

# Regularization of B-Spline Objects \*

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## Abstract

By a  $d$ -dimensional B-spline object (denoted as  $\mathcal{O}^d$ ), we mean a B-spline curve ( $d = 1$ ), a B-spline surface ( $d = 2$ ) or a B-spline volume ( $d = 3$ ). By regularization of a B-spline object  $\mathcal{O}^d$  we mean a process of relocating the control points of  $\mathcal{O}^d$  such that it approximates an isometric map of its definition domain in certain directions and is shape preserving. In this paper we develop an efficient regularization method for  $\mathcal{O}^d$ ,  $d = 1, 2, 3$  based on solving weak form  $L^2$ -gradient flows constructed from the minimization of certain regularizing energy functionals. These flows are integrated via the finite element method using B-spline basis functions. Our experimental results demonstrate that our new regularization method is very effective.

**Key words:** Spline objects, Regularization,  $L^2$ -gradient flow, Finite element method.

**MR (2000) Classification:** 65D17

## 1 Introduction

In this paper we use the term a *d-dimensional B-spline object*  $\mathcal{O}^d$  to imply a B-spline curve ( $\mathcal{O}^1$ ), a B-spline surface ( $\mathcal{O}^2$ ) or a B-spline volume ( $\mathcal{O}^3$ ). Since B-spline basis functions are linearly independent, many people believe that the parametric B-spline representation of a B-spline object is unique. However, this is not true. Two different sets of control points may represent the same  $\mathcal{O}^d$ . To see this explicitly, let us consider a Bézier curve  $\mathbf{x}(t) = \sum_{i=0}^n \mathbf{p}_i B_i^n(t) \in \mathbb{R}^2$ , which is a special case of a B-spline curve, on the interval  $[0, 1]$ . Let  $t = t(s) = \alpha s(1-s) + s^2$ ,  $s \in [0, 1]$ ,  $\alpha \in [0, 2]$ . Then  $t(0) = 0$ ,  $t(1) = 1$  with  $t(s)$  increasing. Substituting  $t(s)$  into  $\mathbf{x}(t)$ , we obtain another representation  $y(s) := \mathbf{x}(t(s))$  for the same curve. In this new representation, there is a free parameter  $\alpha \in [0, 2]$ , which shows that the representation is not unique. Though

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the changing of  $\alpha$  does not change the shape of the curve  $\mathcal{O}^1$ , whenever  $y(s)$  is sampled on a uniform grid of the interval  $[0, 1]$ , different  $\alpha$ 's yield very different distributions of points on the curve. For the plane Bézier curve

$$\mathbf{x}(t) = [0, 1]^T B_1^3(t) + [2, 1]^T B_2^3(t) + [2, 0]^T B_3^3(t), \quad t \in [0, 1], \quad (1.1)$$

Fig 1.1 shows the distributions of the sampling points for three different  $\alpha$ 's, where the sub-figures (a), (b) and (c) are generated using  $\alpha$  values 0, 1 and 2, respectively. It is easy to observe that different  $\alpha$ 's yield different distributions of the sampling points.

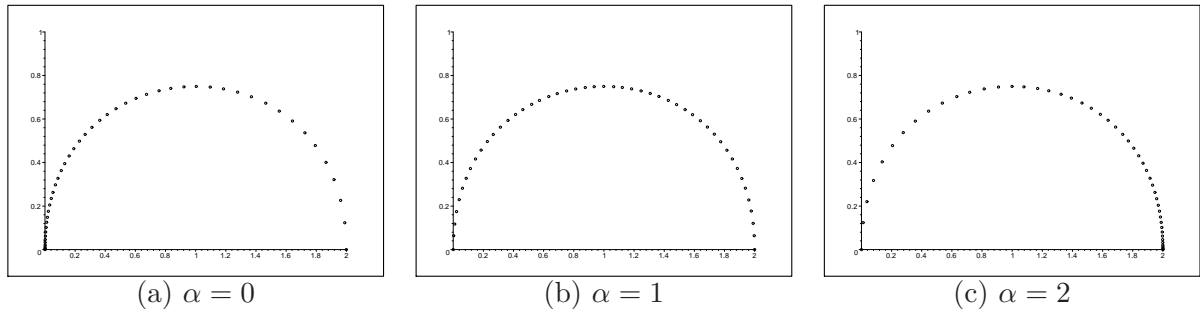


Fig 1.1: (a), (b) and (c) are the sampling points of the curve  $\mathbf{y}(s) = \mathbf{x}(t(s))$  defined by (1.1) at the equal-spaced parameter values on  $[0, 1]$ , using  $\alpha$  values 0, 1 and 2, respectively.

The aim of “regularization of a  $\mathcal{O}^d$ ” is to relocate the control points of  $\mathcal{O}^d$  such that the object shape is not changed and a uniform sampling on the parameter domain leads to  $\mathcal{O}^d$  an approximately uniform sampling. The main idea of the regularization is to construct several  $L^2$ -gradient flows that move the control points of  $\mathcal{O}^d$  in the tangential directions. The foundation of our regularization method is based on a result of Epstein and Gage (see [6]), which states that the tangential motion of a manifold does not change its shape.

The regularization of  $\mathcal{O}^d$  is an interesting problem that has not been directly studied in the past. A direct application of the regularization is to generate a high quality discrete approximation (quality mesh) of  $\mathcal{O}^d$ . Another application perhaps is in the area of computer aided manufacturing (CAM). Quite often, the machine parts to be manufactured are represented by B-splines in a CAM system. When a machine part is manufactured using cutting tools, one often requires the surface motion of the tools or the curve motion relative the tools, be at a uniform rate with respect to the domain parameters.

We arrive at the regularization problem of  $\mathcal{O}^d$  as a result of solving geometric partial differential equations (GPDE) while constructing B-spline GPDE surfaces (see [19]). Certain GPDEs, such as quasi-surface flow (see [20]), contain tangential motion, which cause the evolved surface drift in the tangent directions, yielding an un-isometric surface. Even though the used GPDEs contain no tangential motion, the un-isometric property of the initial surface is inherited during the evolution that follows. Therefore, it becomes necessary to combine a regularization step in the process of surface evolution, especially when the surfaces are being generated for the aforementioned applications.

