New type of KP equation with self-consistent sources and its bilinear Bäcklund transformation

Xing-Biao Hu[1] **and Hong-Yan Wang**[1,2]

[1]Institute of Computational Mathematics and Scientific Engineering Computing, AMSS, Chinese Academy of Sciences, P.O. Box 2719, Beijing 100080, P.R. CHINA

[2]Graduate School of the Chinese Academy of Sciences, Beijing, P.R. CHINA

E-mail: hxb@lsec.cc.ac.cn and wanghy@lsec.cc.ac.cn

Abstract. A new type of the KP equation with self-consistent sources (KPESCS) first found by Melnikov (Lett. Math. Phys. 7(1983) 129-136)is re-constructed via source generation procedure. New feature of the obtained KPESCS is that we allow y-dependence of the arbitrary constants in the determinantal solution for the KP equation while applying the source generation procedure. We also propose a new idea of commutativity of source generation procedure and Bäcklund transformations to generate a BT for the new KPESCS which indicates the integrability of the KPESCS.

1. Introduction

Soliton equations with self-consistent sources (SESCSs) have received considerable attention in recent years. Until now, numerous SESCSs have been found and studied. One typical example is the Kadomtsev-Petviashvili equation with self-consistent sources (KPESCS) [2, 3, 4, 5] which can be expressed as

$$4u_t - u_{xxx} - 6uu_x - 3\int^x u_{yy}\mathrm{d}x + \sum_{j=1}^{K}(\Phi_j\Psi_j)_x = 0, \tag{1}$$

$$\Phi_{j,y} = \Phi_{j,xx} + u\Phi_j, \quad j = 1, 2, \cdots, K \tag{2}$$

$$-\Psi_{j,y} = \Psi_{j,xx} + u\Psi_j, \quad j = 1, 2, \cdots, K \tag{3}$$

In the literature, a variety of methods have been developed to solve SESCSs, such as inverse scattering transform, Darboux transformation, Hirota's bilinear method, and so on (see Refs. [2]-[19]). Very recently, we have proposed a new method called "source generation procedure" to systematically construct and solve SESCSs [21, 22, 23]. One of advantages of this new approach is that SESCSs and their soliton solutions can be generated simultaneously from the procedure. Moreover, different from other methods that we have known, source generation procedure has helped to produce SESCSs in fully discrete case and B-type KP case for the first time. Our procedure consists of three steps:

1. to express N-soliton solutions of a soliton equation without sources in the form of determinant or pfaffian with some arbitrary constants.

2. to construct corresponding determinant or pfaffian with arbitrary functions of one variable.

3. to seek coupled bilinear equations whose solutions are these generalized determinants or pfaffians, and this coupled system is the SESCS.

It is noted that the above step 3 is crucial in source generation procedure and the success of step 3 heavily depends on suitable choice of arbitrary functions involved in step 2. Until now all examples of SESCSs in continuous and semi-discrete cases found by source generation procedure always require time dependence of the arbitrary constants appeared in the determinantal or pfaffian solutions for the equations without sources. The reason for this is that many soliton equations without sources only contain the first order derivative with respect to t, and therefore calculations involved in source generation procedure can be easily managed if we choose arbitrary functions in step 2 as those of temporal variable t. For example, in [5], we have shown that the KPESCS (1)-(3) can be easily constructed from source generation procedure by allowing time dependence of the arbitrary constants in the determinatal solution for the KP equation.

Now, it is natural to ask whether we can still construct other type of SESCSs via source generation procedure if arbitrary functions in step 2 of the procedure are chosen as those of spatial variables. The answer is affirmative. The purpose of this paper is to apply source generation procedure to the KP equation by allowing y-dependence of the arbitrary constants in the determinantal solution for the KP equation. Consequently, a new type of the KPESCS is produced, which is quite different from the known KPESCS (1-3). To our surprise, this new type of KPESCS is nothing but another one previously found by Mel'nikov [1]. In order to further show integrability of novel KPESCS, we propose a new idea of commutativity of source generation procedure and Bäcklund transformations which helps to find a bilinear Bäcklund transformation for this new KPESCS.

The paper is organized as follows. In section 2, a new type of the KPESCS is constructed via source generation procedure by allowing y-dependence of the arbitrary constants in the determinantal solution for the KP equation. Then we present a bilinear Bäcklund transformation for the novel KPESCS with the help of commutativity of source generation procedure and Bäcklund transformations in section 3. Conclusion and discussions are given in section 4. Finally we present a detailed proof of the Proposition 1 and list some bilinear operator identities in the appendices A and B, respectively.

2. New type of the KPESCS

In this section, we will apply the source generation procedure to the KP equation by allowing y-dependence of the arbitrary constants in the determinantal solution for the KP equation. The KP equation can be written as [24]

$$(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0, \tag{4}$$

which can be transformed into the following bilinear equation

$$(D_x^3 - 4D_xD_t + 3D_y^2)\tau \cdot \tau = 0 \tag{5}$$

through the dependent variable transformation

$$u = 2(\ln \tau)_{xx},$$

where D is the Hirota's bilinear operator. The bilinear KP equation (5) has the following Grammian determinant solution [24]

$$\tau = \det(c_{ij} + \int_{-\infty}^{x} f_i \tilde{f}_j \mathrm{d}x)_{1\leqslant i,j\leqslant N}, \qquad c_{ij} = \text{constant},$$

with functions f_i and $\tilde{f}_j$ satisfying

$$\frac{\partial f_i}{\partial x_n} = \frac{\partial^n f_i}{\partial x^n}, \qquad \frac{\partial \tilde{f}_i}{\partial x_n} = (-1)^{n-1}\frac{\partial^n \tilde{f}_i}{\partial x^n}, \quad (x_1 = x, \quad x_2 = y, \quad x_3 = t). \tag{6}$$

