VALUATION AND PARITY FORMULAS FOR EXCHANGE OPTIONS

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Abstract. Valuation and parity formulas for both European-style and American-style exchange options are presented in a general financial model allowing for jumps, possibility of default and bubbles in asset prices. The formulas are given via expectations of auxiliary probabilities using the change-of-numéraire technique. Extensive discussion is provided regarding the way that folklore results such as Merton's no-early-exercise theorem and traditional parity relations have to be altered in this more versatile framework.

Introduction

A multitude of contracts in financial markets can be regarded as options to exchange units of one asset for certain units of another. The first paper to discuss and consider such options in the Black-Scholes-Merton modeling environment is [Mar78]. Building upon the groundbreaking methodology of [BS73] and [Mer73], formulas were provided for the fair value of exchange options when the log-price movement of two no-dividend-paying assets is modeled via (correlated and drifted) Brownian motions. Depending on which of the two assets is chosen as a numéraire in order to denominate wealth, such exchange options can be regarded either of a call or a put type. Under this perspective, and always in the Black-Scholes-Merton model, Merton's no-early-exercise result [Mer73, Theorem 2] can be seen to imply that American-style exchange options have the same value as their European-style counterparts; then, the usual put-call parity translates to a single parity between exchange options of either European or American style.

In recent literature, considerable interest has been placed in financial models where certain "anomalies" exist, the most prominent of which concerns assets which contain bubbles—see, for example, [DS95], [CH05], [PP10], [Hul10], [Ruf11], [KKN12]. (Such bubbles may appear even in the locally riskless bank account, an asset that is traditionally used as a baseline in order to denominate wealth.) When a certain asset contains a bubble, the market allows for arbitrage relative to it; more precisely, there exist free snacks (in the terminology if [LW00]) relative to the asset with the bubble. This last fact prevents the existence of an *equivalent* probability which would render some

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sort of martingale property to wealth processes denominated in units of the asset containing the bubble. Such probability measures are used for valuation of illiquid financial derivative securities; therefore, it would appear that existence of baseline assets containing bubbles presents a hurdle in the development of the theory of financial mathematics. However, a consistent theory of valuation and hedging can still be developed in models where assets with bubbles exist, provided that one utilizes strictly positive local martingale deflators instead of equivalent local martingale measures—the survey article [KF09] is a thorough reference in this respect. Under appropriate assumptions on the underlying stochastic environment which allow for the inference of existence of probability measures in the spirit of Kolmogorov's extension theorem (as explained, for example, in [Par67]) the previous local martingale deflators can still define auxiliary probabilities that can be used for valuation. It should be noted, however, that these valuation probabilities may fail to be even locally (along a sequence of deterministic times converging to infinity) equivalent to the original probability.

It has been argued that several results that are folklore in traditional models fail to hold when bubbles exist in asset prices. Typical examples of such failure include the aforementioned no-early-exercise theorem for American options, as well as certain parity relations. In spite of such claims, it is becoming increasingly understood that an alternative viewpoint concerning such results enables the provision of formulas that are valid in wider-encompassing models. Such viewpoint also facilitates the understanding of the exact attributes of earlier models that resulted in such formulas. The present paper contributes to the existing literature by providing valuation and parity formulas for exchange options via the change-of-numéraire approach in a general modeling environment, allowing for jumps, possible default and bubbles in asset prices. As mentioned previously, in order to provide formulas in terms of expectations under auxiliary valuation probabilities, mild assumptions have to be enforced on the underlying filtered measurable space—canonical examples of such environments are models driven by economic factors, a case that is discussed in detail in the paper. Due to the potential existence of bubbles, the value of American exchange options may be higher than the corresponding value of exchange options of European type; a general formula for the early exercise premium (in terms of explosion probabilities, amongst other elements) is provided that covers all models. The latter discrepancy of American and European option values affects the parity relations as well: several different parity formulas relating European and American exchange option values are provided.

The structure of the paper is as follows. Section 1 presents the underlying financial framework, while Section 2 establishes existence of the valuation probabilities and studies the behavior of ratios of asset prices under these probabilities. In Section 3, several formulas for valuation of European and American exchange options are presented. Finally, Section 4 explores the different parity relations between exchange options of both European and American type.

1. Underlying Framework

1.1. **The set-up.** The underlying financial environment is modeled through a filtered measurable space $(\Omega, \mathbf{F})$, where $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is a right-continuous filtration. In the later development of the paper, the need will arise to infer existence of probabilities arising from local martingale density processes; in order to ensure such existence, the filtration $\mathbf{F}$ is assumed to coincide with the right-continuous augmentation of a continuous-time *standard system*, a notion used in [Föl72] and due to [Par67]. More precisely, it is assumed that $\mathcal{F}_t = \bigcap_{\epsilon > 0} \mathcal{F}^0_{t+\epsilon}$ holds for all $t \in \mathbb{R}_+$, where $(\mathcal{F}^0_t)_{t \in \mathbb{R}_+}$ is a nondecreasing collections of σ-algebras with the following properties:

- For each $t \in \mathbb{R}_+$, $(\Omega, \mathcal{F}^0_t)$ is a standard Borel space, meaning that $\mathcal{F}^0_t$ is σ-isomorphic to the σ-algebra of Borel sets of some complete separable metric space.
- For any $\mathbb{R}_+$-valued nondecreasing sequence $(t_n)_{n \in \mathbb{N}}$ and nonincreasing sequence $(A_n)_{n \in \mathbb{N}}$, where A_n is a *nonempty* atom[1] of $\mathcal{F}^0_{t_n}$ for all $n \in \mathbb{N}$, it holds that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

For both practical and technical reasons, the model will allow for potential "default" of the whole economy. Fix $\zeta \in \mathcal{T}$, where $\mathcal{T}$ will be denoting throughout the set of (possibly infinite-valued) stopping times on $(\Omega, \mathbf{F})$. The following interpretation should be kept in mind: from time ζ onwards, all economic activity ceases and no financial claims are honored.

The concept introduced below will prove useful in localization arguments.

Definition 1.1. A $\mathcal{T}$-valued sequence $(\zeta_n)_{n \in \mathbb{N}}$ will be said to *nicely approximate the default time* if it is nondecreasing, $\zeta_n \leq n$ holds for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \zeta_n(\omega) = \zeta(\omega)$ holds for all $\omega \in \Omega$.

Remark 1.2. Note that $\lim_{n \to \infty} \zeta_n(\omega) = \zeta(\omega)$ in Definition 1.1—as well as the nondecreasing and boundedness property of $(\zeta_n)_{n \in \mathbb{N}}$, in tacit form—is required to hold for *all* $\omega \in \Omega$, and not just in an almost sure sense under some probability (which has not been yet introduced anyway). The requirement $\zeta_n \leq n$ for all $n \in \mathbb{N}$ is hardly a restriction: if a nondecreasing $\mathcal{T}$-valued sequence $(\zeta'_n)_{n \in \mathbb{N}}$ is such that $\lim_{n \to \infty} \zeta'_n(\omega) = \zeta'(\omega)$ holds for all $\omega \in \Omega$, then the sequence $(\zeta_n)_{n \in \mathbb{N}}$ defined via $\zeta_n = \zeta'_n \wedge n$ for all $n \in \mathbb{N}$ nicely approximates the default time in the sense of Definition 1.1.

1.2. **The prototypical example.** The abstract structure of the underlying space mentioned in Subsection 1.1 is valid in a canonical framework, which will be now explained. (For more details regarding the content of the present discussion, the reader is referred to [Mey72] and [Föl72, Example 6.3(2)].) Typically, financial models are build via the introduction of economic factors which affect the movement of asset prices. Mathematically, it is assumed that at each time-point the state of the economy takes value in a topological space E with countable base, which will be further assumed to be locally compact but *not* compact.

[1]Recall that $A \subseteq \Omega$ is called an *atom* of a σ-algebra $\mathcal{F}$ over Ω if $A \in \mathcal{F}$ and the the conditions $B \in \mathcal{F}$ and $B \subseteq A$ imply that either $B = A$ of $B = \emptyset$.

Remark 1.3. Under the previous assumptions on E, it is straightforward to see that there exists a nondecreasing sequence $(E_n)_{n\in\mathbb{N}}$ of open subsets of E such that:

- $\overline{E}_n$ is compact and $\overline{E}_n \subseteq E_{n+1}$ holds for all $n \in \mathbb{N}$.[2]
- $\bigcup_{n\in\mathbb{N}} E_n = E$.

The default time ζ in such framework will correspond to "explosion" of the economy's state. The topological space E is extended via appending an extra point $\triangle$, and the enlarged set $\widetilde{E} := E \cup \{\triangle\}$ is endowed with the one-point (Alexandroff) compactification topology. For any càdlàg (right-continuous with left limits) function $\omega : \mathbb{R}_+ \mapsto \widetilde{E}$, representing a possible path of the time evolution of the economy's state, define the default (or explosion) time[3]

$$\zeta(\omega) := \inf \{ t \in \mathbb{R}_+ \,|\, \omega(t-) = \triangle \text{ or } \omega(t) = \triangle \} . \tag{1.1}$$

The sample space Ω is defined as the set of all càdlàg functions $\omega : \mathbb{R}_+ \mapsto \widetilde{E}$ with the property that $\omega(t) = \triangle$ holds for all $t \geq \zeta(\omega)$. (In words, $\triangle$ becomes an absorbing state for functions in Ω.) The coördinate process $X = (X_t)_{t\in\mathbb{R}_+}$ on Ω, defined via $X_t(\omega) = \omega(t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$, will be modeling the economic factors. Denote by $\mathbf{F}^0 = (\mathcal{F}^0_t)_{t\in\mathbb{R}_+}$ the smallest filtration that makes X an adapted process; then, define $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ via $\mathcal{F}_t = \bigcap_{\epsilon>0} \mathcal{F}^0_{t+\epsilon}$ for $t \in \mathbb{R}_+$. Similar to the argument in [Föl72, Example 6.3(2)], it can be shown that $(\mathcal{F}^0_t)_{t\in\mathbb{R}_+}$ is a standard system. It follows that this framework falls into the set-up of Subsection 1.1. Observe that ζ indeed becomes a stopping time on $(\Omega, \mathbf{F})$.

