Optimization of Truss Vibration with Reduction of Symmetric Semidefinite Programming*

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Abstract A truss vibration optimization problem is to minimize the total weight of truss subject to a given fundamental vibration frequency. This paper focuses on the recent results of matrix algebraic approach to solve the truss vibration optimization problem expressed in terms of symmetric semidefinite programming problem. We derive two sufficient conditions on constructing the symmetric group representation to reduce the problems size. An example of eight-bar truss design problem is given to illustrate how to construct a group representation and to demonstrate its effectiveness.

Keywords Operations research, truss topology optimization, semidefinite programming, symmetry group, group representation

Subject Classification (GB/T13745-92) 110.74

群对称桁架振动设计的半正定模型与降维问题

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摘要 桁架振动优化设计可描述为：在给定振动系统最低频率的约束条件下，设计用材最省的桁架结构．本文针对具有某种结构对称性的桁架，利用有限群描述这一特性，在已有桁架设计的半正定规划模型基础上，运用最近提出的矩阵代数方法对半正定规划问题的决策变量和数据进行降维，给出了构造有限群表示的两个充分条件，并实现了一类群对称桁架振动优化设计的半正定模型降维．基于问题的实际背景，我们又考虑了一个具有八根弹性棒的桁架设计实例，进一步说明在实际问题中根据群对称构造群表示以及对应不可约表示的具体方法．

关键词 运筹学，桁架拓扑优化，半正定优化，群对称，群表示

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1 Introduction

The topology design of trusses is found in a wide variety of natural and engineering sciences, including engineering mechanics, structural engineering, finite element methods and biomedical engineering, etc. A truss is a mechanical construction comprising thin elastic bars linked to each other, such as an electric mast, a railroad bridge, or the Effel tower. The points at which the bars are linked to each other are called the nodes of truss. Usually, the design of the truss is to afford a certain external load – a collection of simultaneous forces acting at the nodes. The main parameters concerned are eigenvalues of free vibration, which are important performance measures of dynamic stiffness of truss. For building a structures of truss, the eigenvalues should be properly assigned to avoid resonance to the seismic motions and wind loads. Therefore, many researches have been presented for optimum design and topology optimization of truss for specified fundamental eigenvalue. It can be expressed as the following optimization problem[1]:

$$
\begin{array}{lll}
(TOP) & \min & \sum_{i=1}^{m} b_i z_i \\
& \text{s.t.} & \Omega_r \geqslant \overline{\Omega} \quad r = 1, 2, ..., n \\
& & z_i \geqslant 0 \quad\;\; i = 1, ..., m
\end{array}
\tag{1.1}
$$

where n is the number of degree freedom of the truss, m is the number of bars in the truss. Ω_r is the r-th eigenvalue of vibration and $\overline{\Omega}$ is a lower bound of the eigenvalues. b_i denotes the bar length of the i-th bar, z_i denotes the cross-sectional areas of the i-th bar. The problem (1.1) seems to be a simple nonlinear optimization problem. However, it is hard to find a solution efficiently[11,14]. Motivated by finding out an efficient methods to solve (1.1), the problem was formed by a semidefinite optimization problem (SDP), (see [4]):

$$
\begin{array}{llll}
(TVDP) & \min & \sum_{i=1}^{m} b_i z_i & \\
& \text{s.t.} & S = \sum_{i=1}^{m} (K_i - \overline{\Omega} M_i) z_i - \overline{\Omega} M_0 & \\
& & z_i \geqslant 0 & i = 1, ..., m \\
& & S \succeq 0 &
\end{array}
\tag{1.2}
$$

where K_i and M_i are the stiffness matrix and the mass matrix for the bar i, $i = 1, 2, ..., m$ and M_0 is a nonstructural mass matrix.