Now following source generation procedure, we generalize τ into the following new function:

$$f = \det(a_{ij})_{1\leqslant i,j\leqslant N} = \mathrm{pf}(1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*) = \mathrm{pf}(\cdot), \tag{7}$$

where pfaffian elements are defined by

$$\mathrm{pf}(i, j^*) = a_{ij} = C_{ij}(y) + \int_{-\infty}^{x} f_i \tilde{f}_j \mathrm{d}x, \quad i, j = 1, 2, \cdots, N,$$

with each function $C_{ij}(y)$ satisfying

$$C_{ij}(y) = \begin{cases} C_i(y), & i = j \text{ and } \ 1 \leqslant i \leqslant M \leqslant N, \quad M, N \in \mathbb{Z}^+, \\ c_{ij}, & \text{otherwise} \end{cases}$$

where each $C_i(y)$ is a function of the variable y. Then we get the following formulas through derivative formulas of pfaffian [24]:

$$f_y = \mathrm{pf}(d_0, d_1^*, \cdot) - \mathrm{pf}(d_1, d_0^*, \cdot) + \sum_{i=1}^{M} k_i, \tag{8}$$

$$\begin{aligned} f_{yy} = {} & \mathrm{pf}(d_0, d_3^*, \cdot) + \mathrm{pf}(d_3, d_0^*, \cdot) - \mathrm{pf}(d_2, d_1^*, \cdot) - \mathrm{pf}(d_1, d_2^*, \cdot) - 2\mathrm{pf}(d_0, d_1, d_0^*, d_1^*, \cdot) \\ & + \sum_{i=1}^{M} k_{i,y} + \sum_{i=1}^{M} \dot{C}_i(y)[\mathrm{pf}(d_0, d_1^*, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*) \\ & - \mathrm{pf}(d_1, d_0^*, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*)], \end{aligned} \tag{9}$$

where the function k_i is defined by

$$k_i = \dot{C}_i(y)\mathrm{pf}(1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*), \quad i = 1, 2, \cdots, M \tag{10}$$

and new paffian elements are defined by:

$$\mathrm{pf}(d_m^*, i) = \frac{\partial^m f_i}{\partial x^m}, \quad \mathrm{pf}(d_m, j^*) = \frac{\partial^m \tilde{f}_j}{\partial x^m},$$
$$\mathrm{pf}(d_m^*, d_l^*) = \mathrm{pf}(d_m, d_l) = \mathrm{pf}(d_m^*, d_l) = \mathrm{pf}(d_m^*, j^*) = \mathrm{pf}(d_m, i) = 0, \quad m, l \in \mathbb{Z}.$$

So we find the function f will never satisfy equation (5). In this case, we need to introduce other new functions defined by

$$g_i = \sqrt{\dot{C}_i(y)}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*), \tag{11}$$

$$h_i = \sqrt{\dot{C}_i(y)}\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*), \tag{12}$$

$$\begin{aligned} P_i = &\frac{C_i''(y)}{2\sqrt{\dot{C}_i(y)}}\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*) \\ &+\sqrt{\dot{C}_i(y)}[\sum_{1\leqslant i<j\leqslant M}\dot{C}_j(y)\mathrm{pf}(d_0^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, \hat{i}^*, \cdots, 1^*) \\ &\quad -\sum_{1\leqslant j<i\leqslant M}\dot{C}_j(y)\mathrm{pf}(d_0^*, 1, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, \hat{j}^*, \cdots, 1^*)], \end{aligned} \tag{13}$$

$$\begin{aligned} Q_i = &\frac{C_i''(y)}{2\sqrt{\dot{C}_i(y)}}\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*) \\ &+\sqrt{\dot{C}_i(y)}[\sum_{1\leqslant i<j\leqslant M}\dot{C}_j(y)\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*) \\ &\quad -\sum_{1\leqslant j<i\leqslant M}\dot{C}_j(y)\mathrm{pf}(d_0, 1, \cdots, \hat{j}, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{j}^*, \cdots, 1^*)], \end{aligned} \tag{14}$$

where the dot denotes the derivative of the function $C_i(y)$ with respect to the variable y, and $''$ denotes the second-order derivative of $C_i(y)$. We can show that the above new functions so defined satisfy the following new bilinear equations:

$$(D_x^4 - 4D_xD_t + 3D_y^2)f\cdot f = 6\sum_{i=1}^{M}(D_yk_i\cdot f - D_xg_i\cdot h_i), \tag{15}$$

$$D_xk_i\cdot f + g_ih_i = 0, \tag{16}$$

$$(D_y - D_x^2)g_i\cdot f = P_if - g_i\sum_{j=1}^{M}k_j, \tag{17}$$

$$(D_y - D_x^2)f\cdot h_i = h_i\sum_{j=1}^{M}k_j - fQ_i, \tag{18}$$

$$(D_x^3 + 3D_xD_y - 4D_t)g_i\cdot f = 3D_x[P_i\cdot f - g_i\cdot(\sum_{j=1}^{M}k_j)], \tag{19}$$

$$(D_x^3 + 3D_xD_y - 4D_t)f\cdot h_i = 3D_x[(\sum_{j=1}^{M}k_j)\cdot h_i - f\cdot Q_i]. \tag{20}$$

Now we prove that these new functions so defined are solutions of equations (15)-(20). At first, we derive the following derivative formulas:

$$\begin{aligned} &f_x = \mathrm{pf}(d_0, d_0^*, \cdot), \quad f_{xx} = \mathrm{pf}(d_1, d_0^*, \cdot) + \mathrm{pf}(d_0, d_1^*, \cdot), \\ &f_t = \mathrm{pf}(d_2, d_0^*, \cdot) - \mathrm{pf}(d_1, d_1^*, \cdot) + \mathrm{pf}(d_0, d_2^*, \cdot), \\ &f_{xxx} = \mathrm{pf}(d_2, d_0^*, \cdot) + 2\mathrm{pf}(d_1, d_1^*, \cdot) + \mathrm{pf}(d_0, d_2^*, \cdot), \end{aligned} \tag{21}$$

