A Weighted Helmholtz Decomposition and Applications to Domain Decomposition for Saddle-point Maxwell Systems

Qiya Hu[1] and Jun Zou[2]

Abstract

We shall establish a discrete weighted Helmholtz decomposition in edge element spaces, which is stable uniformly with respect to the jumps in the discontinuous weight function. The stable decomposition is then applied to show that the preconditioned edge element system for solving saddle-point Maxwell equations by a non-overlapping domain decomposition preconditioner developed in [22] is nearly optimal, i.e., its condition number grows only as the logarithm of the dimension of the local subproblem associated with an individual subdomain; more importantly, the condition number is also independent of the jumps of coefficients across the interfaces between any two subdomains.

Key words. Maxwell's equations, Nédélec finite elements, weighted Helmholtz decomposition, domain decomposition, saddle-point system, preconditioners, condition number

AMS(MOS) subject classification. 65N30, 65N55

1 Introduction

In the numerical simulation of electromagnetic wave propagation, one needs to repeatedly solve the following saddle-point Maxwell system at each time step [9] [10] [14] [23] [28] [29] [30]:

$$\begin{cases} \mathbf{curl}(\alpha\,\mathbf{curl}\,\mathbf{u}) + \gamma_0\beta\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathrm{div}(\beta\mathbf{u}) = g \quad \text{in} \quad \Omega\,. \end{cases} \tag{1.1}$$

The system will be complemented with the following boundary condition:

$$\mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

We shall consider Ω to be a simply-connected open domain in $\mathbf{R}^3$ with a Lipschitz boundary $\partial\Omega$ and the unit outward normal direction $\mathbf{n}$ on $\partial\Omega$. The source functions $\mathbf{f} \in L^2(\Omega)^3$ and $g \in L^2(\Omega)$ satisfy the compatibility condition $\gamma_0\, g = \nabla \cdot \mathbf{f}$. The coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are two positive bounded functions in Ω. In applications, the ratio $\alpha(\mathbf{x})/\beta(\mathbf{x}) = c(\mathbf{x})$ is the speed of light in the concerned medium. It is known that $c(\mathbf{x})$ is a constant in each medium, and it changes only slightly in different media. The constant γ_0 is non-negative, i.e., $\gamma_0 \geq 0$, and it is allowed to be identically zero. It is this extreme case that causes the most troublesome technical difficulty to be dealt with in the subsequent analysis.

Edge finite element methods have been widely used for numerical solution of the system (1.1)-(1.2) in recent years, see, e.g., [9] [10] [24]. As is well-known, the algebraic

[1]LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematical and System Sciences, The Chinese Academy of Sciences, Beijing 100080, China. The work of this author was supported by Natural Science Foundation of China G10771178, The Key Project of Natural Science Foundation of China G10531080, and National Basic Research Program of China G2005CB321702. (hqy@lsec.cc.ac.cn).

[2]Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. The work of this author was substantially supported by Hong Kong RGC grant (Project 404407). (zou@math.cuhk.edu.hk).

systems arising from the discretization by edge element methods are quite different from the ones arising from the discretization by standard nodal finite element methods. Thus the construction of efficient solvers such as multigrid and domain decomposition methods for the nodal element systems, which has been well developed for the second order elliptic problems in the past two decades, does not work for edge element discretization of the equations (1.1)-(1.2) in general, especially in three dimensions. One major technical difficulty in handling the Maxwell system (1.1), compared to the second order elliptic equations, lies in the fact that the curl operator has a much larger null space than the one for the gradient. A fundamental tool, which may treat the larger null space and at the same time take the advantage of some existing methodologies in developing effective multigrid and domain decomposition methods for elliptic equations, is the Helmholtz-type decompositions (see, for example, [11] and [2]). Based on these decompositions, many variants of efficient multigrid and domain decomposition methods have been constructed and analyzed for the edge element systems arising from the discretization of the Maxwell equations; see [2] [12] [14] [15] [21] [22] [29] and the references therein.

However, all the existing Helmholtz-type decompositions do not involve any coefficients in the Maxwell system (1.1), so they may not help analyze how the convergence of the existing methods depend on the coefficients or their jumps across interfaces. In this work we shall establish a weighted Helmholtz decomposition, that is stable uniformly with respect to the concerned discontinuous coefficients. This seems to be the first weighted Helmholtz decomposition in the literature.

The stable Helmholtz decomposition is then applied to investigate the convergence of the non-overlapping domain decomposition preconditioner developed in [22] for solving the saddle-point Maxwell equations (1.1). It has been shown in [22] that the resulting preconditioned system is nearly optimal in the sense that its condition number grows only as the logarithm of the ratio between the subdomain diameter and the finite element mesh size. And it is important for us to emphasize that when the lower order term is present in the Maxwell system (1.1), i.e., $\gamma_0 > 0$, the condition number of the resulting preconditioned system is also independent of the jumps in the coefficients across the interfaces between any two subdomains (see the proof of Theorem 3.2 in [22]). But when the lower order term is missing in the Maxwell system (1.1), i.e., $\gamma_0 = 0$, or when γ_0 is positive but relatively much smaller than α in (1.1), we do not know how the condition number of the global preconditioned system depends on the jumps of coefficients. In the present paper we shall apply the new stable weighted Helmholtz decomposition mentioned above to show that for this case the condition number is still independent of the jumps of the coefficients.

The outline of the paper is as follows. In Section2, we describe a triangulation and the edge elements, and introduce some basic formulae and definitions. The discrete weighted Helmholtz decomposition will be constructed and analyzed in Section 3. The construction of a non-overlapping domain decomposition preconditioner is carried out in Section 4, while the condition number of the preconditioned system is estimated in Sections 4 and 5.

2 Edge elements and domain decomposition

This section shall be devoted to the introduction of the edge elements which will be used in Section 4 for discretization of the system (1.1)-(1.2) and a few fundamental concepts to be used in the subsequent sections for construction and analysis of a weighted Helmholtz decomposition and a non-overlapping domain decomposition preconditioner.

2.1 Edge element discretization

We start with the introduction of some subdomains and the triangulation of domain Ω, as well as the edge elements.

Subdomains in terms of the coefficient. For a given positive discontinuous function $\beta(\mathbf{x})$, we assume that the entire domain Ω is decomposed into the union of N_0 convex polyhedra $\Omega_1^0, \Omega_2^0, \cdots, \Omega_{N_0}^0$ such that the coefficient β does not vary much on each Ω_r^0, i.e., its function values and derivatives are bounded on each Ω_r^0 from above and below by two constants whose ratio is of order $O(1)$.

Without loss of generality, we assume that for $r = 1, \ldots, N_0$,

$$\beta(\mathbf{x}) = \beta_r, \qquad \forall\, \mathbf{x} \in \Omega_r^0\,,$$

where β_r is a constant.

Edge and nodal element spaces. Next, we further divide each Ω_r^0 into smaller tetrahedral elements of size h so that elements from any two neighboring subdomains are consistent with each other on their common face. Let $\mathcal{T}_h$ be the resulting triangulation of the domain Ω, which we assume is quasi-uniform. By $\mathcal{E}_h$ and $\mathcal{N}_h$ we denote the set of edges of $\mathcal{T}_h$ and the set of nodes in $\mathcal{T}_h$ respectively. Then the Nédélec edge element space, of the lowest order, is a subspace of piecewise linear polynomials defined on $\mathcal{T}_h$ (cf. [11] [26]):

$$V_h(\Omega) = \Big\{ \mathbf{v} \in H_0(\mathbf{curl}; \Omega);\ \mathbf{v}\,|_K \in R(K),\ \forall K \in \mathcal{T}_{h_k} \Big\},$$

where $R(K)$ is a subset of all linear polynomials on the element K of the form:

$$R(K) = \Big\{ \mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}, \mathbf{b} \in \mathbf{R}^3,\ \mathbf{x} \in K \Big\}.$$

It is known that for any $\mathbf{v} \in V_h(\Omega)$, its tangential components are continuous on all edges of each element in the triangulation $\mathcal{T}_h$, and $\mathbf{v}$ is uniquely determined by its moments on each edge e of $\mathcal{T}_h$:

$$\Big\{ \lambda_e(\mathbf{v}) = \int_e \mathbf{v} \cdot \mathbf{t}_e ds;\ e \in \mathcal{E}_h \Big\}$$

where $\mathbf{t}_e$ denotes the unit vector on the edge e.

As we will see, the edge element analysis involves frequently the nodal element space. By $Z_h(\Omega)$ we will denote the continuous piecewise linear finite element space of $H_0^1(\Omega)$ on the mesh $\mathcal{T}_h$.

2.2 Some edge and nodal element subspaces

We shall often need to consider the restriction of the edge element space $V_h(\Omega)$ on a subdomain or a part of its boundary.

Let $\hat{\Omega}$ denote a generic polyhedra subdomain of Ω. The faces and vertices of a subdomain $\hat{\Omega}$ will be denoted by F and V respectively. Let G be either the entire boundary $\hat{\Gamma} = \partial\hat{\Omega}$ or a face F of $\hat{\Gamma}$, then we define the restriction space of the tangential components of the functions in $V_h(\Omega)$ on G by

$$V_h(G) = \Big\{ \psi \in L^2(G)^3;\ \psi = \mathbf{v} \times \mathbf{n}\ \text{ on }\ G\ \text{ for some }\ \mathbf{v} \in V_h(\Omega) \Big\}.$$

The restrictions of $V_h(\Omega)$ on each subdomain $\hat{\Omega}$ is denoted by $V_h(\hat{\Omega})$. From now on, the notation e, with $e \subset G \subset \hat{\Gamma}$, always means that e is an edge of $\mathcal{T}_h$ and lies on G. The following local subspaces of $V_h(\hat{\Omega})$ and $V_h(\text{F})$ will be important to our analysis:

$$\begin{aligned} V_h^0(\hat{\Omega}) &= \left\{\mathbf{v} \in V_h(\hat{\Omega});\ \mathbf{v} \times \mathbf{n} = 0 \ \text{ on } \ \hat{\Gamma}\right\}, \\ V_h^0(\text{F}) &= \left\{\Phi = \mathbf{v} \times \mathbf{n} \in V_h(\text{F});\ \lambda_e(\mathbf{v}) = 0, \ \ \forall\, e \subset \partial\text{F} \cap \mathcal{E}_h\right\}. \end{aligned}$$