The specification of Ω as a class of $\widetilde{E}$-valued càdlàg paths allows for a natural identification of practically useful sequences that nicely approximate the default time—their use in a Markovian setting is presented in Subsection 1.4 later on. Choose any nondecreasing sequence $(E_n)_{n\in\mathbb{N}}$ of open subsets of E as in Remark 1.3, and define the sequence $(\zeta_n)_{n\in\mathbb{N}}$ via

$$\zeta_n := \inf \left\{ t \in \mathbb{R}_+ \,|\, X_t \notin \overline{E}_n \right\} \wedge n, \quad \text{for } n \in \mathbb{N}. \tag{1.2}$$

Since $\overline{E}_n$ is closed, ζ_n is a stopping time for all $n \in \mathbb{N}$. It is then straightforward to see that the requirements of Definition 1.1 are satisfied.

Remark 1.4. If a model having factors that change in a continuous fashion is desired, Ω can be chosen to consist of continuous, instead of càdlàg functions. In that case, the default time in (1.1) can be plainly defined via $\zeta(\omega) := \inf \{ t \in \mathbb{R}_+ \,|\, \omega(t) = \triangle \}$ for all continuous functions $\omega : \mathbb{R}_+ \mapsto \widetilde{E}$ and the sequence $(\zeta_n)_{n\in\mathbb{N}}$ in (1.2) is such that $\zeta_n(\omega) < \zeta(\omega)$ holds for all $n \in \mathbb{N}$ and $\omega \in \Omega$.

[2] As usual, $\overline{D}$ denotes the closure of $D \subseteq E$.

[3] The notation $\omega(t-)$ is used for the left limit of ω at $t \in (0, \infty)$; by convention, $\omega(0-)$ is defined to be equal to $\omega(0)$. Furthermore, the infimum of an empty set is defined to be ∞.

1.3. **Assets and stochastic discount factor.** On the filtered measurable space $(\Omega, \mathbf{F})$ satisfying the tenets of Subsection 1.1, we postulate the existence of nonnegative adapted processes S^i for $i \in I$, where I is an arbitrary non-empty index set. Each S^i, $i \in I$, is modeling the price-process of a no-dividend-paying asset in the financial market. To keep in par with the interpretation of ζ as default time of the economy, it shall be assumed that $S^i = 0$ holds on the stochastic interval $[\![\zeta, \infty[\![:= \{(\omega, t) \in \Omega \times \mathbb{R}_+ \,|\, 0 \leq t < \zeta(\omega)\}$ for all $i \in I$.[4] Earlier default for a specific asset is of course also possible in our framework.

The complete probabilistic model for the movement of the asset prices is fulfilled by the introduction of a probability $\mathbb{P}$ on the σ-algebra $\mathcal{F}_\infty := \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t$. The following will be a standing assumption throughout the paper.

Assumption 1.5. For all $i \in I$, $S^i = 0$ holds on $[\![\zeta, \infty[\![$ and S^i_0 is $\mathbb{P}$-a.s. constant and strictly positive. Furthermore, there exist a sequence $(\zeta_n)_{n \in \mathbb{N}}$ which nicely approximates the default time and a nonnegative process Y with $\mathbb{P}[Y_0 = 1] = 1$ such that the process $(Y_{\zeta_n \wedge t} S^i_{\zeta_n \wedge t})_{t \in \mathbb{R}_+}$ is a $\mathbb{P}$-a.s. càdlàg martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $n \in \mathbb{N}$ and $i \in I$.

Before a couple of remarks on Assumption 1.5 are presented, we introduce some notation. The symbol "$\mathbb{E}_\mathbb{P}$" is reserved for expectation with respect to $\mathbb{P}$, with analogous notation used for expectation under other probabilities. Notations of the form $\mathbb{E}[\xi; A]$ for nonnegative $\mathcal{F}_\infty$-measurable random variable ξ and $A \in \mathcal{F}_\infty$ are shorthands for $\mathbb{E}[\xi \mathbb{I}_A]$, where "$\mathbb{I}_A$" denotes the indicator of A.

Remark 1.6. Existence of a process Y with the prescribed properties mentioned in Assumption 1.5 with the additional property that is it is $\mathbb{P}$-a.s. strictly positive is intimately related to market viability; more precisely, to the condition of absence of opportunities of *arbitrage of the first kind*—see [Kar11]. Under an additional *completeness* assumption, the minimal replicating price of a nonnegative $\mathcal{F}_T$-measurable payoff H_T, to be paid at time $T \in \mathcal{T}$, is equal to $\mathbb{E}_\mathbb{P}[Y_T H_T; T < \zeta]$. (The restriction of the expectation on the event $\{T < \zeta\}$ appearing in the valuation formula is due to the interpretation of ζ as default time of the whole economy.) Even if the market is not complete, $\mathbb{E}_\mathbb{P}[Y_T H_T; T < \zeta]$ provides a value for the aforementioned contract that will retain the viability of the market. Such a process Y is commonly referred to as a *stochastic discount factor*, and will be used for the valuation of financial derivatives.

Remark 1.7. The assumption $\zeta_n \leq n$ for all $n \in \mathbb{N}$ for a sequence that nicely approximates the default time in Definition 1.1 has been introduced in order to accommodate the situation where $Y S^i$ is an actual martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $i \in I$, in which case one can choose $\zeta_n = \zeta \wedge n$ for all $n \in \mathbb{N}$. In conjunction with Remark 1.2, note that if a nondecreasing $\mathcal{T}$-valued sequence $(\zeta'_n)_{n \in \mathbb{N}}$ is such that $\lim_{n \to \infty} \zeta'_n(\omega) = \zeta'(\omega)$ holds for all $\omega \in \Omega$ and $(Y_{\zeta'_n \wedge t} S^i_{\zeta'_n \wedge t})_{t \in \mathbb{R}_+}$ is a $\mathbb{P}$-a.s.

[4] This fact is repeated in Assumption 1.5 below.

càdlàg martingale on $(\Omega,\, \mathbf{F},\, \mathbb{P})$ for all $n \in \mathbb{N}$ and $i \in I$, then Assumption 1.5 is valid via use of the sequence $(\zeta_n)_{n \in \mathbb{N}}$ defined via $\zeta_n = \zeta'_n \wedge n$ for all $n \in \mathbb{N}$ that nicely approximates the default time.

1.4. **Markovian continuous factor models.** We discuss here how Assumption 1.5 is valid in a wide range of continuous-time, continuous-path Markovian factor models.

Consider the general framework of Subsection 1.2; in fact, since a continuous-path environment will be utilized, Remark 1.4 is more relevant. Let $m \in \mathbb{N}$ and an open subset E of $\mathbb{R}^m$. Consider functions $a : E \mapsto \mathbb{R}^m$ and $c : E \mapsto \mathbb{S}^m_{++}$, where $\mathbb{S}^m_{++}$ denotes the space of *strictly* positive definite symmetric $m \times m$ matrices. It is assumed that a is locally bounded[5] on E and that c is continuous on E. Under these assumptions, and for a fixed $x_0 \in E$, [Pin95, Chapter 1, Theorem 13.1] implies the existence of a unique solution $\mathbb{P}$ to the martingale problem (with possible explosion) associated with a and c such that $\mathbb{P}\left[X_0 = x_0\right] = 1$. More precisely, with the $\mathcal{T}$-valued sequence $(\zeta_n)_{n \in \mathbb{N}}$ defined as in (1.2), for all $n \in \mathbb{N}$ the process $X - x_0 - \int_0^{\zeta_n \wedge \cdot} a(X_t)\, \mathrm{d}t$ is an m-dimensional martingale on $(\Omega,\, \mathbf{F},\, \mathbb{P})$ with zero initial value and quadratic covariation process given by $\int_0^{\zeta_n \wedge \cdot} c(X_t)\, \mathrm{d}t$.

With $\langle \cdot, \cdot \rangle$ denoting (sometimes, formally) inner product on $\mathbb{R}^m$, let $b^k : E \mapsto \mathbb{R}^m$, $k \in \{1, \ldots, m\}$, be a collection of functions such that $\left\langle b^k,\, b^l \right\rangle = c^{kl}$ holds identically on E for $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, m\}$.[6] Then, there exists an m-dimensional Brownian motion W on $(\Omega,\, \mathbf{F},\, \mathbb{P})$ defined on the stochastic interval $[\![0, \zeta[\![$, such that the formal dynamics

$$\mathrm{d}X^k_t = a^k(X_t)\, \mathrm{d}t + \left\langle b^k(X_t),\, \mathrm{d}W_t \right\rangle, \quad 0 \le t < \zeta, \tag{1.3}$$

are valid for $k \in \{1, \ldots, m\}$. (Note that, under the assumptions previously made, the above dynamics implicitly *define* the process W on $[\![0, \zeta[\![$. Furthermore, W depends on the choice of b only through local orthonormal transformations.)

The m factors will drive the prices of $(d+1)$ financial assets, where $d \in \mathbb{N}$. Let $I = \{0, 1, \ldots, d\}$; the index "0" is reserved for a locally riskless asset, as it typical in the literature. In order to avoid degeneracies in the market, it is assumed that $d \le m$. Consider the "short rate" function $r : E \mapsto \mathbb{R}$, as well as "excess rate of return" functions $\mu^i : E \mapsto \mathbb{R}$, and functions $\sigma^i : E \mapsto \mathbb{R}^m$ for $i \in \{1, \ldots, m\}$. All of the previously-defined functions are assumed locally bounded. For reasons of unifying presentation, set also $\mu^0 : E \mapsto \mathbb{R}$ and $\sigma^0 : E \mapsto \mathbb{R}^m$ to be identically equal to zero. Once again, in order to avoid degeneracy we assume that the collection $\left\{\sigma^i \,|\, i \in \{1, \ldots, d\}\right\}$ is linearly independent on E, which is equivalent to saying that the $(d \times m)$ matrix-valued function $\sigma \equiv (\sigma^{ik})_{i \in \{1, \ldots, d\},\, k \in \{1, \ldots, m\}}$ has (full) rank d on E. Define processes S^i, $i \in I$, satisfying $S^i \equiv 0$ on $[\![\zeta, \infty[\![$ for all $i \in I$, as well as formal dynamics

$$\frac{\mathrm{d}S^i_t}{S^i_t} = \left(r + \mu^i\right)(X_t)\, \mathrm{d}t + \left\langle \sigma^i(X_t),\, \mathrm{d}W_t \right\rangle, \quad 0 \le t < \zeta, \tag{1.4}$$

[5]A function is locally bounded on E is and only if it is bounded on any compact subset of E.