Many efficient algorithms have been proposed to solve SDP problems. However, for computational point of view, the SDP still is a high dimensional problem for the truss topology design problem. For example, the system stiffness matrix K can be with large size if there are large amount of bars in the truss. Recently, a new technique, Group Symmetric Technique, first presented by Kanno et.al.[4], then improved by Bai[2] to reduce the size of SDP problems with the property of group symmetry. Inspired by their work, we further develop this technique and apply it to a class of truss design problems. In this paper, we focus on the recent results of matrix algebraic approach to solve the truss vibration optimization problem expressed in terms of symmetric SDP problem. We derive two sufficient conditions on constructing the symmetric group representation to reduce the problem size. An example of eight-bar truss design problem is given to illustrate how to construct a group represestation and to demonstrate its effectiveness in practice.

2 Notation

The space of $p \times q$ real matrices is denoted by $R^{p\times q}$, and the space of $k \times k$ symmetric matrices is denoted by S_k, and the space of $k \times k$ positive semidefinite matrices by S_k^+. We will sometimes also use the notation $X \geqslant 0$ instead of S_k^+, if the order of matrix is clear from the context.

The Kronecker product $A \otimes B$ of matrices $A \in R^{p\times q}$ and $B \in R^{r\times s}$ is defined as the $pr \times qs$ matrix composed of pq blocks of size $r \times s$,with block ij given by $A_{ij}B$ $(i = 1, ..., p,\ j = 1, ..., q)$.

The following properties of the Kroneccker product will be used in the paper, see e.g. [3],

$$(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}, \quad (A \otimes B)(C \otimes D) = AC \otimes BD$$

for all $A \in R^{p\times q}, B \in R^{r\times s}, C \in R^{q\times k}, D \in R^{s\times l}$.

3 Algebraic preliminaries

Group symmetric technique is a new reduction method on SDP problems. The foundation of this technique is the finite group representation theory. In this section, we recall some basic definitions and theorems that we use.

Definition 3.1 [8] *Let V be a real, n-dimensional vector space and identify $R^{n\times n}$ (respectively, O^n) as the space of all (respectively, orthogonal) $n \times n$ matrices such that $O^n : V \to V$. An orthogonal linear representation of a group $\mathcal{G}$ on V is a group homomorphism $\Gamma : \mathcal{G} \longmapsto R^{n\times n}$ (respectively, $\Gamma : \mathcal{G} \longmapsto O^n$). In other words for each element $g \in \mathcal{G}$ such that $\Gamma(g_1)\Gamma(g_2) = \Gamma(g_1 g_2)$.*

In the rest part of the paper, we consider the images of data matrices $A_i = A_i^{\mathrm{T}} \in R^{n\times n}$ in SDP problem, under Γ_g's. Therefore, we have to restrict our attention to orthogonal representations, as in the usual SDP setting one requires (as it will become clear in what follows) that $B_i = \Gamma_g A_i \Gamma_g^{-1}$ are symmetric, i.e. $B_i = B_i^{\mathrm{T}}$.

In the following theorem, we will find that if one has two orthogonal representations of a finite group, one may obtain a third representation by Kronecker products. In representation theory, this construction is known as tensor product of representations.

Theorem 3.1 [2] *Let $\mathcal{G}$ be a group and it has two orthogonal linear representations of $\mathcal{G}$ denoted by p_i $(i = 1, ..., |\mathcal{G}|)$ and s_i $(i = 1, ..., |\mathcal{G}|)$, such that p_i corresponds to s_i $(i = 1, ..., |\mathcal{G}|)$. Then a third orthogonal linear representation of $\mathcal{G}$ is given by*

$$P_i := p_i \otimes s_i \quad (i = 1, ..., |\mathcal{G}|).$$

The commutant of $\mathcal{G}$ is defined by

$$\mathcal{A}' := \{X \in R^{n\times n} : XP = PX \quad \forall P \in \mathcal{G}\}$$

An alternative, equivalent, definition of the commutant is

$$\mathcal{A}' = \{X \in R^{n\times n} : R(X) = X\}$$

where

$$R(X) := \frac{1}{|\mathcal{G}|} \sum_{P\in\mathcal{G}} PXP^{\mathrm{T}},\ X \in R^{n\times n}$$

is called the *Reynolds operator*(or group average) of $\mathcal{G}$.