Similarly, the restrictions of $Z_h(\Omega)$ in a subdomain $\hat{\Omega}$, on $\hat{\Gamma}$ and on a face F, are written as $Z_h(\hat{\Omega})$, $Z_h(\hat{\Gamma})$, and $Z_h(\text{F})$, respectively. For a subset G of $\hat{\Gamma}$, we define a "local" subspace

$$Z_h^0(G) = \{v \in Z_h(\hat{\Gamma});\ v = 0 \text{ at all nodes on } \hat{\Gamma}\backslash G\}.$$

We end this subsection with two frequently used extension operators related to a subdomain $\hat{\Omega}$. The first is the discrete **curl curl**-extension operator $\hat{\mathbf{R}}_h : V_h(\hat{\Gamma}) \to V_h(\hat{\Omega})$ defined as follows: For any $\Phi \in V_h(\hat{\Gamma})$, $\hat{\mathbf{R}}_h\Phi$ satisfies $\hat{\mathbf{R}}_h\Phi \times \mathbf{n} = \Phi$ on $\hat{\Gamma}$ and solves

$$(\mathbf{curl}\,\hat{\mathbf{R}}_h\Phi, \mathbf{curl}\,\mathbf{v}_h) = 0, \quad \forall\, \mathbf{v}_h \in V_h^0(\hat{\Omega}).$$

The second extension is the discrete harmonic extension operator $\hat{R}_h : Z_h(\hat{\Gamma}) \to Z_h(\hat{\Omega})$. For any $w_h \in Z_h(\hat{\Gamma})$, $\hat{R}_h w_h \in Z_h(\hat{\Omega})$ satisfies $\hat{R}_h w = w_h$ on $\hat{\Gamma}$ and solves

$$(\nabla \hat{R}_h w_h, \nabla v_h) = 0, \quad \forall\, v_h \in Z_h(\hat{\Omega}) \cap H_0^1(\hat{\Omega})\,.$$

3 A stable weighted Helmholtz-type decomposition

We devote this section to the construction of a discrete weighted Helmholtz-type decomposition, that is stable in certain norm, uniformly with respect to the jumps of a given weight coefficient function $\beta(\mathbf{x})$. This decomposition will play a fundamental role in the subsequent analysis of the condition number of the preconditioned edge element system by means of a non-overlapping domain decomposition preconditioner.

From now on, we shall frequently use the notations $\lesssim$ and $\eqsim$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \leq Cy$ for some constant C independent of mesh size h, subdomain size d and the possible large jumps of some related coefficient functions across the interface between any two subdomains. $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

3.1 Assumptions and results

We need to introduce a few concepts in order to describe the relation between different subdomains from $\{\Omega_r^0\}_{r=1}^{N_0}$, which are described in Section 2.1, in terms of the coefficient function $\beta(\mathbf{x})$.

Definition 3.1 *For a polyhedron Ω_r^0, another polyhedron $\Omega_{r'}^0$ is called a "son" of Ω_r^0 if $\Omega_{r'}^0 \cap \Omega_r^0 \neq \emptyset$ and $\beta_{r'} < \beta_r$. In this case, the subdomain Ω_r^0 is called a "mother" of $\Omega_{r'}^0$.*

Now we make an assumption on the coefficients, and the assumption seems mild and reasonable for most applications. For a polyhedron Ω_r^0, we shall assume that they satisfies either of the following two conditions:

Condition A. At most two " mother " subdomains of Ω_r^0 do not intersect each other. Here a " mother " subdomain may be the union of all mother subdomains of Ω_r^0 on which $\beta(\mathbf{x})$ take the same value.

Condition B. The union of the intersection sets of Ω_r^0 with each of its mother subdomains forms a connected set.

Theorem 3.1 *Assume that either* **Condition A** *or* **Condition B** *holds for each subdomain* Ω_r^0*, and that* $\mathbf{v}_h$ *is a function such that* $\mathbf{v}_h \in V_h(\Omega)$ *and*

$$(\beta \mathbf{v}_h, \nabla q_h) = 0, \quad \forall q_h \in Z_h(\Omega)\,, \tag{3.1}$$

then we have the following estimate

$$\|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}^2 \le C \log^{m+1}(1/h) \|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2, \tag{3.2}$$

where the constants m *and* C *are independent of* h *and the possible large jumps of the coefficient* β*, but may depend on the distribution of the polyhedra* $\{\Omega_r^0\}_{r=1}^{N_0}$*.*

Theorem 3.1 implies the following corollary.

Theorem 3.2 *Assume that either* **Condition A** *or* **Condition B** *holds for each subdomain* Ω_r^0*. Then for any* $\mathbf{v}_h \in V_h(\Omega)$*, there exist* $p_h \in Z_h(\Omega)$ *and* $\mathbf{w}_h \in V_h(\Omega)$ *such that*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h$$

and

$$(\beta \mathbf{w}_h, \nabla q_h) = 0, \quad \forall\, q_h \in Z_h(\Omega). \tag{3.3}$$

Moreover, p_h *and* $\mathbf{w}_h$ *have the estimates*

$$\|\beta^{\frac{1}{2}}\nabla p_h\|_{0,\Omega}^2 + \|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}^2 \le C \log^{m+1}(1/h)(\|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}^2 + \|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2), \tag{3.4}$$

where the constants m *and* C *are independent of* h *and the possible large jumps of the coefficient* β*, but may depend on the distribution of the polyhedra* $\{\Omega_r^0\}_{r=1}^{N_0}$*.*

3.2 Several variants of the Helmholtz decomposition

This section is a preparatory section for the establishment of a weighted Helmholtz decomposition. Throughout this subsection, we shall consider a convex polyhedron $\hat{\Omega}$ of size $O(1)$. Let $Z_h(\hat{\Omega})$ and $V_h(\hat{\Omega})$ be the standard nodal and Nedelec finite element space on $\hat{\Omega}$ respectively.

Lemma 3.1 *Let* $\hat{\Gamma}$ *be either an empty set or a (closed) face of* $\hat{\Omega}$ *or the union of several faces of* $\hat{\Omega}$*, and* $\mathbf{v}_h$ *be a function in* $V_h(\hat{\Omega})$ *satisfying* $\mathbf{v}_h \times \mathbf{n} = \mathbf{0}$ *on* $\hat{\Gamma}$*. Then there exist* $p_h \in Z_h(\hat{\Omega})$ *and* $\mathbf{w}_h \in V_h(\hat{\Omega})$ *such that* $p_h = 0$*,* $\mathbf{w}_h \times \mathbf{n} = \mathbf{0}$ *on* $\hat{\Gamma}$*, and*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h$$

where $\mathbf{w}_h$ *satisfies*

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \|\mathbf{curl}\,\mathbf{w}_h\|_{0,\hat{\Omega}}.$$

Proof. Since $\mathbf{v}_h \in V_h(\hat{\Omega})$, we have $\mathbf{v}_h \cdot \mathbf{n} \in L^2(\partial\hat{\Omega})$. Let $p \in H^1(\hat{\Omega})$ be the solution to the system

$$\begin{cases} \triangle p = \mathrm{div}\ \mathbf{v}_h \quad \text{in} \ \ \hat{\Omega}, \\ p = 0 \quad \text{on} \ \ \hat{\Gamma}, \\ \frac{\partial p}{\partial \mathbf{n}} = \mathbf{v}_h \cdot \mathbf{n} \quad \text{on} \ \ \partial\hat{\Omega}\backslash\hat{\Gamma} \end{cases}$$

and $\mathbf{w} = \mathbf{v}_h - \nabla p$. Then we know $\mathbf{w} \in \mathbf{H}(\mathbf{curl};\hat{\Omega}) \cap \mathbf{H}(\mathrm{div};\hat{\Omega})$, and $\mathbf{w}$ satisfies

$$\begin{cases} \mathbf{curl}\ \mathbf{w} = \mathbf{curl}\ \mathbf{v}_h \quad \text{in} \ \ \hat{\Omega}, \\ \mathit{div}\ \mathbf{w} = 0 \quad \text{in} \ \ \hat{\Omega}, \\ \mathbf{w} \times \mathbf{n} = \mathbf{0} \quad \text{on} \ \ \hat{\Gamma}, \\ \mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on} \ \ \partial\hat{\Omega}\backslash\hat{\Gamma}. \end{cases}$$

As in the proof of Theorem 4.3 in [2], we can verify, with some obvious modifications, that

$$\|\mathbf{w}\|_{\delta,\hat{\Omega}} \lesssim \|\mathbf{curl}\ \mathbf{w}\|_{0,\hat{\Omega}} = \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}},$$

where $\delta \in (\frac{1}{2}, 1]$ depends on the geometric shape of $\hat{\Omega}$ only. Now applying the edge element interpolation $\mathbf{r}_h$ on both side of the decomposition $\mathbf{v}_h = \nabla p + \mathbf{w}$, one can see that the desired finite element functions p_h and $\mathbf{w}_h$ in the lemma may be given by $\mathbf{w}_h = r_h \mathbf{w}$ and $\nabla p_h = r_h \nabla p$. ♯

Lemma 3.2 *For any face* F *of* $\hat{\Omega}$*, assume that* $\mathbf{v}_h \in V_h(\hat{\Omega})$ *satisfies* $\mathbf{v}_h \cdot \mathbf{t}_{\partial \mathrm{F}} = 0$ *on* ∂F*. Then there exist* $p_h \in Z_h(\hat{\Omega})$*,* $\mathbf{w}_h \in V_h(\hat{\Omega})$ *such that* $p_h = \mathbf{w}_h \cdot \mathbf{t}_{\partial \mathrm{F}} = 0$ *on* ∂F*, and*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h, \tag{3.5}$$

with the following estimate

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h) \|\mathbf{curl}\ \mathbf{w}_h\|_{0,\hat{\Omega}}. \tag{3.6}$$

The conclusion is also valid for the case when F *is replaced by a union of several faces.*

Proof We separate the proof into two steps.