For the triangular mesh fairing, the regularization problem has been considered in the past in computer graphics literature (see [11, 12, 17, 20]). In these papers, a tangential displacement is imposed at each of the vertices to move it towards its geometric or mass center of the surrounding vertices. In Du et al's publications [4, 5], *Spherical Centroidal Voronoi Tessellation* (SCVT) have been introduced. A Voronoi tessellation  $\{V_i\}_{i=1}^N$  of the sphere  $S$  from the generators  $\{\mathbf{p}_i\}_{i=1}^N \subset S$  is called SCVT if and only if  $\mathbf{p}_i$  is the constrained mass centroid of the Voronoi region  $V_i$  for  $i = 1, \dots, N$ . Recently, a superlinear convergent Centroidal Voronoi Tessellation method for volume has also been presented by Liu, Wang et al. [8]. In [18], a sequence of spherical triangulations are constructed which are optimal in certain sense, leading to smaller truncation error of the discrete Laplace-Beltrami operator.

Another subarea of research that is related to but different from the mesh regularization is mesh re-parametrization, which computes a bijective mapping between a triangular surface mesh and another surface with the same or similar topology. There are several excellent surveys [7, 9, 16] on this topic. For volume data re-parameterization, Martin et al [10] present a method based on discrete volumetric harmonic functions to parameterize a volumetric model in a way that it can be used to fit a single trivariate B-spline to data. We could not find any other prior work that directly addressed the regularization of parametric Bernstein-Bézier or B-spline surfaces.

The rest of our paper is organized as follows: Section 2 introduces some background material on  $\mathcal{O}^d$ . Sections 3 discuss the regularization methods of  $\mathcal{O}^d$ ,  $d = 1, 2, 3$ , respectively. Several experimental examples to illustrate the dramatic effects of  $\mathcal{O}^d$  regularization are given in section 4.

## 2 B-Spline Objects

B-spline representations of curves and surfaces are well-known (see [1, 2, 3, 13, 15]). For easy of understanding and sake of completeness in this paper, we briefly introduce their definition and establish notation, necessary for subsequent use. There are several equivalent approaches to define B-spline functions, including divided differences of truncated power function (see [2, 15]), the blossoming method (see [13]) and the recursive formulas (see [1, 3]). We adopt the approach of recursive formulas which additionally facilitates ease of computer programming.

**Definition.** Given a positive integer  $m$ , nonnegative integer  $k$  and a knot sequence

$$u_0 \leq \dots \leq u_i \leq u_{i+1} \leq u_{i+2} \leq \dots \leq u_{m+2k}.$$

$U = \{u_0, \dots, u_{m+2k}\}$  is referred to as knot vector. Then B-splines basis functions are defined

as follows

$$\left\{ \begin{array}{l} N_{i,0}(u) = \begin{cases} 1, & \text{for } u \in [u_i, u_{i+1}), \\ 0, & \text{otherwise,} \end{cases} \quad i = 0, 1, \dots, m + 2k - 1, \\ N_{i,k}(u) = \frac{u - u_i}{u_{i+k} - u_i} N_{i,k-1}(u) + \frac{u_{i+k+1} - u}{u_{i+k+1} - u_{i+1}} N_{i+1,k-1}(u), \quad i = 0, 1, \dots, m + k - 1, \\ \text{Assume } \frac{0}{0} = 0 \end{array} \right. \quad (2.1)$$

where  $i$  is the index of  $N_{i,k}(u)$ ,  $k$  is the degree. In this paper, we assume that

$$u_0 = u_1 = \dots = u_k = 0, \quad u_k < u_{k+1} < \dots < u_{m+k}, \quad u_{m+k} = \dots = u_{m+2k} = 1.$$

In order to have the B-spline objects to be at least  $C^1$  smooth, we take  $k \geq 2$ .

**B-Spline objects.** For given positive integers  $m_1, m_2, \dots, m_d$ ,  $n \geq d$ , a nonnegative integer  $k$ , and knot vectors

$$U^{(l)} = \{u_0^{(l)}, \dots, u_{m_l+2k}^{(l)}\}, \quad l = 1, \dots, d,$$

the 2d-sided degree  $k$  spline object  $\mathcal{O}^d$  is defined as

$$\mathcal{O}^d = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{x}(\mathbf{u}), \mathbf{u} \in [0, 1]^d\},$$

with

$$\mathbf{x}(\mathbf{u}) = \sum_{i_1=0}^{m_1+k-1} \dots \sum_{i_d=0}^{m_d+k-1} \mathbf{p}_{i_1 \dots i_d} \prod_{j=1}^d N_{i_j, k}(u^{(j)}),$$

where

$$\mathbf{u} = [u^{(1)}, \dots, u^{(d)}]^T \in [0, 1]^d.$$

$\mathbf{p}_{i_1 \dots i_d} \in \mathbb{R}^n$  are called control points of the object  $\mathbf{x}(\mathbf{u})$ . If  $i_l = 0$  or  $m_l + k - 1$ ,  $l = 1, \dots, d$ ,  $\mathbf{p}_{i_1 \dots i_d}$  are called boundary control points. Other control points are called inner control points. Let  $D_l = \frac{\partial}{\partial u^{(l)}}$  be the first order partial derivative operators,  $D_{ij} = D_i D_j$  the second order partial derivative operators. Then  $D_l \mathbf{x}(\mathbf{u})$  are tangential vectors of  $\mathbf{x}(\mathbf{u})$  in the directions  $u^{(l)}$ ,  $l = 1, \dots, d$ . All the vectors perpendicular to the tangent vectors  $D_l \mathbf{x}(\mathbf{u})$  form a normal space.