For the optimization problems, we assume that the feasible set of the problem is contained in some commutant, and we therefore devote one more part to recall some results on the representation of matrix $*$-algebras.

We define the direct sum of matrices X_1 and X_2 as

$$X_1 \oplus X_2 := \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}.$$

An algebra $\mathcal{A}$ is called *basic* if

$$\mathcal{A} = \{\oplus_{i=1}^{\mathrm{T}} M \mid M \in \mathbb{C}^{n\times n}\}$$

for some t and m, where $\mathbb{C}$ is the field of complex numbers. Finally, the direct sum of two algebra $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined as

$$\mathcal{A}_1 \oplus \mathcal{A}_2 := \{X_1 \oplus X_2 \mid X_1 \in \mathcal{A}_1 \ , \ X_2 \in \mathcal{A}_2\}.$$

The following existence theorem gives the so-called completely reduced representation of a matrix $*$-algebra $\mathcal{A}$, which is the algebra fundamental of the reduction technique.

Theorem 3.2 [15] *Each matrix $*$-algebra is equivalent to a direct sum of basic algebras and a zero algebra.*

Remark 3.1 *In general, it is very hard to obtain a decomposition of a matrix $*$-algebra. However,we can obtain the completely reduced representation of a matrix$*$-algebra if it is the commutant of a finite group.*

4 Main stem of group symmetric technique

By using the result of Theorem 3.2, the group symmetric technique realizes the reduction of SDP problems. We give a brief introduction of the method. Consider the standard form of the SDP problem:

$$f^* := \min \{ \ Tr\ (A_0X) \quad : Tr\ (A_iX) \ = b_i, \ X \geqslant 0, \ i = 1, ..., m \ \} \tag{4.1}$$

and its dual problem:

$$d^* := \max \left\{ b^{\mathrm{T}} y : A_0 - \sum_{i=1}^{m} y_i A_i \geqslant 0,\ y \in R^m \right\} \tag{4.2}$$

where $A_i \in S_n$ $(i = 0, ..., m)$. f^* is the optimum value of primal problem and d^* is the optimum value of dual problem. For the primal and dual problems, we have the following assumptions:

Assumption 1 *The SDP problem and its dual problem satisfy the Slater Condition so that both problems have optimal solutions with identical optimal value.*

Assumption 2 [4] *For the SDP problems, we assume that there is a non-trivial multiplicative group of orthogonal matrices $\mathcal{G}$ such that the associated Reynolds operator:*

$$R(X) := \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} P X P^{\mathrm{T}}\ ,\ X \textit{ is in the feasible set.}$$

maps the feasible set of an SDP problem into itself and leaves the objective value invariant, i.e:

$$Tr(A_0 R(X)) = Tr(A_0 X) \tag{4.3}$$

if X is a feasible point of an SDP problem.

Theorem 4.1 [2] *Under these two assumptions, we can easily get the following results:*

$$f^* := \min \{\ Tr\ (R(A_0)X)\ : Tr\ (R(A_i)X)\ = b_i,\ X \geqslant 0,\ i = 1, ..., m\ \} \tag{4.4}$$

$$d^* := \max \left\{ b^{\mathrm{T}} y : R(A_0) - \sum_{i=1}^{m} y_i R(A_i) \geqslant 0,\ y \in R^m \right\} \tag{4.5}$$

According to Theorem 4.1, it implies that the primal and dual problem are invariant when replacing the data matrices A_i with $R(A_i)$ $(i = 0, ..., m)$, respectively. We stem the main steps of group symmetric technique below:

- Find a group representation $\mathcal{P}$ such that $A_0 - \sum_{i=1}^{m} y_i A_i$ is invariant under the action of the *Reynolds operator* defined by $\mathcal{P}$.

- Solve the problem (4.5) instead of the original one (4.2) under the assumption 1 and 2.