[6]In other words, the $(m \times m)$ matrix-valued function $(b^{kl})_{k \in \{1, \ldots, m\}, l \in \{1, \ldots, m\}}$ is a square root of c.

where $S_0^i > 0$ for all $i \in I$. From the dynamics of the factors and the asset-price processes one obtains $\left(\mathrm{d}S_t^i / S_t^i\right)\left(\mathrm{d}X_t^k\right) = \left\langle \sigma^i,\, b^k \right\rangle (X_t)\,\mathrm{d}t$, $0 \le t < \zeta$, for $i \in I$ and $k \in \{1, \ldots, m\}$; naturally, the functions σ^i, $i \in I$, have to be appropriately chosen in order to reflect the local covariance between the asset-price movement and the driving economic factors.

In order to define the stochastic discount factor, consider a locally bounded function $\theta : E \mapsto \mathbb{R}^m$ such that[7] $\left\langle \sigma^i,\, \theta \right\rangle = \mu^i$ holds on E for all $i \in I$. Define the process Y satisfying $Y_0 = 1$, $Y = 0$ on $[\![\zeta, \infty[\![$, and formal dynamics

$$\frac{\mathrm{d}Y_t}{Y_t} = -r(X_t)\,\mathrm{d}t - \left\langle \theta(X_t),\, \mathrm{d}W_t \right\rangle, \quad 0 \le t < \zeta. \tag{1.5}$$

A straightforward use of the integration-by-parts formula shows that

$$\frac{\mathrm{d}\left(Y_t S_t^i\right)}{Y_t S_t^i} = \left\langle \left(\sigma^i - \theta\right)(X_t),\, \mathrm{d}W_t \right\rangle, \quad 0 \le t < \zeta,$$

for all $i \in I$. In particular, $(Y_{\zeta_n \wedge t} S^i_{\zeta_n \wedge t})_{t \in \mathbb{R}_+}$ is a local martingale on $(\Omega,\, \mathbf{F},\, \mathbb{P})$ for all $i \in I$ and $n \in \mathbb{N}$. Furthermore, the local boundedness assumptions on the involved functions coupled with Novikov's condition [KS91, Subsection 3.5.D] imply that the processes $(Y_{\zeta_n \wedge t} S^i_{\zeta_n \wedge t})_{t \in \mathbb{R}_+}$ for all $i \in I$ and $n \in \mathbb{N}$ are actual martingales on $(\Omega,\, \mathbf{F},\, \mathbb{P})$. Therefore, one obtains the validity of Assumption 1.5 in this extremely versatile setting.

Remark 1.8. In the case where $d = m$, the only process Y with $Y_0 = 1$ and $Y_t = 0$ for $t \ge \zeta$ that is capable to render the local martingale property on $(\Omega,\, \mathbf{F},\, \mathbb{P})$ to the processes $(Y_{\zeta_n \wedge t} S^i_{\zeta_n \wedge t})_{t \in \mathbb{R}_+}$ for all $i \in I$ and $n \in \mathbb{N}$ is the one defined via the dynamics in (1.5) with $\theta = \sigma^{-1}\mu$.

2. Valuation Probabilities and Asset Ratios

2.1. **Valuation probabilities.** As mentioned in Remark 1.6, the process Y of Assumption 1.5 plays the role of a stochastic discount factor in the market. As such, it will be used for valuation of securities: the present (time zero) value a contract that pays an $\mathcal{F}_T$-measurable nonnegative amount H_T at time $T \in \mathcal{T}$ is $\mathbb{E}_{\mathbb{P}}\left[Y_T H_T; T < \zeta\right]$. It is customary to write valuation formulas in terms of expectation under auxiliary valuation probabilities. In order to obtain the latter from the representation in terms of expectations under $\mathbb{P}$ and stochastic discounting, a "baseline" (or "numéraire") asset has to be chosen in order to denominate wealth. Section 3 and Section 4 deal with valuation and parity formulas for exchange options; for this reason, we refrain from choosing a single asset to use as baseline; rather, a family of probabilities $(\mathbb{Q}^i)_{i \in I}$ will be introduced, one for each asset indexed by $i \in I$ being used as a baseline. Care has to be exercised in defining these probabilities, since the candidate "density processes" that have to be used in defining them are

[7] A least one function $\theta : E \mapsto \mathbb{R}^m$ satisfying $\left\langle \sigma^i,\, \theta \right\rangle = \mu^i$ on E for all $i \in I$ exists, since the matrix-valued process σ is assumed to have full rank on E; the additional local boundedness is a mild assumption which will hold given local boundedness conditions on μ and c^{-1}.

in general only *local* martingales on $(\Omega, \mathbf{F}, \mathbb{P})$. However, the structure of the filtered probability space described in Subsection 1.1 allows for such construction under Assumption 1.5. Full details are given in the statement and proof of Theorem 2.1 below. Note that results of similar nature have appeared previously—see, for example, [DS95], [PP10], [Ruf11] and [KKN12]. However, since the present setting is more general (involving asset prices with jumps and possibility of individual defaults before time ζ), and since Theorem 2.1 is used extensively in other results of the paper, a complete treatment will be provided. Before the statement of Theorem 2.1, recall that $\mathcal{F}_{\tau-}$ for $\tau \in \mathcal{T}$ represents the σ-algebra over Ω generated by $\mathcal{F}_0$ and the collection of all events of the form $A \cap \{s < \tau\}$ where $s \in \mathbb{R}_+$ and $A \in \mathcal{F}_s$.

Theorem 2.1. *Under Assumption 1.5, for each $i \in I$ there exists a unique probability $\mathbb{Q}^i$ on $\mathcal{F}_{\zeta-}$ such that the following property is valid: for any nonnegative optional process H on $(\Omega, \mathbf{F})$,*

$$\mathbb{E}_{\mathbb{P}}\left[Y_T H_T S^i_T; T < \zeta\right] = S^i_0 \mathbb{E}_{\mathbb{Q}^i}\left[H_T; T < \zeta\right] \quad \text{holds for all } T \in \mathcal{T}. \tag{2.1}$$

Furthermore, the following properties are true:

- *for all $T \in \mathcal{T}$, $\mathbb{Q}^i\left[S^i_T = 0,\, T < \zeta\right] = 0$ holds;*
- *$\mathbb{Q}^i\left[\zeta_n < \zeta,\ \text{for all } n \in \mathbb{N}\right] = 1$.*

Proof. In the course of the proof, fix $i \in I$.

In view of Assumption 1.5, the process $L^i := Y S^i / S^i_0$ is such that $\mathbb{P}\left[L^i_{\zeta_n} \geq 0\right] = 1$ and $\mathbb{E}_{\mathbb{P}}\left[L^i_{\zeta_n}\right] = 1$ hold for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$, there exists a probability $\widetilde{\mathbb{Q}}^i_n$ on $\mathcal{F}_\infty$ such that $L^i_{\zeta_n}$ is the Radon-Nikodým derivative of $\widetilde{\mathbb{Q}}^i_n$ with respect to $\mathbb{P}$. For each $n \in \mathbb{N}$, let $\mathbb{Q}^i_n$ denote the restriction of $\widetilde{\mathbb{Q}}^i_n$ on the σ-algebra $\mathcal{F}_{\zeta_n -}$. Note that $\mathbb{Q}^i_n|_{\mathcal{F}_0} = \mathbb{P}|_{\mathcal{F}_0}$ holds for all $n \in \mathbb{N}$.

For the time being, assume that *some* probability $\mathbb{Q}^i$ on $\mathcal{F}_{\zeta-}$ which satisfies (2.1) for any nonnegative optional process H on $(\Omega, \mathbf{F})$ exists. For $A \in \mathcal{F}_0$, let $H = \mathbb{I}_A \mathbb{I}_{[\![0,0]\!]}$; using (2.1) with $T = 0$ and the facts that S^i_0 is $\mathbb{P}$-a.s. a strictly positive constant (which in particular implies that $\mathbb{P}\left[\zeta > 0\right] = 1$, in view of Assumption 1.5) and $\mathbb{P}\left[Y_0 = 1\right] = 1$, we obtain $\mathbb{P}\left[A\right] = \mathbb{Q}^i\left[A \cap \{\zeta > 0\}\right]$. Since $\mathbb{Q}^i$ is a probability, it follows that $\mathbb{Q}^i$ should coincide with $\mathbb{P}$ on $\mathcal{F}_0$. Furthermore, for $n \in \mathbb{N}$, $s \in \mathbb{R}_+$ and $A \in \mathcal{F}_s$, (2.1) combined with the fact that $\{s < \zeta_n\} \subseteq \{s < \zeta\}$ gives

$$\mathbb{Q}^i\left[A \cap \{s < \zeta_n\}\right] = \mathbb{E}_{\mathbb{P}}\left[L^i_s \mathbb{I}_A; s < \zeta_n\right] = \mathbb{E}_{\mathbb{P}}\left[L^i_{\zeta_n} \mathbb{I}_A; s < \zeta_n\right],$$

where the last equality follows from the fact that $(L^i_{\zeta_n \wedge t})_{t \in \mathbb{R}_+}$ is a uniformly integrable martingale on $(\Omega, \mathbf{F}, \mathbb{P})$. Since $\mathcal{F}_{\zeta_n -}$ is generated by $\mathcal{F}_0$ and the class of sets $A \cap \{s < \zeta_n\}$ for all $s \in \mathbb{R}_+$ and $A \in \mathcal{F}_s$, it follows that the restriction of $\mathbb{Q}^i$ (if the latter probability can be defined) on $\mathcal{F}_{\zeta_n -}$ coincides with $\mathbb{Q}^i_n$ for all $n \in \mathbb{N}$. The previous discussion implies that the sequence $(\mathbb{Q}^i_n)_{n \in \mathbb{N}}$ will have to be utilized is order to *construct* $\mathbb{Q}^i$. This is done in the next paragraph.