For a fixed  $\mathbf{u}_l \in [0, 1]^{d-1}$ , let  $L^{(l)}(\mathbf{u}_l)$  be the arc length of the space curve  $\mathbf{x}(\mathbf{u})$  over  $[0, 1]$ , where

$$\mathbf{u}_l = [u^{(1)}, \dots, u^{(l-1)}, u^{(l+1)}, \dots, u^{(d)}]^T \in \mathbb{R}^{d-1}.$$

If  $\|D_l \mathbf{x}(\mathbf{u})\| = L^{(l)}(\mathbf{u}_l)$ ,  $\forall \mathbf{u}_l \in [0, 1]^{d-1}$ , then we say the object is isometric in the  $u^{(l)}$  direction.

In this paper, our attention is focused on the cases  $n = 3$ ,  $d = 1, 2, 3$ , where the B-spline objects  $\mathcal{O}^d$  are space curves  $\mathcal{O}^1$ , surfaces  $\mathcal{O}^2$  and volumes  $\mathcal{O}^3$  in  $\mathbb{R}^3$ , respectively.

### 3 Regularization of B-Spline Objects Using $L^2$ -Gradient Flows

In this section, we give a set of  $L^2$ -gradient flows, and a method for discretization, and finally the solution via a linear system solver. The full derivation of these flows from an energy functional is given in the appendix.

#### 3.1 $L^2$ -Gradient Flows

For a given B-spline object  $\mathcal{O}^d$  defined by  $\mathbf{x}(\mathbf{u})$ , let

$$L^{(l)}(\mathbf{u}_l) = \int_0^1 \|D_l \mathbf{x}(\mathbf{u})\| du^{(l)}, \quad \mathbf{u} \in [0, 1]^d, \quad l = 1, \dots, d.$$

For a fixed  $\mathbf{u}_l \in [0, 1]^{d-1}$ ,  $L^{(l)}(\mathbf{u}_l)$  is the arc length of the space curve  $\mathbf{x}(\mathbf{u})$ ,  $u^{(l)} \in [0, 1]$ , in the  $u^{(l)}$  direction. Let

$$\mathcal{E}^{(l)}(\mathcal{O}^d) = \int_{\mathcal{O}^d} \left[ \|D_l \mathbf{x}(\mathbf{u})\| - L^{(l)}(\mathbf{u}_l) \right]^2 dV, \quad l = 1, \dots, d. \quad (3.1)$$

To regularize the B-spline object  $\mathbf{x}(\mathbf{u})$ , we minimize the energies  $\mathcal{E}^{(l)}(\mathcal{O}^d)$  by moving the object point in the  $D_l \mathbf{x}$  directions, respectively. This goal is achieved by solving  $d$   $L^2$ -gradient flows derived from (3.1). Consider the general form energy functional

$$\mathcal{E}(\mathcal{O}^d) = \int_{\mathcal{O}^d} f(D_1 \mathbf{x}, \dots, D_d \mathbf{x}) dV, \quad (3.2)$$

where  $f$  is a given  $C^1$  smooth function with respect to its arguments.

**Theorem 3.1** *Let  $\mathcal{E}(\mathcal{O}^d)$  be an energy functional defined by (3.2) for the B-spline object  $\mathcal{O}^d$ . Then the  $L^2$ -gradient flow of  $\mathcal{E}(\mathcal{O}^d)$  in the direction  $D_l \mathbf{x}$  is given by*

$$\int_{\mathcal{O}^d} \frac{\partial \mathbf{x}}{\partial t} \phi dV = - \int_{\mathcal{O}^d} \sum_{k=1}^d \left[ \left( (D_{kl} \mathbf{x})(\alpha_k^d)^T + (D_{kl} \mathbf{x})^T \alpha_k^d \mathbf{I}_n \right) (D_l \mathbf{x}) \phi + (D_l \mathbf{x})^T \alpha_k^d \mathbf{I}_n (D_l \mathbf{x}) D_k \phi \right] dV, \quad (3.3)$$

where  $\mathbf{I}_n$  stands for the  $n \times n$  unit matrix, for  $d = 1, 2, 3$ ,  $\alpha_k^d$  are defined by

$$\begin{aligned} \alpha_1^1 &= \nabla_{D_1 \mathbf{x}} f + \frac{D_1 \mathbf{x}}{\|D_1 \mathbf{x}\|}, \\ \alpha_1^2 &= \nabla_{D_1 \mathbf{x}} f + \frac{1}{\sqrt{g}} [g_{22}(D_1 \mathbf{x} - g_{12} D_2 \mathbf{x})], \quad \alpha_2^2 = \nabla_{D_2 \mathbf{x}} f + \frac{1}{\sqrt{g}} [g_{11}(D_2 \mathbf{x} - g_{12} D_1 \mathbf{x})], \\ \alpha_1^3 &= \nabla_{D_1 \mathbf{x}} f + D_2 \mathbf{x} \times D_3 \mathbf{x}, \quad \alpha_2^3 = \nabla_{D_2 \mathbf{x}} f + D_3 \mathbf{x} \times D_1 \mathbf{x}, \quad \alpha_3^3 = \nabla_{D_3 \mathbf{x}} f + D_1 \mathbf{x} \times D_2 \mathbf{x}. \end{aligned}$$

Taking  $l = 1, \dots, d$  in (3.3), we obtain a sequence of flows. For  $n = 3$  and  $d = 1, 2, 3$ , these flows are for B-spline curves, surfaces and volumes, respectively.

### 3.2 Numerical Solutions

To regularize the B-spline object  $\mathcal{O}^d$  in the  $D_l \mathbf{x}$  directions, we solve (3.3) interchangeably using finite element method in the spacial discretization and semi-implicit Euler scheme in temporal discretization. More specifically, the terms

$$(D_{kl} \mathbf{x})(\alpha_k^d)^T + (D_{kl} \mathbf{x})^T \alpha_k^d \mathbf{I}_n, \quad (D_l \mathbf{x})^T \alpha_k^d \mathbf{I}_n, \quad (3.4)$$

in (3.3) are treated as known quantities which are computed from the previous step of the approximation. The term  $D_l \mathbf{x}$  is treated as the unknowns.