- Decompose $A_0 - \sum\limits_{i=1}^{m} y_i A_i$ into the direct sum of basic algebras. In other words, makes $A_0 - \sum\limits_{i=1}^{m} y_i A_i$ block-dialogized.
- Caculate the bases of these irreducible representations and form a orthogonal matrix Q, then we rewrite the problem as follows:

$$\begin{aligned}
\min \quad & \sum_{\text{o an orbit}} \xi_o \sum_{i\in o} b_i \\
\text{s.t.} \quad & S = \sum_{\text{o an orbit}} \xi_o Q^{\mathrm{T}}\Big(\sum_{i\in o}(K_i - \overline{\Omega} M_i)\Big)Q - \overline{\Omega} Q^{\mathrm{T}} R(M_0) Q \\
& \xi_o \geqslant 0 \quad o \text{ is an orbit} \\
& S \geqslant 0
\end{aligned} \tag{4.6}$$

  where S is a block-dialogized matrix. Thus, the large dimensional semidefinite constraint has been reduced into some smaller ones.

5 Apply to truss design problems

By using the group symmetric technique to reducing the size of SDP, the key is to construct a group representation $\mathcal{P}$ satisfying the condition that the semidefinite constraint $A_0 - \sum\limits_{i=1}^{m} y_i A_i$ is in the commutant of it. In this section, we give two sufficient conditions on how to construct such a $\mathcal{P}$ and apply it to a class of truss design problems.

Theorem 5.1 *If each data matrix A_i $(i = 0, ..., m)$ is in the commutant of a group representation $\mathcal{P}$, i.e: $R(A_i) = A_i$, then $S = A_0 - \sum\limits_{i=1}^{m} y_i A_i$ is also in the commutant of the representation $\mathcal{P}$.*

Proof If each data matrix A_i $(i = 0, ..., m)$ is in the commutant of a group representation $\mathcal{P}$, that is

$$R(A_i) = A_i \qquad i = 0, ..., m.$$

$$R(S) = R\Big(A_0 - \sum_{i=1}^{m} y_i A_i\Big) = R(A_0) - \sum_{i=1}^{m} y_i R(A_i) = A_0 - \sum_{i=1}^{m} y_i A_i = S.$$

In fact, it is very difficult to construct such a $\mathcal{P}$. Given a matrix A, it may be not difficult for us to find a matrices P commuting with it, i.e: $PA = AP$. However, given a series of matrices A_i $(i = 0, ..., m)$, it may be high demanding work to find a matrix P commuting with them, especially m is large. Therefore, we need a more practical condition.

Theorem 5.2 *If the following conditions hold, then* $S = A_0 - \sum_{i=1}^{m} y_i A_i$ *is in the commutant of a representation* $\mathcal{P}$.

1. *The data matrix* A_0 *is in the commutant of the finite group representation* $\mathcal{P}$.
2. *The data matrices* A_i $(i = 1, ..., m)$ *can form the orbits* O_j $(j = 1, ..., w)$, *when the group* $\mathcal{P}$ *acts on the set* A_i $(i = 1, ..., m)^*$.
3. *The corresponding decision variables* y_i *belonging to one orbit can be regarded equivelantly. Therefore, the variables belonging to one orbit can be repalced by one variable. (Denoted as* y_{o_j} , $j = 1, ..., w$.*)*

Proof

$$
\begin{aligned}
R(S) &= R\Big(A_0 - \sum_{i=1}^{m} y_i A_i\Big) \\
&= R(A_0) - \sum_{i=1}^{m} y_i R(A_i) \\
&= A_0 - \sum_{i \in o_1} y_{o_1} R(A_i) - \cdots - \sum_{i \in o_w} y_{o_w} R(A_i) \\
&= A_0 - y_{o_1} R\Big(\sum_{i \in o_1} A_i\Big) - \cdots - y_{o_w} R\Big(\sum_{i \in o_w} A_i\Big). \qquad (5.1)
\end{aligned}
$$