$$
\begin{aligned}
g_{i,x} &= \sqrt{\dot{C}_i(y)}\mathrm{pf}(d_1^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*) \triangleq \sqrt{\dot{C}_i(y)}\mathrm{pf}(d_1^*, \star), \\
g_{i,xx} &= \sqrt{\dot{C}_i(y)}[\mathrm{pf}(d_2^*, \star) + \mathrm{pf}(d_0, d_0^*, d_1^*, \star)], \\
g_{i,y} &= P_i + \sqrt{\dot{C}_i(y)}[\mathrm{pf}(d_2^*, \star) - \mathrm{pf}(d_0, d_0^*, d_1^*, \star)], \\
g_{i,xy} &= P_{i,x} + \sqrt{\dot{C}_i(y)}[\mathrm{pf}(d_3^*, \star) - \mathrm{pf}(d_1, d_0^*, d_1^*, \star)], \\
g_{i,t} &= \sqrt{\dot{C}_i(y)}[\mathrm{pf}(d_3^*, \star) + \mathrm{pf}(d_0, d_0^*, d_1^*, \star) - \mathrm{pf}(d_0, d_0^*, d_2^*, \star)].
\end{aligned}
\tag{22}
$$

Substituting (7)-(9) and (11)-(12) into equation (15) yields the sum of the determinant identities:

$$
\begin{aligned}
0 = \sum_{i=1}^{M} &\sqrt{\dot{C}_i(y)}[\mathrm{pf}(d_0, d_1^*, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*)f \\
&-\mathrm{pf}(1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*)\mathrm{pf}(d_0, d_1^*, \cdot) \\
&+\mathrm{pf}(d_1^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*)\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*) \\
&-\mathrm{pf}(d_1, d_0^*, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*)f \\
&+\mathrm{pf}(1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*)\mathrm{pf}(d_1, d_0^*, \cdot) \\
&-\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*)\mathrm{pf}(d_1, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*)],
\end{aligned}
$$

which indicates equation (15) holds. In the same way, substitution of (7), (10) and (11)-(12) into equation (16) leads to the Jacobi identity of determinants:

$$
\begin{aligned}
&\dot{C}_i(y)[\mathrm{pf}(d_0, d_0^*, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*)f \\
&-\mathrm{pf}(1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 2^*, 1^*)\mathrm{pf}(d_0, d_0^*, \cdot) \\
&+\mathrm{pf}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*)\mathrm{pf}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*)] = 0,
\end{aligned}
$$

then equation (16) holds. Similarly, substituting (21)-(22) into equation (17), we get the following determinant identity:

$$
\sqrt{\dot{C}_i(y)}[\mathrm{pf}(d_0^*, \star)\mathrm{pf}(d_0, d_1^*, \cdot) + \mathrm{pf}(d_1^*, \star)\mathrm{pf}(d_0, d_0^*, \cdot) - \mathrm{pf}(d_0, d_0^*, d_1^*, \star)f] = 0.
$$

So equation (17) holds. In an analogous way, we can prove new functions in (7), (10) and (11)-(14) are determinant solutions of bilinear equations (15)-(20). And equations (15)-(20) just constitute a new type of KPESCS in the bilinear forms.

Applying the dependent variable transformations:

$$
\begin{aligned}
&u = 2(\ln f)_{xx}, \quad \varphi_i = h_i/f, \quad \psi_i = g_i/f, \\
&\chi_i = k_i/f \quad \phi_{1,i} = P_i/f, \quad \phi_{2,i} = Q_i/f,
\end{aligned}
\tag{23}
$$

the bilinear equations (15)-(20) are transformed into nonlinear equations:

$$\begin{aligned}
&u_{xxx}+6uu_x-4u_t+3\int_{-\infty}^{x}u_{yy}\mathrm{d}x=6\sum_{i=1}^{M}(\chi_{i,y}+\varphi_{i,x}\psi_i-\varphi_i\psi_{i,x})_x,\\
&\chi_{i,x}+\varphi_i\psi_i=0,\\
&\varphi_{i,y}=-\varphi_{i,xx}-u\varphi_i+\phi_{2,i}-\varphi_i\sum_{j=1}^{M}\chi_j,\\
&\psi_{i,y}=\psi_{i,xx}+u\psi_i+\phi_{1,i}-\psi_i\sum_{j=1}^{M}\chi_j,\\
&\varphi_{i,xxx}-4\varphi_{i,t}-3\varphi_{i,xy}+3u\varphi_{i,x}-3\varphi_i\int_{-\infty}^{x}u_y\mathrm{d}x=-3\phi_{2,i,x}+3\sum_{j=1}^{M}(\varphi_{i,x}\chi_j-\varphi_i\chi_{j,x}),\\
&\psi_{i,xxx}-4\psi_{i,t}+3\psi_{i,xy}+3u\psi_{i,x}+3\psi_i\int_{-\infty}^{x}u_y\mathrm{d}x=3\phi_{1,i,x}-3\sum_{j=1}^{M}(\psi_{i,x}\chi_j-\psi_i\chi_{j,x}),
\end{aligned}\tag{24}$$

which can be further simplified into the following nonlinear equations:

$$\begin{aligned}
&u_{xxx}+6uu_x-4u_t+3\int_{-\infty}^{x}u_{yy}\mathrm{d}x=6\sum_{i=1}^{M}[\varphi_{i,xx}\psi_i-\varphi_i\psi_{i,xx}-(\varphi_i\psi_i)_y],\\
&4\varphi_{i,xxx}-4\varphi_{i,t}+6u\varphi_{i,x}+3u_x\varphi_i-3\varphi_i\int_{-\infty}^{x}u_y\mathrm{d}x=6\varphi_i\sum_{j=1}^{M}\varphi_j\psi_j,\\
&4\psi_{i,xxx}-4\psi_{i,t}+6u\psi_{i,x}+3u_x\psi_i+3\psi_i\int_{-\infty}^{x}u_y\mathrm{d}x=-6\psi_i\sum_{j=1}^{M}\varphi_j\psi_j.
\end{aligned}\tag{25}$$

