An Explicit Harmonic Extension for The Constant-like Basis And Its Application to Domain Decompositions[1]

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Abstract

In this paper we are concerned with substructuring methods for the second-order elliptic problems in three-dimensional domains. We first design a simple and completely explicit nearly harmonic extension for the *constant-like* basis function (i.e., the face basis function), and then define a coarse subspace based on this nearly harmonic extension. Own to the resulting coarse solver, we develop a kind of substructuring preconditioner with inexact solvers. We show that the condition number of the preconditioned system grows only as the logarithm of the dimension of the local problem associated with an individual substructure, and is independent of possible jumps of the coefficient in the elliptic equation. Numerical experiments confirms the theoretical results.

Key words. domain decomposition, coarse subspace, constant-like basis, nearly harmonic extension, preconditioner, inexact solvers, condition number

AMS(MOS) subject classification 65F10, 65N30, 65N55

1 Introduction

Non-overlapping domain decomposition methods (DDMs) have been shown to be powerful techniques for solving partial differential equations, especially for the case with large jump coefficients. One's main task in non-overlapping DDMs is the construction of an efficient substructuring preconditioner for discretization system associated with the underlying partial differential equations. The construction of this preconditioner has been investigated from various ways and to various models in literature, see, for example, [1]-[11], [13]-[18], [20]-[24], [26]-[27], [29].

Most non-overlapping DDMs studied so far require exact subdomain solvers; we refer [28] and [32] (and the references cited therein). Such a requirement severely degrade the efficiency of the methods. There are only a few works studying substructuring methods with inexact subdomain solvers [3], [4], [11] and [6]. The essential difficulty is that discrete harmonic extensions on each subdomain are used in non-overlapping domain decomposition methods. In [3], analysis and numerical experiments with inexact algorithms of Neumann-Dirichlet type was done under the additional assumption of high accuracy of the inexact solvers. In [4], the harmonic extension on a subdomain was replaced by a simple *average* extension, and substructuring preconditioners with the average extension are constructed. Because of such *average* extension, nearly optimal convergence can not be gotten for these substructuring preconditioners. To avoid harmonic extensions, [11] considered so called *approximate* harmonic basis functions, which still involve high accuracy of the inexact solvers. Another way to construct substructuring preconditioner with inexact solvers was considered in [6] (mainly for two dimensions). In this preconditioner, overlapping face subspaces are used, and harmonic extension is used only in the definition of coarse subspace. How to design *nearly* harmonic extensions involved in the coarse subspace is the core problem in

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the construction of such substructuring preconditioner. In the case with three dimensions, the coarse subspace may consist of the edge basis and face basis, which are *constant-like* functions. Then, one needs only to define suitable extensions for *constant-like* functions on faces. Such extensions, which depend on the geometric shapes of the subdomains, was first studied in [8].

In the present paper we introduce a new variant of substructuring preconditioner with inexact solvers considered in [6] (see also [32]). The main contribution of the paper is the design of a simple and completely explicit extension for the *constant-like* basis function. This extension plays a key role in the substructuring preconditioner with inexact solvers. We show that the new substructuring preconditioner possesses nearly optimal convergence, which is independent of possible large jumps of the coefficient across the interface. For the new method, no additional assumption is required.

The outline of the remainder of the paper is as follows. In Section 2, we introduce some notation and our motive. In Section 3, we present an explicit extension of the constant-like function. The results on the substructuring preconditioner with inexact solvers are described in Section 4. In Section 5, we prove the stability of the extension of *constant-like* function, which is used in Section 4. Some numerical results are reported in Section 6.

2 Preliminaries

2.1 Domain decomposition

Let Ω be a bounded polyhedron in $\mathcal{R}^3$. Consider the model problem

$$\begin{cases} -div(\omega\nabla u) = f, \quad in \;\; \Omega, \\ \quad u = 0, \quad on \;\; \partial\Omega, \end{cases} \tag{2.1}$$

where $\omega \in L^\infty(\Omega)$ is a positive function.

Let $H_0^1(\Omega)$ denote the standard Sobolev space, and let $(\cdot,\cdot)$ denote the $L^2(\Omega)$-inner product. The weak formulation of (2.1) in $H_0^1(\Omega)$ is then given by the following.

Find $u \in H_0^1(\Omega)$ *such that*

$$\mathcal{A}(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$, and

$$\mathcal{A}(u,v) = \int_\Omega \omega\nabla u\cdot\nabla v dp.$$

We will apply a kind of non-overlapping domain decomposition method to solving (2.2). For simplicity of exposition, we consider only the case with matching grids in this paper.

Let $\mathcal{T}_h = \{\tau_i\}$ be a regular and quasi-uniform triangulation of Ω with $\tau_i's$ being non-overlapping simplexes of size h $(\in (0,1])$. The set of notes of $\mathcal{T}_h$ is denoted by $\mathcal{N}_h$. We then define $V_h(\Omega)$ to be the piecewise linear finite element subspace of $H_0^1(\Omega)$ associated with $\mathcal{T}_h$:

$$V_h(\Omega) = \{v \in H_0^1(\Omega): \; v|_\tau \in \mathcal{P}_1 \;\; \forall\tau \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ is the space of linear polynomials. Then the finite element approximation for (2.2) is to find $u_h \in V_h(\Omega)$ such that

$$\mathcal{A}(u_h,v_h) = (f,v_h), \quad \forall v_h \in V_h(\Omega). \tag{2.3}$$

Let Ω be decomposed into the union of N polyhedrons $\Omega_1,\cdots,\Omega_\text{N}$, which satisfy $\Omega_i\cap\Omega_j = \emptyset$ when $i \neq j$. We assume that each $\partial\Omega_k$ can be written as a union of boundaries of elements in $\mathcal{T}_h$, and all Ω_k are of size H in the usual sense (see [5] and [32]). Without loss of generality, we assume that the coefficient $\omega(p)$ is piecewise constant, then each subdomain Ω_k is chosen such that $\omega(p)$ equals to a constant ω_k in Ω_k. Note that $\{\Omega_k\}$ may not constitute a triangulation of Ω.

The common part of two neighboring subdomains Ω_i and Ω_j may be a vertex, an edge or a face. In particular, we denote by Γ_{ij} the common face of two neighboring subdomains Ω_i and Ω_j (i.e., $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$). The union of all Γ_{ij} is denoted by Γ, which is called the interface. In this paper, we choose Dirichlet data as the interface unknown.

Define the operator $A_h : V_h(\Omega) \to V_h(\Omega)$ by

$$(A_h v,\ w) = \mathcal{A}(v, w) = \sum_{k=1}^{N} \omega_k \int_{\Omega_k} \nabla v \cdot \nabla w dx, \quad v \in V_h(\Omega),\ \forall w \in V_h(\Omega).$$

The equation (2.3) can be written as

$$A_h u_h = f_h, \quad u_h \in V_h(\Omega). \tag{2.4}$$

The goal of this paper is to construct a substructuring preconditioner for A_h based on the domain decomposition described above.

2.2 Notations

To introduce the new method, we need some more notations. Throughout this paper, a subset G of Ω are always understood as an open set.

• subdomain spaces

For subdomain Ω_k, define

$$V_h(\Omega_k) = \{v|_{\Omega_k} :\ \forall v \in V_h(\Omega)\},$$

and

$$V_h^p(\Omega_k) = \{v_h \in V_h(\Omega) :\ supp\ v_h \subset \Omega_k\}.$$

Set

$$\Omega_{ij} = \Omega_i \cup \Gamma_{ij} \cup \Omega_j,$$

and define

$$V_h^p(\Omega_{ij}) = \{v_h \in V_h(\Omega) :\ supp\ v_h \subset \Omega_{ij}\}.$$

• interface space and face spaces

As usual, we define the (global) interface space by

$$W_h(\Gamma) = \{v|_\Gamma :\ \forall v \in V_h(\Omega)\}.$$

For each $\partial\Omega_k$, set

$$W_h(\partial\Omega_k) = \{v|_{\partial\Omega_k} :\ \forall v \in W_h(\Gamma)\}.$$

For a face $\mathrm{F} = \Gamma_{ij}$, define

$$\tilde{W}_h(\mathrm{F}) = \{\phi_h \in W_h(\Gamma) :\ supp\ \phi_h \subset \mathrm{F}\}.$$

• interpolation-type operator and *constant-like* basis

For a subset G of Γ, define the interpolation-type operator $I_G^0 : W_h(\Gamma) \to W_h(\Gamma)$ as

$$(I_G^0 \phi_h)(p) = \begin{cases} \phi_h(p), & \text{if } p \in \mathcal{N}_h \cap G, \\ 0, & \text{if } p \in \mathcal{N}_h \cap (\Gamma \backslash G). \end{cases}$$

In particular, we have

$$(I_G^0 1)(p) = \begin{cases} 1, & \text{if } p \in \mathcal{N}_h \cap G, \\ 0, & \text{if } p \in \mathcal{N}_h \cap (\Gamma \backslash G). \end{cases}$$

If G is an edge or a face generated by the domain decomposition, we call $\phi_G = I_G^0 1$ to be *constant-like* basis function on G, which will be used repeatedly.

• integration average and algebraic average

For a function $\varphi_h \in W_h(\Gamma)$, let $\gamma_G(\varphi_h)$ denote the integration average of φ_h on G, and let $\gamma_{h,G}(\varphi)$ denote the algebraic average of the values of φ on the nodes in G.

• sets of faces, edges, vertices and subdomains

For convergence, let $\mathcal{F}_\Gamma$ denote the set of all the faces Γ_{ij}. Besides, let $\mathcal{E}_\Gamma$ and $\mathcal{V}_\Gamma$ denote the set of the interior edges and the set of interior vertices generated by the decomposition

$$\bar{\Omega} = \bigcup \bar{\Omega}_k,$$

respectively. For an edge $\mathrm{E} \in \mathcal{E}_\Gamma$, let $\mathcal{Q}_\mathrm{E}$ denote the set of the indices k of the subdomains Ω_k which contain E as an edge. Namely,

$$\mathcal{Q}_\mathrm{E} = \{k : \ \mathrm{E} \subset \partial\Omega_k\}.$$

Define

$$\Omega_\mathrm{E} = \bigcup_{k \in \mathcal{Q}_\mathrm{E}} \Omega_k, \quad \mathrm{E} \in \mathcal{E}_\Gamma.$$

• face inner-products, scaling norm and interface norm

For a subset G of Γ, let $\langle \cdot, \cdot \rangle_G$ denote the L^2 inner product on G. In particular, the $\langle \cdot, \cdot \rangle_\Gamma$ is abbreviated as $\langle \cdot, \cdot \rangle$. Let $\| \cdot \|_{0,\ G}$ denote the norm induced from $\langle \cdot,\ \cdot \rangle_G$.