From the properties of the orbit, we have that $\sum_{i \in o_j} A_i$ $(j = 1, ..., w)$ is in the commutant of the representation $\mathcal{P}$, i.e:

$$
R\Big(\sum_{i \in o_j} A_i\Big) = \sum_{i \in o_j} A_i \quad (j = 1, ..., w).
$$

Therefore, we rewrite $R(S)$ as:

$$
\begin{aligned}
R(S) &= A_0 - y_{o_1}\Big(\sum_{i \in o_1} A_i\Big) - \cdots - y_{o_w}\Big(\sum_{i \in o_w} A_i\Big) \\
&= A_0 - \sum_{i=1}^{m} y_i A_i = S. \qquad (5.2)
\end{aligned}
$$

*We regard the group representation $\mathcal{P}$ as a group and define the action that the group acts on a set as $P(A_i) = PA_iP^{\mathrm{T}}$ and the orbit of A_i as $\{PA_iP^{\mathrm{T}} \mid P \in \mathcal{P}\}$.

It is easy to verify that theorem 5.1 is a special case of theorem 5.2 (When $\mathcal{P}$ acts on A_i, there are m orbits). Roughly speaking, in Theorem 5.2, we divide the data matrices set $A = \{A_i \mid i = 0, ...m\}$ into several non-intersection subsets denoted as O_j , $(j = 1, ..., w)$ and find a representation $\mathcal{P}$ with $\sum_{l \in O_j} A_l$, $(j = 1, ..., w, \ \ w \leqslant m)$ in its commutant. When compared with Theorem 5.1, Theorem 5.2 is more practical. As an application of Theorem 5.2, we will construct a representation $\mathcal{P}$ for a class of truss design problems.

We consider a class of truss design problems with n free nodes, n fixed nodes and $2n$ bars. (In Figure 1, the darker circles are free nodes, and the rest are fixed ones.)

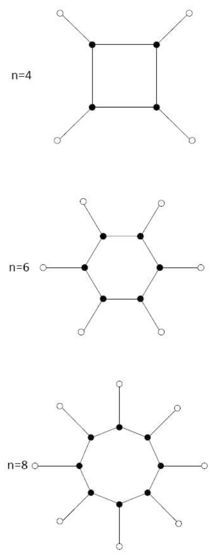

Figure 1

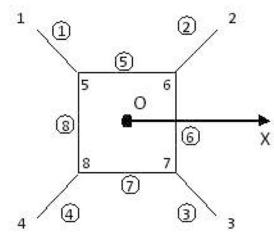

Figure 2

The 3-dimension Euclidean coordinates is constructed as follows:

i). the$x$-axis horizon and towards right.

ii). the z-axis vertical and outside the paper in the geometric center.

Figure 2 is the 3-dimension Euclidean coordinates for the truss when n equals four. O is the z-axis. The circled number i is a label of the i-th bar. We define bars 1,2,3,4 as outer circle bars and bars 5,6,7,8 as inner circle bars.

Obviously, the structural properties of these trusses are invariant under two operations. One is the rotation corresponding to z-axis by $\frac{2\pi k}{n}$, $(k = 0, 1, ..., n-1)$. The other is the reflection corresponding to the $x - z$ plane. Algebraically, the dihedral group is often used to describe such rotations and reflections. The definition of dihedral group D_n is

$$D_n = \left\{ R\Big(\frac{2k\pi}{n}\Big),\ FR\Big(\frac{2k\pi}{n}\Big)\ \Big|\ k = 0, 1, ..., n-1 \right\}$$

where R is a rotation and F is any reflections.

Since there are n free nodes in the truss, the dimension of data matrices is $3n$. Therefore, we need to find a $3n$-dimension representation of D_n and use the Kronecker product to get it.