Utilizing the expressions (7), (10), (11)-(14) and the relation (23), we can give N-soliton ($N\geqslant M$) solution of the new type of KPESCS (25). For example, when $M=1$, we take

$$C_1(y)=\frac{1}{p+q}e^{2\alpha(y)},$$

$$f_1=e^{\xi}=e^{px+p^2y+p^3t},\quad \tilde{f}_1=e^{\eta}=e^{qx-q^2y+q^3t},\quad p,q\in\mathbb{R},$$

where $\alpha(y)$ is an arbitrary function of the variable y. Then the 1-soliton solution can be expressed in the following forms:

$$\begin{aligned}
&u=2\frac{\partial^2}{\partial x^2}\ln(1+e^{\xi+\eta-2\alpha(y)}),\\
&\psi=\frac{\sqrt{2(p+q)\dot{\alpha}(y)}e^{\xi-\alpha(y)}}{1+e^{\xi+\eta-2\alpha(y)}},\quad \varphi=\frac{\sqrt{2(p+q)\dot{\alpha}(y)}e^{\eta-\alpha(y)}}{1+e^{\xi+\eta-2\alpha(y)}}.
\end{aligned}$$

When $M=2$, we take

$$C_i(y)=\frac{1}{p_i+q_i}e^{2\alpha_i(y)},\quad i=1,2$$

$$f_i=e^{\xi_i}=e^{p_ix+p_i^2y+p_i^3t},\quad \tilde{f}_i=e^{\eta_i}=e^{q_ix-q_i^2y+q_i^3t},\quad p_i,q_i\in\mathbb{R},\quad i=1,2$$

then the 2-soliton solution of the system has the following form:

$$\begin{aligned}
&u=2\frac{\partial^2}{\partial x^2}\ln[1+e^{\xi_1+\eta_1-2\alpha_1(y)}+e^{\xi_2+\eta_2-2\alpha_2(y)}+Ae^{\xi_1+\eta_1+\xi_2+\eta_2-2\alpha_1(y)-2\alpha_2(y)}],\\
&\psi_1=\frac{\sqrt{2(p_1+q_1)\dot{\alpha}_1(y)}e^{\xi_1-\alpha_1(y)}[1+a_1e^{\xi_2+\eta_2-2\alpha_2(y)}]}{1+e^{\xi_1+\eta_1-2\alpha_1(y)}+e^{\xi_2+\eta_2-2\alpha_2(y)}+Ae^{\xi_1+\eta_1+\xi_2+\eta_2-2\alpha_1(y)-2\alpha_2(y)}},
\end{aligned}$$

$$\psi_2 = -\frac{\sqrt{2(p_2+q_2)\dot{\alpha}_2(y)}e^{\xi_2-\alpha_2(y)}[1+a_2e^{\xi_1+\eta_1-2\alpha_1(y)}]}{1+e^{\xi_1+\eta_1-2\alpha_1(y)}+e^{\xi_2+\eta_2-2\alpha_2(y)}+Ae^{\xi_1+\eta_1+\xi_2+\eta_2-2\alpha_1(y)-2\alpha_2(y)}},$$
$$\varphi_1 = \frac{\sqrt{2(p_1+q_1)\dot{\alpha}_1(y)}e^{\eta_1-\alpha_1(y)}[1+b_1e^{\xi_2+\eta_2-2\alpha_2(y)}]}{1+e^{\xi_1+\eta_1-2\alpha_1(y)}+e^{\xi_2+\eta_2-2\alpha_2(y)}+Ae^{\xi_1+\eta_1+\xi_2+\eta_2-2\alpha_1(y)-2\alpha_2(y)}},$$
$$\varphi_2 = -\frac{\sqrt{2(p_2+q_2)\dot{\alpha}_2(y)}e^{\eta_2-\alpha_2(y)}[1+b_2e^{\xi_1+\eta_1-2\alpha_2(y)}]}{1+e^{\xi_1+\eta_1-2\alpha_1(y)}+e^{\xi_2+\eta_2-2\alpha_2(y)}+Ae^{\xi_1+\eta_1+\xi_2+\eta_2-2\alpha_1(y)-2\alpha_2(y)}},$$

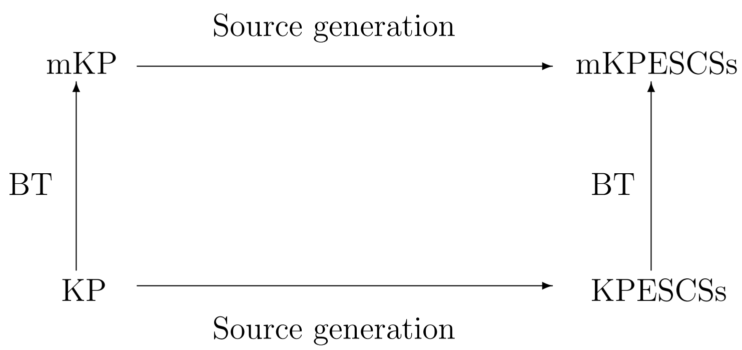
where

$$a_1=\frac{p_1-p_2}{p_1+q_2},\quad a_2=\frac{p_2-p_1}{p_2+q_1},\quad b_1=\frac{q_1-q_2}{p_2+q_1},\quad b_2=\frac{q_2-q_1}{p_1+q_2},\quad A=\frac{(p_1-p_2)(q_1-q_2)}{(p_1+q_2)(p_2+q_1)}.$$

From the expressions of above solutions, we can find these solutions of new KPESCS (25) include arbitrary functions of the spatial variable y, which are different from the solutions of previous KPESCS which are related with arbitrary functions of the temporal variable t [2, 3, 4, 5].