For easy of description, we reorder the control points  $\mathbf{p}_{i_1 \dots i_d}$  of the B-spline object into a 1-dimensional array and represent them as:

$$\mathbf{x}_0, \dots, \mathbf{x}_{n_0}, \mathbf{x}_{n_0+1}, \dots, \mathbf{x}_{n_1},$$

where  $\mathbf{x}_0, \dots, \mathbf{x}_{n_0}$  are inner control points,  $\mathbf{x}_{n_0+1}, \dots, \mathbf{x}_{n_1}$  are boundary control points, and

$$n_0 = \prod_{i=1}^d (m_i + k - 2) - 1, \quad n_1 = \prod_{i=1}^d (m_i + k) - 1.$$

The B-spline basis functions  $\prod_{j=1}^d N_{i_j, k}(u^{(j)})$  are correspondingly reordered and represented as

$$\phi_0, \dots, \phi_{n_0}, \phi_{n_0+1}, \dots, \phi_{n_1}.$$

Using this ordering of the basis functions and control points,  $\mathcal{O}^d$  can be represented as

$$\mathbf{x}(\mathbf{u}) = \sum_{j=0}^{n_0} \mathbf{x}_j \phi_j(\mathbf{u}) + \sum_{j=n_0+1}^{n_1} \mathbf{x}_j \phi_j(\mathbf{u}). \quad (3.5)$$

Substituting  $\mathbf{x}(u, v)$  into (3.3), and taking the test function  $\phi$  as  $\phi_i$ , for  $i = 0, \dots, n_0$ , we can discretize (3.3) as a set of linear systems of ordinary differential equations (ODE) with the inner control points  $\mathbf{x}_i$ ,  $i = 0, \dots, n_0$ , as unknowns.

$$\sum_{j=0}^{n_0} m_{ij}^{(l)} \frac{d\mathbf{x}_j(t)}{dt} + \sum_{j=0}^{n_0} q_{ij}^{(l)} \mathbf{x}_j = \sum_{j=n_0+1}^{n_1} q_{ij}^{(l)} \mathbf{x}_j, \quad i = 0, \dots, n_0, \quad l = 1, \dots, d,$$

where

$$m_{ij}^{(l)} = \int_{\mathcal{O}^d} \phi_i \phi_j \mathbf{I}_n \, dV,$$

$$q_{ij}^{(l)} = \int_{\mathcal{O}^d} \sum_{k=1}^d \left[ \left( (D_{kl} \mathbf{x})(\alpha_k^d)^T + (D_{kl} \mathbf{x})^T \alpha_k^d \mathbf{I}_n \right) (D_l \phi_j) \phi_i + (D_l \mathbf{x})^T \alpha_k^d \mathbf{I}_n (D_l \phi_j) D_k \phi_i \right] dV,$$

For the temporal direction discretization of these ODE systems, we use a forward Euler scheme which finally leads to a set of linear systems of algebraic equations.

$$\sum_{j=0}^{n_0} \left( m_{ij}^{(l)} + \tau q_{ij}^{(l)} \right) \mathbf{x}_j^{(s)} = \sum_{j=0}^{n_0} m_{ij}^{(l)} \mathbf{x}_j^{(s-1)} + \sum_{j=n_0+1}^{n_1} q_{ij}^{(l)} \mathbf{x}_j^{(0)}, \quad i = 0, \dots, n_0, \quad l = 1, \dots, d, \quad (3.6)$$

where  $\tau$  is a temporal step-size. Solving these linear systems via GMRES (see [14]) for  $l = 1, \dots, d$ , we obtained the new inner control points.

## 4 Algorithm Steps and Illustrative Examples

This section outline the steps of regularization algorithms for B-spline curves, surfaces and volumes. Illustrative examples are also given.

### 4.1 Curve Regularization

Let  $\mathcal{O}^1$  be the B-spline curve to be regularized and defined by  $\mathbf{x}(u) = \sum_{i_1=0}^{m_1+k-1} \mathbf{p}_{i_1} N_{i_1,k}(u)$ .

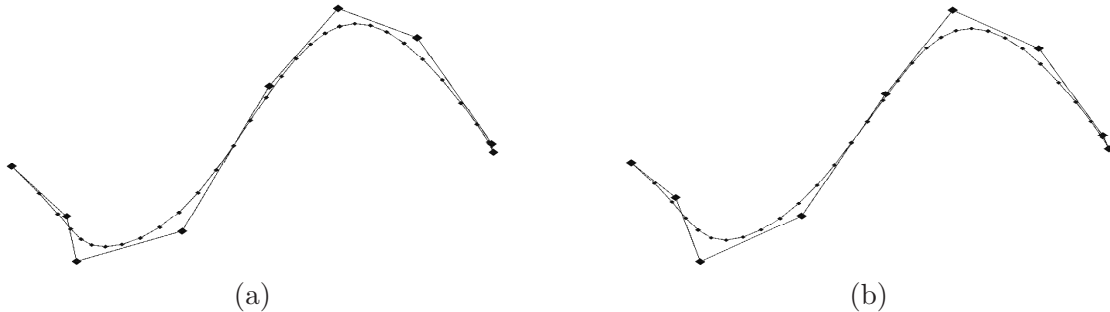


Fig 4.1: (a) and (b) are control polygons as well as the sampled curves of a spline curve before and after regularization. The larger and smaller diamonds represent the control points and sampled curve points, respectively.

#### Algorithm 4.1. Curve Regularization

1. Set the temporal step-size  $\tau$ , and set  $s = 0$ ,  $\mathbf{p}_{i_1}^{(0)} = \mathbf{p}_{i_1}$ .
2. For  $n = 3$  and  $d = 1$ , form the linear system (3.6) and solve it.
3. Check the stopping conditions if  $s \geq 1$

$$\begin{aligned} \max \|\mathbf{p}_{i_1}^{(s)} - \mathbf{p}_{i_1}^{(s-1)}\| &\leq \varepsilon_1 \tau, \\ \text{dis}((\mathcal{O}^1)^{(s)}, (\mathcal{O}^1)^{(s-1)}) &\geq \varepsilon_2, \end{aligned}$$

where  $\text{dis}(\cdot, \cdot)$  stands for Hausdorff distance between two curves. If one of the two conditions is satisfied, stop the iteration, otherwise set  $s$  to be  $s + 1$ , go back to the previous step.