Step 1: Establish the desired decomposition. First on the two-dimensional face F, we can establish a Hodge-type decomposition. Define $\mathbf{v}_{h,\mathrm{F}} \in V_h(\mathrm{F})$ such that

$$\mathbf{v}_{h,\mathrm{F}} = \begin{cases} \mathbf{n} \times (\mathbf{v}_h \times \mathbf{n}), & \text{on } \mathrm{F}, \\ \mathbf{0}, & \text{on } \partial\hat{\Omega}\backslash\mathrm{F}. \end{cases}$$

Then, by Lemma 7.12 of [30], there exist $p_{h,\mathrm{F}} \in Z_h(\hat{\Omega})$ and $\mathbf{w}_{h,\mathrm{F}} \in V_h(\hat{\Omega})$, such that

$$\mathbf{v}_{h,\mathrm{F}} = \nabla_S\, p_{h,\mathrm{F}} + \mathbf{w}_{h,\mathrm{F}}, \quad \text{on } \mathrm{F},$$

where ∇_S is the two-dimensional surface gradient, $p_{h,\mathrm{F}}$ and $\mathbf{w}_{h,\mathrm{F}}$ satisfy $p_{h,\mathrm{F}} = \mathbf{w}_{h,\mathrm{F}} = 0$ on $\partial\hat{\Omega}\backslash$F, and have the estimate

$$\|\mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} + \|\mathbf{curl}\ \mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} \lesssim \|\mathrm{div}_\tau(\mathbf{n} \times \mathbf{v}_{h,\mathrm{F}})\|_{-\frac{1}{2},\mathrm{F}}, \tag{3.7}$$

where div_τ is the so-called tangential divergence; see [1] and [2] for its definition.

Then we define

$$\mathbf{v}^{\partial}_{h,\mathrm{F}} = \mathbf{v}_h - (\nabla p_{h,\mathrm{F}} + \mathbf{w}_{h,\mathrm{F}}). \tag{3.8}$$

One can easily see $\mathbf{v}^{\partial}_{h,\mathrm{F}} \times \mathbf{n} = \mathbf{0}$ on F. By Lemma 3.1, there are $p^{\partial}_h \in Z_h(\hat{\Omega})$ and $\mathbf{w}^{\partial}_h \in V_h(\hat{\Omega})$ such that $p^{\partial}_h = 0$ and $\mathbf{w}^{\partial}_h \times \mathbf{n} = 0$ on F, and we have the following decomposition

$$\mathbf{v}^{\partial}_{h,\mathrm{F}} = \nabla p^{\partial}_h + \mathbf{w}^{\partial}_h \tag{3.9}$$

and the estimate

$$\|\mathbf{w}^{\partial}_h\|_{0,\hat{\Omega}} \lesssim \|\mathbf{curl}\ \mathbf{w}^{\partial}_h\|_{0,\hat{\Omega}}. \tag{3.10}$$

Now by defining

$$p_h = p_{h,\mathrm{F}} + p^{\partial}_h \quad \text{and} \quad \mathbf{w}_h = \mathbf{w}_{h,\mathrm{F}} + \mathbf{w}^{\partial}_h,$$

we get the expected decomposition

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h \tag{3.11}$$

where p_h and $\mathbf{w}_h$ satisfy $p_h = \mathbf{w}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on $\partial\mathrm{F}$.

Step 2: Verify the desired estimate (3.6) for the decomposition (3.11).

Using the inequality (3.7), the known face $H^{-\frac{1}{2}}$-extension (refer to [17] and [30]) and the trace theorem, we obtain

$$\begin{aligned} \|\mathbf{w}_{h,\mathrm{F}}\|_{0,\,\hat{\Omega}} + \|\mathbf{curl}\,\mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} &\lesssim \|\mathrm{div}_\tau(\mathbf{v}_h \times \mathbf{n})\|_{-\frac{1}{2},\,\mathrm{F}} \\ &\lesssim \log(1/h)\|\mathrm{div}_\tau(\mathbf{v}_h \times \mathbf{n})\|_{-\frac{1}{2},\,\partial\hat{\Omega}} \\ &\lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\hat{\Omega}}. \end{aligned} \tag{3.12}$$

Now using the definition of $\mathbf{w}_h$ and the triangle inequality, we have

$$\|\mathbf{w}_h\|_{0,\,\hat{\Omega}} \lesssim \|\mathbf{w}_{h,\mathrm{F}}\|_{0,\,\hat{\Omega}} + \|\mathbf{w}_h^{\partial}\|_{0,\,\hat{\Omega}}.$$

This, together with (3.10)-(3.12) and (3.8)-(3.9), leads to

$$\begin{aligned} \|\mathbf{w}_h\|_{0,\,\hat{\Omega}} &\lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}} + \|\mathbf{curl}\ \mathbf{w}_h^{\partial}\|_{0,\hat{\Omega}} \\ &\leq 2\log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}} + \|\mathbf{curl}\ \mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} \\ &\lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}}. \end{aligned}$$

♯

Lemma 3.3 *Let* E *be a (closed) edge of* $\hat{\Omega}$, *and* $\mathbf{v}_h$ *be a finite element function in* $V_h(\hat{\Omega})$ *and satisfies* $\mathbf{v}_h \cdot \mathbf{t}_\mathrm{E} = 0$ *on* E. *Then* $\mathbf{v}_h$ *admits a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h, \quad p_h \in Z_h(\hat{\Omega}),\ \mathbf{w}_h \in V_h(\hat{\Omega})$$

such that $p_h = \mathbf{w}_h \cdot \mathbf{t}_\mathrm{E} = 0$ *on* E *and*

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{w}_h\|_{0,\hat{\Omega}}. \tag{3.13}$$

Proof. We separate the proof into three steps.

Step 1: Establish an edge decomposition.

Let F be a face containing the edge E, and we first consider a decomposition of the normal component $\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}}$ of $\mathbf{v}_h$ on $\partial\mathrm{F}$. For the ease of notation, we shall write $\mathrm{E}^{\partial} = \partial\mathrm{F}\backslash\mathrm{E}$. Following [30], we can construct a decomposition on E^{∂}:

$$\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}^\partial} = \phi'_{\mathrm{E}^\partial} + C_{\mathrm{E}^\partial},$$

where C_{E^∂} and $\phi_{\mathrm{E}^\partial}$ are respectively a constant and a one-variable function given by

$$C_{\mathrm{E}^\partial} = \frac{1}{|\mathrm{E}^\partial|}\int_{\mathrm{E}^\partial} \mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}^\partial} ds\,, \quad \phi_{\mathrm{E}^\partial}(t) = \int_{\mathrm{E}^\partial} (\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}^\partial} - C_{\mathrm{E}^\partial}) ds\,, \quad t \in [0,\ |\mathrm{E}^\partial|].$$

Here $t = 0$ and $t = |\mathrm{E}^\partial|$ correspond to the two endpoints V_1 and V_2 of E, and we see

$$\phi_{\mathrm{E}^\partial}(\mathrm{V}_1) = \phi_{\mathrm{E}^\partial}(\mathrm{V}_2) = 0.$$

Then we shall naturally extend $\phi_{\mathrm{E}^\partial}$ and C_{E^∂} by zero into E, then extend by zero into $\hat{\Omega}$ such that $\tilde{\phi}_{\mathrm{E}^\partial} \in Z_h(\hat{\Omega})$ and $\tilde{C}_{\mathrm{E}^\partial} \in V_h(\hat{\Omega})$. One can verify that

$$\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = (\nabla\tilde{\phi}_{\mathrm{E}^\partial}) \cdot \mathbf{t}_{\partial\mathrm{F}} + \tilde{C}_{\mathrm{E}^\partial} \cdot \mathbf{t}_{\partial\mathrm{F}}. \tag{3.14}$$

Step 2: Construct the desired decomposition in the lemma. For the ease of notation, we set

$$\mathbf{v}_{h,\mathrm{E}}^{\partial} = \mathbf{v}_h - (\nabla\tilde{\phi}_{\mathrm{E}^\partial} + \tilde{C}_{\mathrm{E}^\partial}).$$

By (3.14) we know $\mathbf{v}_{h,\mathrm{E}}^{\partial} \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on $\partial\mathrm{F}$. Applying Lemma 3.2 for $\mathbf{v}_{h,\mathrm{E}}^{\partial}$, one can find two functions $p_h^\partial \in Z_h(\hat{\Omega})$ and $\mathbf{w}_h^\partial \in V_h(\hat{\Omega})$ such that $p_h^\partial = \mathbf{w}_h^\partial \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on $\partial\mathrm{F}$, and

$$\mathbf{v}_{h,\mathrm{E}}^{\partial} = \nabla p_h^\partial + \mathbf{w}_h^\partial,$$

with the following estimate

$$\|\mathbf{w}_h^\partial\|_{0,\hat{\Omega}} \lesssim \|\mathbf{curl}\ \mathbf{w}_h^\partial\|_{0,\hat{\Omega}}. \tag{3.15}$$

Now define

$$p_h = \tilde{\phi}_{\mathrm{E}^\partial} + p_h^\partial \ \text{ and } \ \mathbf{w}_h = \tilde{C}_{\mathrm{E}^\partial} + \mathbf{w}_h^\partial\,,$$

which gives the final decomposition

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h \tag{3.16}$$

such that $p_h = \mathbf{w}_h \cdot \mathbf{t}_\mathrm{E} = 0$ on E.

Step 3: Derive the desired estimate in Lemma 3.3 for the decomposition (3.16).