For a sub-faces G of Γ, let H_G denote the "size" of G. Define the scaled norm

$$\|\phi\|_{\frac{1}{2},\ G} = (|\phi|^2_{\frac{1}{2},\ G} + H_G^{-1} \|\phi\|^2_{0,\ G})^{\frac{1}{2}}, \quad \forall \phi \in H^{\frac{1}{2}}(G).$$

For convenience, define

$$\|\phi_h\|_{*,\Gamma} = (\sum_{k=1}^{\mathrm{N}} \omega_k |\phi_h|^2_{\frac{1}{2},\partial\Omega_k})^{\frac{1}{2}}, \quad \forall \phi_h \in W_h(\Gamma).$$

• discrete norms

Discrete norms (or semi-norms) of finite element functions will be used repeatedly in this paper, since the discrete norms are defined on a set of nodes only, and do not depend on the geometric shape of the underlying domain.

We first give definitions of two well known discrete norms (refer to [32]), which are equivalent to their respective continuous norms. For $v_h \in V_h(\Omega_k)$, the discrete H^1 semi-norm is defined by

$$|v_h|^2_{1,\ h,\ \Omega_k} = h^3 \sum_{p_i,\ p_j \in \mathcal{N}_h \cap \Omega_k} |v_h(p_i) - v_h(p_j)|^2,$$

where p_i and p_j denote two neighboring nodes. Similarly, the discrete L^2 norm on an edge E of Ω_k is defined by

$$\|v_h\|^2_{0,\ h,\ \mathrm{E}} = h \sum_{p \in \mathcal{N}_h \cap \mathrm{E}} |v_h(p)|^2.$$

• $H_{00}^{\frac{1}{2}}$ norms

For $\varphi_h \in \tilde{W}_h(\mathrm{F})$ with $\mathrm{F} = \Gamma_{ij}$, define

$$\|\varphi_h\|^2_{H^{\frac{1}{2}}_{00}(\mathrm{F})} = |\varphi_h|^2_{\frac{1}{2},\ \mathrm{F}} + \int_{\mathrm{F}} \frac{|\varphi_h(x)|^2}{dist(x,\ \partial\mathrm{F})} ds(x).$$

Hereafter, $dist(x,\ \partial\mathrm{F})$ denotes the shortest distance from a point $x \in \mathrm{F}$ to the boundary $\partial\mathrm{F}$. It is known that

$$\|\varphi_h\|^2_{H^{\frac{1}{2}}_{00}(\mathrm{F})} \eqsim |\tilde{\varphi}_h|^2_{\frac{1}{2},\ \Omega_i} \eqsim |\tilde{\varphi}_h|^2_{\frac{1}{2},\ \Omega_j},$$

where $\tilde{\varphi}_h \in W_h(\Gamma)$ denotes the zero extension of φ_h. Moreover, we have

$$\|\varphi_h\|^2_{H^{\frac{1}{2}}_{00}(\mathrm{F})} \eqsim \int_{\mathrm{F}} \frac{|\varphi_h(x)|^2}{dist(x,\ \partial\mathrm{F})} ds(x).$$

The corresponding discrete semi-norm is defined by

$$\|\varphi_h\|^2_{h,\ H^{\frac{1}{2}}_{00}(\mathrm{F})} = h^2 \sum_{p \in \mathcal{N}_h \cap \mathrm{F}} \frac{|\varphi_h(p)|^2}{dist(p,\ \partial\mathrm{F})}.$$

• spectrally equivalences

For simplicity, we will frequently use the notations $\lesssim$ and $\eqsim$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \leq Cy$ for some constant C independent of mesh size h, subdomain size d and the related parameters. $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

2.3 Motivation

We first recall the main ideas of the existing substucturing preconditioners.

Let $E_k : W_h(\partial\Omega_k) \rightarrow V_h(\Omega_k)$ be the discrete harmonic extension. Define the *harmonic* subspace

$$V_h^{\Gamma}(\Omega) = \{v_h \in V_h(\Omega) :\ v_h|_{\Omega_k} = E_k(\phi_h|_{\partial\Omega_k})\ (k = 1, \cdots, \mathrm{N}) \ \text{ for some } \phi_h \in W_h(\Gamma)\}.$$

Then, we have the initial space decomposition

$$V_h(\Omega) = \sum_{k=1}^{\mathrm{N}} V_h^p(\Omega_k) + V_h^{\Gamma}(\Omega).$$

Let $A_{h,k} : V_h^p(\Omega_k) \rightarrow V_h^p(\Omega_k)$ be the restriction of the operator A_h on the local space $V_h^p(\Omega_k)$, and let $B_{h,\Gamma} : V_h^{\Gamma}(\Omega) \rightarrow V_h^{\Gamma}(\Omega)$ be a symmetric and positive definite operator which is spectrally equivalent to the restriction of the operator A_h on the harmonic subspace $V_h^{\Gamma}(\Omega)$. Then, the classical substructuring preconditioner (refer to [5]) can be defined in the rough form

$$B_{old}^{-1} = \sum_{k=1}^{N} A_{h,k}^{-1} Q_k + B_{h,\Gamma}^{-1} Q_{\Gamma}, \tag{2.5}$$

where Q_k and Q_{Γ} denote the standard L^2 projectors into their respective subspaces. For the preconditioner B_{old}, we have (see [5])

$$cond(B_{old}^{-1} A_h) \lesssim \log^2(H/h). \tag{2.6}$$

In many applications, the subspaces $V_h^p(\Omega_k)$ still have high dimensions, so it is expensive to use the exact solvers $A_{h,k}^{-1}$.

It was shown in [3] that substructuring preconditioners with inexact solvers $B_{h,k}^{-1}$ still possess nearly optimal convergence, if each $B_{h,k}$ has some spectrally *approximation* to $A_{h,k}$ (the usual spectrally equivalence is not enough). Hereafter, "inexact" means that $B_{h,k}$ is only spectrally equivalent to $A_{h,k}$, for example, $B_{h,k}$ is a multigrid preconditioner for $A_{h,k}$. It seems difficult to design an efficient substructuring preconditioner with completely inexact solvers $B_{h,k}^{-1}$, instead of $A_{h,k}^{-1}$ itself or its approximation. In essence, one has to modify the harmonic subspace $V_h^{\Gamma}(\Omega)$ by replacing each harmonic extension E_k with another extension.

In [4], a substructuring preconditioner B_{bpv}^{-1} with inexact solvers was been designed by replacing each harmonic extension E_k with a simple *average* extension. It has been shown that the condition number of the resulting preconditioned system can be estimated by

$$cond(B_{bpv}^{-1}A_h) \lesssim H/h. \tag{2.7}$$

In [11], another substructuring preconditioner B_{hlm}^{-1} with inexact solvers was been designed by replacing each harmonic extension E_k with an *approximate* harmonic extension. If the approximate harmonic extension is exact enough, then

$$cond(B_{blm}^{-1}A_h) \lesssim \log^2(H/h). \tag{2.8}$$

The approximate harmonic extension can be defined by *approximate* harmonic basis functions. It is not practical to compute all the *approximate* harmonic basis functions. Because of this, an alternative method, which still require high accuracy of $B_{h,k}$, was considered in [11].

Another way to construct substructuring preconditioner with inexact solvers was considered in [6] (mainly for two dimensions).

Let $\hat{W}_h^0(\Gamma)$ be a subspace of $W_h(\Gamma)$, such that a function $\varphi \in \hat{W}_h^0(\Gamma)$ equals a constant C_{F} at every nodes in each face F of Γ. Define the *coarse* subspace

$$\hat{V}_h^0(\Omega) = \{v_h \in V_h(\Omega):\ v_h|_{\Omega_k} = E_k(\phi_h|_{\partial\Omega_k})\ (k = 1, \cdots, \mathrm{N})\ \text{ for some } \phi_h \in \hat{W}_h^0(\Gamma)\}.$$

Then, we have the space decomposition

$$V_h(\Omega) = \hat{V}_h^0(\Omega) + \sum_{\Gamma_{ij}} V_h^p(\Omega_{ij}).$$

Let $B_0 : \hat{V}_h^0(\Omega) \to \hat{V}_h^0(\Omega)$ and $B_{ij} : V_h^p(\Omega_{ij}) \to V_h^p(\Omega_{ij})$ be symmetric and positive definite operators which are spectrally equivalent to the restrictions of the operator A_h on the coarse subspace $\hat{V}_h^0(\Omega)$ and the local subspace $V_h^p(\Omega_{ij})$, respectively. Then, we can define a substructuring preconditioner as (refer to [6] and [32])

$$B_{dw}^{-1} = B_0^{-1}Q_0 + \sum_{\Gamma_{ij}} B_{ij}^{-1}Q_{ij}, \tag{2.9}$$

where Q_0 and Q_{ij} denote the standard L^2 projectors into their respective subspaces. For the preconditioner B_{dw}, we have (refer to [6] and [32])

$$cond(B_{dw}^{-1}A_h) \lesssim \log^2(H/h). \tag{2.10}$$

As pointed out in [32], the advantage of the preconditioner B_{dw} is that one can use inexact solver B_{ij} for the restriction of A_h on $V_h^p(\Omega_{ij})$, but an inexact solver B_0^{-1} is hard to come by as harmonic extensions in the definition of $\hat{V}_h^0(\Omega)$ mean an exact solver. As an alternate method, one can use a *nearly* harmonic extension to replace the exact harmonic extension E_k. However, the design of such approximate harmonic extension is also difficult for the general case that either the subdomain Ω_k is a general polyhedron or $\mathcal{T}_h$ is a general quasi-uniform triangulation (an attempt for two dimension is given in [19]). From the definition of the coarse subspace $\hat{V}_h^0(\Omega)$, we know that one needs only to design such extension for functions in the subspace $\hat{W}_h^0(\Gamma)$. In essence, one needs only to define a suitable extension for the *constant-like* basis function on each face of Γ. This kind of extension was first studied in [8]. For the purpose of applications, the definition of such extension would be sufficiently simple, otherwise, the action of B_0^{-1} is difficult to implement.

In this paper, we construct a completely explicit extension for the *constant-like* basis function on each face of Γ. As we will see, the implementation of the new extensions is very convenient and cheap. We can define a new variant of the coarse subspace $\hat{V}_h^0(\Omega)$ based on the extensions. Then, we use the new coarse subspace to construct a substructuring preconditioner with inexact solvers.

2.4 Basic tools

In the analysis later, we will use some basic estimates on $H^{\frac{1}{2}}$ norms.

The following results are well-known (see, for example, [32]).

Lemma 2.1 *Let* E *and* F *be an edge and a face of* Ω_k. *Then,*

$$\|\phi_h\|_{0,\partial\mathrm{F}} \lesssim \log^{\frac{1}{2}}(H/h)\|\phi_h\|_{\frac{1}{2},\partial\Omega_k}, \quad \forall\phi_h \in W_h(\partial\Omega_k), \tag{2.11}$$

$$\|I_{\mathrm{E}}^0\phi_h\|_{\frac{1}{2},\partial\Omega_k} \lesssim \log^{\frac{1}{2}}(H/h)\|\phi_h\|_{\frac{1}{2},\partial\Omega_k}, \quad \forall\phi_h \in W_h(\partial\Omega_k) \tag{2.12}$$

and

$$\|I_{\mathrm{F}}^0\phi_h\|_{H_{00}^{\frac{1}{2}}(\mathrm{F})} \lesssim \log(d/h)\|\phi_h\|_{\frac{1}{2},\partial\Omega_k}, \quad \forall\phi_h \in W_h(\partial\Omega_k). \tag{2.13}$$

□

The following result can be proved as in Lemma 6.2 in [18], together with the standard technique.