3. New type of modified KP ESCS

In section 2, we constructed the new type of KPESCS and gave its Grammian determinant solutions. In order to show the integrability of this new coupled system, we will give a bilinear Bäcklund transformation (BT) for the KPESCS (24). However it is difficult to obtain the bilinear BT directly from the bilinear KPESCS (15)-(20). Motivated by the fact that commutativity of the pfaffianization procedure and bilinear BT has helped to find a bilinear BT for the coupled KP equation[25], we now propose a similar idea of commutativity of the source generation procedure and BTs to try to find a bilinear BT for the KPESCS (24). This idea can be described more clearly using the diagram:



According to this scheme, the problem of finding a bilinear BT for the KPESCS (24) becomes the problem of constructing the corresponding mKP ESCS if the commutativity of source generation procedure and BTs is true. So in the following, we will first give the modified KP (mKP) ESCS through the source generation procedure. The bilinear mKP equation is [24]:

$$\begin{aligned}&(D_y+D_x^2)\tau\cdot\tau'=0,\\&(3D_xD_y-D_x^3+4D_t)\tau\cdot\tau'=0.\end{aligned}\tag{26}$$

In Ref. [24], the Grammian determinant solution of (26) was given in the following form

$$\tau = \det(c_{ij} + \int_{-\infty}^{x} f_i \tilde{f}_j \mathrm{d}x)_{1\leqslant i,j\leqslant N}, \quad c_{ij} = \text{constant},$$
$$\tau^{'} = \det(c_{ij} - \int_{-\infty}^{x} \frac{\partial f_i}{\partial x} \frac{\partial^{-1} \tilde{f}_j}{\partial x^{-1}} \mathrm{d}x)_{1\leqslant i,j\leqslant N},$$

where ∂^{-1} denotes the integral of a function with respect to the variable x, and the functions f_i, $\tilde{f}_i$ still satisfy the relation (6).

Now we change the functions τ and $\tau^{'}$ into the following forms:

$$F = \det(C_{ij}(y) + \int_{-\infty}^{x} f_i \tilde{f}_j \mathrm{d}x)_{1\leqslant i,j\leqslant N}, \tag{27}$$

$$F^{'} = \det(C_{ij}(y) - \int_{-\infty}^{x} \frac{\partial f_i}{\partial x} \frac{\partial^{-1} \tilde{f}_j}{\partial x^{-1}} \mathrm{d}x)_{1\leqslant i,j\leqslant N}, \tag{28}$$

where $C_{ij}(y)$ is the same as in the section 2. They can be expressed as the form of paffian:

$$F = (1, 2, \cdots, N, N^*, \cdots, 1^*)_1 = (\cdot)_1, \quad F^{'} = F - (d_{-1}, d_0^*, \cdot)_1,$$

where pfaffian elements are defined by:

$$(i, j^*)_1 = C_{ij}(y) - \int_{-\infty}^{x} \frac{\partial f_i}{\partial x} \frac{\partial^{-1} \tilde{f}_j}{\partial x^{-1}} \mathrm{d}x,$$
$$(d_m^*, i)_1 = \frac{\partial^m f_i}{\partial x^m}, \quad (d_m, j^*)_1 = \frac{\partial^m \tilde{f}_j}{\partial x^m},$$
$$(d_m^*, d_l^*)_1 = (d_m, d_l)_1 = (d_m^*, d_l)_1 = (d_m^*, j^*)_1 = (d_m, i)_1 = 0, \quad m, l \in \mathbb{Z}.$$

According to the source generation procedure, we introduce other new functions defined as follows:

$$G_i = \sqrt{\dot{C}_i(y)}(d_0^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*)_1, \tag{29}$$
$$H_i = \sqrt{\dot{C}_i(y)}(d_0, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*)_1, \tag{30}$$
$$G_i^{'} = \sqrt{\dot{C}_i(y)}[(d_{-1}, d_0^*, d_1^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*)_1 - (d_1^*, 1, \cdots, N, N^*, \cdots, \hat{i}^*, \cdots, 1^*)_1], \tag{31}$$
$$H_i^{'} = \sqrt{\dot{C}_i(y)}(d_{-1}, 1, \cdots, \hat{i}, \cdots, N, N^*, \cdots, 1^*)_1, \tag{32}$$

where $i=1,2,\cdots,M$. Then these new functions so defined satisfy the following bilinear equations:

$$
\begin{aligned}
&(D_y+D_x^2)F\cdot F' = -\sum_{i=1}^{M} G_iH_i',\\
&(3D_xD_y-D_x^3+4D_t)F\cdot F' = 3\sum_{i=1}^{M} D_xG_i\cdot H_i',\\
&D_xG_i\cdot F'+FG_i'=0,\\
&D_xF\cdot H_i'+H_iF'=0,\\
&(D_x^3-4D_t)G_i\cdot F'-3D_x^2F\cdot G_i'=0,\\
&(D_x^3-4D_t)F\cdot H_i'-3D_x^2H_i\cdot F'=0,\\
&FK_i'-K_iF'=G_iH_i',\\
&D_xG_i\cdot H_i'-D_xK_i\cdot F'+D_xF\cdot K_i'=0.
\end{aligned}
\tag{33}
$$

where the last two equations are auxiliary equations, and K_i, K_i' are expressed as

$$
\begin{aligned}
K_i &= \dot{C}_i(y)(1,\cdots,\hat{i},\cdots,N,N^*,\cdots,\hat{i}^*,\cdots,2^*,1^*)_1,\\
K_i' &= \dot{C}_i(y)[(1,\cdots,\hat{i},\cdots,N,N^*,\cdots,\hat{i}^*,\cdots,2^*,1^*)_1\\
&\quad -(d_{-1},d_0^*,1,\cdots,\hat{i},\cdots,N,N^*,\cdots,\hat{i}^*,\cdots,2^*,1^*)_1].
\end{aligned}
$$