Noting that $\mathbf{v}_h \cdot \mathbf{t}_\mathrm{E} = 0$ on E, i.e., $\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on E, one can follow [30] to obtain the estimate for the constant C_{E^∂}:

$$|C_{\mathrm{E}^\partial}| \lesssim \|\mathrm{div}_\tau(\mathbf{v}_h \times \mathbf{n})\|_{-\frac{1}{2},\ \mathrm{F}} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}}.$$

Using this estimate and the definition of $\tilde{C}_{\mathrm{E}^\partial}$ we deduce

$$\|\tilde{C}_{\mathrm{E}^\partial}\|_{0,\hat{\Omega}} + \|\mathbf{curl}\ \tilde{C}_{\mathrm{E}^\partial}\|_{0,\hat{\Omega}} \lesssim \|\tilde{C}_{\mathrm{E}^\partial} \cdot \mathbf{t}_{\partial\mathrm{F}}\|_{0,\partial\mathrm{F}} \lesssim |C_{\mathrm{E}^\partial}| \lesssim \log(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\ \hat{\Omega}}. \tag{3.17}$$

Now by the triangle inequality, we have

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \|\tilde{C}_{\mathrm{E}^\partial}\|_{0,\hat{\Omega}} + \|\mathbf{w}_h^\partial\|_{0,\hat{\Omega}}.$$

This, together with (3.17) and (3.15), leads to

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\hat{\Omega}} + \|\mathbf{curl}\mathbf{w}_h^\partial\|_{0,\hat{\Omega}}. \tag{3.18}$$

Noting that

$$\mathbf{curl}\ \mathbf{w}_h^\partial = \mathbf{curl}\ \mathbf{w}_h - \mathbf{curl}\ \tilde{C}_{\mathrm{E}^\partial} = \mathbf{curl}\ \mathbf{v}_h - \mathbf{curl}\ \tilde{C}_{\mathrm{E}^\partial},$$

we obtain by using (3.17) that

$$\begin{aligned}\|\mathbf{curl}\ \mathbf{w}_h^\partial\|_{0,\hat{\Omega}} &\lesssim \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}} + \|\mathbf{curl}\ \tilde{C}_{\mathrm{E}^\partial}\|_{0,\hat{\Omega}}\\ &\lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\ \hat{\Omega}}.\end{aligned}$$

Combining this with (3.18) leads readily to the desired estimate (3.13). ♯

Lemma 3.4 *Let* V *be a vertex of* $\hat{\Omega}$ *and* $\mathbf{v}_h$ *be a function in* $V_h(\hat{\Omega})$. *Then one can decompose* $\mathbf{v}_h$ *as*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h, \quad p_h \in Z_h(\hat{\Omega}),\ \mathbf{w}_h \in V_h(\hat{\Omega})$$

such that $p_h(\mathrm{V}) = 0$ *and*

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{w}_h\|_{0,\hat{\Omega}}.$$

Proof. Consider a face F containing V as a vertex. Then following [30], one can decompose $\mathbf{v}_h \cdot \mathbf{t}_{\partial F}$ into the sum $\phi'_{\partial F} + C_{\partial F}$ on ∂F, such that the piecewise linear function $\phi_{\partial F}$ on ∂F satisfies $\phi_{\partial F}(V) = 0$. Then, the desired decomposition can be built as in Lemma 3.3. $\sharp$

Lemma 3.5 *Let* E *be a (closed) edge of* $\hat{\Omega}$*, and* V *be a vertex of* $\hat{\Omega}$ *but* V $\notin$ E*. Assume that* $\mathbf{v}_h \in V_h(\hat{\Omega})$ *satisfies* $\lambda_e(\mathbf{v}_h) = \mathbf{0}$ *for* $e \subset$ E*. Then, there is a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h, \quad p_h \in Z_h(\hat{\Omega}), \;\; \mathbf{w}_h \in V_h(\hat{\Omega})$$

such that

$$p_h(\mathrm{V}) = 0\,, \quad \textit{and} \;\; p_h = 0 \;\; \textit{on} \;\; \mathrm{E}, \qquad \lambda_e(\mathbf{w}_h) = \mathbf{0} \;\; \forall\, e \subset \mathrm{E},$$

and $\mathbf{w}_h$ *has the following estimate*

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h) \|\mathbf{curl}\mathbf{w}_h\|_{0,\hat{\Omega}}. \tag{3.19}$$

Proof. Let F be a (closed) face, which has V as one of its vertices, but does not have E as one of its edges. We first split $\mathbf{v}_h \cdot \mathbf{t}_{\partial F}$ into the sum $\phi'_{\partial F} + C_{\partial F}$ on ∂F such that the $\phi_{\partial F}$ is continuous on ∂F, and linear on each edge of F and satisfies $\phi_{\partial F}(V) = 0$. Let $\tilde{\phi}_{\partial F} \in Z_h(\hat{\Omega})$ and $C_{\partial F} \in V_h(\hat{\Omega})$ be the natural extensions of $\phi_{\partial F}$ and $C_{\partial F}$ by zero, respectively.

We will treat the problem separately according to two different cases.

(i) There is a (closed) face F$'$ such that F$' \cap$ F $= \emptyset$ and F$'$ has E as one of its edges. It is the case when $\hat{\Omega}$ is a hexahedron.

In this case, we set $\mathrm{E}^\partial = \partial\mathrm{F}'\backslash\mathrm{E}$, and directly decompose $(\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}^\partial})|_{\mathrm{E}^\partial}$ into the sum $\phi'_{\mathrm{E}^\partial} + C_{\mathrm{E}^\partial}$ as in Lemma 3.3. Let $\tilde{\phi}_{\mathrm{E}^\partial} \in Z_h(\hat{\Omega})$ and $\tilde{C}_{\mathrm{E}^\partial} \in V_h(\hat{\Omega})$ be the zero extensions of $\phi_{\mathrm{E}^\partial}$ and C_{E^∂}, respectively. Then we define

$$\mathbf{v}_h^\partial = \mathbf{v}_h - (\nabla\tilde{\phi}_{\partial F} + \nabla\tilde{\phi}_{\mathrm{E}^\partial} + \tilde{C}_{\partial F} + \tilde{C}_{\mathrm{E}^\partial}).$$

It is clear to see

$$(\mathbf{v}_h^\partial \cdot \mathbf{t}_{\partial F})|_{\partial F} = (\mathbf{v}_h^\partial \cdot \mathbf{t}_{\partial F'})|_{\partial F'} = 0.$$

Now applying Lemma 3.2 for $\mathbf{v}_h^\partial$, one can get a decomposition of $\mathbf{v}_h^\partial$ based on the two faces F and F$'$, and further construct the desired decomposition of $\mathbf{v}_h$.

(ii) The edge E has a common vertex with a (closed) edge E$'$ of F. This is the case when $\hat{\Omega}$ is a tetrahedron. Then we set

$$\mathbf{v}_{h,\mathrm{V}}^\partial = \mathbf{v}_h - (\nabla\tilde{\phi}_{\partial F} + \tilde{C}_{\partial F}).$$

By the assumption, we know $\mathbf{v}_{h,\mathrm{V}}^\partial \cdot \mathbf{t}_\Gamma = 0$ on Γ, with $\Gamma = \mathrm{E} \cup \mathrm{E}'$. Let F$'$ be the face with E and E$'$ as two of its neighboring edges, and set $\Gamma^\partial = \partial\mathrm{F}'\backslash\Gamma$. As in Lemma 3.3, we can build a decomposition of $\mathbf{v}_{h,\mathrm{V}}^\partial \cdot \mathbf{t}_{\Gamma^\partial}$ as follows:

$$\mathbf{v}_{h,\mathrm{V}}^\partial \cdot \mathbf{t}_{\Gamma^\partial} = \phi'_{\Gamma^\partial} + C_{\Gamma^\partial} \quad \text{on } \Gamma^\partial,$$

where ϕ_{Γ^∂} vanishes at the two endpoints of Γ^∂. Let $\tilde{\phi}_{\Gamma^\partial} \in Z_h(\hat{\Omega})$ and $C_{\Gamma^\partial} \in V_h(\hat{\Omega})$ be the zero extensions of ϕ_{Γ^∂} and C_{Γ^∂}, respectively. Then we set

$$\mathbf{v}_h^\partial = \mathbf{v}_h - (\nabla\tilde{\phi}_{\partial F} + \nabla\tilde{\phi}_{\Gamma^\partial} + \tilde{C}_{\partial F} + \tilde{C}_{\Gamma^\partial}).$$

One can easily check that

$$(\mathbf{v}_h^\partial \cdot \mathbf{t}_{\partial F})|_{\partial F} = (\mathbf{v}_h^\partial \cdot \mathbf{t}_{\partial F'})|_{\partial F'} = 0.$$

Now applying Lemma 3.2 for $\mathbf{v}_h^\partial$, one can get a decomposition of $\mathbf{v}_h^\partial$ based on the two faces F and F$'$, and further get the desired decomposition of $\mathbf{v}_h$ as in Lemma 3.3. $\sharp$

Following the same argument as that used for Lemma 3.5, we can show

Lemma 3.6 *Let $\hat{\Gamma}$ be a connected subset of $\hat{\Omega}$, which is formed by several (closed) edges and (closed) faces of $\hat{\Omega}$, and $\mathbf{v}_h$ be a function in $V_h(\hat{\Omega})$ satisfying $\lambda_e(\mathbf{v}_h) = \mathbf{0}$ for $e \subset \hat{\Gamma}$. Then $\mathbf{v}_h$ admits a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h, \quad p_h \in Z_h(\hat{\Omega}), \ \mathbf{w}_h \in V_h(\hat{\Omega}),$$

such that

$$p_h = 0 \ \text{ on } \ \hat{\Gamma} \ \text{ and } \ \lambda_e(\mathbf{w}_h) = 0 \ \ \forall\, e \subset \hat{\Gamma},$$

and $\mathbf{w}_h$ meets the estimate

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h) \|\mathbf{curl}\mathbf{w}_h\|_{0,\hat{\Omega}}.$$

♯

3.3 A stable decomposition for any function $\mathbf{v}_h$ in $V_h(\Omega)$

With the help of the preliminary results from Section 3.2, we are now ready to construct an important weighted Helmholz-type decomposition for any function $\mathbf{v}_h$ in $V_h(\Omega)$. We start with a classification of all the polyhedra $\{\Omega_r^0\}_{r=1}^{N_0}$ based on the corresponding values of the coefficient $\beta(\mathbf{x})$ on them.

Let Σ_1 be the set of all polyhedra Ω_r^0 which have no any mother subdomain. Namely, $\Omega_r^0 \in \Sigma_1$ if and only if it holds that for any subdomain $\Omega_{r'}^0$ with $r' \neq r$, either $\bar{\Omega}_{r'}^0 \cap \bar{\Omega}_r^0 = \emptyset$ or $\beta_{r'} \leq \beta_r$.

Let Σ_2 denote a subset of the sons of all polyhedra belonging to Σ_1 such that each polyhedron in Σ_2 has no mother subdomain in $\{\Omega_r^0\}_{r=1}^{N_0} \backslash \Sigma_1$.

Similarly, let Σ_3 be a subset of the sons of all polyhedra belonging to $\Sigma_1 \cup \Sigma_2$ such that each polyhedron in Σ_3 has no mother subdomain in $\{\Omega_r^0\}_{r=1}^{N_0} \backslash (\Sigma_1 \cup \Sigma_2)$.