Note the following consistency property: whenever $\mathbb{N} \ni n \leq m \in \mathbb{N}$, $\mathbb{Q}^i_m|_{\mathcal{F}_{\zeta_n -}} = \mathbb{Q}^i_n$ holds. Therefore, one can indeed define a finitely-additive probability $\mathbb{Q}^i$ on the algebra $\bigcup_{n \in \mathbb{N}} \mathcal{F}_{\zeta_n -}$ with the property that $\mathbb{Q}^i|_{\mathcal{F}_{\zeta_n -}} = \mathbb{Q}^i_n$ holds for all $n \in \mathbb{N}$. As the sequence of σ-algebras $(\mathcal{F}_{\zeta_n -})_{n \in \mathbb{N}}$ is a discrete-time standard system [Föl72, Remark 6.1], it follows from [Par67, Theorem V.4.1] that $\mathbb{Q}^i$ is countably additive on $\bigcup_{n \in \mathbb{N}} \mathcal{F}_{\zeta_n -}$; therefore, it can be extended uniquely into a probability on $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{\zeta_n -}$, the σ-algebra generated by $\bigcup_{n \in \mathbb{N}} \mathcal{F}_{\zeta_n -}$. In view of Definition 1.1 and [HWY92, Theorem III.3.4(10)], it holds that $\bigvee_{n \in \mathbb{N}} \mathcal{F}_{\zeta_n -} = \mathcal{F}_{\zeta -}$. The previous discussion shows that, under the prescribed properties of Theorem 2.1, a unique candidate for $\mathbb{Q}^i$ exists on $\mathcal{F}_{\zeta -}$. However, the properties claimed by Theorem 2.1 have yet to be established for $\mathbb{Q}^i$. (The validity of one property was rather assumed in order to show how $\mathbb{Q}^i$ should be defined.) The properties will be discussed in the next couple of paragraphs.

Fix $T \in \mathcal{T}$ and a nonnegative optional process H on $(\Omega, \mathbf{F})$. Since H_T is $\mathcal{F}_T$-measurable, for all $n \in \mathbb{N}$ the random variable $H_T \mathbb{I}_{\{T < \zeta_n\}}$ is $\mathcal{F}_{\zeta_n -}$-measurable. Therefore, $\mathbb{E}_{\mathbb{P}}\left[Y_T S^i_T H_T; T < \zeta_n\right] = S^i_0 \mathbb{E}_{\mathbb{Q}^i_n}\left[H_T; T < \zeta_n\right] = S^i_0 \mathbb{E}_{\mathbb{Q}^i}\left[H_T; T < \zeta_n\right]$ holds for all $n \in \mathbb{N}$. Taking limits as n goes to infinity and using Assumption 1.5, (2.1) readily follows. Using (2.1) with $H = \mathbb{I}_{\{S^i = 0\}}$ gives that $\mathbb{Q}^i\left[S^i_T = 0, \, T < \zeta\right] = 0$.

It remains to show that $\mathbb{Q}^i\left[\zeta_n < \zeta\right] = 1$ holds for all $n \in \mathbb{N}$, which is equivalent to the equality $\mathbb{Q}^i\left[\zeta_n < \zeta, \text{ for all } n \in \mathbb{N}\right] = 1$. Fix $n \in \mathbb{N}$. For $m \in \mathbb{N}$ with $n \leq m$, note that $\{\zeta_n = \zeta_m\} = \Omega \setminus \{\zeta_n < \zeta_m\} \in \mathcal{F}_{\zeta_m -}$ holds; therefore, $\mathbb{Q}^i\left[\zeta_n = \zeta_m\right] = \mathbb{E}_{\mathbb{P}}\left[L_{\zeta_m} \mathbb{I}_{\{\zeta_n = \zeta_m\}}\right] = \mathbb{E}_{\mathbb{P}}\left[L_{\zeta_n} \mathbb{I}_{\{\zeta_n = \zeta_m\}}\right]$ holds, where the last equality follows from the fact that $(L_{\zeta_m \wedge t})_{t \in \mathbb{R}_+}$ is a uniformly integrable martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ and $\{\zeta_n = \zeta_m\} \in \mathcal{F}_{\zeta_n}$. It follows that $\mathbb{Q}^i\left[\zeta_n = \zeta\right] = \lim_{m \to \infty} \mathbb{Q}^i\left[\zeta_n = \zeta_m\right] = \lim_{m \to \infty} \mathbb{E}_{\mathbb{P}}\left[L_{\zeta_n} \mathbb{I}_{\{\zeta_n = \zeta_m\}}\right] = \mathbb{E}_{\mathbb{P}}\left[L_{\zeta_n} \mathbb{I}_{\{\zeta_n = \zeta\}}\right] = \mathbb{E}_{\mathbb{P}}\left[L_{\zeta} \mathbb{I}_{\{\zeta_n = \zeta\}}\right]$ Since $\zeta_n \leq n$ and $L^i_\zeta = Y_\zeta S^i_\zeta / S^i_0 = 0$ holds on $\{\zeta < \infty\}$, we obtain $\mathbb{Q}^i\left[\zeta_n = \zeta\right] = 0$, which completes the proof. □

Remark 2.2. Assumption 1.5 and a straightforward application of the conditional version of Fatou's lemma implies that $Y S^i$ is a (nonnegative) supermartingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $i \in I$. Using $H \equiv 1$ in (2.1) and taking $T \in \mathcal{T}$ to be equal to $t \in \mathbb{R}_+$, it follows that $S^i_0 \mathbb{Q}^i\left[t < \zeta\right] = \mathbb{E}_{\mathbb{P}}\left[Y_t S^i_t; t < \zeta\right] = \mathbb{E}_{\mathbb{P}}\left[Y_t S^i_t\right]$ holds for all $t \in \mathbb{R}_+$ and $i \in I$, where the last equation follows from the fact that $S^i_t = 0$ holds on $\{t < \zeta\}$. It then follows in a straightforward way that $\mathbb{Q}^i\left[\zeta = \infty\right] = 1$ holds for some $i \in I$ if and only if the process $(Y_t S^i_t)_{t \in \mathbb{R}_+}$ is an actual martingale on $(\Omega, \mathbf{F}, \mathbb{P})$.

Remark 2.3. In the notation used in the proof of Theorem 2.1, $L^i_{\zeta_n} = L^i_{\zeta_n} \mathbb{I}_{\{\zeta_n < \zeta\}}$ holds for all $i \in I$ and $n \in \mathbb{N}$. Therefore, $L^i_{\zeta_n}$ is the density of $\mathbb{Q}^i$ with respect to $\mathbb{P}$ on $\mathcal{F}_{\zeta_n} \cap \mathcal{F}_{\zeta -}$ for all $i \in I$ and $n \in \mathbb{N}$. This fact can help in obtaining the behavior of processes under $\mathbb{Q}^i$ for $i \in I$; see Example 2.6 below for an illustration.

Although $\mathbb{Q}^i$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_{\zeta_n} \cap \mathcal{F}_{\zeta -}$ for all $i \in I$ and $n \in \mathbb{N}$, it should be noted that there is no general relationship between $\mathbb{Q}^i$ and $\mathbb{P}$ on $\mathcal{F}_{\zeta -}$.

Remark 2.4. In the setting of Subsection 1.2, there is a natural σ-isomorphism between the σ-algebras $\mathcal{F}_{\zeta -}$ and $\mathcal{F}_\infty$. In this case, and under Assumption 1.5, it follows that the probability $\mathbb{Q}^i$ in Theorem 2.1 is uniquely defined on $\mathcal{F}_\infty$ for all $i \in I$.

Remark 2.5. In general, ζ is *not* a predictable time on $(\Omega, \mathbf{F})$. (An exception is the situation described in Remark 1.4.) However, as Theorem 2.1 implies, the $\mathcal{T}$-valued sequence $(\zeta_n)_{n\in\mathbb{N}}$ $\mathbb{Q}^i$-a.s. announces the default time ζ for all $i \in I$.

We proceed with an example that illustrates Theorem 2.1.

Example 2.6. Retain the framework and all notation of Subsection 1.4. Let Y satisfy the formal dynamics in (1.5), and fix $j \in I$. A straightforward use of Girsanov's theorem on each stochastic interval $[\![0, \zeta_n]\!]$, $n \in \mathbb{N}$, shows that the coördinate factor process X under $\mathbb{Q}^j$ solves the martingale problem associated with the diffusion function c and drift function $a_{\mathbb{Q}^j} : E \mapsto \mathbb{R}^m$, where $a^k_{\mathbb{Q}^j} = a^k + \langle b^k, \sigma^j - \theta\rangle$ holds for all $k \in \{1, \dots, m\}$. More precisely, for $k \in \{1, \dots, m\}$ it holds that

$$\mathrm{d}X^k_t = \Big(a^k + \langle b^k, \sigma^j - \theta\rangle\Big)(X_t)\,\mathrm{d}t + \langle b^k(X_t),\, \mathrm{d}W^{\mathbb{Q}^j}_t\rangle, \quad 0 \le t < \zeta,$$

where $W^{\mathbb{Q}^j}$ is an m-dimensional Brownian motion (defined on $[\![0, \zeta[\![$) on $(\Omega, \mathbf{F}, \mathbb{Q}^j)$. In fact, comparing the above formal factor dynamics with (1.3), the relationship between $W^{\mathbb{Q}^j}$ and W on $[\![0, \zeta[\![$ becomes immediate. Then, recalling (1.4), we obtain the formal dynamics

$$\frac{\mathrm{d}S^i_t}{S^i_t} = \big(r + \langle \sigma^i,\, \sigma^j\rangle\big)(X_t)\,\mathrm{d}t + \langle \sigma^i(X_t),\, \mathrm{d}W^{\mathbb{Q}^j}_t\rangle, \quad 0 \le t < \zeta,$$

for $i \in I$. The choice of the "risk-premium" function θ affects the stochastic behavior of each S^i, $i \in I$, on $(\Omega, \mathbf{F}, \mathbb{Q}^j)$ indirectly through the dynamics of X.