Lemma 2.2 *Let* E *be an edge of* Ω_k. *Then,*

$$\|\phi_h - \gamma_{h,\mathrm{E}}(\phi_h)\|_{\frac{1}{2},\partial\Omega_k} \lesssim \log^{\frac{1}{2}}(H/h)|\phi_h|_{\frac{1}{2},\partial\Omega_k}, \quad \forall\phi_h \in W_h(\partial\Omega_k). \tag{2.14}$$

□

3 Extensions for particular functions

As in Subsection 2.3, the desired coarse subspace of $V_h(\Omega)$ is always associated with an interface coarse subspace. Let $I_{\mathrm{F}}^0 1$ and $I_{\mathrm{E}}^0 1$ denote the *constant-like* basis functions on the face F and the coarse edge E, respectively (see Subsection 2.2). In this paper, we consider the interface coarse subspace

$$W_h^0(\Gamma) = span\{I_F^0 1,\ I_{\mathrm{E}}^0 1,\ \varphi_p :\ F \in \mathcal{F}_\Gamma,\ \mathrm{E} \in \mathcal{E}_\Gamma,\ p \in \mathcal{V}_\Gamma\}.$$

Hereafter, φ_p denotes the nodal basis function on the node $p \in \mathcal{N}_h \cap \Omega$.

In this section, we construct an approximation for the restriction of the discrete harmonic extension E_k on $W_h^0(\Gamma)$. For convenience, let F denote a general face Γ_{ij}, and let $\phi_{\mathrm{F}} = I_{\mathrm{F}}^0 1 \in \tilde{W}_h(F)$ be the *constant-like* basis function on F.

3.1 An explicit extension of $\phi_{\rm F}$

In this subsection, we define a stable extension $E_{\rm F}\phi_{\rm F} \in V_h^p(\Omega_{ij})$, which satisfies $(E_{\rm F}\phi_{\rm F})(p) = \phi_{\rm F}(p)$ when $p \in \bar{\rm F}$.

For a node p in $\Omega_{ij} \backslash {\rm F}$, let p' denote the projection of p on the plane containing F. Besides, when $p' \in {\rm F}$, we use p'' to denote a point on the boundary $\partial{\rm F}$, such that $|p'p''| = dist(p', \partial{\rm F})$, which is the shortest distance from p' to the boundary $\partial{\rm F}$ (see Figure 1).

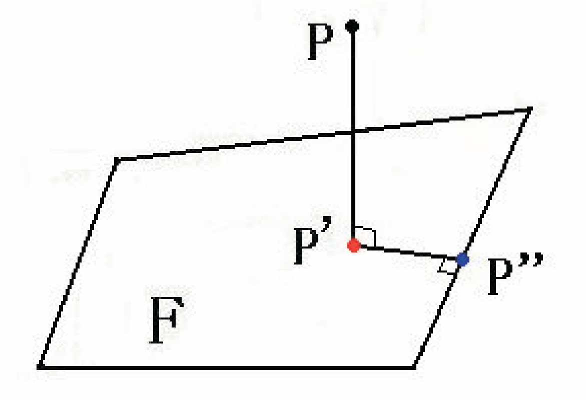

Figure 1: illustration for the notations

In short words, we define the extension $E_{\rm F}\phi_{\rm F}$ such that the values $(E_{\rm F}\phi_{\rm F})(p)$ decrease gradually when the lengths $|pp'|$ increase, or the lengths $|p'p''|$ decrease. Although this property holds also for the extension designed in [8], we will use a different idea from [8].

For simplicity of exposition, we make an assumption:

Assumption 3.1: when $p \in \partial\Omega_{ij}$ with $p' \in {\rm F}$, $|pp'| \geq |p'p''|$.

This assumption means that each subdomain Ω_k is not too thin. As we will see in Appendix, we need only to revise slightly the definition of the extension for the general situation without this assumption. To give the exact definition of $E_{\rm F}\phi_{\rm F}$ with ${\rm F} = \Gamma_{ij}$, we set

$$\Lambda_{\rm F} = \{p \in \mathcal{N}_h \cap (\Omega_{ij} \backslash {\rm F}) : \ p' \in {\rm F}, \ |pp'| \leq |p'p''|\}.$$

For a face ${\rm F} = \Gamma_{ij}$, define the extension $E_{\rm F}\phi_{\rm F} \in V_h^p(\Omega_{ij})$ as follows:

$$(E_{\rm F}\phi_{\rm F})(p) = \begin{cases} 1, & \text{if } p \in {\rm F}, \\ 1 - \frac{|pp'|}{|p'p''|}, & \text{if } p \in \Lambda_{\rm F}, \\ 0, & \text{otherwise}. \end{cases} \tag{3.15}$$

In particular, we have $(E_{\rm F}\phi_{\rm F})(p) = \phi_{\rm F}(p)$ when $p \in \bar{\rm F}$.

For rectangular face F with uniform triangulation, the values of $E_{\rm F}\phi_{\rm F}$ at some nodes are given in Figure 2.

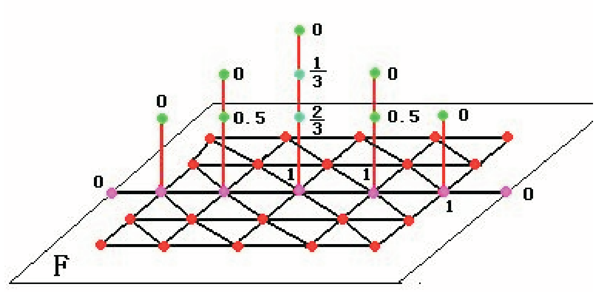

Figure 2: extension of $\phi_{\rm F}$ for particular case

Set

$$d_{\mathrm{F}} = \max_{q \in \mathrm{F}} \; dist(q, \partial \mathrm{F}).$$

It is clear that d_{F} is just the radium of the largest circle contained in F, and $d_{\mathrm{F}} \eqsim H$.

It is easy to see that the extension operator E_{F} possesses the properties:

(i) the support set Ω_{F} of $E_{\mathrm{F}}\phi_{\mathrm{F}}$ is a simply connected domain with the size d_{F}. In fact, Ω_{F} like a cone with the bottom F.

(ii) the calculation of $E_{\mathrm{F}}\phi_{\mathrm{F}}$ possesses the optimal complexity $O(n_{\mathrm{F}})$ with $n_{\mathrm{F}} = (d_{\mathrm{F}}/h)^3$ being the number of the nodes in Ω_{F}.

Besides, the extension E_{F} is stable in the following sense

Theorem 3.1 *The extension E_{F} defined by (3.15) satisfies the stability condition*

$$|E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_i}, \; |E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_j} \lesssim H \log(H/h) \eqsim |\phi_{\mathrm{F}}|^2_{H^{\frac{1}{2}}_{00}(\mathrm{F})} \qquad (\mathrm{F} = \Gamma_{ij}). \tag{3.16}$$

Since proof of the result is technical, we prove this theorem in Section 5.

Remark 3.1 *The definition of the extension E_{F} not only is simple and completely explicit, but also is identical to different geometric shape (tetrahedron, hexahedron or other polyhedrons) of the two subdomains containing F as a face. Thus, the action of E_{F} is easy to implement.*

3.2 A nearly harmonic extension of $\phi \in W_h^0(\Gamma)$

Let $R_k^0 : W_h(\partial\Omega_k) \to V_h(\Omega_k)$ denote the zero extension operator in the sense that $R_k^0\phi = \phi$ on $\partial\Omega_k$, and $R_k^0\phi$ vanishes at all internal nodes of Ω_k for $\phi \in W_h(\partial\Omega_k)$. For $\psi \in W_h^0(\Gamma)$, we have

$$\psi = \gamma_{h,\partial\Omega_k}(\psi) + I^0_{\mathcal{W}_k}(\psi - \gamma_{h,\partial\Omega_k}(\psi)) + \sum_{\mathrm{F}\subset\partial\Omega_k} \gamma_{h,\mathrm{F}}(\psi - \gamma_{h,\partial\Omega_k}(\psi))\phi_{\mathrm{F}}, \quad \text{on} \;\; \partial\Omega_k.$$

Hereafter, $\mathcal{W}_k$ denotes the *wire-basket* set of Ω_k, i.e., the union of all the edges of Ω_k. Note that $\gamma_{h,\mathrm{F}}(\psi - \gamma_{h,\partial\Omega_k}(\psi))$ is just the (constant) value of $\psi - \gamma_{h,\partial\Omega_k}(\psi)$ at the interior nodes of F. Then, we define on each Ω_k

$$E_0\psi = \gamma_{h,\partial\Omega_k}(\psi) + R_k^0[I^0_{\mathcal{W}_k}(\psi - \gamma_{h,\partial\Omega_k}(\psi))] + \sum_{\mathrm{F}\subset\partial\Omega_k} \gamma_{h,\mathrm{F}}(\psi - \gamma_{h,\partial\Omega_k}(\psi))E_{\mathrm{F}}\phi_{\mathrm{F}}. \tag{3.17}$$

It is easy to see that $E_0\psi = \psi$ on Γ.

The extension E_0 is nearly harmonic in the following sense

Proposition 3.1 The global coarse extension E_0 satisfies

$$\sum_{k=1}^{\mathrm{N}} \omega_k |E_0\phi_0|^2_{1,\Omega_k} \lesssim \log(H/h)\|\phi_0\|^2_{*,\Gamma}, \quad \forall \phi_0 \in W_h^0(\Gamma). \tag{3.18}$$

Proof. By the definition of E_0, we have

$$\begin{aligned} |E_0\phi_0|^2_{1,\Omega_k} \;&\lesssim\; |R_k^0[I^0_{\mathcal{W}_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0))]|^2_{1,\Omega_k} \\ &+\; \sum_{\mathrm{F}\subset\partial\Omega_k} |\gamma_{h,\mathrm{F}}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0))|^2 \cdot |E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_k} \\ &\lesssim\; \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\mathcal{W}_k} \end{aligned}$$

$$+\sum_{\mathrm{F}\subset\partial\Omega_k}(H^{-2}\|\phi_0-\gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\partial\Omega_k}\cdot|E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_k}). \tag{3.19}$$

Using (2.11) and Friedrichs' inequality, yields

$$\|\phi_0-\gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\mathcal{W}_k}\lesssim\log(H/h)|\phi_0|^2_{\frac{1}{2},\partial\Omega_k}. \tag{3.20}$$

On the other hand, we deduce by (3.16) and Friedrichs' inequality

$$\begin{aligned}H^{-2}\|\phi_0-\gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\partial\Omega_k}\cdot|E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_k} &\lesssim H^{-1}\log(H/h)\|\phi_0-\gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\partial\Omega_k}\\ &\lesssim \log(H/h)|\phi_0|^2_{\frac{1}{2},\partial\Omega_k}.\end{aligned}$$

Substituting (3.20) and the above inequality into (3.19), we get (3.18).
□

4 A substructuring method with inexact solvers

This section is devoted to construction of a substructuring preconditioner with inexact solvers. The new preconditioner is based on the extension designed in the last section.