By the definition of Σ_i ($i = 1, 2, 3$) above, one can easily see that any two polyhedra in Σ_i either do not intersect each other or the coefficients $\beta(\mathbf{x})$ on both subdomains are equal

Repeating the above procedure, one can define a series of Σ_l ($l = 2, \cdots, m$), which satisfy the condition: (1) Σ_l consists of some sons of polyhedra belonging to $\cup_{i=1}^{l-1} \Sigma_i$; (2) each polyhedron in Σ_l has no mother subdomain in $\{\Omega_r^0\}_{r=1}^{N_0} \backslash (\cup_{i=1}^{l-1} \Sigma_i)$; (3) any two polyhedra in Σ_l either do not intersect each other or the coefficients $\beta(\mathbf{x})$ on both subdomains are equal. We can readily see that

$$\{\Omega_r^0\}_{r=1}^{N_0} = \bigcup_{l=1}^{m} \Sigma_l.$$

Next, we set $n_0 = 0$. Without loss of generality, we assume that

$$\Sigma_l = \{\Omega^0_{n_{l-1}+1}, \ \Omega^0_{n_{l-1}+2}, \cdots, \Omega^0_{n_l}\} \quad (l = 1, \cdots, m)\,,$$

and $n_l > n_{l-1}$. Clearly, we see $n_m = N_0$ and that Σ_l contains $(n_l - n_{l-1})$ polyhedra.

We are now ready to construct a desired decomposition for any $\mathbf{v}_h$ in $V_h(\Omega)$, and will achieve this by three steps.

Step 1: Decompose $\mathbf{v}_h$ on all the polyhedra in Σ_1.

We shall write $\mathbf{v}_{h,r} = \mathbf{v}_h|_{\Omega_r^0}$. For $r = 1, 2, \cdots, n_1$, we can follow the arguments of Lemma 3.1 to decompose $\mathbf{v}_{h,r}$ as follows:

$$\mathbf{v}_{h,r} = \nabla p_r + \mathbf{w}_r = \nabla p_{h,r} + r_h \mathbf{w}_r := \nabla p_{h,r} + \mathbf{w}_{h,r}, \tag{3.20}$$

where $p_r \in H^1(\Omega_r^0)$ and $\mathbf{w}_r \in \mathbf{H}(\mathbf{curl};\Omega_r^0) \cap \mathbf{H}_0(\mathrm{div}_0;\Omega_r^0)$. Moreover, we have

$$\|\mathbf{w}_{h,r}\|_{0,\Omega_r^0} + \|\mathbf{curl}\ \mathbf{w}_{h,r}\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\ \mathbf{v}_{h,r}\|_{0,\Omega_r^0}, \quad r = 1, \cdots, n_1. \tag{3.21}$$

Let $\tilde{p}_{h,r} \in Z_h(\Omega)$ be the standard extensions of $p_{h,r}$ by zero onto Ω, and $\tilde{\mathbf{w}}_{h,r} \in V_h(\Omega)$ be an extension of $\mathbf{w}_{h,r}$ such that $\lambda_e(\tilde{\mathbf{w}}_{h,r}) = 0$ for every $e \subset \partial\Omega_l^0 \backslash \partial\Omega_r^0$ with all l such that $l \neq r$, and $\tilde{\mathbf{w}}_{h,r}$ is the discrete **curl curl**-extension on each Ω_l^0. Then we define

$$\tilde{\mathbf{v}}_{h,r} = \nabla\tilde{p}_{h,r} + \tilde{\mathbf{w}}_{h,r} \quad \text{for all } r \text{ with } \Omega_r^0 \in \Sigma_1. \tag{3.22}$$

We remark that if a subdomain Ω_r^0 in Σ_1 intersects one or more than one other subdomains in Σ_1, then $\beta(\mathbf{x})$ must take the same values in all these subdomains. In this case, we should take the union of all these subdomains to replace Ω_r^0 when we do the extensions for $\tilde{p}_{h,r}$ and $\tilde{\mathbf{w}}_{h,r}$ above.

Step 2: Decompose $\mathbf{v}_h$ on all the polyhedra in Σ_2.

Consider a subdomain Ω_r^0 from Σ_2. For the sake of exposition, we assume that Ω_r^0 satisfies **Condition A** and has just two mother subdomains in Σ_1, and both do not intersect each other, say $\Omega_{r_1}^0$ and $\Omega_{r_2}^0$. The case with **Condition B** will be handled in Step 3. Without loss of generality, we assume that $\Omega_r^0 \cap \Omega_{r_1}^0 = \mathrm{V}$ (a vertex) and $\Omega_r^0 \cap \Omega_{r_2}^0 = \mathrm{E}$ (an edge). Set

$$\mathbf{v}_{h,r}^{\partial} = \mathbf{v}_{h,r} - \sum_{l=1}^{2} \tilde{\mathbf{v}}_{h,r_l} \quad \text{on } \Omega_r^0.$$

It is easy to see that $\lambda_e(\mathbf{v}_{h,r}^{\partial}) = 0$ for $e \subset \mathrm{E}$. Then by Lemma 3.5, there exist $p_{h,r}^{\partial} \in Z_h(\Omega_r^0)$ and $\mathbf{w}_{h,r}^{\partial} \in V_h(\Omega_r^0)$ such that

$$\mathbf{v}_{h,r}^{\partial} = \nabla p_{h,r}^{\partial} + \mathbf{w}_{h,r}^{\partial} \quad \text{on} \quad \Omega_r^0\,, \tag{3.23}$$

and

$$p_{h,r}^{\partial}(\mathrm{V}) = 0\,,\ p_{h,r}^{\partial} = 0 \quad \text{on} \quad \mathrm{E}, \quad \text{and} \quad \lambda_e(\mathbf{w}_{h,r}^{\partial}) = 0 \quad \text{for} \quad e \subset \mathrm{E}. \tag{3.24}$$

Moreover, for $r = n_1 + 1, \cdots, n_2$, i.e., for all indices r with $\Omega_r^0 \in \Sigma_2$, it follows from (3.23) and (3.22) that

$$\begin{aligned}\|\mathbf{curl}\ \mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0} &= \|\mathbf{curl}\ \mathbf{v}_{h,r}^{\partial}\|_{0,\Omega_r^0} \\ &= \|\mathbf{curl}\ (\mathbf{v}_{h,r} - \sum_{l=1}^{2} \tilde{\mathbf{w}}_{h,r_l})\|_{0,\Omega_r^0} \\ &\lesssim \|\mathbf{curl}\ \mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{l=1}^{2} \|\mathbf{curl}\ \tilde{\mathbf{w}}_{h,r_l}\|_{0,\Omega_r^0}.\end{aligned} \tag{3.25}$$

We further get by (3.19)

$$\begin{aligned}\|\mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0} &\lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0} \\ &\lesssim \log(1/h)(\|\mathbf{curl}\ \mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{l=1}^{2} \|\mathbf{curl}\ \tilde{\mathbf{w}}_{h,r_l}\|_{0,\Omega_r^0}).\end{aligned} \tag{3.26}$$

Now we can define the decomposition of $\mathbf{v}_h$ on $\Omega_r^0 \in \Sigma_2$ as

$$\mathbf{v}_{h,r} = \nabla(p_{h,r}^{\partial} + \sum_{l=1}^{2} \tilde{p}_{h,r_l}) + \mathbf{w}_{h,r}^{\partial} + \sum_{l=1}^{2} \tilde{\mathbf{w}}_{h,r_l}\,, \tag{3.27}$$

where $\tilde{\mathbf{w}}_{h,r_1} = 0$ on Ω_r^0, by noting that $\Omega_r^0 \cap \Omega_{r_1}^0 = \mathrm{V}$, with V being a single vertex.

For the functions $p_{h,r}^\partial$ and $\mathbf{w}_{h,r}^\partial$ in (3.23), we shall extend them onto the entire domain Ω. Let $\tilde{p}_{h,r}^\partial \in Z_h(\Omega)$ be the standard extensions of $p_{h,r}^\partial$ by zero onto Ω, and $\tilde{\mathbf{w}}_{h,r}^\partial \in V_h(\Omega)$ be an extension of $\mathbf{w}_{h,r}^\partial$ such that $\lambda_e(\tilde{\mathbf{w}}_{h,r}^\partial) = 0$ for every $e \subset \partial\Omega_l^0 \backslash \partial\Omega_r^0$ with all l such that $l \neq r$, and $\tilde{\mathbf{w}}_{h,r}$ is the discrete **curl curl**-extension on each Ω_l^0. Then we set

$$\tilde{\mathbf{v}}_{h,r}^\partial = \nabla \tilde{p}_{h,r}^\partial + \tilde{\mathbf{w}}_{h,r}^\partial \quad \text{for all } r \text{ with } \ \Omega_r^0 \in \Sigma_2. \tag{3.28}$$

We remark that if a subdomain Ω_r^0 in Σ_2 intersects one or more than one other subdomains in Σ_2, then $\beta(\mathbf{x})$ must take the same value in all these subdomains. In this case, we should take the union of all these subdomains to replace Ω_r^0 when we do the extensions for $\tilde{p}_{h,r}^\partial$ and $\tilde{\mathbf{w}}_{h,r}^\partial$ above.

Step 3: Obtain the final desired decomposition of $\mathbf{v}_h$.

We now consider the index $l \geq 3$, and assume that the decompositions of $\mathbf{v}_h$ on all polyhedra belonging to Σ_1 $\Sigma_2, \cdots, \Sigma_{l-1}$ are done as in Steps 1 and 2. Next, we will build up a decomposition of $\mathbf{v}_h$ in all subdomains $\Omega_r^0 \in \Sigma_l$.