Figure 4.1 shows an regularization example of a B-spline curve, which is defined on the equal-spaced knots with  $k = 3$ ,  $m_1 = 6$ .

## 4.2 Surface Regularization

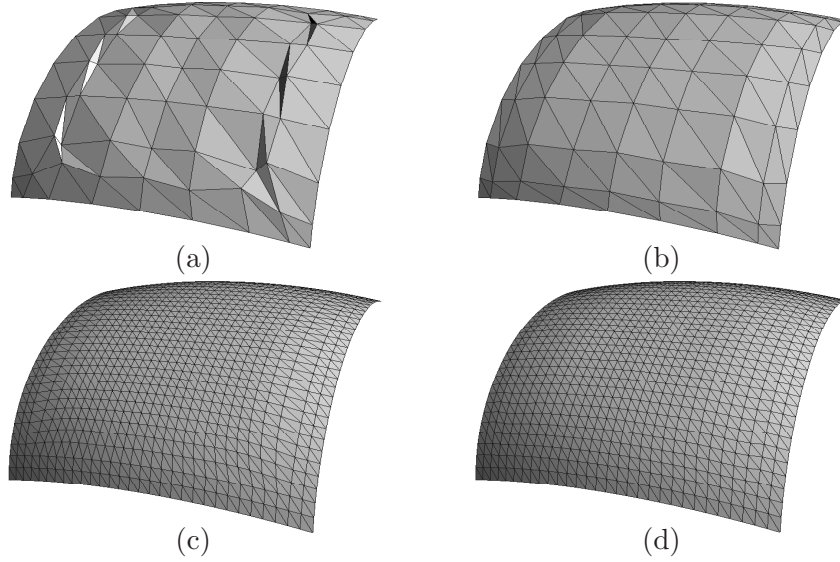


Fig 4.2: Convex B-spline surface: (a) Control mesh of a spline surface without regularization. (b) Control mesh of the same spline surface after regularization. (c) Spline surface mesh without regularization. (d) Spline surface mesh after regularization.

Let  $\mathcal{O}^2$  be given B-spline surface to be regularized and defined by

$$\mathbf{x}(u, v) = \sum_{i_1=0}^{m_1+k-1} \sum_{i_2=0}^{m_2+k-1} \mathbf{p}_{i_1 i_2} N_{i_1, k}(u) N_{i_2, k}(v).$$

### Algorithm 4.2. Surface Regularization

1. Regularize each of the four boundary curves using the curve regularization algorithm.
2. Regularize the interior of the surface as follows
  - (a) Set the temporal step-size  $\tau$ , and set  $s = 0$ ,  $\mathbf{p}_{i_1 i_2}^{(0)} = \mathbf{p}_{i_1 i_2}$ .
  - (b) For  $n = 3$  and  $d = 2$ , form the systems (3.6) and solve them for  $l = 1, 2$ .
  - (c) Check the stopping conditions

$$\max \|\mathbf{p}_{i_1 i_2}^{(s)} - \mathbf{p}_{i_1 i_2}^{(s-1)}\| \leq \varepsilon_1 \tau, \quad (4.1)$$

$$\text{dis}((\mathcal{O}^2)^{(s)}, (\mathcal{O}^2)^{(s-1)}) \geq \varepsilon_2, \quad (4.2)$$



where  $\text{dis}(\cdot, \cdot)$  stands for Hausdorff distance between two surfaces. If one of the two conditions is satisfied, stop the iteration, otherwise set  $s$  to be  $s + 1$ , go back to step (b).

**Remark 4.1** *Though the tangential motion does not change the shape of the surface, but since the PDEs to be solved are nonlinear, the semi-implicit discretization yields certain amount of normal motion. This motion is small relative to the tangential motion. However the accumulation of long time evolution will make it significant. The stopping condition (4.2) is used to control this movement within a given allowable bound.*

**Remark 4.2** *If a B-spline surface representation is not unique, then there are degrees of freedom to relocate the control points to regularize the surface. If the B-spline surface representation is unique, then there will be no freedom for the tangential motion. However we can still regularize the surface within a given error bound  $\varepsilon_2$ .*

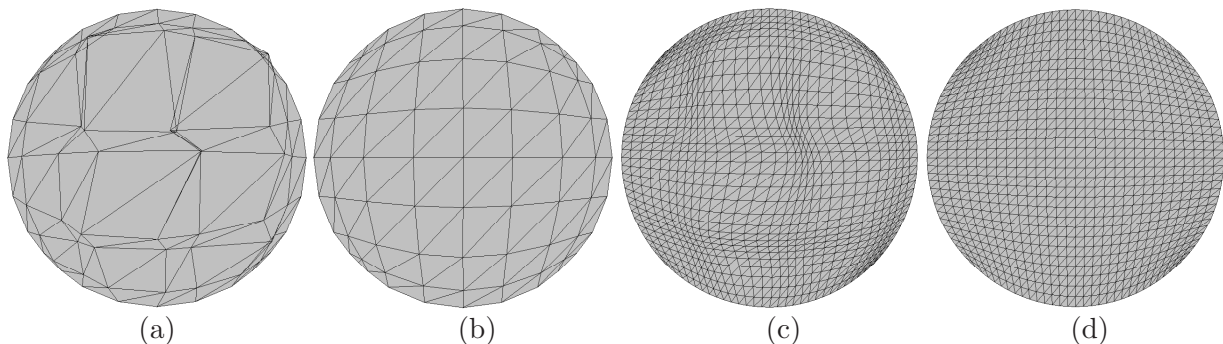


Fig 4.3: (a) Control mesh of a spline surface without regularization. (b) Control mesh of the same spline surface after regularization. (c) Spline surface mesh without regularization. (d) Spline surface mesh after regularization.