4.1 Space decomposition for $V_h(\Omega)$

In this subsection, we define a space decomposition of $V_h(\Omega)$.

For each subdomain Ω_{ij} containing Γ_{ij}, let $V_h^p(\Omega_{ij})$ be the subspace given in Subsection 2.2. Namely,

$$V_h^p(\Omega_{ij})=span\{\varphi_p:\ p\in\mathcal{N}_h\cap\Omega_{ij}\}.$$

For the extension $E_0: W_h^0(\Gamma)\to V_h(\Omega)$ described in the last section, define

$$V_h^0(\Omega)=\{v_h\in V_h(\Omega):\ v_h=E_0\phi \text{ for some } \phi_h\in W_h^0(\Gamma)\}.$$

For $\mathrm{E}\in\mathcal{E}_\Gamma$, define

$$V_h(\Omega_{\mathrm{E}})=span\{\varphi_p:\ p\in \mathrm{E}\cap\mathcal{N}_h\}.$$

Then, we have the space decomposition

$$V_h(\Omega)=V_h^0(\Omega)+\sum_{\mathrm{E}\in\mathcal{E}_\Gamma}V_h(\Omega_{\mathrm{E}})+\sum_{\Gamma_{ij}}V_h^p(\Omega_{ij}).$$

Remark 4.1 *Since the new extension E_0 described in the last section is used in $V_h^0(\Omega)$, the coarse subspace $V_h^0(\Omega)$ is different from the one in [8].*

4.2 A substructuring preconditioner

Based on the space decomposition in the last subsection, we can define a preconditioner in the standard way.

Define symmetric and positive definite operators as follows:

• the global coarse solver $B_0: V_h^0(\Omega)\to V_h^0(\Omega)$ satisfies:

$$(B_0v_h,v_h)\eqsim(A_hv_h,v_h),\quad \forall v_h\in V_h^0(\Omega); \tag{4.1}$$

• the edge solver $B_{\mathrm{E}}: V_h(\Omega_{\mathrm{E}})\to V_h(\Omega_{\mathrm{E}})$ satisfies:

$$(B_{\mathrm{E}}v_h,v_h)\eqsim(A_hv_h,v_h),\quad \forall v_h\in V_h(\Omega_{\mathrm{E}}); \tag{4.2}$$

• the interface solver $B_{ij} : V_h^p(\Omega_{ij}) \to V_h^p(\Omega_{ij})$ satisfies:

$$(B_{ij} v_h, v_h) \eqsim (A_h v_h, v_h), \quad \forall v_h \in V_h^p(\Omega_{ij}). \tag{4.3}$$

In applications, the interface solver B_{ij} is usually chosen as a symmetric multigrid preconditioner for the restriction of A_h on $V_h^p(\Omega_{ij})$. Since all the subspaces $V_h^0(\Omega)$ and $V_h(\Omega_{\mathrm{E}})$ have low dimensions, the solvers B_0 and B_{E} can be simply defined as the restriction operators of A_h on their respective subspaces.

Now, the desired domain decomposition preconditioner for A_h is defined as

$$B_h^{-1} = B_0^{-1} Q_0 + \sum_{\mathrm{E} \in \mathcal{E}_\Gamma} B_{\mathrm{E}}^{-1} Q_{\mathrm{E}} + \sum_{\Gamma_{ij}} B_{ij}^{-1} Q_{ij}, \tag{4.4}$$

where $Q_0 : V_h(\Omega) \to V_h^0(\Omega)$, $Q_{\mathrm{E}} : V_h(\Omega) \to V_h(\Omega_{\mathrm{E}})$ and $Q_{ij} : V_h(\Omega) \to V_h^p(\Omega_{ij})$ denote L^2 projectors.

4.3 Implementation

Before describing our algorithms, we give exact definitions of the edge solver B_{E} and the coarse solver B_0.

• the edge solver

It is known that

$$|\nabla v|_{0,\Omega_k} \eqsim \|v\|_{0,\mathrm{E}}, \quad \forall v \in V_h(\Omega_{\mathrm{E}}) \ \ (\mathrm{E} \subset \partial\Omega_k).$$

Then,

$$(A_h v,\ v) \eqsim \sum_{k \in \mathcal{Q}_{\mathrm{E}}} \omega_k \|v\|_{0,\mathrm{E}}^2, \quad \forall v \in V_h(\Omega_{\mathrm{E}}).$$

We can define $B_{\mathrm{E}} : V_h(\Omega_{\mathrm{E}}) \to V_h(\Omega_{\mathrm{E}})$ by

$$(B_{\mathrm{E}} v, w) = (\sum_{k \in \mathcal{Q}_{\mathrm{E}}} \omega_k) \langle v, w \rangle_{\mathrm{E}}, \quad \forall w \in V_h(\Omega_{\mathrm{E}}).$$

The stiffness of B_{E} is just a scaling of the mass matrix associated with E.

• the coarse solver

As we will see in Section 5, we have

$$|E_0 \phi_{\mathrm{F}}|_{1,\Omega_k}^2 \eqsim H \log(H/h).$$

As in Proposition 3.1, one can verify that

$$|v_h^0|_{1,\Omega_k}^2 \eqsim \|v_h^0 - \gamma_{h,\partial\Omega_k}(v_h^0)\|_{0,\mathcal{W}_k}^2 + H \log(H/h) \sum_{\mathrm{F} \subset \partial\Omega_k} |\gamma_{h,\mathrm{F}}(v_h^0 - \gamma_{h,\partial\Omega_k}(v_h^0))|^2, \ \ \forall v_h^0 \in V_h^0(\Omega).$$

Hence, we have for any $v_h^0 \in V_h^0(\Omega)$

$$(A_h v_h^0, v_h^0) \eqsim \sum_{k=1}^{N} \omega_k \{ \|v_h^0 - \gamma_{h,\partial\Omega_k}(v_h^0)\|_{0,\mathcal{W}_k}^2 + H \log(H/h) \sum_{\mathrm{F} \subset \partial\Omega_k} |\gamma_{h,\mathrm{F}}(v_h^0 - \gamma_{h,\partial\Omega_k}(v_h^0))|^2 \}.$$

This means that B_0 can be defined as

$$\begin{aligned} (B_0 v_h^0, w_h^0) = \sum_{k=1}^{N} \omega_k \{ \langle v_h^0 - \gamma_{h,\partial\Omega_k}(v_h^0), w_h^0 - \gamma_{h,\partial\Omega_k}(w_h^0) \rangle_{h,\mathcal{W}_k} \\ + H \log(H/h) \sum_{\mathrm{F} \subset \partial\Omega_k} \gamma_{h,\mathrm{F}}(v_h^0 - \gamma_{h,\partial\Omega_k}(v_h^0)) \cdot \gamma_{h,\mathrm{F}}(w_h^0 - \gamma_{h,\partial\Omega_k}(w_h^0)) \}, \\ v_h^0 \in V_h^0(\Omega), \forall w_h^0 \in V_h^0(\Omega). \end{aligned}$$

Then, the stiffness matrix of B_0 is calculated by $\mathbf{B}_0 = (b_{rl})_{L_0\times L_0}$ with

$$b_{rl} = (B_0(E_0\psi_r^0), E_0\psi_l^0) = \sum_{k=1}^{N} \omega_k\{\langle \psi_r^0 - \gamma_{h,\partial\Omega_k}(\psi_r^0), \psi_l^0 - \gamma_{h,\partial\Omega_k}(\psi_l^0)\rangle_{h,\mathcal{W}_k}$$
$$+H\log(H/h) \sum_{\mathrm{F}\subset\partial\Omega_k} \gamma_{h,\mathrm{F}}(\psi_r^0 - \gamma_{h,\partial\Omega_k}(\psi_r^0)) \cdot \gamma_{h,\mathrm{F}}(\psi_l^0 - \gamma_{h,\partial\Omega_k}(\psi_l^0))\}.$$
$$(r,\ l = 1,\cdots,L_0)$$

Remark 4.2 *Note that our coarse solver B_0 is different from the one described in Algorithm 6.10 of [8]. The coarse solver in [8] involves N extra unknowns (see also [5]), which need to be solved by a special technique (see [32]). The coarse solver B_0 is similar with the one given in Subsection 4.3 of [7], in which stability of the coarse solver was not proved (We do not know why the article [7] cited directly the stability result of the coarse solver in [8]).*

The action of B_h^{-1} can be described by the following algorithm
Algorithm 4.1. For $g \in V_h(\Omega)$, the solution $u_g \in V_h(\Omega)$ satisfying

$$(B_h u_g, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega)$$

can be gotten as follows:

Step 1. Computing $u_0 \in V_h^0(\Omega)$ by

$$(B_0 u_0, v_h) = (g, v_h), \quad \forall v_h \in V_h^0(\Omega);$$

Step 2. Computing $u_\mathrm{E} \in V_h(\Omega_\mathrm{E})$ in parallel by

$$(B_\mathrm{E} u_\mathrm{E}, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega_\mathrm{E});$$

Step 3. Computing $u_{ij}^p \in V_h^p(\Omega_{ij})$ in parallel by

$$(B_{ij} u_{ij}^p, v_h) = (g, v_h), \quad \forall v_h \in V_h^p(\Omega_{ij});$$

Step 4. Set

$$u_g = u_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} u_\mathrm{E} + \sum_{\Gamma_{ij}} u_{ij}^p.$$

Remark 4.3 *The efficiency of the preconditioner B_h strongly depends on the action of the coarse solver B_0. The face basis extensions $E_\mathrm{F}\phi_\mathrm{F}$ play a key role in the definition of B_0. This is why we focus on the design of the extension E_F in this paper.*

In the rest of this subsection, we consider an acceleration of Algorithm 4.1 (see [16] for the detailed discussion on this kind of acceleration). For convenience, set $V = V_h(\Omega)$, and define

$$V_1 = V_h^0(\Omega) + \sum_{\Gamma_{ij}} V_h^p(\Omega_{ij}) \quad \text{and} \quad V_2 = \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} V_h(\Omega_\mathrm{E}).$$

Let $V_2^\perp$ denote the orthogonal complement of V_2 with respect to the inner product $(A_h\cdot,\cdot)$. It is clear that

$$V = V_1 + V_2.$$

Let $A_\mathrm{E} : V_h(\Omega_\mathrm{E}) \to V_h(\Omega_\mathrm{E})$ denote the restriction of A_h on $V_h(\Omega_\mathrm{E})$. Define

$$B_1^{-1} = B_0^{-1}Q_0 + \sum_{\Gamma_{ij}} B_{ij}^{-1}Q_{ij}.$$

The following algorithm can be viewed as a combination between the CG method and the multiplicative Schwarz method with the above decomposition for solving (2.4).