The above equations can also be proved through identities of determinants. Here Taking the sixth equation in (33) as an example, we prove that functions in (27)-(32) satisfy that equation. In fact we have the following formulas:

$$
\begin{aligned}
H_{i,x}' &= \sqrt{\dot{C}_i(y)}[(d_0,*)_1+(d_{-1},d_0,d_0^*,*)_1],\\
H_{i,xx}' &= \sqrt{\dot{C}_i(y)}[(d_1,*)_1+(d_{-1},d_1,d_0^*,*)_1+(d_{-1},d_0,d_1^*,*)_1],\\
H_{i,t}' &= \sqrt{\dot{C}_i(y)}[(d_2,*)_1+(d_{-1},d_2,d_0^*,*)_1-(d_{-1},d_1,d_1^*,*)_1+(d_{-1},d_0,d_2^*,*)_1],
\end{aligned}
$$

where $*$ denotes $\{1,\cdots,\hat{i},\cdots,N,N^*,\cdots,1^*\}$. Substituting the above results into the sixth equation in (33), we get the the sum of six determinant identities

$$
\begin{aligned}
0 = &\,6[(d_{-1},*)_1(d_1,d_1^*,\cdot)_1-(d_1,*)_1(d_{-1},d_1^*,\cdot)_1-F(d_{-1},d_1,d_1^*,*)_1]\\
&-3[(d_{-1},*)_1(d_2,d_0^*,\cdot)_1-(d_2,*)_1(d_{-1},d_0^*,\cdot)_1-F(d_{-1},d_2,d_0^*,*)_1]\\
&-3[(d_{-1},*)_1(d_0,d_2^*,\cdot)_1-(d_0,*)_1(d_{-1},d_2^*,\cdot)_1-F(d_{-1},d_0,d_2^*,*)_1]\\
&-3[(d_0,*)_1(d_1,d_0^*,\cdot)_1-(d_1,*)_1(d_0,d_0^*,\cdot)_1-F(d_0,d_1,d_0^*,*)_1]\\
&+3[(d_0,d_0^*,\cdot)_1(d_{-1},d_0,d_1^*,*)_1-(d_0,d_1^*,\cdot)_1(d_{-1},d_0,d_0^*,*)_1+(d_0,*)_1(d_{-1},d_0,d_0^*,d_1^*,\cdot)_1\\
&\quad+(d_0,d_0^*,\cdot)_1(d_{-1},d_1,d_0^*,*)_1-(d_1,d_0^*,\cdot)_1(d_{-1},d_0,d_0^*,*)_1-(d_0,d_1,d_0^*,*)_1(d_{-1},d_0^*,\cdot)_1],
\end{aligned}
$$

which indicates the sixth equation in (33) holds. So equations in (33) constitute the other type of mKP ESCS in the bilinear form, and functions in (27)-(32) are the determinant solutions of the mKP ESCS (33). This type of mKP ESCS are different from those given in [11] and [19].

If we apply the set of dependent variable transformation:

$$p = 2(\ln F)_{xx}, \quad q = \frac{F'}{F}, \quad \Psi_i = \frac{G_i}{F}, \quad \Phi_i = \frac{H_i}{F}, \quad \bar{\Psi}_j = \frac{G_i'}{F}, \quad \bar{\Phi}_i = \frac{H_i'}{F},$$

the bilinear equations (33) are transformed into the nonlinear equations:

$$\begin{aligned} &q_{xxx} - 4q_t + 3q_{xy} + 3q^{-1}q_x(q_y - q_{xx} - \sum_{j=1}^{M} \Psi_j\bar{\Phi}_j) - 3\sum_{j=1}^{M}(\Psi_{j,x}\bar{\Phi}_j - \Psi_j\bar{\Phi}_{j,x}) \\ &\quad +3q\int_{-\infty}^{x} q^{-2}[qq_{yy} - qq_{xxy} - q\sum_{j=1}^{M}(\Psi_{j,y}\bar{\Phi}_j + \Psi_j\bar{\Phi}_{j,y}) \\ &\qquad -q_y(q_y - q_{xx} - \sum_{j=1}^{M}\Psi_j\bar{\Phi}_j)]\mathrm{d}x = 0, \end{aligned} \tag{34}$$

$$\begin{aligned} &2(\Psi_{i,xxx}q - \Psi_i q_{xxx} - \Psi_{i,t}q + \Psi_i q_t) \\ &\qquad +3q^{-1}(\Psi_{i,x}q - \Psi_i q_x)(q_y - q_{xx} - \sum_{j=1}^{M}\Psi_j\bar{\Phi}_j) = 0, \end{aligned} \tag{35}$$

$$\begin{aligned} &2\bar{\Phi}_{i,xxx} - 6q_x q^{-2}(\bar{\Phi}_{i,xx}q - \bar{\Phi}_{i,x}q_x) \\ &\qquad -2\bar{\Phi}_{i,t} + 3q^{-1}\bar{\Phi}_{i,x}(q_y - q_{xx} - \sum_{j=1}^{M}\Psi_j\bar{\Phi}_j) = 0. \end{aligned} \tag{36}$$

According to what we explained at the beginning of this section, if commutativity of the source generation procedure and BTs is true for the KP equation, the bilinear mKP ESCS (33) should provide with us a bilinear Bäcklund transformation for the new KPESCS (25). In the Appendix A, we have shown that the system (33) does constitute a bilinear BT for the new KPESCS (25). This result can be described as the following proposition:

Proposition 1. *The bilinear KPESCS (15)-(20) has the following bilinear BT:*

$$(D_y + D_x^2)f \cdot f' = -\sum_{i=1}^{M} g_i h_i', \tag{37}$$

$$(3D_xD_y - D_x^3 + 4D_t)f \cdot f' = 3\sum_{i=1}^{M} D_x g_i \cdot h_i', \tag{38}$$

$$D_x g_i \cdot f' + f g_i' = 0, \tag{39}$$

$$D_x f \cdot h_i' + h_i f' = 0, \tag{40}$$

$$(D_x^3 - 4D_t)g_i \cdot f' - 3D_x^2 f \cdot g_i' = 0, \tag{41}$$

$$(D_x^3 - 4D_t)f \cdot h_i' - 3D_x^2 h_i \cdot f' = 0, \tag{42}$$

$$f k_i' - k_i f' = g_i h_i', \tag{43}$$

$$D_x g_i \cdot h_i' - D_x k_i \cdot f' + D_x f \cdot k_i' = 0. \tag{44}$$

4. Conclusion and discussions

In this paper, we have constructed a new type of KPESCS through the source generation procedure and given its Grammian determinant solution. For the previously known KPESCS (1)-(3), its determinant solutions are connected with arbitrary functions which are dependent on the time variable t. However for the new type of KPESCS, we allow y-dependence of the arbitrary functions in its determinantal solution. If we set each arbitrary function $C_i(y)$ be a constant, the new KPESCS (25) is reduced to the KP equation, and its determinant solutions (7) and (10)-(14) are transformed into the solution of the KP equation. In addition, as another important part of this paper, we have also obtained a new type of mKP ESCS, and proved that the bilinear mKP ESCS (33) just constitutes a bilinear BT for the new KPESCS. This result indicates that the commutativity of the source generation procedure and the bilinear BT is valid for the KP equation.