Now we give a few examples to illustrate that the surface regularization method proposed is efficient and gives good results. Fig 4.2–4.4 show the regularization effects for B-spline surfaces with different shaped boundaries. The B-spline basis are defined on the equal-spaced knots with  $k = 3$  and  $m_1 = m_2 = 6$ . Sub-figures (a) and (b) show the control meshes of the spline surfaces on the domain  $[0, 1]^2$  without and with using the regularization step, respectively. Sub-figures (c) and (d) show the uniform sampling of the spline surfaces on the domain  $[0, 1]^2$  without and with using the regularization step, respectively. It is easy to observe that the surface shapes are unchanged after the regularization, but the quality of the sampling surface meshes are significantly improved.

### 4.3 Volume Regularization

Let  $\mathcal{O}^3$  be the given B-spline volume to be regularized and defined by

$$\mathbf{x}(u, v, w) = \sum_{i_1=0}^{m_1+k-1} \sum_{i_2=0}^{m_2+k-1} \sum_{i_3=0}^{m_3+k-1} \mathbf{p}_{i_1 i_2 i_3} N_{i_1, k}(u) N_{i_2, k}(v) N_{i_3, k}(w).$$

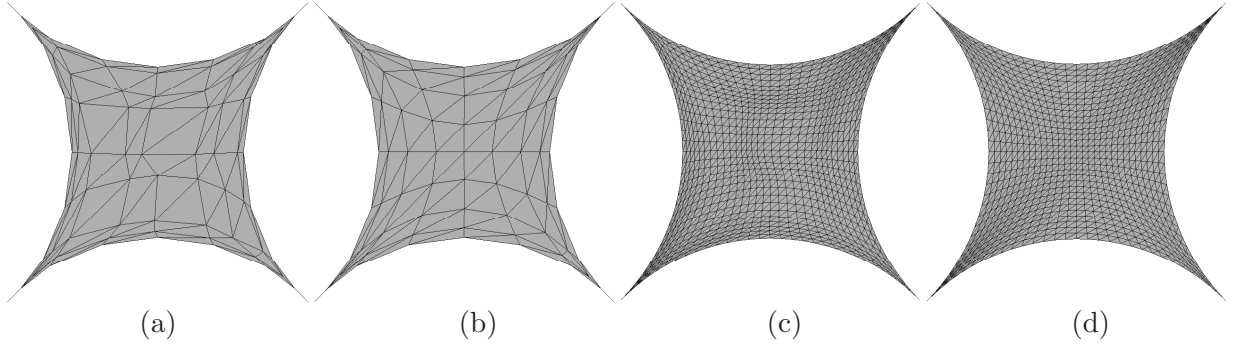


Fig 4.4: (a) Control mesh of a spline surface without regularization. (b) Control mesh of the same spline surface after regularization. (c) Spline surface mesh without regularization. (d) Spline surface mesh after regularization.

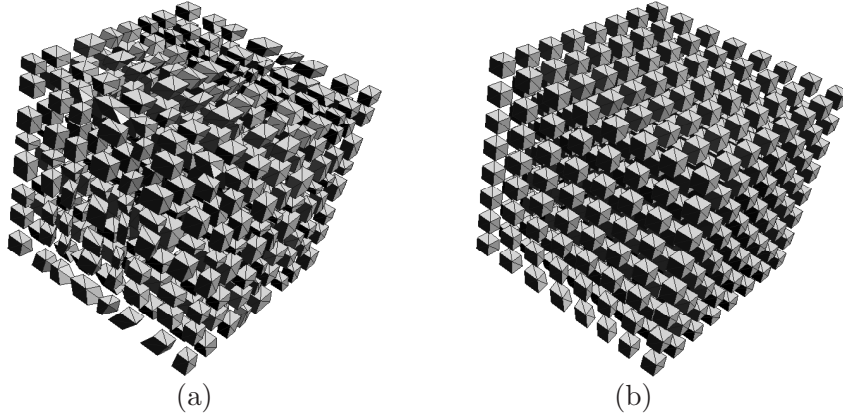


Fig 4.5: Regularization of a cube: (a) Sampling of the initial B-spline volume with irregularly distributed control points. (b) Sampling of the B-spline volume after the regularization.

**Algorithm 4.3.** *Volume Regularization*

1. Regularize each of 12 boundary curves of the volume using the curve regularization algorithm.
2. Regularize each of 6 boundary faces of the volume using the surface regularization algorithm.
3. Regularize the interior of the volume as follows:
  - (a) Set the temporal step-size  $\tau$ , and set  $s = 0$ ,  $\mathbf{p}_{i_1 i_2 i_3}^{(0)} = \mathbf{p}_{i_1 i_2 i_3}$ .
  - (b) For  $n = 3$  and  $d = 3$ , form the systems (3.6) and then solve them for  $l = 1, 2, 3$ .
  - (c) Check the stopping conditions if  $s \geq 1$

$$\max \|\mathbf{p}_{i_1 i_2 i_3}^{(s)} - \mathbf{p}_{i_1 i_2 i_3}^{(s-1)}\| \leq \varepsilon_1 \tau, \quad (4.3)$$

If this condition is satisfied, stop the iteration, otherwise set  $s$  to be  $s + 1$ , go back to step (b).