Algorithm 4.2 (multiplicative Schwarz-CG) Let $u_1 \in V$ be an initial guess such that the error $u_h - u_1 \in V_2^{\perp}$ (for example, u_1 can be chosen as $u_1 = \sum\limits_{\mathrm{E}\in\mathcal{E}_\Gamma} A_{\mathrm{E}}^{-1}Q_{\mathrm{E}}f_h \in V_2$). When an approximation $u_n \in V$ has been gotten, we look for $u_{n+1} \in V$ as follows ($n \geq 1$):

Step 1. Solve $\varepsilon_{n1} \in V_1$ by

$$\varepsilon_{n1} = B_1^{-1}(f_h - A_h u_n).$$

If $\varepsilon_{n1} = 0$, then the iteration is terminated; otherwise, goto Step 2:

Step 2. Compute $u_{\mathrm{E}} \in V_h(\Omega_{\mathrm{E}})$ in parallel by

$$(A_{\mathrm{E}}u_{\mathrm{E}}, v) = (f_h - A_h(u_n + \varepsilon_{n1}), v), \quad \forall v \in V_h(\Omega_{\mathrm{E}}).$$

Define $\varepsilon_{n2} \in V_2$ by

$$\varepsilon_{n2} = \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} u_{\mathrm{E}},$$

and set

$$\varepsilon_n = \varepsilon_{n1} + \varepsilon_{n2}.$$

Step 3. Compute

$$\alpha_n = \varepsilon_n - \frac{(\varepsilon_n, A_h\alpha_{n-1})}{\|\alpha_{n-1}\|_{A_h}^2}\alpha_{n-1}, \quad (\alpha_0 = 0)$$

and

$$u_{n+1} = u_n + \frac{(f_h - A_h u_n, \alpha_n)}{\|\alpha_n\|_{A_h}^2}\alpha_n.$$

Remark 4.4 *Step 1 in Algorithm 4.2 would be implemented by Step 1 and Step 3 in Algorithm 4.1. Since each $V_h(\Omega_{\mathrm{E}})$ has a very low dimension, which equals the number of nodes in the coarse edge* E*, Step 2 in Algorithm 4.2 is also very cheap.*

4.4 Convergence

The following result gives an estimate of $cond(B_h^{-1}A_h)$.

Theorem 4.1 *Let the extension be defined by Subsection 3.1. For the preconditioner B_h defined in Subsection 4.3, we have*

$$cond(B_h^{-1}A_h) \leq C\log^2(H/h), \tag{4.5}$$

where C is a constant independent of h, H and the jumps of the coefficient ω across the faces Γ_{ij}.

The convergence of Algorithm 4.2 involves the subspace $V_2^{\perp}$. Define $T^* = (B_1^{-1}A_h)|_{V_2^{\perp}}$. It is clear that T^* is symmetric and positive definite with respect to the inner product $(A_h\cdot, \cdot)$.

Theorem 4.2 [16] *Let the sequence u_{n+1} be defined by Algorithm 4.2. Then*

$$\|u_h - u_{n+1}\|_{A_h} < 2\Big(\frac{\sqrt{\kappa(T^*)} - 1}{\sqrt{\kappa(T^*)} + 1}\Big)^n \|u_h - u_1\|_{A_h}, \tag{4.6}$$

where $\kappa(T^*)$ denotes the condition number of the operator T^*. Moreover, we have $\kappa(T^*) < cond(B_h^{-1}A_h)$.

Remark 4.5 *Algorithm 4.2 is as cheap as PCG method for solving (2.4) with the preconditioner B_h. However, Theorem 4.2 indicates that the former has a faster convergence speed than the later.*

Before proving Theorem 4.1, we need to derive a new estimate of the extension E_0. For $\phi \in W_h(\Gamma)$, define $\phi_0 \in W_h^0(\Gamma)$ by

$$\phi_0 = \sum_{\mathrm{F}\in\mathcal{F}_\Gamma} \gamma_{h,\mathrm{F}}(\phi) I^0_{\mathrm{F}}1 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \gamma_{h,\mathrm{E}}(\phi) I^0_{\mathrm{E}}1 + \sum_{p\in\mathcal{V}_\Gamma} \phi(p)\varphi_p. \tag{4.7}$$

Lemma 4.1 *Let $\phi \in W_h(\Gamma)$, and let ϕ_0 be defined by (4.7). Then,*

$$\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\mathcal{W}_k} \lesssim \|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\mathcal{W}_k}, \tag{4.8}$$

and

$$|\gamma_{h,\partial\Omega_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi))|^2 \lesssim H^{-2}\|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k}. \tag{4.9}$$

Proof. Since

$$1 = \sum_{\mathrm{F}\subset\partial\Omega_k} I^0_{\mathrm{F}}1 + \sum_{\mathrm{E}\subset\partial\Omega_k} I^0_{\mathrm{E}}1 + \sum_{p\in\partial\Omega_k\cap\mathcal{V}_\Gamma} \phi_p \quad \text{on } \partial\Omega_k,$$

we have

$$\gamma_{h,\partial\Omega_k}(\phi) = \sum_{\mathrm{F}\subset\partial\Omega_k} \gamma_{h,\partial\Omega_k}(\phi) I^0_{\mathrm{F}}1 + \sum_{\mathrm{E}\subset\partial\Omega_k} \gamma_{h,\partial\Omega_k}(\phi) I^0_{\mathrm{E}}1 + \sum_{p\in\partial\Omega_k\cap\mathcal{V}_\Gamma} \gamma_{h,\partial\Omega_k}(\phi)\varphi_p \quad \text{on } \partial\Omega_k.$$

Then,

$$\begin{aligned}
\phi_0 - \gamma_{h,\partial\Omega_k}(\phi) &= \sum_{\mathrm{F}\subset\partial\Omega_k} (\gamma_{h,\mathrm{F}}(\phi) - \gamma_{h,\partial\Omega_k}(\phi)) I^0_{\mathrm{F}}1 + \sum_{\mathrm{E}\subset\partial\Omega_k} (\gamma_{h,\mathrm{E}}(\phi) - \gamma_{h,\partial\Omega_k}(\phi)) I^0_{\mathrm{E}}1 \\
&+ \sum_{p\in\partial\Omega_k\cap\mathcal{V}_\Gamma} (\phi(p) - \gamma_{h,\partial\Omega_k}(\phi))\varphi_p \\
&= \sum_{\mathrm{F}\subset\partial\Omega_k} \gamma_{h,\mathrm{F}}(\phi - \gamma_{h,\partial\Omega_k}(\phi)) I^0_{\mathrm{F}}1 + \sum_{\mathrm{E}\subset\partial\Omega_k} \gamma_{h,\mathrm{E}}(\phi - \gamma_{h,\partial\Omega_k}(\phi)) I^0_{\mathrm{E}}1 \\
&+ \sum_{p\in\partial\Omega_k\cap\mathcal{V}_\Gamma} (\phi(p) - \gamma_{h,\partial\Omega_k}(\phi))\varphi_p \quad \text{on } \partial\Omega_k.
\end{aligned} \tag{4.10}$$

Furthermore, we get by the direct calculation and the inverse estimates

$$\begin{aligned}
\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\mathcal{W}_k} &\lesssim H \sum_{\mathrm{E}\subset\partial\Omega_k} |\gamma_{h,\mathrm{E}}(\phi - \gamma_{h,\partial\Omega_k}(\phi))|^2 + h\|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\infty,\mathcal{W}_k} \\
&\lesssim \|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\mathcal{W}_k},
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k} &\lesssim H^2 \sum_{\mathrm{F}\subset\partial\Omega_k} |\gamma_{h,\mathrm{F}}(\phi - \gamma_{h,\partial\Omega_k}(\phi))|^2 + h^2\|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\infty,\partial\Omega_k} \\
&\lesssim \|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k}.
\end{aligned} \tag{4.12}$$

It can be verified that

$$|\gamma_{h,\partial\Omega_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi))|^2 \lesssim H^{-2}\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k},$$

which, together with (4.12), gives (4.9).
□

Lemma 4.2 *Let $\phi \in W_h(\Gamma)$, and let ϕ_0 be defined by (4.7). Then,*

$$\sum_{k=1}^{\mathrm{N}} \omega_k |E_0\phi_0|^2_{1,\Omega_k} \lesssim \log(H/h)\|\phi\|^2_{*,\Gamma}. \tag{4.13}$$

Proof. It suffices to estimate (3.19) more carefully. It is easy to see that

$$\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0) = (\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)) - \gamma_{h,\partial\Omega_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)).$$

By the triangle inequality, we deduce

$$\begin{aligned} \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\mathcal{W}_k} &\lesssim \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\mathcal{W}_k} + \|\gamma_{h,\partial\Omega_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi))\|^2_{0,\mathcal{W}_k} \\ &\lesssim \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\mathcal{W}_k} + H|\gamma_{h,\partial\Omega_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi))|^2, \end{aligned}$$

and

$$\begin{aligned} \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\partial\Omega_k} &\lesssim \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k} + \|\gamma_{h,\partial\Omega_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi))\|^2_{0,\partial\Omega_k} \\ &\lesssim \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k} + H^2|\gamma_{h,\partial\Omega_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi))|^2. \end{aligned}$$

Substituting (4.8)-(4.9) and (4.12) into the above two inequalities, yields

$$\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\mathcal{W}_k} \lesssim \|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\mathcal{W}_k} + H^{-1}\|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k}, \tag{4.14}$$

and

$$\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\partial\Omega_k} \lesssim \|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{0,\partial\Omega_k}. \tag{4.15}$$

Combining (4.14) with (2.11), leads to

$$\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\mathcal{W}_k} \lesssim \log(H/h)H^{-1}\|\phi - \gamma_{h,\partial\Omega_k}(\phi)\|^2_{\frac{1}{2},\partial\Omega_k}. \tag{4.16}$$

Plugging (4.16) and (4.15) in (3.19), and using Friedrichs' inequality, leads to (4.13). □

Proof of Theorem 4.2. The idea of the proof is standard. But, for readers' convenience, we still give a complete proof of this theorem below. One needs to establish a suitable decomposition for $v_h \in V_h(\Omega)$

$$v_h = v_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} v_{\mathrm{E}} + \sum_{\Gamma_{ij}} v_{ij}, \tag{4.17}$$

with

$$v_0 \in V_h^0(\Omega), \quad v_{\mathrm{E}} \in V_h(\Omega_{\mathrm{E}}) \quad \text{and} \quad v_{ij} \in V_h^p(\Omega_{ij}).$$

This decomposition should satisfy the stability condition

$$(B_0 v_0,\ v_0) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} (B_{\mathrm{E}} v_{\mathrm{E}},\ v_{\mathrm{E}}) + \sum_{\Gamma_{ij}} (B_{ij} v_{ij},\ v_{ij}) \lesssim \log^2(H/h)(A_h v_h, v_h). \tag{4.18}$$