Without loss of generality, we assume that Ω_r^0 satisfies **Condition B**. Then by **Condition B**, we use Γ_r to denote the corresponding connected set, which is the union of some edges and faces. For the ease of notation, we introduce two index sets:

$$\Lambda_r^1 = \{\ i\ ;\ 1 \leq i \leq n_1 \text{ such that } \ \partial\Omega_i^0 \cap \partial\Omega_r^0 \neq \emptyset\}$$

and

$$\Lambda_r^{l-1} = \{\ i\ ;\ n_1 + 1 \leq i \leq n_{l-1} \text{ such that } \ \partial\Omega_i^0 \cap \partial\Omega_r^0 \neq \emptyset\}.$$

Define

$$\mathbf{v}_{h,r}^\partial = \mathbf{v}_{h,r} - \sum_{i\in\Lambda_r^1} \tilde{\mathbf{v}}_{h,i} - \sum_{i\in\Lambda_r^{l-1}} \tilde{\mathbf{v}}_{h,i}^\partial \quad \text{on } \ \Omega_r^0. \tag{3.29}$$

By the definitions of $\tilde{\mathbf{v}}_{h,i}$ and $\tilde{\mathbf{v}}_{h,i}^\partial$, we know $\lambda_e(\mathbf{v}_{h,r}^\partial) = 0$ for all $e \subset \Gamma_r$. By Lemma 3.6, one can find $p_{h,r}^\partial \in Z_h(\Omega_r^0)$ and $\mathbf{w}_{h,r}^\partial \in V_h(\Omega_r^0)$ such that

$$\mathbf{v}_{h,r}^\partial = \nabla p_{h,r}^\partial + \mathbf{w}_{h,r}^\partial \quad \text{on } \ \Omega_r^0\,, \tag{3.30}$$

and

$$p_{h,r}^\partial = 0 \quad \text{on} \quad \Gamma_r \quad \text{and} \quad \lambda_e(\mathbf{w}_{h,r}^\partial) = 0 \ \text{ for all } e \subset \Gamma_r. \tag{3.31}$$

Using (3.29) and (3.30), we have the following decomposition for $\mathbf{v}_h$ on each $\Omega_r^0 \in \Sigma_l$:

$$\mathbf{v}_{h,r} = \nabla(p_{h,r}^\partial + \sum_{i\in\Lambda_r^1} \tilde{p}_{h,i} + \sum_{i\in\Lambda_r^{l-1}} \tilde{p}_{h,i}^\partial) + \ \mathbf{w}_{h,r}^\partial + \sum_{i\in\Lambda_r^1} \tilde{\mathbf{w}}_{h,i} + \sum_{i\in\Lambda_r^{l-1}} \tilde{\mathbf{w}}_{h,i}^\partial \quad \text{on } \ \Omega_r^0. \tag{3.32}$$

Using (3.30) and the estimate for $\mathbf{w}_{h,r}^\partial$ in Lemma 3.6, one can verify for all $\Omega_r^0 \in \Sigma_l$ that (refer to Step 2)

$$\|\mathbf{curl}\ \mathbf{w}_{h,r}^\partial\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\ \mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{i\in\Lambda_r^1} \|\mathbf{curl}\ \tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0} + \sum_{i\in\Lambda_r^{l-1}} \|\mathbf{curl}\ \tilde{\mathbf{w}}_{h,i}^\partial\|_{0,\Omega_r^0}, \tag{3.33}$$

and

$$\begin{aligned} \|\mathbf{w}_{h,r}^\partial\|_{0,\Omega_r^0} \ \ \lesssim \ \ & \log(1/h)(\|\mathbf{curl}\ \mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{i\in\Lambda_r^1} \|\mathbf{curl}\ \tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0} \\ & + \sum_{i\in\Lambda_r^{l-1}} \|\mathbf{curl}\ \tilde{\mathbf{w}}_{h,i}^\partial\|_{0,\Omega_r^0}). \end{aligned} \tag{3.34}$$

As it was done in Steps 1 and 2, we can extend $p^{\partial}_{h,r}$ and $\mathbf{w}^{\partial}_{h,r}$ by zero onto the entire domain Ω to get $\tilde{p}^{\partial}_{h,r}$ and $\tilde{\mathbf{w}}^{\partial}_{h,r}$. Then we define

$$\tilde{\mathbf{v}}^{\partial}_{h,r} = \nabla \tilde{p}^{\partial}_{h,r} + \tilde{\mathbf{w}}^{\partial}_{h,r} \quad \text{for all } r \text{ with } \ \Omega^0_r \in \Sigma_l. \tag{3.35}$$

By the definition of $\tilde{\mathbf{v}}^{\partial}_{h,r}$ and the property (3.31), we know $\lambda_e(\tilde{\mathbf{v}}^{\partial}_{h,r}) = 0$ for all $e \in \Gamma_r$.

Continuing with the above procedure for all l's till $l = m$, we will have built up the decomposition of $\mathbf{v}_h$ over all the subdomains Ω^0_1, Ω^0_2, …, $\Omega^0_{N_0}$ such that

$$\mathbf{v}_h = \sum_{r=1}^{n_1} \tilde{\mathbf{v}}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{\mathbf{v}}^{\partial}_{h,r} = \nabla p_h + \mathbf{w}_h \tag{3.36}$$

where $p_h \in Z_h(\Omega)$ and $\mathbf{w}_h \in V_h(\Omega)$ are given by

$$p_h = \sum_{r=1}^{n_1} \tilde{p}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{p}^{\partial}_{h,r} \quad \text{and} \quad \mathbf{w}_h = \sum_{r=1}^{n_1} \tilde{\mathbf{w}}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{\mathbf{w}}^{\partial}_{h,r}. \tag{3.37}$$

3.4 Proof of the key auxiliary results

This section is devoted to the proof of Theorem 3.1. For this purpose a few important concepts about the relation between different subdomains will be first introduced. It is reminded that all the subdomains Ω^0_1,Ω^0_2, …, $\Omega^0_{N_0}$, to be addressed below, are the same as those described in Subsect. 3.1.

Definition 3.2 *A mother of subdomain Ω^0_r is called a level-1 ancestor of Ω^0_r, and a mother of a level-1 ancestor of Ω^0_r is called a level-2 ancestor of Ω^0_r. In general, a mother of a level-j ancestor of Ω^0_r is called a level-$(j+1)$ ancestor of Ω^0_r.*

Definition 3.3 *A son of Ω^0_r is called a level-1 offspring of Ω^0_r, and a son of a level-1 offspring of Ω^0_r is called a level-2 offspring of Ω^0_r. In general, a son of a level-l offspring of Ω^0_r is called a level-$(l+1)$ offspring of Ω^0_r.*

By $\Lambda^{(j)}_r(a)$ we shall denote the set of all level-j ancestors of Ω^0_r, and $L_r(a)$ the number of all the levels of the ancestors of Ω^0_r. By $\Lambda^{(l)}_r(o)$ we shall denote the set of all l-level offsprings of Ω^0_r, and $L_r(o)$ the number of all the levels of the offsprings of Ω^0_r.

The following auxiliary estimates will be used in the proof of Theorem 3.1.

Lemma 3.7 *For any subdomain Ω^0_r from Σ_l $(l \geq 2)$, let $\mathbf{w}^{\partial}_{h,r}$ be defined as in Steps 2 and 3 for the construction of the decomposition of any $\mathbf{v}_h \in V_h(\Omega)$ in Subsect. 3.3. Then $\mathbf{w}^{\partial}_{h,r}$ admits the following estimate*

$$\|\mathbf{curl}\, \mathbf{w}^{\partial}_{h,r}\|_{0,\Omega^0_r} \lesssim \|\mathbf{curl}\, \mathbf{v}_h\|_{0,\Omega^0_r} + \sum_{j=1}^{L_r(a)} \log^j(1/h) \sum_{i \in \Lambda^{(j)}_r(a)} \|\mathbf{curl}\, \mathbf{v}_h\|_{0,\Omega^0_i}. \tag{3.38}$$

Proof. We prove this lemma by induction, and start with the case of $l = 2$. It follows from (3.25) that

$$\|\mathbf{curl}\ \mathbf{w}^{\partial}_{h,r}\|_{0,\Omega^0_r} \lesssim \|\mathbf{curl}\ \mathbf{v}_{h,r}\|_{0,\Omega^0_r} + \sum_{l=1}^{2} \|\mathbf{curl}\ \tilde{\mathbf{w}}_{h,r_l}\|_{0,\Omega^0_r}. \tag{3.39}$$

Since $\tilde{\mathbf{w}}_{h,r_2}$ is the discrete **curl curl**-extension in Ω_r^0, we deduce by Lemmata 4.5 and 6.10 in [21] that

$$\begin{aligned}\|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,r_2}\|_{0,\Omega_r^0} &\lesssim \log^{\frac{1}{2}}(1/h)\|\tilde{\mathbf{w}}_{h,r_2}\times\mathbf{n}\|_{0,\mathrm{E}}\\ &\lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{w}_{h,r_2}\|_{0,\Omega_{r_2}^0}\\ &= \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_{r_2}^0}.\end{aligned}$$

This, combining with (3.39) and the fact that $\tilde{\mathbf{w}}_{h,r_1}=0$ on Ω_r^0, yields

$$\begin{aligned}&\|\mathbf{curl}\,\mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0}\\ \lesssim\ &\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}+\log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_{r_2}^0}\\ \lesssim\ &\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}+\log(1/h)\sum_{i\in\Lambda_r^{(1)}(a)}\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}.\end{aligned}\tag{3.40}$$

So (3.38) is verified for all the subdomains Ω_r^0 in Σ_2.

Now, assume that (3.38) is true for all subdomains $\Omega_r^0\in\Sigma_l$ with $l\le n$. Then we need to verify (3.38) for all subdomains $\Omega_r^0\in\Sigma_{n+1}$. It follows from (3.33) that

$$\begin{aligned}\|\mathbf{curl}\,\mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0} &\lesssim \|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r^0}+\sum_{i\in\Lambda_r^1}\|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0}\\ &\quad+\sum_{i\in\Lambda_r^n}\|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}^{\partial}\|_{0,\Omega_r^0}.\end{aligned}\tag{3.41}$$

Similarly, as it was done earlier one can check that for each $i\in\Lambda_r^n$,

$$\|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0}\lesssim\log(1/h)\|\mathbf{curl}\,\mathbf{w}_{h,i}\|_{0,\Omega_i^0}=\log(1/h)\|\mathbf{curl}\,\mathbf{v}_{h,i}\|_{0,\Omega_i^0},$$

and

$$\|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}^{\partial}\|_{0,\Omega_r^0}\lesssim\log(1/h)\|\mathbf{curl}\,\mathbf{w}_{h,i}^{\partial}\|_{0,\Omega_i^0}.$$

Combining these estimates with (3.41) gives

$$\begin{aligned}\|\mathbf{curl}\,\mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0} &\lesssim \|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r^0}+\log(1/h)\sum_{i\in\Lambda_r^1}\|\mathbf{curl}\,\mathbf{v}_{h,i}\|_{0,\Omega_i^0}\\ &\quad+\ \log(1/h)\sum_{i\in\Lambda_r^n}\|\mathbf{curl}\,\mathbf{w}_{h,i}^{\partial}\|_{0,\Omega_i^0}.\end{aligned}\tag{3.42}$$