In the setting of this example, note that

$$\frac{\mathrm{d}\big(S^i_t/S^j_t\big)}{S^i_t/S^j_t} = \big\langle (\sigma^i - \sigma^j)(X_t),\, \mathrm{d}W^{\mathbb{Q}^j}_t\big\rangle, \quad 0 \le t < \zeta,$$

holds for all $i \in I$, which implies that the processes S^i, when denominated in units of the asset $j \in I$, become local martingales on $(\Omega, \mathbf{F}, \mathbb{Q}^j)$ and the stochastic interval $[\![0, \zeta[\![$. The behavior of asset-price ratios in a general setting is taken up in Subsection 2.2 below.

2.2. **Asset-price ratio processes.** Define the family of nonnegative processes

$$R^{ij} := \left(\frac{S^i}{S^j}\right)\mathbb{I}_{\{S^j>0\}}, \quad i \in I \text{ and } j \in I. \tag{2.2}$$

In words, R^{ij} represents the asset-price process $i \in I$ denominated in units of the asset-price process $j \in I$, as long as the latter asset has not defaulted . In the setting of Theorem 2.1, for any $i \in I$, $j \in I$, and nonnegative optional process H on $(\Omega, \mathbf{F})$ and any $T \in \mathcal{T}$, it holds that

$$S^j_0 \mathbb{E}_{\mathbb{Q}^j}\left[R^{ij}_T H_T; T < \zeta\right] = \mathbb{E}_{\mathbb{P}}\left[S^i_T H_T; S^j_T > 0,\, T < \zeta\right] = S^i_0 \mathbb{E}_{\mathbb{Q}^i}\left[H_T; S^j_T > 0,\, T < \zeta\right]. \tag{2.3}$$

The next is a result in the spirit of the supermartingale optional sampling theorem.

Proposition 2.7. *Let $\sigma \in \mathcal{T}$ and $\tau \in \mathcal{T}$ with $\sigma \le \tau$. Under Assumption 1.5,*

$$\mathbb{E}_{\mathbb{Q}^j}\big[R^{ij}_\tau; \tau < \zeta\big] \le \mathbb{E}_{\mathbb{Q}^j}\big[R^{ij}_\sigma; \sigma < \zeta\big] \quad \textit{holds for all } i \in I \textit{ and } j \in I. \tag{2.4}$$

Proof. In the course of the proof, fix $i \in I$ and $j \in I$, as well as $\sigma \in \mathcal{T}$ and $\tau \in \mathcal{T}$ with $\sigma \le \tau$. The first equality in (2.3) applied twice gives $S^j_0 \mathbb{E}_{\mathbb{Q}^j}\big[R^{ij}_\sigma; \sigma < \zeta\big] = \mathbb{E}_{\mathbb{P}}\big[Y_\sigma S^i_\sigma; S^j_\sigma > 0,\, \sigma < \zeta\big]$ and $S^j_0 \mathbb{E}_{\mathbb{Q}^j}\big[R^{ij}_\tau; \tau < \zeta\big] = \mathbb{E}_{\mathbb{P}}\big[Y_\tau S^i_\tau; S^j_\tau > 0,\, \tau < \zeta\big]$. Therefore, the inequality (2.4) is equivalent to $\mathbb{E}_{\mathbb{P}}\big[Y_\tau S^i_\tau; S^j_\tau > 0,\, \tau < \zeta\big] \le \mathbb{E}_{\mathbb{P}}\big[Y_\sigma S^i_\sigma; S^j_\sigma > 0,\, \sigma < \zeta\big]$. Recall from Remark 2.2 that, under Assumption 1.5, YS^j is a nonnegative supermartingale on $(\Omega,\, \mathbf{F},\, \mathbb{P})$; therefore, it follows that $\mathbb{P}\big[S^j_\sigma = 0,\, Y_\tau > 0,\, S^j_\tau > 0,\, \tau < \zeta\big] = 0$. The last fact combined with $\{\tau < \zeta\} \subseteq \{\sigma < \zeta\}$ implies the string of inequalities $Y_\tau \mathbb{I}_{\{S^j_\tau > 0,\, \tau < \zeta\}} \le Y_\tau \mathbb{I}_{\{S^j_\sigma > 0,\, \tau < \zeta\}} \le Y_\tau \mathbb{I}_{\{S^j_\sigma > 0,\, \sigma < \zeta\}}$, holding modulo $\mathbb{P}$. In turn, the last fact implies the first inequality in

$$\mathbb{E}_{\mathbb{P}}\big[Y_\tau S^i_\tau; S^j_\tau > 0,\, \tau < \zeta\big] \le \mathbb{E}_{\mathbb{P}}\big[Y_\tau S^i_\tau; S^j_\sigma > 0,\, \sigma < \zeta\big] \le \mathbb{E}_{\mathbb{P}}\big[Y_\sigma S^i_\sigma; S^j_\sigma > 0,\, \sigma < \zeta\big],$$

where the second equality follows from the fact that the process YS^i is a supermartingale on $(\Omega,\, \mathbf{F},\, \mathbb{P})$. The proof is complete. □

In Section 3, we shall make use of the family of random variables

$$\rho^{ij} := \liminf_{n\to\infty} R^{ij}_{\zeta_n}, \quad i \in I \text{ and } j \in I. \tag{2.5}$$

By Definition 1.1 and Assumption 1.5, $R^{ij}_{\zeta_n} = R^{ij}_{\zeta_n} \mathbb{I}_{\{\zeta_n < \zeta\}}$ holds; in particular, $R^{ij}_{\zeta_n}$ is $(\mathcal{F}_{\zeta_n} \cap \mathcal{F}_{\zeta-})$-measurable for all $n \in \mathbb{N}$, $i \in I$ and $j \in I$. Since $\mathbb{Q}^j\left[\zeta_n < \zeta, \text{ for all } n \in \mathbb{N}\right] = 1$ holds for all $j \in I$ under Assumption 1.5 by Theorem 2.1, it follows in a straightforward way from Proposition 2.7 that $(R^{ij}_{\zeta_n})_{n\in\mathbb{N}}$ is a nonnegative supermartingale on the discrete-time stochastic basis $\big(\Omega,\, (\mathcal{F}_{\zeta_n} \cap \mathcal{F}_{\zeta-})_{n\in\mathbb{N}},\, \mathbb{Q}^j\big)$ for all $i \in I$ and $j \in I$. Therefore, in view of the nonnegative supermartingale convergence theorem, for all $i \in I$ and $j \in I$, on $(\Omega,\, \mathbf{F},\, \mathbb{Q}^j)$ the $\mathcal{F}_{\zeta-}$-measurable random variable ρ^{ij} is $\mathbb{R}_+$-valued and the "$\liminf$" in (2.5) is an actual limit.

3. Valuation Formulas for Exchange Options

3.1. **Valuation formulas for European-style exchange options.**

Given the stochastic discount factor Y of Assumption 1.5, define the value of a European option to exchange asset $i \in I$ for asset $j \in I$ at time $T \in \mathcal{T}$ as

$$\mathsf{EX}^{ij}(T) := \mathbb{E}_{\mathbb{P}}\left[Y_T (S^j_T - S^i_T)_+; T < \zeta\right]. \tag{3.1}$$

In view of Theorem 2.1, under Assumption 1.5 note the validity of the relationships $\mathsf{EX}^{ij}(T) \le \mathbb{E}_{\mathbb{P}}\big[Y_T S^j_T; T < \zeta\big] = S^j_0 \mathbb{Q}^j\left[T < \zeta\right] \le S^j_0$ for all $i \in I$, $j \in I$ and $T \in \mathcal{T}$.

Remark 3.1. Under Assumption 1.5, $S^i_T = 0$ holds on $\{\zeta \le T\}$ for all $i \in I$. It follows that the indicator of the event $\{T < \zeta\}$ inside the expectation in (3.1) may be omitted. The same holds for several equations that will appear below (although not all); we choose to keep the indicator in order to explicitly reinforce the convention that no claims are honored from time ζ onwards.

The next result gives several representations for the value of European-style exchange options. Recall from (2.2) the definition of the collection of processes R^{ij} for $i \in I$ and $j \in I$.

Proposition 3.2. *For all $i \in I$, $j \in I$ and $T \in \mathcal{T}$, the following formulas are valid:*

$$\begin{aligned} \mathsf{EX}^{ij}(T) &= S^j_0 \mathbb{Q}^j \left[S^i_T < S^j_T,\ T < \zeta \right] - S^i_0 \mathbb{Q}^i \left[S^i_T < S^j_T,\ T < \zeta \right] \\ &= S^j_0 \mathbb{Q}^j \left[S^i_T \leq S^j_T,\ T < \zeta \right] - S^i_0 \mathbb{Q}^i \left[S^i_T \leq S^j_T,\ T < \zeta \right] \\ &= S^j_0 \mathbb{E}_{\mathbb{Q}^j} \left[\left(1 - R^{ij}_T\right)_+ ; T < \zeta \right] \\ &= S^i_0 \mathbb{E}_{\mathbb{Q}^i} \left[\left(R^{ji}_T - 1\right)_+ ; T < \zeta \right] + S^j_0 \mathbb{Q}^j \left[S^i_T = 0,\ T < \zeta \right]. \end{aligned}$$

Proof. Fix $i \in I$ and $j \in I$. Since $(S^j - S^i)_+ = S^j \mathbb{I}_{\{S^i < S^j\}} - S^i \mathbb{I}_{\{S^i < S^j\}} = S^j \mathbb{I}_{\{S^i \leq S^j\}} - S^i \mathbb{I}_{\{S^i \leq S^j\}}$, the first two equalities follow in a straightforward way from (2.1). Continuing note that $(S^j - S^i)_+ = (S^j - S^i)_+ \mathbb{I}_{\{S^j > 0\}} = S^j (1 - R^{ij})_+$ holds. Using $H = (1 - R^{ij})_+$ in (2.1) (with j replacing i there), the third equality follows immediately. Furthermore, upon noting that $(S^j - S^i)_+ = (S^j - S^i)_+ \mathbb{I}_{\{S^i > 0\}} + S^j \mathbb{I}_{\{S^i = 0\}} = S^i (R^{ji} - 1)_+ + S^j \mathbb{I}_{\{S^i = 0\}}$ and using (2.1) twice, once with $H = (R^{ji} - 1)_+$ and another time with $H = \mathbb{I}_{\{S^i = 0\}}$ (and j replacing i there), the last equality follows. □

Remark 3.3. Fix $j \in I$ and suppose that $\mathbb{Q}^j [\zeta < \infty] = 0$ holds, which in view of Remark 2.2 is equivalent to the process $(Y_t S^j_t)_{t \in \mathbb{R}_+}$ being an actual martingale on $(\Omega, \mathbf{F}, \mathbb{P})$. In that case, since $\mathbb{Q}^j [T < \zeta] = 1$ holds for all $T \in \mathcal{T}$ with $T < \infty$, a combination of Proposition 2.7 and Proposition 3.2, the convexity of the function $\mathbb{R} \ni x \mapsto x_+ \in \mathbb{R}_+$ and Jensen's inequality give $\mathsf{EX}^{ij}(\sigma) \leq \mathsf{EX}^{ij}(\tau)$ whenever $\sigma \in \mathcal{T}$ and $\tau \in \mathcal{T}$ are such that $\sigma \leq \tau < \infty$ holds. It follows that the value $\mathsf{EX}^{ij}(T)$ of the European exchange option is non-decreasing for finite maturities $T \in \mathcal{T}$.