Set $\phi_h = v_h|_\Gamma$, and define $\phi_0 \in W_h^0(\Gamma)$ as in (4.7). For $\mathrm{E} \in \mathcal{E}_\Gamma$ and $\mathrm{F} = \Gamma_{ij} \in \mathcal{F}_\Gamma$, set

$$\phi_{\mathrm{E}} = I^0_{\mathrm{E}}[\phi_h - \gamma_{h,\mathrm{E}}(\phi_h)], \text{ and } \phi_{ij} = I^0_{\mathrm{F}}[\phi_h - \gamma_{h,\mathrm{F}}(\phi_h)].$$

It is clear that $\phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E})$ and $\phi_{ij} \in \tilde{W}_h(\mathrm{F})$. It is easy to see that

$$\phi_h = \phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_{\mathrm{E}} + \sum_{\Gamma_{ij}} \phi_{ij}. \tag{4.19}$$

Let $v_{\rm E}$ be the zero extension of $\phi_{\rm E}$, and set $v_0 = E_0\phi_0$. Then,

$$v_{\rm E} \in V_h(\Omega_{\rm E}) \quad \text{and} \quad v_0 \in V_h^0(\Omega).$$

Let $v_{ij}^H \in V_h^p(\Omega_{ij})$ be defined such that $v_{ij}^H|_{\Gamma_{ij}} = \phi_{ij}$, and v_{ij}^H is discrete harmonic on Ω_i and Ω_j. Set

$$v_k^p = (v_h - v_0 - \sum_{{\rm E}\in\mathcal{E}_\Gamma} v_{\rm E} - \sum_{\Gamma_{ij}} v_{ij}^H)|_{\Omega_k}. \tag{4.20}$$

Then, we have $v_k \in V_h^p(\Omega_k)$ by (4.19). For each k, let m_k be the number of faces that belong to $\partial\Omega_k$. Define

$$v_{ij} = v_{ij}^H + v_i^p/m_i + v_j^p/m_j.$$

It suffices to verify (4.18) for the functions defined above. For convenience, set

$$\mathcal{G}(v_h) = (B_0v_0, v_0) + \sum_{{\rm E}\in\mathcal{E}_\Gamma} (B_{\rm E}v_{\rm E},\ v_{\rm E}) + \sum_{\Gamma_{ij}} (B_{ij}v_{ij}^H, v_{ij}^H).$$

Using (4.1), (4.13) and the trace Theorem, we get

$$(B_0v_0, v_0) \lesssim \log(H/h)(A_hv_h, v_h). \tag{4.21}$$

Moreover, we have by (4.2), (2.12) and (2.14)

$$(B_{\rm E}v_{\rm E},\ v_{\rm E}) \lesssim \|v_{\rm E}\|_{0,{\rm E}}^2 \lesssim \log^2(H/h) \sum_{k\in\mathcal{Q}_{\rm E}} \omega_k \int_{\Omega_k} |\nabla v_h|^2 dp. \tag{4.22}$$

Here, we have used the discrete norm to derive the first inequality for $v_{\rm E}$ vanishing at all nodes outside E. Besides, we deduce by (4.3), the definition of v_{ij}^H and (2.13)

$$(B_{ij}v_{ij}^H, v_{ij}^H) \lesssim (\omega_i + \omega_j)\|\phi_{ij}\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} \lesssim \log^2(H/h)(\omega_i|v_h|_{1,\Omega_i}^2 + \omega_j|v_h|_{1,\Omega_j}^2). \tag{4.23}$$

This, together with (4.21) and (4.22), leads to

$$\mathcal{G}(v_h) \lesssim \log^2(H/h)(A_hv_h, v_h). \tag{4.24}$$

On the other hand, it follows by (4.20) that

$$|v_k^p|_{1,\Omega_k}^2 \lesssim |v_h|_{1,\Omega_k}^2 + |v_0|_{1,\Omega_k}^2 + \sum_{{\rm E}\in\mathcal{E}_\Gamma} |v_{\rm E}|_{1,\Omega_k}^2 + \sum_{\Gamma_{ij}} |v_{ij}^H|_{1,\ \Omega_k}^2.$$

Combing this with (4.1)-(4.3), yields

$$\sum_{k=1}^{\rm N} \omega_k |v_k^p|_{1,\Omega_k}^2 \lesssim (A_hv_h, v_h) + \mathcal{G}(v_h).$$

This, together with (4.24), leads to

$$\sum_{\Gamma_{ij}} (B_{ij}(v_i^p/m_i + v_j^p/m_j), v_i^p/m_i + v_j^p/m_j) \lesssim \sum_{k=1}^{\rm N} \omega_k |v_k^p|_{1,\Omega_k}^2 \lesssim \log^2(H/h)(A_hv_h, v_h).$$

By the definition of v_{ij}, (4.23) and the above inequality, we further get

$$\sum_{\Gamma_{ij}} (B_{ij}v_{ij}, v_{ij}) \lesssim \log^2(H/h)(A_hv_h, v_h).$$

Now, (4.18) is a direct consequence of the above inequality, together with (4.24).
□

5 Analysis for the stability of the extension $E_{\rm F}$

This section is devoted to verification of the stability condition (3.16), which has been used in Lemma 4.2. Since quasi-uniform meshes are considered, the analysis is a bit technical. Our basic idea is to introduce a suitable "approximate" extension of $E_{\rm F}$. This auxiliary extension is defined by positive integers, so that its stability can be verified more easily by estimating two finite sums associated with the discrete H^1 semi-norm.

5.1 An auxiliary extension

For face $\mathrm{F} = \Gamma_{ij}$ and a node in Ω_{ij}, let p', p'', $d_{\rm F}$ and $\Lambda_{\rm F}$ be defined as in Subsection 4.1. Without loss of generality, we assume that $h \leq \min\{|pp'|:\ p \in \mathcal{N}_h \cap (\Omega_{ij} \backslash \mathrm{F})\}$. For a positive number x, let $[x]$ denote the integer part of x. Set $n_{\rm F} = [\frac{d_{\rm F}}{h}]$. For any node $p \in \Omega_{ij} \backslash \mathrm{F}$ with $p' \in \mathrm{F}$, define

$$m_p = [\frac{|pp'|}{h}] \qquad \text{and} \quad n_p = [\frac{|p'p''|}{h}].$$

To understand the meaning of the integers m_p and n_p more intuitively, we image that the segments pp' and $p'p''$ are divided into some smaller segments with the size h. Then, m_p and n_p can be viewed roughly as the numbers of the division points on the segments pp' and $p'p''$, respectively.

By the definition of the set $\Lambda_{\rm F}$, it is easy to see that the positive integers m_p and n_p possess the properties:

Property A. For each node $p \in \Lambda_{\rm F}$, we have $1 \leq m_p \leq n_p \leq n_{\rm F}$;

Property B. For two integers r and k satisfying $1 \leq r \leq k \leq n_{\rm F}$, there are at most $O(n_{\rm F})$ nodes $p \in \Lambda_{\rm F}$, such that these nodes p define the same $m_p = r$ and $n_p = k$;

Property C. For an integer k satisfying $1 \leq k \leq n_{\rm F}$, there are at most $O(m_p n_{\rm F})$ nodes $p \in \Lambda_{\rm F}$, such that these nodes p define the same $n_p = k$.

For the *constant-like* basis function $\phi_{\rm F} = I^0_{\rm F} 1$, define the auxiliary extension

$$(E'_{\rm F}\phi_{\rm F})(p) = \begin{cases} 1, & \text{if } p \in \mathrm{F}, \\ 1 - \frac{m_p}{n_p}, & \text{if } p \in \Lambda_{\rm F}, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

For the verification of the stability (3.16), one needs to prove the following two inequalities

$$|(E_{\rm F} - E'_{\rm F})\phi_{\rm F}|^2_{1,\Omega_i} \lesssim d_{\rm F} \log(d_{\rm F}/h) \tag{5.2}$$

and

$$|E'_{\rm F}\phi_{\rm F}|^2_{1,\Omega_i} \lesssim d_{\rm F} \log(d_{\rm F}/h). \tag{5.3}$$

5.2 Auxiliary results

In this subsection, we estimate two finite sums, which will be used to verify the inequality (5.3).

In the rest of this section, p_1 and p_2 always denote two neighboring nodes. Since the triangulation $\mathcal{T}_h$ is quasi-uniform, there exists a (fixed) positive integer k_0, such that $|p_1p_2| \leq k_0 h$ for any two neighboring nodes p_1 and p_2. Then, it can be verified, by the definitions of m_p and n_p, that

$$|m_{p_1} - m_{p_2}|, \quad |n_{p_1} - n_{p_2}| \leq k_0 + 1. \tag{5.4}$$

Lemma 5.1 *Let $\Lambda_{\rm F}^{\partial}$ be the set of nodes p, at which the extension $E_{\rm F}\phi_{\rm F}$ vanish. Namely,*

$$\Lambda_{\rm F}^{\partial} = \{p \in \mathcal{N}_h \cap \bar{\Omega}_{ij} : \quad p \notin {\rm F}, \quad p \notin \Lambda_{\rm F}\}.$$

Then,

$$h \sum_{p_1 \in \Lambda_{\rm F}^{\partial}} \sum_{p_2 \in \Lambda_{\rm F}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 \lesssim d_{\rm F}. \tag{5.5}$$

Note that p_1 and p_2 above denote two neighboring nodes.

Proof. For a node $p_2 \in \Lambda_{\rm F}$, there are at most finite nodes $p_1 \in \Lambda_{\rm F}^{\partial}$, such that p_1 and p_2 are neighboring each other. Then,

$$h \sum_{p_1 \in \Lambda_{\rm F}^{\partial}} \sum_{p_2 \in \Lambda_{\rm F}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 \lesssim h \sum_{p_2 \in \tilde{\Lambda}_{\rm F}} (1 - \frac{m_{p_2}}{n_{p_2}})^2, \tag{5.6}$$

where

$$\tilde{\Lambda}_{\rm F} = \{p \in \Lambda_{\rm F} : \text{ there is } p_* \in \Lambda_{\rm F}^{\partial} \text{ such that } p \text{ and } p_* \text{ are neighboring}\}.$$

For $p \in \tilde{\Lambda}_{\rm F}$, let $p_* \in \Lambda_{\rm F}^{\partial}$ denote a neighboring node with p, and let p'_* be the projection of p_* on the plane containing F. Set

$$\tilde{\Lambda}_{\rm F}^{(1)} = \{p \in \tilde{\Lambda}_{\rm F} : \quad p'_* \in {\rm F}\} \quad \text{and} \quad \tilde{\Lambda}_{\rm F}^{(2)} = \{p \in \tilde{\Lambda}_{\rm F} : \quad p'_* \notin {\rm F}\}.$$

Then,

$$\tilde{\Lambda}_{\rm F} = \tilde{\Lambda}_{\rm F}^{(1)} \cup \tilde{\Lambda}_{\rm F}^{(2)}.$$