With respect to the 3-dimension Euclidean space, an anti-clockwise rotation of $\frac{2k\pi}{n}$ $(k = 0, 1, ..., n-1)$ around the z-axis has a matrix representation:

$$s_k = \begin{pmatrix} \cos\dfrac{2k\pi}{n} & -\sin\dfrac{2k\pi}{n} & 0 \\ \sin\dfrac{2k\pi}{n} & \cos\dfrac{2k\pi}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, a reflection across the $x-z$ plane can be represented by:

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, we obtain a 3-dimension representation of D_n:

$$s = \{s_k,\ fs_k\ \ |\ \ k = 0, 1, ..., n-1\}$$

Furthermore, considering the permutation of n free nodes, we have an n-dimension representation of D_n denoted as p.

Using the Kronecker product, we obtain the $3n$-dimension representation $\mathcal{P}$ of D_n

$$\mathcal{P}_k = p_k \otimes s_k, \quad (k = 0, 1, ..., n-1)$$

$$\mathcal{P}_k = p_k \otimes fs_k, \quad (k = n, n+1, ..., 2n-1)$$

We explain that the representation $\mathcal{P}$ is true forthe conditions Theorem 5.2:

1. We consider the case that each free node is added by the same mass, so the data matrix A_0 is a diagonal matrix and the diagonal elements are the same. Therefore, the data matrix A_0 is in the commutant.

2. Under the action of D_n, the physical structural properties of these trusses are invariant. Therefore, the structure of inner circle bars are invariant. In other words, the sum of the inner bars' stiffness and mass matrix is invariant. Similary, the sum of outer bars' stiffness and mass matrix is also invariant. Obviously, two orbits are formed.

3. The decision variations are the size of the cross-sectional areas, so the variables in one orbit can be regarded the same.

According to group symmetric techqinue, the class of truss design problems can be rewrited as:

$$\begin{aligned} \min \quad & \xi_{o_1}\sum_{i\in o_1} b_i + \xi_{o_2}\sum_{j\in o_2} b_j \\ \text{s.t.} \quad & S = \xi_{o_1}\Big(\sum_{i\in o_1}(K_i - \bar{\Omega}M_i)\Big) + \xi_{o_2}\Big(\sum_{j\in o_2}(K_j - \bar{\Omega}M_j)\Big) - \bar{\Omega}R(M_0) \\ & \xi_{o_w} \geqslant 0 \quad w = 1,2 \quad o_1 \text{ is inner circle bars, } o_2 \text{ is outer ones} \\ & S \geqslant 0 \end{aligned} \tag{5.3}$$

Thus, we have reduced the number of decision variables. Moreover, by finding all the bases of representation of $\mathcal{P}$, we can reduce the size of semidefinite constraint and obtain the reduction result in table 1.

Table 1: the result of decomposing

N	**B. N.**	**V. N.**	**origin dim**	**reduction conclusion**
$n \quad odd$	$2n$	2	$3n \times 3n$	$1-dim:1; 2-dim:1; 3-dim:\frac{n-1}{2}$
$n \quad even$	$2n$	2	$3n \times 3n$	$1-dim:2; 2-dim:2; 3-dim:\frac{n}{2}-1$

Remark 5.1 In table 1, 'B.N.' denotes the bar number, 'V.N.' denotes the scalar variation after reduction, 'origin dim' denotes the size of original semidefinite constraint, 'reduction conclusion' denotes the size and the number of the constraints after reduction.

6 Numerical implementation

In this section, we present an example of 8-bar truss design problem to illustrates how to construct a group representation. Assume that S is consisted by 8 nodes and 8 bars(Figure 2). The SDP form of this truss optimization problem for special fundamental eigenvalue can be written as

$$\begin{aligned} \min \quad & \sum_{i=1}^{8} b_i z_i \\ s.t: \quad & S = \sum_{i=1}^{8}(K_i - \overline{\Omega}M_i)z_i - \overline{\Omega}M_0 \\ & S \geqslant 0, \\ & z_i \geqslant 0 \quad i = 1,...,8 \end{aligned} \tag{6.1}$$

The material of the members is steel where κ=205.8GPa and $\rho = 7.86 \times 10^{-3} kg/cm^3$. The specified eigenvalue is 1000.0 rad^2/s^2 for all cases. And nonstructural masses of 2.1×10^4 kg

are located at each free nodes†.