In contrast to the situation where ζ is $\mathbb{Q}^j$-a.s. infinite for some $j \in I$, when $\mathbb{Q}^j [\zeta < \infty] > 0$ the previous monotonicity property need not hold, due to the non-triviality of the indicator of the event $\{T < \zeta\}$ in the expression $\mathsf{EX}^{ij}(T) = S^j_0 \mathbb{E}_{\mathbb{Q}^j} \left[\left(1 - R^{ij}_T\right)_+ ; T < \zeta \right]$. The latter event is nonincreasing in T and may result in reversal of the inequality $\mathsf{EX}^{ij}(\sigma) \leq \mathsf{EX}^{ij}(\tau)$ whenever $\sigma \in \mathcal{T}$ and $\tau \in \mathcal{T}$ are such that $\sigma \leq \tau < \infty$ holds. In fact, an example presented in [PP10] shows a case where the function $\mathbb{R}_+ \ni T \mapsto \mathsf{EX}^{ij}(T)$ is initially strictly increasing and then strictly decreasing.

Remark 3.4. The representation $\mathsf{EX}^{ij}(T) = S^j_0 \mathbb{E}_{\mathbb{Q}^j} \left[\left(1 - R^{ij}_T\right)_+ ; T < \zeta \right]$ gives the value of the exchange option in terms of a put option on the asset $i \in I$ by considering asset $j \in I$ as a numéraire. Similarly, the expression $\mathsf{EX}^{ij}(T) = S^i_0 \mathbb{E}_{\mathbb{Q}^i} \left[\left(R^{ji}_T - 1\right)_+ ; T < \zeta \right] + S^j_0 \mathbb{Q}^j \left[S^i_T = 0,\ T < \zeta \right]$ follows from the use of asset $i \in I$ as a numéraire, in terms of a call option on asset $j \in I$. Note however, an asymmetry between the two representations, since the equality $\mathsf{EX}^{ij}(T) = S^i_0 \mathbb{E}_{\mathbb{Q}^i} \left[\left(R^{ji}_T - 1\right)_+ ; T < \zeta \right]$ is actually valid only if $\mathbb{Q}^j \left[S^i_T = 0,\ T < \zeta \right] = 0$ for $T \in \mathcal{T}$.

3.2. **Valuation formulas for American-style exchange options.** For $T \in \mathcal{T}$ define $\mathcal{T}_{[0,T]}$ as the class of all $\tau \in \mathcal{T}$ such that $0 \leq \tau \leq T$ holds. Given the process Y of Assumption 1.5, the value of an American option to exchange asset $i \in I$ for asset $j \in I$ up to time T is defined to be

$$\mathsf{AX}^{ij}(T) := \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}_{\mathbb{P}}\left[Y_\tau (S^j_\tau - S^i_\tau)_+ ; \tau < \zeta\right] = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathsf{EX}^{ij}(\tau). \tag{3.2}$$

The inequalities $\mathsf{EX}^{ij}(T) \leq \mathsf{AX}^{ij}(T) \leq S^j_0$ hold for all $i \in I$, $j \in I$ and $T \in \mathcal{T}$. Proposition 3.5 provides, *inter alia*, a formula for the early exercise premium $\mathsf{AX}^{ij}(T) - \mathsf{EX}^{ij}(T)$ of the American versus the European option. Recall from (2.5) the random variables ρ^{ij} for $i \in I$ and $j \in I$.

Proposition 3.5. *Fix $i \in I$, $j \in I$ and $T \in \mathcal{T}$. Under Assumption 1.5, the following are true:*

(1) The sequence $\left(\mathsf{EX}^{ij}(T \wedge \zeta_n)\right)_{n \in \mathbb{N}}$ is nondecreasing. Furthermore,

$$\mathsf{AX}^{ij}(T) = \lim_{n \to \infty} \mathsf{EX}^{ij}(T \wedge \zeta_n). \tag{3.3}$$

(2) The early exercise premium is given by

$$\mathsf{AX}^{ij}(T) - \mathsf{EX}^{ij}(T) = S^j_0 \mathbb{E}_{\mathbb{Q}^j}\left[\left(1 - \rho^{ij}\right)_+ ; \zeta \leq T\right]. \tag{3.4}$$

Proof. In the course of the proof, fix $i \in I$, $j \in I$ and $T \in \mathbb{T}$.
(1). Let $\tau \in \mathcal{T}_{[0,T]}$. By Proposition 3.2, and since $\mathbb{Q}^j\left[\zeta_n < \zeta\right] = 1$ holds by Theorem 2.1,

$$\mathsf{EX}^{ij}(\tau \wedge \zeta_n) = S^j_0 \mathbb{E}_{\mathbb{Q}^j}\left[\left(1 - R^{ij}_{\tau \wedge \zeta_n}\right)_+ ; \tau \wedge \zeta_n < \zeta\right] = S^j_0 \mathbb{E}_{\mathbb{Q}^j}\left[\left(1 - R^{ij}_{\tau \wedge \zeta_n}\right)_+\right].$$

The fact $\mathbb{Q}^j\left[\zeta_n < \zeta\right] = 1$ and Proposition 2.7 imply the inequality $\mathbb{E}_{\mathbb{Q}^j}\left[R^{ij}_{\tau \wedge \zeta_m}\right] \leq \mathbb{E}_{\mathbb{Q}^j}\left[R^{ij}_{\tau \wedge \zeta_n}\right]$ whenever $\mathbb{N} \ni n \leq m \in \mathbb{N}$. The convexity of the function $\mathbb{R} \ni x \mapsto x_+ \in \mathbb{R}_+$ and Jensen's inequality imply that $\mathsf{EX}^{ij}(\tau \wedge \zeta_n) \leq \mathsf{EX}^{ij}(\tau \wedge \zeta_m)$ holds whenever $\mathbb{N} \ni n \leq m \in \mathbb{N}$, which shows that the sequence $\left(\mathsf{EX}^{ij}(\tau \wedge \zeta_n)\right)_{n \in \mathbb{N}}$ is nondecreasing. Furthermore, in view of the fact that $\lim_{n \to \infty} \zeta_n = \zeta$, it $\mathbb{Q}^j$-a.s. holds that $(1 - R^{ij}_\tau)_+ \mathbb{I}_{\{\tau < \zeta\}} \leq \liminf_{n \to \infty} \left((1 - R^{ij}_{\tau \wedge \zeta_n})_+\right)$. This fact, coupled with Fatou's lemma, implies that

$$\mathsf{EX}^{ij}(\tau) = \mathbb{E}_{\mathbb{Q}^j}\left[(1 - R^{ij}_\tau)_+ ; \tau < \zeta\right] \leq \mathbb{E}_{\mathbb{Q}^j}\left[\liminf_{n \to \infty}\left((1 - R^{ij}_{\tau \wedge \zeta_n})_+\right)\right] \leq \lim_{n \to \infty} \mathsf{EX}^{ij}(\tau \wedge \zeta_n).$$

In a similar way as was reasoned above, Proposition 2.7 and the facts that $\mathbb{Q}^j\left[\zeta_n < \zeta\right] = 1$ for all $n \in \mathbb{N}$ and $\tau \leq T$ give $\mathsf{EX}^{ij}(\tau \wedge \zeta_n) \leq \mathsf{EX}^{ij}(T \wedge \zeta_n)$ for all $n \in \mathbb{N}$; therefore, $\mathsf{EX}^{ij}(\tau) \leq \lim_{n \to \infty} \mathsf{EX}^{ij}(T \wedge \zeta_n)$ holds for all $\tau \in \mathcal{T}_{[0,T]}$. Equation (3.3) immediately follows.
(2). Since $\lim_{n \to \infty} R^{ij}_{T \wedge \zeta_n} = \rho^{ij} \mathbb{I}_{\{\zeta \leq T\}} + R^{ij}_T \mathbb{I}_{\{T < \zeta\}}$ holds $\mathbb{Q}^j$-a.s., the dominated convergence theorem gives

$$\mathsf{AX}^{ij}(T) = \lim_{n \to \infty} \mathsf{EX}^{ij}(T \wedge \zeta_n) = S^j_0 \mathbb{E}_{\mathbb{Q}^j}\left[\left(1 - \rho^{ij}\right)_+ ; \zeta \leq T\right] + S^j_0 \mathbb{E}_{\mathbb{Q}^j}\left[\left(1 - R^{ij}_T\right)_+ ; T < \zeta\right].$$

By Proposition 3.2, the second term in the right-hand-side of the the above equation is equal to $\mathsf{EX}^{ij}(T)$; therefore, (3.4) has been established. □

Remark 3.6. Proposition 3.5 implies that, for any $T \in \mathcal{T}$, the supremum in (3.2) for $\mathsf{AX}^{ij}(T)$ is monotonically achieved through the sequence $(T \wedge \zeta_n)_{n \in \mathbb{N}}$ of stopping times in $\mathcal{T}_{[0,T]}$, this being true for all combinations of $i \in I$ and $j \in I$. This fact has the important consequence that a parity relation for American exchange options follows from the corresponding parity relation for European options—see the statement and proof of Proposition 4.1.

