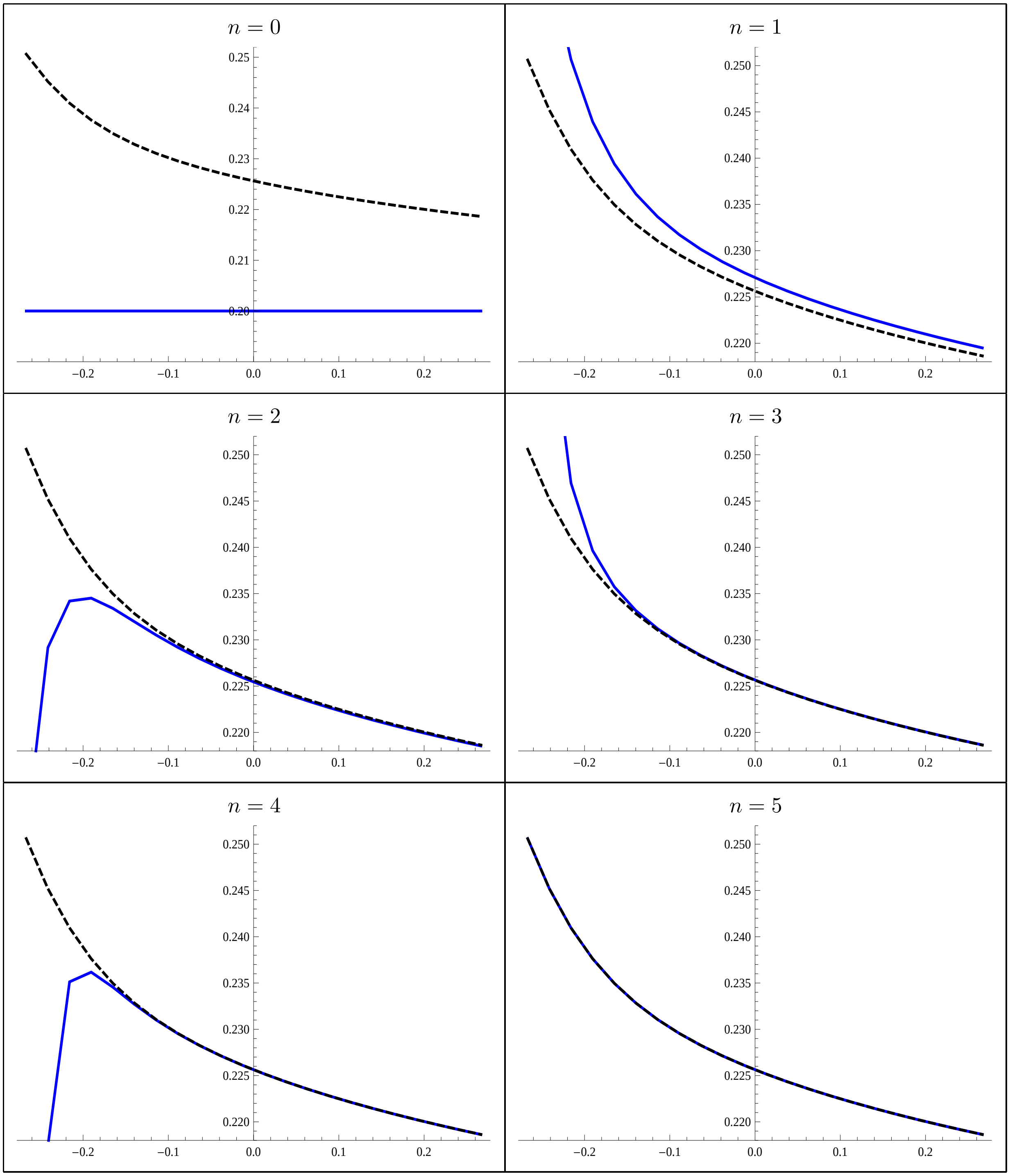

Figure 3: We plot $\sigma^{(n)}$ (solid blue) and σ^{ε} (dashed black) as a function of LMMR. The following parameters are used in these plots: $a = 0.20$, $\sqrt{\varepsilon} = 0.10$, $\beta = -0.75$, $t = 3.0$ $y = -0.01$.