We use the D_4 group to reduce the dimension of the SDP problem. Denote the 3-dimensional Euclidean representation of D_4 as s and the permutation representation of D_4 as p. Therefore, by using the Kronecker product, we can form a 12-dimension representation $\mathcal{P}$ of D_4 as:

$$P_k = p_k \otimes s_k, \quad (k = 0, 1, ..., 7).$$

When acting the group $\mathcal{P}$ on the set of data matrices in problem (6.1), two orbits are formed as $O_1 = \{1, 2, 3, 4\}$ and $O_2 = \{5, 6, 7, 8\}$. (Each number corresponds to the data matrix of the i-th bar). By mergine all variables of one orbit into an variable, the problem (6.1) can be rewrited as follows:

$$\begin{aligned} \min \quad & 2\sqrt{3}\ \xi_{O_1} \ + \ 4\ \xi_{O_2} \\ \text{s.t}: \quad & S = \xi_{O_1} \sum_{i \in O_1} (K_i - \overline{\Omega} M_i) + \xi_{O_2} \sum_{j \in O_2} (K_j - \overline{\Omega} M_j) \\ & S \geqslant 0, \quad \xi_{O_1} \geqslant 0, \quad \xi_{O_2} \geqslant 0 \\ & O_1 = \{1, 2, 3, 4\}, \quad O_2 = \{5, 6, 7, 8\} \end{aligned} \tag{6.2}$$

Therefore, the number of the variables in (6.1) is reduced as the munber of the orbits. In the rest part of the section, we focus on computing the orthogonal matrix Q such that the size of $A_0 - \sum_{i=1}^{m} y_i A_i$ is reduced. In fact, it means that we need to calculate the bases of linear representation $\mathcal{P}$.

By using the result of [2], the permutation representation p of D_4 can be decomposed into

$$p = \Psi_1 \oplus \Psi_3 \oplus \rho^1$$

where Ψ_1 is the trivial representation, Ψ_3 is the 1-dimensional representation given by:

$$\Psi_3(r^k) = (-1)^k, \quad \Psi_3(fr^k) = (-1)^k, \quad (k = 0, 1, 2, 3)$$

ρ^1 is the 2-dimensional irreducible representation given by:

$$\rho^1(r^k) = \begin{pmatrix} \cos\dfrac{\pi k}{2} & -\sin\dfrac{\pi k}{2} \\ \sin\dfrac{\pi k}{2} & \cos\dfrac{\pi k}{2} \end{pmatrix} \quad \rho^1(f) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \rho^1(fr^k) = \rho^1(f) \cdot \rho^1(r^k)$$

Note the dimension of the permutation representation p is 4, so each base of the representation p can be written as the linear combination of the nature bases e_1, e_2, e_3, e_4. (e_i is the column vector that i-th element is 1 and 0 the others.) Using the relation between the representation p and its irreducible representation Ψ_1, Ψ_3, ρ^1, we obtain the bases of p, denoted as V_1, V_2, V_3, V_4. Similarly, we obtain the bases of the representation s through the relation $s = \rho^1 \oplus \Psi_1$, denoted as $\alpha_1, \alpha_2, \alpha_3$. Then, we get the bases of $\mathcal{P}$ by: $\overline{Q} = (V_1, V_2, V_3, V_4) \otimes (\alpha_1, \alpha_2, \alpha_3)$.

†We only consider the case that the same nonstructural mass is added to each free node.

Remark 6.1 The bases formed by Kroneccker product $(V_3, V_4) \otimes (\alpha_1, \alpha_2)$ may be not the bases of the representation $\mathcal{P}$, so we have to compute the four bases of the representation $\mathcal{P}$ in the 4-dimensional space $Span(V_3, V_4) \otimes Span(\alpha_1, \alpha_2)$ once more. The method is similar to the previous, and only the difference is that the properties of Kroneccker product are used. Denote these four new bases as $\beta_1,\ \beta_2,\ \beta_3,\ \beta_4$.