Remark 3.7. While in the Black-Scholes-Merton modeling environment discussed in [Mar78] it is never optimal to exercise an American-style exchange option before a *finite* maturity $T \in \mathcal{T}$, Proposition 3.5 implies that, if $\mathbb{Q}^j[\zeta \leq T] > 0$ holds, it is not optimal to keep an American option to exchange any asset $i \in I$ for some asset $j \in I$ until maturity $T \in \mathcal{T}$. Instead, (3.3) reasonably suggests that one should keep the option until maturity $T \in \mathcal{T}$ *provided* that default of the whole economy does not appear imminent; otherwise, early exercise may be preferable.

Remark 3.8. Using the (self-explanatory) notation $R^{ij}_{T \wedge (\zeta -)} = R^{ij}_T \mathbb{I}_{\{T < \zeta\}} + \rho^{ij} \mathbb{I}_{\{\zeta \leq T\}}$ for $i \in I$, $j \in I$ and $T \in \mathcal{T}$, it follows by a combination of Proposition 3.2 and Proposition 3.5 that

$$\mathsf{AX}^{ij}(T) = S^j_0 \mathbb{E}_{\mathbb{Q}^j}\left[\left(1 - R^{ij}_{T \wedge (\zeta -)}\right)_+\right],$$

which provides a direct representation for the value of American-style exchange options.

An interesting special case in Proposition 3.5 is when $\mathbb{Q}^j\left[\rho^{ij} = 0\right] = 1$ holds for some $i \in I$ and $j \in I$; this is, for example, true in the case in the Black-Scholes-Merton model where the logarithms of asset-price processes are (not perfectly) correlated drifted Brownian motions. When $\mathbb{Q}^j\left[\rho^{ij} = 0\right] = 1$ holds for $i \in I$ and $j \in I$, the simpler formula $\mathsf{AX}^{ij}(T) - \mathsf{EX}^{ij}(T) = S^j_0 \mathbb{Q}^j[\zeta \leq T]$ for the early exercise premium holds for all $T \in \mathcal{T}$. The next result gives several equivalent formulations of the latter condition.

Proposition 3.9. *Fix $i \in I$ and $j \in I$. Under Assumption 1.5, the following statements are equivalent:*

(1) $\lim_{n \to \infty} \mathsf{AX}^{ij}(\zeta_n) = S^j_0$.
(2) $\lim_{n \to \infty} \mathsf{EX}^{ij}(\zeta_n) = S^j_0$.
(3) $\lim_{n \to \infty} \mathbb{Q}^j\left[S^j_{\zeta_n} \leq S^i_{\zeta_n}\right] = 0$ *and* $\lim_{n \to \infty} \mathbb{Q}^i\left[S^i_{\zeta_n} \leq S^j_{\zeta_n}\right] = 0$.
(4) $\mathbb{Q}^j\left[\rho^{ij} = 0\right] = 1$.
(5) $\mathsf{AX}^{ij}(T) - \mathsf{EX}^{ij}(T) = S^j_0 \mathbb{Q}^j[\zeta \leq T]$ *holds for all* $T \in \mathcal{T}$.

Proof. Fix $i \in I$ and $j \in I$. By Proposition 3.5, $\mathsf{AX}^{ij}(\zeta_n) = \lim_{m \to \infty} \mathsf{EX}^{ij}(\zeta_n \wedge \zeta_m) = \mathsf{EX}^{ij}(\zeta_n)$ holds for all $n \in \mathbb{N}$. This shows the equivalence of statements (1) and (2). Furthermore, since $\mathbb{Q}^i[\zeta_n < \zeta] = 1$ and $\mathbb{Q}^j[\zeta_n < \zeta] = 1$ holds by Theorem 2.1, Proposition 3.2 gives $\mathsf{EX}^{ij}(\zeta_n) = S^j_0 \mathbb{Q}^j\left[S^i_{\zeta_n} < S^j_{\zeta_n}\right] - S^i_0 \mathbb{Q}^i\left[S^i_{\zeta_n} < S^j_{\zeta_n}\right]$. Therefore, $\lim_{n \to \infty} \mathsf{EX}^{ij}(\zeta_n) = S^j_0$ is equivalent to the validity of both $\lim_{n \to \infty} \mathbb{Q}^j\left[S^j_{\zeta_n} \leq S^i_{\zeta_n}\right] = 0$ and $\lim_{n \to \infty} \mathbb{Q}^i\left[S^i_{\zeta_n} < S^j_{\zeta_n}\right] = 0$. Since

$$S^j_0 \mathbb{Q}^j\left[S^j_{\zeta_n} = S^i_{\zeta_n}\right] = \mathbb{P}\left[S^j_{\zeta_n} = S^i_{\zeta_n}, \, \zeta_n < \zeta\right] = S^i_0 \mathbb{Q}^j\left[S^j_{\zeta_n} = S^i_{\zeta_n}\right]$$

holds in view of Theorem 2.1, $\lim_{n\to\infty} \mathsf{EX}^{ij}(\zeta_n) = S_0^j$ is equivalent to $\lim_{n\to\infty} \mathbb{Q}^j\big[S^j_{\zeta_n} \le S^i_{\zeta_n}\big] = 0$ and $\lim_{n\to\infty} \mathbb{Q}^i\big[S^i_{\zeta_n} \le S^j_{\zeta_n}\big] = 0$. This shows the equivalence of (2) and (3). Therefore, the equivalence of conditions (1), (2) and (3) has been established. Continuing, a combination of Proposition 3.2 and the dominated convergence theorem give $\lim_{n\to\infty} \mathsf{EX}^{ij}(\zeta_n) = S_0^j \mathbb{E}_{\mathbb{Q}^j}\left[(1-\rho^{ij})_+\right]$. Therefore, conditions (2) and (4) are equivalent. The fact that condition (4) implies condition (5) follows from (3.4). Furthermore, if (5) holds then (3.4) with $T = \zeta$ gives $\mathbb{E}_{\mathbb{Q}^j}\left[\left(1-\rho^{ij}\right)_+\right] = 1$, which is equivalent to $\mathbb{Q}^j\left[\rho^{ij} = 0\right] = 1$, i.e., condition (4). □

Remark 3.10. Note that condition (3) of Proposition 3.9 is symmetric in $i \in I$ and $j \in I$. This means that conditions (1), (2), (4) and (5) of Proposition 3.9 are also equivalent to the corresponding conditions where the roles of i and j are interchanged.

Remark 3.11. Fix $i \in I$ and $j \in I$. Under any of the equivalent conditions of Proposition 3.9, the equality $\mathsf{AX}^{ij}(T) = S_0^j$ holds whenever $T \in \mathcal{T}$ is such that $T \ge \zeta$. In fact, one can get a nice expression for the difference $S_0^j - \mathsf{AX}^{ij}(T)$ for all $T \in \mathcal{T}$. Assuming any of the equivalent conditions of Proposition 3.9, $S_0^j - \mathsf{AX}^{ij}(T) = S_0^j \mathbb{Q}^j\left[T < \zeta\right] - \mathsf{EX}^{ij}(T)$ holds for all $T \in \mathcal{T}$. Since $\mathsf{EX}^{ij}(T) = S_0^j \mathbb{E}_{\mathbb{Q}^j}\left[(1-R_T^{ij})_+; T < \zeta\right]$ holds by Proposition 3.2, we obtain

$$S_0^j - \mathsf{AX}^{ij}(T) = S_0^j \mathbb{E}_{\mathbb{Q}^j}\left[1 \wedge R_T^{ij}; T < \zeta\right] = S_0^j \mathbb{Q}^j\left[R_T^{ij} \ge 1,\, T < \zeta\right] + S_0^j \mathbb{E}_{\mathbb{Q}^j}\left[R_T^{ij}; R_T^{ij} < 1,\, T < \zeta\right].$$

Now, $\mathbb{Q}^j\big[R_T^{ij} \ge 1,\, T < \zeta\big] = \mathbb{Q}^j\big[S_T^j \le S_T^i,\, S_T^j > 0,\, T < \zeta\big] = \mathbb{Q}^j\big[S_T^j \le S_T^i,\ T < \zeta\big]$, the last equality following from $\mathbb{Q}^j\big[S_T^j = 0, T < \zeta\big] = 0$ in Theorem 2.1. Furthermore, note that (2.3) gives

$$S_0^j \mathbb{E}_{\mathbb{Q}^j}\left[R_T^{ij}; R_T^{ij} < 1,\, T < \zeta\right] = S_0^i \mathbb{Q}^i\big[S_T^i < S_T^j,\, S_T^j > 0,\, T < \zeta\big] = S_0^i \mathbb{Q}^i\big[S_T^i < S_T^j,\, T < \zeta\big],$$

the last equality following from the nonnegativity of S^i. It follows that

$$S_0^j - \mathsf{AX}^{ij}(T) = S_0^j \mathbb{Q}^j\big[S_T^j \le S_T^i,\, T < \zeta\big] + S_0^i \mathbb{Q}^i\big[S_T^i < S_T^j,\, T < \zeta\big].$$

4. Parities Involving Exchange Options

4.1. **Parity formulas.** The following result gives two parity relations—one regarding European-style and another regarding American-style exchange options.