It is easy to see that the set $\tilde{\Lambda}_{\rm F}^{(2)}$ contains $O(n_{\rm F})$ nodes at most. It follows by (5.6) that

$$\begin{aligned} h \sum_{p_1 \in \Lambda_{\rm F}^{\partial}} \sum_{p_2 \in \Lambda_{\rm F}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 &\lesssim h \sum_{p \in \tilde{\Lambda}_{\rm F}^{(1)}} (1 - \frac{m_p}{n_p})^2 + h \sum_{p \in \tilde{\Lambda}_{\rm F}^{(2)}} (1 - \frac{m_p}{n_p})^2 \\ &\lesssim h \sum_{p \in \tilde{\Lambda}_{\rm F}^{(1)}} (1 - \frac{m_p}{n_p})^2 + d_{\rm F}. \end{aligned} \tag{5.7}$$

Let $p \in \tilde{\Lambda}_{\rm F}^{(1)}$. From the definition of $\Lambda_{\rm F}^{\partial}$, we know that either $p_* \in \partial\Omega_{ij} \backslash \partial{\rm F}$ or $|p_* p'_*| > |p'_* p''_*|$, where p''_* is defined as in Subsection 3.1. By Assumption 3.1, we have $m_{p_*} \geq n_{p_*}$ in any case. Then, we deduce, by (5.4), that

$$m_p \geq m_{p_*} - (k_0 + 1) \geq n_{p_*} - (k_0 + 1) \geq n_p - 2(k_0 + 1).$$

This implies that

$$\max\{1, \ n_p - k^*\} \leq m_p \leq n_p, \quad \forall p \in \tilde{\Lambda}_{\rm F}^{(1)},$$

with $k^* = 2(k_0 + 1)$. Thus, we have from **Property A** and **Property B**

$$\begin{aligned} \sum_{p \in \tilde{\Lambda}_{\rm F}^{(1)}} (1 - \frac{m_p}{n_p})^2 &\lesssim n_{\rm F} \left\{ \sum_{n_p=1}^{k^*} \sum_{m_p=1}^{k^*} (1 - \frac{m_p}{n_p})^2 + \sum_{n_p=k^*+1}^{n_{\rm F}} \sum_{m_p=n_p-k^*}^{n_p} (1 - \frac{m_p}{n_p})^2 \right\} \\ &\lesssim n_{\rm F} \left\{ (k^*)^2 + \sum_{n_p=k^*+1}^{n_{\rm F}} \sum_{m_p=n_p-k^*}^{n_p} (1 - \frac{m_p}{n_p})^2 \right\}. \end{aligned} \tag{5.8}$$

It is easy to see that

$$\sum_{m_p=n_p-k^*}^{n_p}(1-\frac{m_p}{n_p})^2 \le \sum_{m_p=n_p-k^*}^{n_p}(1-\frac{n_p-k^*}{n_p})^2 \le (k^*)^3\cdot\frac{1}{n_p^2}.$$

Plugging this in (5.8), and note that k^* is a constant, leads to

$$h\sum_{p\in\tilde{\Lambda}_{\mathrm{F}}^{(1)}}(1-\frac{m_p}{n_p})^2 \lesssim d_{\mathrm{F}}\left((k^*)^2+(k^*)^3\sum_{n_p=k^*+1}^{n_{\mathrm{F}}}\frac{1}{n_p^2}\right)\lesssim d_{\mathrm{F}}.$$

Combining (5.7) with the above inequality, gives the desired result.
□

Lemma 5.2 *The following inequality holds for two neighboring nodes p_1 and p_2*

$$h\sum_{p_1,p_2\in\Lambda_{\mathrm{F}}}(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_2}}{n_{p_2}})^2 \lesssim d_{\mathrm{F}}\log(d_{\mathrm{F}}/h). \tag{5.9}$$

Proof. For $p_1\in\Lambda_{\mathrm{F}}$, define

$$\Lambda_{\mathrm{F},p_1}=\{p\in\Lambda_{\mathrm{F}}:\ p \text{ is neighboring with } p_1\}.$$

Let k_0 be defined by (5.4). Then, we have from **Property A** and **Property B**

$$\begin{aligned}
h\sum_{p_1,p_2\in\Lambda_{\mathrm{F}}}(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_2}}{n_{p_2}})^2 &\lesssim h\cdot\frac{d_{\mathrm{F}}}{h}\sum_{n_{p_1}=1}^{n_{\mathrm{F}}}\sum_{m_{p_1}=1}^{n_{p_1}}\sum_{p_2\in\Lambda_{\mathrm{F},p_1}}(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_2}}{n_{p_2}})^2\\
&= d_{\mathrm{F}}\sum_{n_{p_1}=1}^{k_0+1}\sum_{m_{p_1}=1}^{n_{p_1}}\sum_{p_2\in\Lambda_{\mathrm{F},p_1}}(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_2}}{n_{p_2}})^2\\
&+ d_{\mathrm{F}}\sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}}\sum_{m_{p_1}=1}^{k_0}\sum_{p_2\in\Lambda_{\mathrm{F},p_1}}(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_2}}{n_{p_2}})^2\\
&+ d_{\mathrm{F}}\sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}}\sum_{m_{p_1}=k_0+1}^{n_{p_1}}\sum_{p_2\in\Lambda_{\mathrm{F},p_1}}(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_2}}{n_{p_2}})^2\\
&= I_1+I_2+I_3.
\end{aligned}\tag{5.10}$$

It is clear that $I_1\lesssim d_{\mathrm{F}}$. When $m_{p_1}\le k_0$, we deduce from (5.4) that $m_{p_2}\le 2k_0+1$ for $p_2\in\Lambda_{\mathrm{F},p_1}$. Note that k_0 is a constant, we get

$$\begin{aligned}
I_2 &\lesssim d_{\mathrm{F}}\sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}}\sum_{p_2\in\Lambda_{\mathrm{F},p_1}}(\frac{1}{n_{p_1}^2}+\frac{1}{n_{p_2}^2})\\
&\lesssim d_{\mathrm{F}}\sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}}\left(\frac{1}{n_{p_1}^2}+\frac{1}{(n_{p_1}-k_0-1)^2}\right)\lesssim d_{\mathrm{F}}.
\end{aligned}\tag{5.11}$$

It follows by (5.4) that

$$I_3 \lesssim d_{\mathrm{F}}\sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}}\sum_{m_{p_1}=k_0+1}^{n_{p_1}}\sum_{k=-k_0-1}^{k_0+1}\sum_{r=-k_0-1}^{k_0+1}\left(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_1}+r}{n_{p_1}+k}\right)^2$$

$$
\begin{aligned}
&= d_{\mathrm{F}} \sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}} \sum_{m_{p_1}=k_0+1}^{n_{p_1}} \sum_{k=-k_0-1}^{k_0+1} \sum_{r=-k_0-1}^{k_0+1} \left(\frac{km_{p_1} - rn_{p_1}}{n_{p_1}(n_{p_1}+k)}\right)^2 \\
&\lesssim d_{\mathrm{F}} \sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}} \sum_{m_{p_1}=k_0+1}^{n_{p_1}} \frac{m_{p_1}^2 + n_{p_1}^2}{n_{p_1}^2 (n_{p_1} - k_0 - 1)^2} \\
&\lesssim d_{\mathrm{F}} \sum_{n_{p_1}=k_0+2}^{n_{\mathrm{F}}} \frac{n_{p_1}}{(n_{p_1} - k_0 - 1)^2} \lesssim d_{\mathrm{F}} \log n_{\mathrm{F}}.
\end{aligned}
$$

Plugging (5.11) and the above inequality in (5.10), leads to

$$
h \sum_{p_1, p_2 \in \Lambda_{\mathrm{F}}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2 \lesssim d_{\mathrm{F}} \log(d_{\mathrm{F}}/h).
$$

□

5.3 Proof of Theorem 4.1

It suffices to verify that

$$
|E_{\mathrm{F}}\phi_{\mathrm{F}}|_{1,\Omega_i}^2, \quad |E_{\mathrm{F}}\phi_{\mathrm{F}}|_{1,\Omega_j}^2 \lesssim d_{\mathrm{F}} \log(d_{\mathrm{F}}/h) \eqsim |\phi_{\mathrm{F}}|^2_{H_{00}^{\frac{1}{2}}(\mathrm{F})}. \tag{5.12}
$$

Let $E'_{\mathrm{F}}\phi_{\mathrm{F}}$ be the auxiliary extension defined by Subsection 5.1.

Step 1. Verify the inequality (5.2)

Let $p \in \Lambda_{\mathrm{F}}$. Then, we have

$$
|(E_{\mathrm{F}}\phi_{\mathrm{F}} - E'_{\mathrm{F}}\phi_{\mathrm{F}})(p)| = |\frac{|pp'|}{|p'p''|} - [\frac{|pp'|}{h}]/[\frac{|p'p''|}{h}]|. \tag{5.13}
$$

Since

$$
\frac{|pp'|}{h}/(\frac{|p'p''|}{h}+1) \le [\frac{|pp'|}{h}]/[\frac{|p'p''|}{h}] \le (\frac{|pp'|}{h}+1)/\frac{|p'p''|}{h},
$$

we get

$$
\frac{|pp'|}{|p'p''|+h} \le [\frac{|pp'|}{h}]/[\frac{|p'p''|}{h}] \le \frac{|pp'|+h}{|p'p''|}.
$$

Plugging this in (5.13), leads to

$$
|(E_{\mathrm{F}}\phi_{\mathrm{F}} - E'_{\mathrm{F}}\phi_{\mathrm{F}})(p)| \le \frac{h}{|p'p''|} \le \frac{1}{n_p}. \tag{5.14}
$$

By the inverse estimate and the discrete L^2-norm, we get

$$
\begin{aligned}
|(E_{\mathrm{F}} - E'_{\mathrm{F}})\phi_{\mathrm{F}}|_{1,\Omega_i}^2 &\lesssim h^{-2}\|(E_{\mathrm{F}} - E'_{\mathrm{F}})\phi_{\mathrm{F}}|_{0,\Omega_i}^2 \\
&\eqsim h \sum_{p\in\Lambda_{\mathrm{F}}} (E_{\mathrm{F}}\phi_{\mathrm{F}} - E'_{\mathrm{F}}\phi_{\mathrm{F}})^2(p).
\end{aligned}
$$

This, together with (5.14), yields

$$
|(E_{\mathrm{F}} - E'_{\mathrm{F}})\phi_{\mathrm{F}}|_{1,\Omega_i}^2 \lesssim h \sum_{p\in\Lambda_{\mathrm{F}}} \frac{1}{n_p^2}. \tag{5.15}
$$

By **Property A** and **Property C**, we get

$$
\sum_{p\in\Lambda_{\mathrm{F}}} \frac{1}{n_p^2} \lesssim n_{\mathrm{F}} \sum_{n_p=1}^{n_{\mathrm{F}}} \frac{m_p}{n_p^2} \le n_{\mathrm{F}} \sum_{n_p=1}^{n_{\mathrm{F}}} \frac{n_p}{n_p^2} \lesssim n_{\mathrm{F}} \log n_{\mathrm{F}}.
$$

Plugging this in (5.15), gives (5.2).