By replacing the four bases $(V_3, V_4) \otimes (\alpha_1, \alpha_2)$ with $\beta_1,\ \beta_2,\ \beta_3,\ \beta_4$, we obtain the bases of the representation $\mathcal{P}$ denoted as Q. Therefore, the 8-bar truss design problem can be writted as follows:

$$\begin{aligned} \min \quad & 2\sqrt{3}\,\xi_{O_1} \ + \ 4\,\xi_{O_2} \\ \text{s.t}: \quad & Y = \xi_{O_1} Q^{\mathrm{T}} S_1 Q + \xi_{O_2} Q^{\mathrm{T}} S_2 Q \\ & Y \geqslant 0, \quad \xi_{O_1} \geqslant 0, \quad \xi_{O_2} \geqslant 0 \\ & O_1 = \{1,2,3,4\}, \quad O_2 = \{5,6,7,8\}. \end{aligned} \tag{6.3}$$

where $Q^{\mathrm{T}} S_1 Q$ and $Q^{\mathrm{T}} S_2 Q$ are shown below, respectively.

$$\begin{pmatrix} 57.1404 & 59.4093 & -59.4093 & & & & & & & & & \\ 59.4093 & 57.1404 & -59.4093 & & & & & & & & & \\ -59.4093 & -59.4093 & 57.1404 & & & & & & & & & \\ & & & 57.1404 & 59.4093 & -59.4093 & & & & & & \\ & & & 59.4093 & 57.1404 & -59.4093 & & & & & & \\ & & & -59.4093 & -59.4093 & 57.1404 & & & & & & \\ & & & & & & 57.1404 & 118.8187 & & & & \\ & & & & & & 118.8187 & 233.0994 & & & & \\ & & & & & & & & 57.1404 & -118.8187 & & \\ & & & & & & & & -118.8187 & 233.0994 & & \\ & & & & & & & & & & -4.5380 & \\ & & & & & & & & & & & -4.5380 \end{pmatrix}$$

$$\begin{pmatrix} -5.24 & & & & & & & & & & & \\ & 197.94 & & & & & & & & & & \\ & & 203.18 & & & & & & & & & \\ & & & -5.24 & & & & & & & & \\ & & & & 197.94 & & & & & & & \\ & & & & & 203.18 & & & & & & \\ & & & & & & -7.86 & & & & & \\ & & & & & & & 401.12 & & & & \\ & & & & & & & & -2.62 & & & \\ & & & & & & & & & 401.12 & & \\ & & & & & & & & & & 401.12 & \\ & & & & & & & & & & & 401.12 \end{pmatrix}$$

From the result of $Q^{\mathrm{T}} S_1 Q$ and $Q^{\mathrm{T}} S_2 Q$, the original semidefinite constraint are reduced into two 1-dimension, two 2-dimension and two 3-dimension semidefinite constraints. In other words, the size of the semidefinite constraint $A_0 - \sum\limits_{i=1}^{m} y_i A_i$ is reduced sufficiently.

7 Conclusions and remarks

We have established two sufficient conditions for constructing the group representation $\mathcal{P}$. It is based on analyzing the group symmetric technique[2]. Furthermore, We have applied it to a family of truss design problems with 8-bar structure as its special case. Through our work, the number of decision variables in the problem have been reduced to the number of the orbit, and the size of the semidefinite constraint $A_0 - \sum_{i=1}^{m} y_i A_i$ have been reduced, too. In practice, many SDP problems are with the property of the group symmetric defined by Kanno et al[4]. Therefore, the two sufficient conditions proposed in our paper may privide a wide application to deal with the reduction of the SDP problems. However, the challenges are existed so far. The first one is how to structure a proper representation $\mathcal{P}$ for a SDP problem. And the seocnd is how to find the irreducible representations of $\mathcal{P}$ efficiently.

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