Proposition 4.1. *Let $i \in I$ and $j \in I$, as well as $T \in \mathcal{T}$. Under Assumption 1.5, the following parity relations hold:*

$$\mathsf{EX}^{ij}(T) + S_0^i \mathbb{Q}^i\left[T < \zeta\right] = \mathsf{EX}^{ji}(T) + S_0^j \mathbb{Q}^j\left[T < \zeta\right], \tag{4.1}$$

$$\mathsf{AX}^{ij}(T) + S_0^i = \mathsf{AX}^{ji}(T) + S_0^j. \tag{4.2}$$

Proof. Combining the relationships $\mathsf{EX}^{ij}(T) = S_0^j \mathbb{Q}^j\big[S_T^i < S_T^j,\, T < \zeta\big] - S_0^i \mathbb{Q}^i\big[S_T^i < S_T^j,\, T < \zeta\big]$ and $\mathsf{EX}^{ji}(T) = S_0^i \mathbb{Q}^i\big[S_T^j \le S_T^i,\, T < \zeta\big] - S_0^j \mathbb{Q}^j\big[S_T^j \le S_T^i,\, T < \zeta\big]$, both following from Proposition 3.2, one obtains $\mathsf{EX}^{ij}(T) - \mathsf{EX}^{ji}(T) = S_0^j \mathbb{Q}^j\left[T < \zeta\right] - S_0^i \mathbb{Q}^i\left[T < \zeta\right]$, which shows (4.1). Replacing

T by $T \wedge \zeta_n$ and using the fact that $\mathbb{Q}^i[\zeta_n < \zeta] = 1 = \mathbb{Q}^j[\zeta_n < \zeta]$ holds for all $n \in \mathbb{N}$ that follows from Theorem 2.1, we obtain $\mathsf{EX}^{ij}(T \wedge \zeta_n) + S^i_0 = \mathsf{EX}^{ji}(T \wedge \zeta_n) + S^j_0$. Sending n to infinity and using (3.3), (4.2) follows. □

Remark 4.2. An alternative, more direct proof of (4.1) utilizes the equality

$$(S^j - S^i)_+ + S^i = (S^i - S^j)_+ + S^j, \quad \text{for } i \in I \text{ and } j \in I. \tag{4.3}$$

Applying (4.3) with the processes sampled at $T \in \mathcal{T}$ on the event $\{T < \zeta\}$, multiplying both sides by Y_T and taking expectation with respect to $\mathbb{P}$, one obtains (4.1) by Proposition 3.2, given the equalities $\mathbb{E}_{\mathbb{P}}\big[Y_T S^i_T; T < \zeta\big] = S^i_0 \mathbb{Q}^i[T < \zeta]$ and $\mathbb{E}_{\mathbb{P}}\big[Y_T S^j_T; T < \zeta\big] = S^j_0 \mathbb{Q}^j[T < \zeta]$ that follow from (2.1).

For $i \in I$, the quantity $S^i_0 \mathbb{Q}^i[T < \zeta]$ is the value of the contract that pays S^i_T at time $T \in \mathcal{T}$ when $T < \zeta$. Being a European-style contract, its value may be *strictly* less than S^i_0, which happens exactly when $\mathbb{Q}^i[\zeta \leq T] > 0$. In contrast, the value of the corresponding "American" option that pays S^i_τ at any chosen time $\tau \in \mathcal{T}_{[0,T]}$ for $T \in \mathcal{T}$ would be

$$\sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}_{\mathbb{P}}\left[Y_\tau S^i_\tau; \tau < \zeta\right] = \sup_{\tau \in \mathcal{T}_{[0,T]}} S^i_0 \mathbb{Q}^i[\tau < \zeta] = S^i_0 \mathbb{Q}^i[\zeta > 0] = S^i_0, \tag{4.4}$$

since $\mathbb{Q}^i[\zeta > 0] = 1$ holds in view of Theorem 2.1. In models where no "bubbles" exist, in the sense that $\mathbb{Q}^i[\zeta < \infty] = 0$ is valid for all $i \in I$, $\mathsf{EX}^{ij}(T) = \mathsf{AX}^{ij}(T)$ holds for all $i \in I$, $j \in I$ and $T \in \mathcal{T}$ with $T < \infty$. Then, (4.2) becomes a parity relation for both European-style and American-style exchange options. The fact that (4.1), instead of (4.2), holds for European options has sometimes lead to claims that the "usual" parity is not valid in markets where bubbles exist. Of course, in order for a parity relation to hold, the contracts used have to be of similar type. In this sense, (4.1) is the correct and perfectly valid parity relation for European options; this has already been made clear in [Hul10], in the setting of the example of Subsection 4.2 below. On the other hand, when American-style exchange options are involved, American-style contracts that pay off the stock price have to be used in both sides; in view of (4.4), (4.2) is the parity relation to be expected. As noted in Remark 3.6 and demonstrated in the proof of Proposition 4.1, the American parity relationship (4.2) follows from the validity of (4.1) and the fact that the approximating sequence $(T \wedge \zeta_n)_{n \in \mathbb{N}}$ is the same for all choices of $i \in I$ and $j \in I$.

In the special case where any of the equivalent conditions of Proposition 3.9 hold, two more parity relations are valid, mixing European and American options.

Proposition 4.3. *Under Assumption 1.5 and the validity of any of the equivalent conditions of Proposition 3.9, the following parity relations hold:*

$$\begin{aligned} \mathsf{AX}^{ij}(T) + S^i_0 \mathbb{Q}^i[T < \zeta] &= \mathsf{EX}^{ji}(T) + S^j_0, \\ \mathsf{EX}^{ij}(T) + S^i_0 &= \mathsf{AX}^{ji}(T) + S^j_0 \mathbb{Q}^j[T < \zeta]. \end{aligned}$$

Proof. Since Proposition 3.9 gives $\mathsf{AX}^{ij}(T) = \mathsf{EX}^{ij}(T) + S_0^j \mathbb{Q}^j[T < \zeta]$ and $\mathsf{AX}^{ji}(T) = \mathsf{EX}^{ji}(T) + S_0^i \mathbb{Q}^i[T < \zeta]$, both relationships follow directly from (4.1). □

4.2. **An illustrative example involving the three-dimensional Bessel process.** In the setting of Subsection 1.4 (continued in Example 2.6), let $m = 1$, $E = (0, \infty)$ and suppose that $\mathbb{P}[X_0 = 1] = 1$, $a(x) = 1/x$ and $c(x) = 1$ holds for $x \in (0, \infty)$. One can choose $E_n = (1/n, n+1)$ for all $n \in \mathbb{N}$. In this case, X under $\mathbb{P}$ is behaving like a three-dimensional Bessel process with unit initial value. Note that $\mathbb{P}[\zeta < \infty] = 0$. Let $I = \{0, 1\}$, and suppose that $S^0 = K\mathbb{I}_{[\![0,\zeta[\![}$ for some $K \in (0, \infty)$ and $S^1 \equiv X\mathbb{I}_{[\![0,\zeta[\![}$. It can be shown in a straightforward way that $Y = (1/X)\mathbb{I}_{[\![0,\zeta[\![}$ is the (essentially) unique process such that YS^i is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for $i \in I$. In fact, if one defines $(\zeta_n)_{n \in \mathbb{N}}$ through (1.2), it is easily seen that $\left(Y_{\zeta_n \wedge t} S^i_{\zeta_n \wedge t}\right)_{t \in \mathbb{R}_+}$ is an actual martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $n \in \mathbb{N}$ and $i \in I$. Clearly $\mathbb{Q}^1 = \mathbb{P}$, while $\mathbb{Q}^0$ can be seen to coincide with the probability on $\mathcal{F}_\infty$ such that X is Brownian motion starting from one and stopped when it reaches zero. The equality $\mathbb{Q}^1\left[\rho^{01} = 0\right] = \mathbb{P}[\lim_{t\to\infty} X_t = \infty] = 1$ follows from the fact that X behaves like three-dimensional Bessel process under $\mathbb{P}$. In particular, we obtain all relations of Proposition 3.9 when $i = 0$ and $j = 1$, as well as when $i = 1$ and $j = 0$.

As $\mathbb{Q}^1[\zeta < \infty] = \mathbb{P}[\zeta < \infty] = 0$, it follows that $\mathsf{AX}^{01}(T) = \mathsf{EX}^{01}(T)$ holds for all $T \in \mathbb{R}_+$. Furthermore, Proposition 3.2 gives $\mathsf{EX}^{01}(T) = \mathbb{P}[X_T > K] - K\mathbb{Q}^0[X_T > K, \zeta > T]$ for $T \in \mathbb{R}_+$. Let $\Phi : \mathbb{R} \mapsto (0, 1)$ denote the cumulative distribution function of the standard normal law, and set $\overline{\Phi} = 1 - \Phi$. The joint distribution of Brownian motion and its minimum gives

$$\mathbb{Q}^0[X_T > K, \zeta > T] = \Phi\left(\frac{1-K}{\sqrt{T}}\right) - \overline{\Phi}\left(\frac{1+K}{\sqrt{T}}\right), \quad T \in \mathbb{R}_+.$$

Furthermore, from properties of the non-central chi-squared distribution one can obtain that

$$\mathbb{P}[X_T > K] = \Phi\left(\frac{1-K}{\sqrt{T}}\right) + \overline{\Phi}\left(\frac{1+K}{\sqrt{T}}\right) + \sqrt{\frac{2T}{\pi}} \exp\left(-\frac{1+K^2}{2T}\right) \sinh\left(\frac{K}{T}\right), \quad T \in \mathbb{R}_+.$$

(For the last formula see also [Hul10, Proposition 1].) It then follows that

$$\mathsf{EX}^{01}(T) = (1+K)\overline{\Phi}\left(\frac{1+K}{\sqrt{T}}\right) + (1-K)\Phi\left(\frac{1-K}{\sqrt{T}}\right) + \sqrt{\frac{2T}{\pi}} \exp\left(-\frac{1+K^2}{2T}\right) \sinh\left(\frac{K}{T}\right), \quad T \in \mathbb{R}_+,$$

with the same equality valid for $\mathsf{AX}^{01}(T)$. Equation (4.2) gives $\mathsf{AX}^{10}(T) = \mathsf{AX}^{01}(T) - (1-K)$, i.e.,

$$\mathsf{AX}^{10}(T) = (1+K)\overline{\Phi}\left(\frac{1+K}{\sqrt{T}}\right) - (1-K)\overline{\Phi}\left(\frac{1-K}{\sqrt{T}}\right) + \sqrt{\frac{2T}{\pi}} \exp\left(-\frac{1+K^2}{2T}\right) \sinh\left(\frac{K}{T}\right), \quad T \in \mathbb{R}_+.$$

Furthermore, the law of the minimum of Brownian motion gives $\mathbb{Q}^0[\zeta \leq T] = 2\overline{\Phi}(1/\sqrt{T})$ holding $T \in \mathbb{R}_+$, which implies that $\mathsf{EX}^{10}(T) = \mathsf{AX}^{10}(T) - 2K\overline{\Phi}(1/\sqrt{T})$ holds for $T \in \mathbb{R}_+$.

Note that the previous closed-form expressions give $\lim_{T\to\infty} \mathsf{EX}^{01}(T) = 1 = \lim_{T\to\infty} \mathsf{AX}^{01}(T)$, as well as $\lim_{T\to\infty} \mathsf{EX}^{10}(T) = 0 < K = \lim_{T\to\infty} \mathsf{AX}^{10}(T)$.

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