**Remark 4.3** *Since the tangential space for the B-spline volume is three dimensional, the normal space is null. Hence there is no normal motion problem as mentioned in Remark 4.1. For this reason, we do not need the control of normal motion. We further point out that the regularization for B-spline volume could always be performed. This is different from the cases of curves and surfaces.*

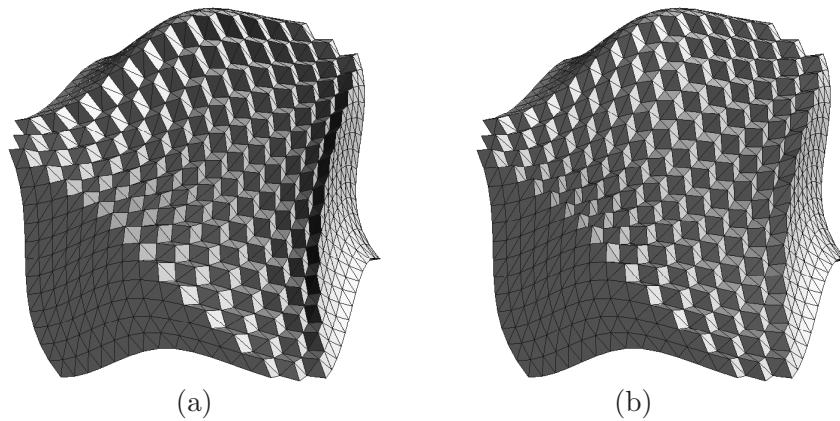


Fig 4.6: Regularizing a B-spline volume with curved boundary faces: (a) Sampling of the initial B-spline volume. (b) Sampling of the B-spline volume after the regularization.

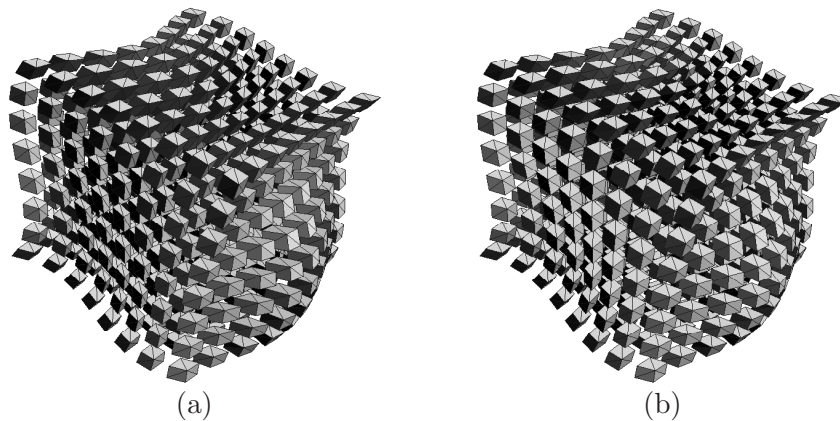


Fig 4.7: Regularizing a B-spline volume with curved boundary faces: (a) Sampling of the initial B-spline volume. (b) Sampling of the B-spline volume after the regularization.

Now we present a few test examples to illustrate the regular effect of B-spline volumes. To see the insides of the sampled volumes, we use both spacing and cutoff mesh display techniques. Figure 4.5 shows the regularization of a cube, where the initial inner control points are not

regularly distributed. The left figure shows this irregularities. The right figure is the result after regularization. Figure 4.6 and Figure 4.7 show the regularizing results for a volume with curved boundary faces. These figures clearly exhibit the regularization effects. In these examples, the B-spline volume are defined on equal-spaced knots with  $m_1 = m_2 = m_3 = 6$  and  $k = 3$ . The sampling is on equal-spaced grid of the domain  $[0, 1]^3$  with sampling rate  $16^3$ .

## 5 Conclusions and Extensions

We have presented a novel and efficient regularization technique for multi-dimensional B-spline objects. Our new technique consists of first devising appropriate energy functionals, then constructing  $L^2$ -gradient flows via the minimization of the regularizing energy functionals and finally solving the flows using the finite element method. Our regularization method can be used to generate uniform polygonal approximations for curves, near regular and quality quadrilateral surface meshes for surfaces and near regular and quality hexahedral volumetric meshes for volumes. The implementation results show that the proposed method is effective and yields our desired results. This research could be extended in several directions. Here we list a few possibilities.

1. Apply the regularizing method to regular curve, surface and volume meshes. This could be done by first interpolating the mesh with the B-spline objects, and then regularizing the B-spline objects and finally resampling them to get the regularized meshes. We are planing to regularize the meshes from segmentation of virus image data.
2. The regularization method for B-spline objects could be applied to other type objects, such as Bézier objects, NURBS objects as well as subdivision objects (subdivision curves, surfaces and volumes) and so on.
3. The regularizing energies used could be replaced with other type functionals to meet different requirement, such as the energy functional measuring the angle distribution of the tangent vectors aiming at to achieve uniform distribution of angles. Since the flows for surface and volume are derived for the general form functionals, these extensions are straightforward.
4. The spatial discretization of the used flows by finite element method we use involves integral computations for forming the coefficient matrices of the ODE systems. For high dimension problem, such as volume regularization ( $9(m_1 + k - 2)^2(m_2 + k - 2)^2(m_3 + k - 2)^2$  3D integrals need to be computed), the computation is intensive if  $m_1, m_2 \dots$  are large. To get a more efficient evolution process, one may use divided difference method to solve the flows. In order to do that the weak form flows need to be reformulated into non-weak forms using Green's formulas.

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## A The Derivation of $L^2$ -gradient Flows

We construct an  $L^2$ -gradient flow for a general form energy functional

$$\mathcal{E}(\mathcal{O}^d) = \int_{\mathcal{O}^d} f(D_1\mathbf{x}, \dots, D_d\mathbf{x}) \, dV = \int_0^1 \cdots \int_0^1 f(D_1\mathbf{x}, \dots, D_d\mathbf{x}) J^d \, d\mathbf{u}, \quad (\text{A.1})$$

where  $f$  is a given  $C^1$  smooth function with respect to its arguments,  $J^d$  is the metric of the object. For  $n = 3$  and  $d = 1, 2, 3$ ,

$$\begin{aligned} J^1 &= \|D_1\mathbf{x}\|, \\ J^2 &= \sqrt{g}, \quad g = g_{11}g_{22} - g_{12}^2, \\ &\quad g_{11} = (D_1\mathbf{x})^T D_1\mathbf{x}, \quad g_{12} = (D_1\mathbf{x})^T D_2\mathbf{x}, \quad g_{22} = (D_2\mathbf{x})^T D_2\mathbf{x}, \\ J^3 &= \det[D_1\mathbf{x}, D_2\mathbf{x}, D_3\mathbf{x}], \end{aligned}$$

and  $\det[\cdot]$  stands for the determinant of a matrix. Let

$$\underline{\mathcal{O}}^d(\Phi, \varepsilon) = \{\underline{\mathbf{x}}(\mathbf{u}, \varepsilon) = \mathbf{x} + \varepsilon\Phi(\mathbf{u}) : \mathbf{u} \in [0, 1]^d\}, \quad \Phi \in C_0^1([0, 1]^d)^n.$$