Noting that for $i\in\Lambda_r^n$, we have $\Omega_i^0\in\Sigma_l$ for some $l\le n$. Thus by the inductive assumption,

$$\begin{aligned}\sum_{i\in\Lambda_r^n}\|\mathbf{curl}\,\mathbf{w}_{h,i}^{\partial}\|_{0,\Omega_i^0} &\lesssim \sum_{i\in\Lambda_r^n}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i^0}\\ &\quad+\sum_{i\in\Lambda_r^n}\sum_{j=1}^{L_i(a)}\log^j(1/h)\sum_{k\in\Lambda_i^{(j)}(a)}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_k^0}.\end{aligned}\tag{3.43}$$

But for all subdomains $\Omega_r^0\in\Sigma_{n+1}$ and $i\in\Lambda_r^n$, we know $L_i(a)\le L_r(a)$ and $\Lambda_i^{(j)}(a)=\emptyset$ for $j>L_i(a)$ by definition, so we have the relation

$$\sum_{i\in\Lambda_r^n}\sum_{j=1}^{L_i(a)}\sum_{k\in\Lambda_i^{(j)}(a)}=\sum_{j=1}^{L_r(a)}\sum_{i\in\Lambda_r^n}\sum_{k\in\Lambda_i^{(j)}(a)}.\tag{3.44}$$

It is easy to see that

$$\sum_{i\in\Lambda_r^n}\sum_{k\in\Lambda_i^{(j)}(a)} = \sum_{k\in\Lambda_r^{(j+1)}(a)} .$$

Combining this with (3.44), and use the fact that $\Lambda_r^{(j+1)}(a)=\emptyset$ for $j\geq L_r(a)$, we get

$$\sum_{i\in\Lambda_r^n}\sum_{j=1}^{L_i(a)}\sum_{k\in\Lambda_i^{(j)}(a)} = \sum_{j=1}^{L_r(a)-1}\sum_{k\in\Lambda_r^{(j+1)}(a)} .$$

Thus, it follows from (3.43) that

$$\begin{aligned}\sum_{i\in\Lambda_r^n}\|\mathbf{curl}\,\mathbf{w}_{h,i}^{\partial}\|_{0,\Omega_i^0} &\lesssim \sum_{i\in\Lambda_r^n}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i^0}\\ &+\sum_{j=2}^{L_r(a)}\log^j(1/h)\sum_{k\in\Lambda_r^{(j)}(a)}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_k^0}. \end{aligned} \tag{3.45}$$

Substituting this into (3.42), and using the relation

$$\sum_{i\in\Lambda_r^1}+\sum_{i\in\Lambda_r^n} = \sum_{i\in\Lambda_r^{(1)}(a)} \quad (\Omega_r^0\in\Sigma_{n+1}),$$

we can immediately derive that

$$\|\mathbf{curl}\,\mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0}\lesssim\|\mathbf{curl}\ v_h\|_{0,\Omega_r^0}+\sum_{j=1}^{L_r(a)}\log^j(1/h)\sum_{k\in\Lambda_r^{(j)}(a)}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_k^0}. \tag{3.46}$$

This proves the validness of (3.38) for all subdomains $\Omega_r^0\in\Sigma_{n+1}$, thus completes the proof of Lemma 3.7 by the mathematical induction. ♯

Stability of the weighted Helmholtz-type decomposition. We are now ready to demonstrate Theorem 3.1. The construction of the desired weighted Helmholtz-type decomposition for any $\mathbf{v}_h\in V_h(\Omega)$ was given in (3.36), so it remains to show the stability (3.2) of the decomposition.

By means of (3.36) and (3.1), we first see

$$(\beta\mathbf{v}_h,\mathbf{v}_h)=(\beta\mathbf{w}_h,\mathbf{w}_h)-(\beta\nabla p_h,\nabla p_h)\leq(\beta\mathbf{w}_h,\mathbf{w}_h)=\sum_{r=1}^{N_0}\|\beta^{\frac12}\mathbf{w}_h\|_{0,\Omega_r^0}^2. \tag{3.47}$$

It suffices to estimate $\|\beta^{\frac12}\mathbf{w}_h\|_{0,\Omega_r^0}^2$ for each subdomain Ω_r^0.

We start with the estimate of $\|\beta^{\frac12}\mathbf{w}_h\|_{0,\Omega_r^0}^2$ for each subdomain Ω_r^0 in Σ_1, i.e., $1\leq r\leq n_1$.

By the definition of $\tilde{\mathbf{w}}_{h,i}^{\partial}$ in (3.37), we have $\lambda_e(\tilde{\mathbf{w}}_{h,i}^{\partial})=0$ for $e\in\partial\Omega_r^0$. Moreover, any two of the subdomains $\Omega_1^0,\cdots,\Omega_{n_1}^0$ do not intersect, so we have

$$\|\beta^{\frac12}\mathbf{w}_h\|_{0,\Omega_r^0}^2=\|\beta^{\frac12}\tilde{\mathbf{w}}_{h,r}\|_{0,\Omega_r^0}^2=\beta_r\|\mathbf{w}_{h,r}\|_{0,\Omega_r^0}^2.$$

This, along with (3.21), yields the following estimate for $r=1,\cdots,n_1$,

$$\|\beta^{\frac12}\mathbf{w}_h\|_{0,\Omega_r^0}^2\lesssim\beta_r\|\mathbf{curl}\ \mathbf{v}_{h,r}\|_{0,\Omega_r^0}^2=\|\beta_r^{\frac12}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}^2. \tag{3.48}$$

Next, we consider all the subdomains Ω_r^0 in Σ_2. As in Step 2 of the construction of the stable decomposition for $\mathbf{v}_h$, we assume that Ω_r^0 satisfies **Condition A** and has just two mother subdomains in Σ_1, $\Omega_{r_1}^0$ and $\Omega_{r_2}^0$, which satisfy that $\Omega_r^0 \cap \Omega_{r_1}^0 = \mathrm{V}$ (a vertex) and $\Omega_r^0 \cap \Omega_{r_2}^0 = \mathrm{E}$ (an edge). Then we have

$$\mathbf{w}_h|_{\Omega_r^0} = \mathbf{w}_{h,r}^{\partial} + \tilde{\mathbf{w}}_{h,r_2}|_{\Omega_r^0}.$$

By the triangle inequality,

$$\|\mathbf{w}_h\|_{0,\Omega_r^0} \lesssim \|\mathbf{w}_{h,r}^{\partial}\|_{0,\Omega_r^0} + \|\tilde{\mathbf{w}}_{h,r_2}\|_{0,\Omega_r^0}. \tag{3.49}$$

Noting that $\tilde{\mathbf{w}}_{h,r_2}$ is the discrete $\mathbf{curl}\,\mathbf{curl}$-extension in Ω_r^0, we can deduce by using Lemmata 4.5 and 6.10 in [21] that

$$\|\tilde{\mathbf{w}}_{h,r_2}\|_{0,\Omega_r^0} \lesssim \log^{\frac{1}{2}}(1/h)\|\tilde{\mathbf{w}}_{h,r_2} \times \mathbf{n}\|_{0,\mathrm{E}} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_{r_2}^0}.$$

Using this estimate, (3.26) and (3.40) we derive from (3.49) that

$$\|\mathbf{w}_h\|_{0,\Omega_r^0} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0} + \log^2(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_{r_2}^0}. \tag{3.50}$$

Then by inserting the coefficient β, we readily have for all subdomains $\Omega_r^0 \in \Sigma_2$ that

$$\begin{aligned}\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega_r^0} &\lesssim \log(1/h)\|\beta_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r^0} + \log^2(1/h)\|\beta_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_{r_2}^0} \\ &\lesssim \log(1/h)\|\beta_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r^0} + \log^2(1/h)\sqrt{\frac{\beta_r}{\beta_{r_2}}}\|\beta_{r_2}^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_{r_2}^0}\end{aligned} \tag{3.51}$$

Finally we consider all the subdomains Ω_r^0 from the general class Σ_l with $l \geq 3$. By the definition of $\mathbf{w}_h$ we know the following relation holds on Ω_r^0:

$$\mathbf{w}_h = \mathbf{w}_{h,r}^{\partial} + \sum_{i\in\Lambda_r^1}\tilde{\mathbf{w}}_{h,i} + \sum_{i\in\Lambda_r^{l-1}}\tilde{\mathbf{w}}_{h,i}^{\partial}.$$

In an analogous way as deriving (3.50), one can verify by using (3.34) that

$$\begin{aligned}\|\mathbf{w}_h\|_{0,\Omega_r^0} &\lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0} + \log^2(1/h)\sum_{i\in\Lambda_r^1}\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0} \\ &+ \log^2(1/h)\sum_{i\in\Lambda_r^{l-1}}\|\mathbf{curl}\ \mathbf{w}_{h,i}^{\partial}\|_{0,\Omega_i^0}.\end{aligned} \tag{3.52}$$

But it follows from Lemma 3.7 that

$$\|\mathbf{curl}\ \mathbf{w}_{h,i}^{\partial}\|_{0,\Omega_i^0} \lesssim \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0} + \sum_{j=1}^{L_i}\log^j(1/h)\sum_{k\in\Lambda_r^{(j)}(a)}\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_k^0}.$$

Then we further deduce from (3.52) that

$$\|\mathbf{w}_h\|_{0,\Omega_r^0} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0} + \sum_{j=1}^{L_r}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}.$$

Inserting the coefficient β gives

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega_r^0} \lesssim \log(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0} + \sum_{j=1}^{L_r(a)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}(\frac{\beta_r}{\beta_i})^{\frac{1}{2}}\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}.$$

Summing up this estimate with the ones in (3.48) and (3.51), we come to

$$\begin{aligned} \|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}^2 &\lesssim \log(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2 \\ &\quad + \sum_{r=n_1+1}^{N_0}\sum_{j=1}^{L_r(a)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}\frac{\beta_r}{\beta_i}\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}^2. \end{aligned} \tag{3.53}$$

By the definitions of $L_r(a)$, $\Lambda_r^{(j)}(a)$ and $\Lambda_r^{(j)}(o)$, we can verify that

$$\begin{aligned} &\sum_{r=n_1+1}^{N_0}\sum_{j=1}^{L_r(a)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}\frac{\beta_r}{\beta_i}\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}^2 \\ &= \sum_{r=1}^{N_0}\frac{\sum_{j=1}^{L_r(o)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(o)}\beta_i}{\beta_r}\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}^2 \\ &\le \log^{m+1}(1/h)\sum_{r=1}^{N_0}C_r\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}^2, \end{aligned} \tag{3.54}$$

where C_r is a constant given by

$$C_r = \frac{\sum_{j=1}^{L_r(p)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(p)}\beta_i}{\beta_r}.$$

Noting the facts that $\beta_i < \beta_r$ for all $i \in \Lambda_r^{(j)}(o)$, $L_r(p)$ is a finite number and the set $\Lambda_r^{(j)}(o)$ contains only a few elements, the constant C_r must be uniformly bounded for all r's. Applying (3.54) to (3.53), we obtain

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}^2 \lesssim C\log^{m+1}(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2$$

where C is a constant given by $C = \max_{1\le r\le N_0} C_r$. $\sharp$

4 A non-overlapping domain decomposition method

In this section we shall apply the discrete weighted Helmholtz decomposition developed in Section 3 to prove that the condition number of the preconditioned edge element system by the non-overlapping domain decomposition preconditioner proposed in [22] is nearly optimal, i.e., its condition number grows only as the logarithm of the dimension of the local subproblem associated with an individual subdomain; more importantly, the condition number is also independent of the jumps of coefficients across the interfaces between any two subdomains.