Step 2. Verify the inequality (5.3)

For simplicity of exposition, let $\Lambda_{\rm F}^{*}$ denote the set of neighboring node paring (p_1, p_2), which satisfies $(E'_{\rm F}\phi_{\rm F})(p_1) - (E'_{\rm F}\phi_{\rm F})(p_2) \neq 0$. By the discrete H^1 semi-norm, we have

$$|E'_{\rm F}\phi_{\rm F}|^2_{1,\Omega_i} \eqsim h \sum_{(p_1,\ p_2)\in\Lambda_{\rm F}^{*}} |(E'_{\rm F}\phi_{\rm F})(p_1) - (E'_{\rm F}\phi_{\rm F})(p_2)|^2. \tag{5.16}$$

For ease of notation, define

$$\Lambda_{{\rm F},\ b} = \{p \in \mathcal{N}_h \cap {\rm F} :\ p \ \text{ closes } \ \partial{\rm F}\}.$$

It is easy to see that the set $\Lambda_{\rm F}^{*}$ can be decomposed into several groups: (a) $p_1 \in \Lambda_{{\rm F},b}$ and $p_2 \in \partial{\rm F}$; (b) $p_1 \in {\rm F}$ and $p_2 \in \Lambda_{\rm F}$; (c) $p_1 \in \Lambda_{\rm F}^{\partial}$ and $p_2 \in \Lambda_{\rm F}$; (d) $p_1, p_2 \in \Lambda_{\rm F}$. It is certain that one can also consider the inverse situation with exchanging the positions of p_1 and p_2, but this will not affect the result.

It follows by (5.16) that

$$\begin{aligned} |E'_{\rm F}\phi_{\rm F}|^2_{1,\Omega_i} \ &\eqsim\ h \sum_{p_1\in\Lambda_{{\rm F},b}} \sum_{p_2\in\partial{\rm F}} (1-0)^2 + h \sum_{p_1\in{\rm F}} \sum_{p_2\in\Lambda_{\rm F}} (\frac{m_{p_2}}{n_{p_2}})^2 \\ &+\ h \sum_{p_1\in\Lambda_{\rm F}^{\partial}} \sum_{p_2\in\Lambda_{\rm F}} (1-\frac{m_{p_2}}{n_{p_2}})^2 + h \sum_{p_1,p_2\in\Lambda_{\rm F}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2. \end{aligned} \tag{5.17}$$

It is clear that the set $\Lambda_{{\rm F},b}$ contains only $O(n_{\rm F})$ nodes. Then, we get for two neighboring nodes p_1 and p_2

$$h \sum_{p_1\in\Lambda_{{\rm F},b}} \sum_{p_2\in\partial{\rm F}} (1-0)^2 \lesssim h \cdot \frac{d_{\rm F}}{h} = d_{\rm F}. \tag{5.18}$$

When $p_2 \in \Lambda_{\rm F}$ is neighboring with some $p_1 \in {\rm F}$, we have

$$m_{p_2} \leq |p_2 p'_2|/h \leq |p_1 p_2|/h \leq k_0.$$

Besides, for any $p_2 \in \Lambda_{\rm F}$, there are at most finite nodes $p_1 \in {\rm F}$, such that p_1 and p_2 are neighboring each other. Thus, we deduce by **Property A** and **Property B**

$$h \sum_{p_1\in{\rm F}} \sum_{p_2\in\Lambda_{\rm F}} (\frac{m_{p_2}}{n_{p_2}})^2 \lesssim h \cdot \frac{d_{\rm F}}{h} \sum_{l=1}^{k_0} \sum_{n_{p_2}=1}^{n_{\rm F}} (\frac{l}{n_{p_2}})^2 \lesssim d_{\rm F}. \tag{5.19}$$

Substituting (5.18)-(5.19), (5.5) and (5.9) into (5.17), yields (5.3).

Step 3. Prove the desired result (5.12)

Combining (5.2) and (5.3), yields

$$|E_{\rm F}\phi_{\rm F}|^2_{1,\Omega_i} \lesssim d_{\rm F} \log(d_{\rm F}/h).$$

In the same way, we can prove that

$$|E_{\rm F}\phi_{\rm F}|^2_{1,\Omega_j} \lesssim d_{\rm F} \log(d_{\rm F}/h).$$

On the other hand, it can be verified, by the discrete semi-norm in Subsection 2.2, that

$$|\phi_{\rm F}|^2_{H_{00}^{\frac{1}{2}}({\rm F})} \eqsim d_{\rm F} \log(d_{\rm F}/h).$$

□

6 Numerical experiments

In this section, we give some numerical results to confirm our theoretical results described in section 4.

Consider the elliptic problem (2.1) with Ω being the cube $\Omega = [0,\ 1]^3$, and the coefficient $a(x, y, z)$ being defined by

$$\begin{aligned} a(x,y,z) &= 10^{-5}, \quad \text{if } x,y \le 0.5 \text{ or } x,y \ge 0.5; \\ a(x,y,z) &= 1, \quad \text{otherwise.} \end{aligned}$$

The source function f is chosen in a suitable manner.

Let Ω be decomposed into N cube subdomains with the edge length H. To illustrate wide practicality of the new method, we consider tetrahedron elements instead of hexahedron elements. Let each subdomain be divided into tetrahedron elements with the size h in the standard way, and use the usual P_1 finite element approximate space.

We solve the algebraic system associated with the equation (2.4) by PCG iteration with the preconditioner B_h defined in Section 4 or Algorithm 4.2. Here, each local solver B_{ij} is chosen as the symmetric multigrid preconditioner for the restriction of A_h on the subspace $V_h^0(\Omega_{ij})$. The iteration terminates when the relative remainder is less than $1.0D - 5$. The iteration counts are listed as the following tables.

Table 6.1
iteration counts for PCG with B_h

H/h	$H = 1/4$	$H = 1/6$	$H = 1/8$
8	29	32	32
16	40	42	42
32	50	51	52

Table 6.2
iteration counts for Algorithm 4.2

H/h	$H = 1/4$	$H = 1/6$	$H = 1/8$
8	24	26	26
16	31	33	33
32	39	41	41

These numerical results indicates that the convergence of the new preconditioner is stable with the subdomain number N, and depends slightly on the ratio H/h. These are just predicted by Theorem 4.1 and Theorem 4.2. Besides, the numerical results show that Algorithm 4.2 is indeed more efficient than the standard PCG.

7 Appendix

In this Appendix, we revise the definition of the extension $E_{\rm F}$ for the general case without Assumption 3.1.

Let $o_{\rm F} \in$ F denote the center of the largest circle[2] contained in F. Then, $o_{\rm F}$ satisfies $dist(o_{\rm F}, \partial{\rm F}) = d_{\rm F}$. Through the center $o_{\rm F}$, we draw the perpendicular line L of F. Let $q_1^{\partial},\ q_2^{\partial} \in \partial\Omega_{ij}$ denote the two intersection points of L with $\partial\Omega_{ij}$. Set

$$s_{\rm F} = \min\{|o_{\rm F}q_1^{\partial}|,\ |o_{\rm F}q_2^{\partial}|\}.$$

Set $r_{\rm F} = d_{\rm F}/s_{\rm F}$. Since Ω_i and Ω_j are regular and convex domains, the ratio $r_{\rm F}$ is uniformly bounded for every F. With the same notations in Subsection 3.1, set

$$\Lambda_{\rm F} = \{p \in \mathcal{N}_h \cap (\Omega_{ij}\backslash{\rm F}) :\ p' \in {\rm F},\ r_{\rm F}|pp'| \leq |p'p''|\}.$$

For a face ${\rm F} = \Gamma_{ij}$, we revise the extension $E_{\rm F}$ as :

$$(E_{\rm F}\phi_{\rm F})(p) = \begin{cases} 1, & \text{if } p \in {\rm F}, \\ 1 - \frac{r_{\rm F}|pp'|}{|p'p''|}, & \text{if } p \in \Lambda_{\rm F}, \\ 0, & \text{otherwise.} \end{cases} \tag{7.1}$$

For such revision, the proof of Theorem 3.1 is almost the same with that in Section 5.

We revise the definition of the positive integer m_p for $p \in \Omega_{ij}\backslash{\rm F}$ ($p' \in$ F) as

$$m_p = [\frac{r_{\rm F}|pp'|}{h}],$$

but reserve the definition of the auxiliary extension $E'_{\rm F}$ given in (5.1). We need only to revise slightly the proof of Lemma 5.1. Below is the changes needed to make.

The inequality (5.4) now becomes to be

$$|m_{p_1} - m_{p_2}| \leq r_{\rm F}k_0 + 1,\ |n_{p_1} - n_{p_2}| \leq k_0 + 1, \tag{7.2}$$

for two neighboring nodes p_1 and p_2.

Let $\Lambda_{\rm F}^{\partial}$ and $\tilde{\Lambda}_{\rm F}$ be defined in Lemma 5.1. For $p \in \tilde{\Lambda}_{\rm F}$, let $p_* \in \Lambda_{\rm F}^{\partial}$ be some neighboring node with p. When $p_* \in \partial\Omega_{ij}$, we deduce that $r_{\rm F}|p_*p'_*| \geq |p'_*p''_*|$ by the definition of $r_{\rm F}$ and the convexity of Ω_i and Ω_j. Then, we have $m_{p_*} \geq n_{p_*}$ for $p_* \in \Lambda_{\rm F}^{\partial}$. For $p \in \tilde{\Lambda}_{\rm F}$, we get by (7.2)

$$m_p \geq m_{p_*} - (r_{\rm F}k_0 + 1) \geq n_{p_*} - (r_{\rm F}k_0 + 1) \geq n_p - (r_{\rm F} + 1)k_0 - 2.$$

This implies that

$$\max\{1,\ n_p - k^*\} \leq m_p \leq n_p, \quad \forall p \in \tilde{\Lambda}_{\rm F},$$

with $k^* = (r_{\rm F} + 1)k_0 + 2$.

Remark 7.1 *It is easy to see that the above revision is needed only for the case with $r_{\rm F} > 1$, since $r_{\rm F} \leq 1$ implies that Assumption 3.1 is satisfied. When $r_{\rm F} \leq 1$, the use of the original definition of $E_{\rm F}$ (see Subsection 3.1) can reduce the cost for calculating $E_{\rm F}\phi_{\rm F}$.*

8 Conclusions

We have developed a kind of simple nearly harmonic extension for the *constant-like* basis function. This extension is used to define a coarse subspace, and then is applied to construct a substructuring preconditioner with *inexact* subdomain solvers. The simplicity of the extension guarantees that the resulting preconditioner is easy to implement.

[2] If F possesses two parallel edges, then there may exist infinite largest circles contained in F. For this case, one needs to consider two such circles, which are tangent with three or more edges of F. Besides, the point $o_{\rm F}$ can be also chosen as any "central" point of F. Different choices of $o_{\rm F}$ will result in different values of the constant in the right side of (5.5).

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