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Grant no. 10471139), the knowledge innovation program of the Institute of Computational Math., AMSS and Hong Kong RGC Grant No. HKBU2016/05P.

Appendix A. Proof of the Proposition 1

Proof. Let $(f_n, g_i, h_i, k_i, P_i, Q_i)$ be a solution of eqs.(15)-(20), and $(f', g_i', h_i', k_i', P_i', Q_i')$ satisfies relations (37)-(44). We only need to prove that $(f', g_i', h_i', k_i', P_i', Q_i')$ is also a solution of eqs.(15)-(20). In fact, through the relations (37)-(44) and the bilinear operator identities in the appendix B, we have

$$\begin{aligned}
P_{1,i} &= (D_x k_i \cdot f + g_i h_i)(f')^2 - f^2(D_x k_i' \cdot f' + g_i' h_i') \\
&= D_x(k_i f' - f k_i') \cdot f f' + g_i h_i (f')^2 - f^2 g_i' h_i' \\
&= -D_x g_i \cdot h_i' + g_i h_i (f')^2 - f^2 g_i' h_i' \\
&= g_i f'(D_x f \cdot h_i' + h_i f') - (D_x g_i \cdot f' + f g_i') f h_i' \equiv 0,
\end{aligned}$$

$$\begin{aligned}
P_{3,i} &= [(D_y - D_x^2) g_i \cdot f - P_i f + g_i \sum_{j=1}^{M} k_j] g_i' f' - g_i f[(D_y - D_x^2) g_i' \cdot f' - P_i' f' + g_i' \sum_{j=1}^{M} k_j'] \\
&= [(D_y + D_x^2) g_i \cdot g_i'] f f' - g_i g_i'[(D_y + D_x^2) f \cdot f'] - 2D_x(D_x g_i \cdot f') \cdot f g_i' \\
&\quad - P_i g_i' f f' + g_i P_i' f f' + g_i g_i' (\sum_{j=1}^{M} k_j) f' - g_i g_i' f (\sum_{j=1}^{M} k_j') \\
&= [(D_y + D_x^2) g_i \cdot g_i' - P_i g_i' + g_i P_i'] f f' - g_i g_i'[(D_y + D_x^2) f \cdot f' + \sum_{j=1}^{M} (f k_j' - k_j f')] \\
&\equiv 0.
\end{aligned}$$

$$
\begin{aligned}
P_2 =& [(D_x^4 - 4D_xD_t + 3D_y^2)f\cdot f - 6\sum_{i=1}^{M}(D_yk_i\cdot f - D_xg_i\cdot h_i)](f^{'})^2\\
&- f^2[(D_x^4 - 4D_xD_t + 3D_y^2)f^{'}\cdot f^{'} - 6\sum_{i=1}^{M}(D_yk_i^{'}\cdot f^{'} - D_xg_i^{'}\cdot h_i^{'})]\\
=& 2D_x(D_x^3f\cdot f)\cdot ff^{'} - 6D_x(D_x^2f\cdot f^{'})\cdot(D_xf\cdot f^{'}) - 8D_x(D_tf\cdot f^{'})\cdot ff^{'}\\
&+ 6D_y(D_yf\cdot f^{'})\cdot ff^{'} - 6D_y\sum_{i=1}^{M}(h_if^{'} - fh_i^{'})\cdot ff^{'} + 6\sum_{i=1}^{M}D_x(g_if^{'}\cdot h_if^{'} - fg_i^{'}\cdot fh_i^{'})\\
=& 2D_x[(D_x^3 - 4D_t)f\cdot f^{'}]\cdot ff^{'} - 6D_x(D_x^2f\cdot f^{'})\cdot(D_xf\cdot f^{'})\\
&- 6D_y(D_x^2f\cdot f^{'})\cdot ff^{'} + 6\sum_{i=1}^{M}D_x(g_if^{'}\cdot h_if^{'} - fg_i^{'}\cdot fh_i^{'})\\
=& -6\sum_{i=1}^{M}D_x[(D_xf\cdot h_i^{'})\cdot g_if^{'} + (D_xg_i\cdot f^{'})\cdot fh_i^{'}] + 6\sum_{i=1}^{M}D_x(g_if^{'}\cdot h_if^{'} - fg_i^{'}\cdot fh_i^{'})\\
\equiv& 0,
\end{aligned}
$$

The above results indicates that $(f^{'}, g_i^{'}, h_i^{'}, k_i^{'}, P_i^{'}, Q_i^{'})$ satisfies eqs. (15)-(17). Much in the same way, eqs. (18)-(20) hold. So we have completed the proof of the proposition. □

Appendix B. Hirota's bilinear operator identities.