Then we have

$$\delta(\mathcal{E}(\mathcal{O}^d), \Phi) = \frac{d}{d\varepsilon} \mathcal{E}(\underline{\mathcal{O}}^d(\cdot, \varepsilon)) \Big|_{\varepsilon=0},$$

where

$$\delta(\mathcal{E}(\mathcal{O}^d), \Phi) = \int_0^1 \cdots \int_0^1 \left[ \sum_{k=1}^d (\nabla_{D_k\mathbf{x}} f)^T \delta(D_k\mathbf{x}) + \frac{f\delta(J^d)}{J^d} \right] J^d \, d\mathbf{u}. \quad (\text{A.2})$$

Now we compute  $\delta(D_k\mathbf{x})$  and  $\delta(J^d)$  in (A.2). It follows from

$$\underline{\mathbf{x}} = \mathbf{x} + \varepsilon\Phi, \quad (\text{A.3})$$

$$D_k\underline{\mathbf{x}} = D_k\mathbf{x} + \varepsilon D_k\Phi, \quad (\text{A.4})$$

we have

$$\delta(D_k\mathbf{x}) = D_k\Phi, \quad k = 1, \dots, d,$$

and

$$\begin{aligned}\delta(J^1) &= \frac{(D_1\Phi)^T D_1\mathbf{x}}{\|D_1\mathbf{x}\|}, \\ \delta(J^2) &= \frac{1}{\sqrt{g}} [g_{22}(D_1\Phi)^T D_1\mathbf{x} + g_{11}(D_2\Phi)^T D_2\mathbf{x} - g_{12}[(D_1\Phi)^T D_2\mathbf{x} + (D_2\Phi)^T D_1\mathbf{x}]] . \\ \delta(J^3) &= \det[D_1\Phi, D_2\mathbf{x}, D_3\mathbf{x}] + \det[D_1\mathbf{x}, D_2\Phi, D_3\mathbf{x}] + \det[D_1\mathbf{x}, D_2\mathbf{x}, D_3\Phi] \\ &= (D_1\Phi)^T (D_2\mathbf{x} \times D_3\mathbf{x}) + (D_2\Phi)^T (D_3\mathbf{x} \times D_1\mathbf{x}) + (D_3\Phi)^T (D_1\mathbf{x} \times D_2\mathbf{x}),\end{aligned}$$

where  $\times$  denotes cross product of two vectors in  $\mathbb{R}^3$ . Hence  $\delta(J^d)$  could be written as

$$\delta(J^d) = \sum_{k=1}^d (D_k\Phi)^T \beta_k^d, \quad \beta_k^d \in \mathbb{R}^n.$$

Substituting these into (A.2), we have

$$\begin{aligned}\delta(\mathcal{E}(\mathcal{O}^d), \Phi) &= \int_{\mathcal{O}^d} \sum_{k=1}^d \left[ (D_k\Phi)^T \left( \nabla_{D_k\mathbf{x}} f + \beta_k^d \right) \right] dV \\ &= \int_{\mathcal{O}^d} \sum_{k=1}^d \left( (D_k\Phi)^T \alpha_k^d \right) dV,\end{aligned}\tag{A.5}$$

where

$$\alpha_k^d = \nabla_{D_k\mathbf{x}} f + \beta_k^d \in \mathbb{R}^n,$$

To construct  $L^2$ -gradient flows moving the volume in the tangent  $D_l\mathbf{x}$  direction,  $l = 1, \dots, d$ , we take

$$\Phi = (D_l\mathbf{x})(D_l\mathbf{x})^T \phi, \quad \phi \in C_0^1([0, 1]^d).$$

Then

$$D_k\Phi = (D_{kl}\mathbf{x})(D_l\mathbf{x})^T \phi + (D_l\mathbf{x})(D_{kl}\mathbf{x})^T \phi + (D_l\mathbf{x})(D_l\mathbf{x})^T D_k\phi.$$

Therefore, we construct the following weak form  $L^2$ -gradient flows moving the surface in the  $D_l\mathbf{x}$  direction

$$\begin{aligned}\int_{\mathcal{O}^d} \frac{\partial \mathbf{x}}{\partial t} \phi \, dV &= - \int_{\mathcal{O}^d} \sum_{k=1}^d \left[ \left( (D_{kl}\mathbf{x})(\alpha_k^d)^T + (D_{kl}\mathbf{x})^T \alpha_k^d \mathbf{I}_n \right) (D_l\mathbf{x})\phi + (D_l\mathbf{x})^T \alpha_k^d \mathbf{I}_n (D_l\mathbf{x}) D_k\phi \right] dV, \\ & \quad l = 1, \dots, d.\end{aligned}\tag{A.6}$$

For the energy functional defined by (3.1),  $f = [\|D_l\mathbf{x}(\mathbf{u})\| - L^l(\mathbf{u}_l)]^2$ , it is easy to derive that

$$\nabla_{D_l\mathbf{x}} f = 2 \left[ 1 - \frac{L^l(\mathbf{u}_l)}{\|D_l\mathbf{x}(\mathbf{u})\|} \right] D_l\mathbf{x}(\mathbf{u}), \quad \nabla_{D_k\mathbf{x}} f = \mathbf{0}, \quad \text{for } k \neq l.$$