The following bilinear operator identities hold for arbitrary functions $a, b, a^{'}, b^{'}, c$ and d.

$$D_x(a\cdot b)c^2 - b^2(D_yd\cdot c) = D_x(ac - bd)\cdot bc; \tag{B1}$$

$$(D_xa\cdot b)a^{'}b^{'} - ab(D_xa^{'}\cdot b^{'}) = (D_xa\cdot a^{'})bb^{'} - aa^{'}(D_xb\cdot b^{'}); \tag{B2}$$

$$(D_x^2a\cdot b)a^{'}b^{'} - ab(D_x^2a^{'}\cdot b^{'}) = (D_x^2a\cdot a^{'})bb^{'} - aa^{'}(D_x^2b\cdot b^{'}) - 2D_xab^{'}\cdot(D_xb\cdot a^{'}); \tag{B3}$$

$$(D_x^4a\cdot a)b^2 - a^2(D_x^4b\cdot b) = 2D_x(D_x^3a\cdot b)\cdot ab - 6D_x(D_x^2a\cdot b)\cdot(D_xa\cdot b); \tag{B4}$$

$$(D_xD_ya\cdot a)b^2 - a^2(D_xD_yb\cdot b) = 2D_x(D_ya\cdot b)\cdot ab = 2D_y(D_xa\cdot b)\cdot ab; \tag{B5}$$

$$
\begin{aligned}
&(D_x^3a\cdot b)a^{'}b^{'} - ab(D_x^3a^{'}\cdot b^{'}) - 3(D_x^2a\cdot b)(D_xa^{'}\cdot b^{'}) + 3(D_xa\cdot b)(D_x^2a^{'}\cdot b^{'})\\
&\quad = (D_x^3a\cdot a^{'})bb^{'} - aa^{'}(D_x^3b\cdot b^{'})\\
&\qquad -3(D_x^2a\cdot a^{'})(D_xb\cdot b^{'}) + 3(D_xa\cdot a^{'})(D_x^2b\cdot b^{'});
\end{aligned} \tag{B6}
$$

$$D_x(D_xD_ya\cdot b)\cdot ab = D_y(D_x^2a\cdot b)\cdot ab - D_x(D_ya\cdot b)\cdot(D_xa\cdot b); \tag{B7}$$

$$D_x[(D_ya\cdot b)\cdot cd + (D_yc\cdot d)\cdot ab] = D_y[(D_xa\cdot d)\cdot cb - ad\cdot(D_xc\cdot b)]; \tag{B8}$$

$$
\begin{aligned}
&D_x[(D_ya\cdot b)\cdot cd + (D_yc\cdot d)\cdot ab] + (D_xD_ya\cdot b)cd - (D_xD_yc\cdot d)ab\\
&\quad = (D_xa\cdot b)(D_yc\cdot d) - (D_ya\cdot b)(D_xc\cdot d) + D_y(D_xa\cdot d)\cdot cb.
\end{aligned} \tag{B9}
$$

References

[1] Mel'nikov V K 1983 On equations for wave interactions. Lett. Math. Phys. 7 129-136

[2] Mel'nikov V K 1989 Interaction of solitary waves in the system described by the Kadomtsev-Petviashvili equation with a self-consistent source Commun. Math.Phys. **126** 201-215

[3] Xiao T and Zeng Y B 2004 Generalized Darboux transformations for the KP equation with self-consistent sources J. Phys. A **37** 7143-7162

[4] Dneg S F, Chen D Y and Zhang D J 2003 The multisoliton solutions of the KP equation with self-consistent sources J. Phys. Soc. Jpn. **72** 2184-2192.

[5] Wang H Y, Hu X B and Tam H W 2006 On the pfaffianized-KP equation with self-consistent sources submitted

[6] Mel'nikov V K 1989 Capture and confinement of solitons in nonlinear integrable systems, Commun. Math. Phys. **120** 451-468

[7] Mel'nikov V K 1990 Integration of the Korteweg-de Vries equation with a source Inverse Problems **6** 233-246

[8] Mel'nikov V K 1992 Integration of the nonlinear Schrödinger equation with a source. Inverse Problems **8** 133-147

[9] Zeng Y B, Ma W X and Lin R L 2000 Integration of the soliton hierarchy with self-consistent sources, J. Math. Phys. **41** 5453-5489

[10] Zeng Y B, Ma W X and Shao Y J 2001 Two binary Darboux transformations for the KdV hierarchy with self-consistent sources, J. Math. Phys. **42** 2113-2128

[11] Xiao T and Zeng Y B 2006 Bäcklund transformations for the KP and mKP hierarchies with self-consistent sources J. Phys. A: Math. Gen. **39** 139-156

[12] Antonowicz M and Rauch-Wojciechowski S 1993 Soliton hierarchies with sources and Lax representation for restricted flows, Inverse Problems **9** 201-215

[13] Leon J and Latifi A 1990 Solution of an initial-boundary value problem for coupled nonlinear waves, J. Phys. A **23** 1385-1403

[14] Ma W X and Strampp W 1994 An explicit symmetry constraint for the Lax pairs and the adjoint Lax pairs of AKNS systems, Phys. Lett. A **185** 277-286

[15] Hu X B 1996 The higher-order KdV equation with a source and nonlinear superposition formula, Chaos Solitons Fractals **7** 211-215

[16] Zhang D J 2003 The N-soliton solutions of some soliton equations with self-consistent sources, Chaos, Solitons and Fractals **18** 31-43

[17] Gegenhasi and Hu X B 2006 On a integrable differential-difference equation with a source, J. Nonlinear Math. Phys. 13(2), 183-192

[18] Liu X J and Zeng Y B 2005 On the Toda lattice equation with self-consistent sources. J. Phys. A **38** 8951-8965

[19] Deng S F 2006 The multisoliton solutions for the mKPI equation with self-consistent sources **39** 14929-14945

[20] Wang H Y, Hu X B and Gegengasi, 2D Toda lattice equation with self-consistent sources: Casoratian type solutions, bilinear Bäcklund transformation and Lax pair J. Comp. Appl. Math. in press (2007)

[21] Hu X B and Wang H Y 2006 Construction of dKP and BKP equations with self-consistent sources, Inverse Problems **22** 1903-1920

[22] Wang H Y Integrability of the semi-discrete Toda equation with self-consistent sources, J. Math. Anal. Appl. in press (2007).

[23] Wang H Y Hu X B and H.W. Tam 2006 Construction of q-discrete Toda lattice equation with self-consistent sources accepted.

[24] Hirota R *Direct method in soliton theory (In English)*, (Edited and Translated by Nagai A, Nimmo J and Gilson C Cambridge University Press, 2004. 6).

[25] Hu X B and Zhao J X 2005 Commutativity of pfaffianization and Bäcklund transformations: the

KP equation Inverse Problems **21** 1461-1472