Let us start with the weak formulation of the concerned equations (1.1)-(1.2). For this, we define the following subspace of $H(\mathbf{curl};\Omega)$:

$$H_0(\mathbf{curl};\Omega) = \left\{\mathbf{v}\in H(\mathbf{curl};\Omega);\ \mathbf{v}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\right\}.$$

By introducing a Lagrange multiplier p to deal with the divergence condition in (1.1) and then by integration by parts, one can easily derive the variational saddle-point problem associated with the system (1.1)-(1.2):

Find $(\mathbf{u}, p) \in H_0(\mathbf{curl}; \Omega) \times H_0^1(\Omega)$ such that

$$\begin{cases} (\alpha \mathbf{curl}\ \mathbf{u}, \mathbf{curl}\ \mathbf{v}) + \gamma_0(\beta\mathbf{u}, \mathbf{v}) + (\nabla p, \beta\mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall\, \mathbf{v} \in H_0(\mathbf{curl}; \Omega) \\ (\beta\mathbf{u}, \nabla q) = (g, q), \qquad \forall\, q \in H_0^1(\Omega) \end{cases} \tag{4.1}$$

Let $\mathcal{T}_h$ be the triangulation of Ω, and $V_h(\Omega)$ and $Z_h(\Omega)$ be respectively the $H(\mathbf{curl})$-conforming edge element space and the H^1-conforming nodal element space on $\mathcal{T}_h$; see Section 2 for the definitions of these discrete concepts as well as some others in the remaining part of this section. Then the saddle-point system (4.1) may be approximated as follows: Find $(\mathbf{u}_h, p_h) \in V_h(\Omega) \times Z_h(\Omega)$ such that

$$\begin{cases} (\alpha \mathbf{curl}\ \mathbf{u}_h, \mathbf{curl}\ \mathbf{v}_h) + \gamma_0(\beta\mathbf{u}_h, \mathbf{v}_h) + (\nabla p_h, \beta\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), & \forall\, \mathbf{v}_h \in V_h(\Omega) \\ (\beta\mathbf{u}_h, \nabla q_h) = (g, q_h), \qquad \forall\, q_h \in Z_h(\Omega). \end{cases} \tag{4.2}$$

Efficient substructuring preconditioners was proposed in [22] for solving the saddle-point system (4.2), in combination with a preconditioned iterative Uzawa method. One may easily observe that the block stiffness matrix corresponding to the first two terms in (4.2) is singular when $\gamma_0 = 0$. To overcome this difficulty, the saddle-point system was first transformed in [22] into another equivalent saddle-point problem whose block stiffness matrix corresponding to the prime variable $\mathbf{u}$ is positive definite. A substructuring preconditioner for such equivalent saddle-point system was constructed in [22]. As pointed out in Section 1, the resulting preconditioned system is nearly optimal in the sense that its condition number grows only as the logarithm of the ratio between the subdomain diameter and the finite element mesh size, but no conclusion has been achieved in [22] about how the condition number of the global preconditioned system depends on the jumps of coefficients (for the case of $\gamma_0 = 0$). The unique task of this section is to demonstrate that the condition number is also independent of the jumps of the coefficients. The fundamental tool leading to the successful demonstration is the new stable weighted Helmholtz decomposition developed in Section 3.

4.1 Augmented saddle-point system and Uzawa methods

In this and next subsection, we shall discuss how to solve the saddle-point problem (4.2) effectively by making use of the non-overlapping domain decomposition preconditioner. For the purpose, we write the system into an equivalent operator form by introducing the operators $\bar{A} : V_h(\Omega) \to V_h(\Omega)$ and $B : Z_h(\Omega) \to V_h(\Omega)$ by

$$\begin{aligned} (\bar{A}\mathbf{u}_h, \mathbf{v}_h) &= (\alpha\, \mathbf{curl}\ \mathbf{u}_h, \mathbf{curl}\ \mathbf{v}_h), \quad \forall\, \mathbf{u}_h, \mathbf{v}_h \in V_h(\Omega)\,, \\ (Bp_h, \mathbf{v}_h) &= (\nabla p_h, \beta\mathbf{v}_h), \quad \forall\, p_h \in Z_h(\Omega),\ \mathbf{v}_h \in V_h(\Omega), \end{aligned}$$

and the dual operator $B^t : V_h(\Omega) \to Z_h(\Omega)$ of B by

$$(B^t\mathbf{u}_h, q_h) = (\beta\, \mathbf{u}_h, \nabla q_h), \quad \forall q_h \in Z_h(\Omega).$$

Let $\bar{\mathbf{f}}_h \in V_h(\Omega)$ and $g_h \in Z_h(\Omega)$ be the L^2-projections of $\mathbf{f}$ and g. Then, the system (4.2) can be written as

$$\begin{cases} (\bar{A} + \gamma_0\, \beta\, I)\mathbf{u}_h + Bp_h = \bar{\mathbf{f}}_h \\ \quad B^t\mathbf{u}_h = g_h. \end{cases} \tag{4.3}$$

In the past decade, there is an rapidly increasing interest in finding effective iterative methods for solving saddle-point problems like (4.3), see, for example, [4] [5] [19] [20] [27]. But most existing methods require the stiffness matrix corresponding to the primal variable $\mathbf{u}_h$ above to be nonsingular, so can not be applied to solve the saddle-point system (4.3) with $\gamma_0 = 0$, due to the singularity of the operator $\bar{A}$ in the space $V_h(\Omega)$. On the other hand, even when $\gamma_0 \neq 0$, the stiffness matrix corresponding to the primal variable $\mathbf{u}_h$ above may be still nearly singular, by noting the fact that the speed $c(x)$ of light is very large, i.e., $\beta(\mathbf{x})/\alpha(\mathbf{x}) = 1/c(\mathbf{x}) \ll 1$. To overcome this difficulty, we shall introduce another saddle-point system which has the same solution as the problem (4.3), but can be solved by existing preconditioned iterative methods.

For the sake of exposition, we shall characterize the following two cases:

Property A. $\beta(\mathbf{x})/\alpha(\mathbf{x}) = 1/c(\mathbf{x}) \ll 1$;

Property B. $\beta(\mathbf{x})/\alpha(\mathbf{x}) = 1/c(\mathbf{x})$ does not vary largely in the global Ω.

Let $\hat{C} : Z_h(\Omega) \to Z_h(\Omega)$ be symmetric and positive definite, which should be chosen as a preconditioner for the discrete Laplace operator on $Z_h(\Omega)$, as we will see below. Then we define

$$A = \bar{A} + \gamma_0\, \beta\, I + r_0 B \hat{C}^{-1} B^t \quad \text{and} \quad \mathbf{f}_h = \bar{\mathbf{f}}_h + r_0 B \hat{C}^{-1} g_h\,, \tag{4.4}$$

where r_0 is some positive constant. One of the possible choices for r_0 is the average value of $c(\mathbf{x}) = \alpha(\mathbf{x})/\beta(\mathbf{x})$.

Clearly, the system (4.3) has the same solution as the augmented saddle-point problem:

$$\begin{cases} A\mathbf{u}_h + Bp_h = \mathbf{f}_h \\ B^t \mathbf{u}_h = g_h. \end{cases} \tag{4.5}$$

Let $\hat{A}$ be a preconditioner for the operator A. Since A is symmetric and positive definite, the system (4.5) can be solved by many existing iterative methods. Below is a recently developed Uzawa-type algorithm with variable relaxation parameters (see [19] and [20]):

Step 1. Choose a parameter ω_i and compute

$$\mathbf{u}_h^{i+1} = \mathbf{u}_h^i + \omega_i \hat{A}^{-1}[f_h - (A\mathbf{u}_h^i + Bp_h^i)];$$

Step 2. Choose a parameter τ_i and compute

$$p_h^{i+1} = p_h^i + \tau_i \hat{C}^{-1}(B^t \mathbf{u}_h^{i+1} - g_h).$$

Remark 4.1 *Some simple choices of parameters ω_i and τ_i are given in [19] and [20] to ensure the convergence of the algorithm. Noting the fact that*

$$f_h - A\mathbf{u}_h^i = \bar{f}_h - \bar{A}\mathbf{u}_h^i - r_0 B[\hat{C}^{-1}(B^t\mathbf{u}_h^i - g_h)] - \gamma_0\, \beta\, I$$

in the case of $\gamma_0 = 0$, so the value $\hat{C}^{-1}(B^t\mathbf{u}_h^i - g_h)$ computed in Step 2 of the i-th iteration can be used in Step 1 of the $(i+1)$-th iteration. That is, the newly added term $r_0 B\hat{C}^{-1}B^t$ in the augmented saddle-point system (4.5) does not cause any extra cost in this Uzawa algorithm as the action of $\hat{C}^{-1}$ is needed only once at each iteration.

As shown in [19] [20], the convergence rate of the above Uzawa algorithm is completely determined by the condition numbers $\kappa(\hat{A}^{-1}A)$ and $\kappa(\hat{C}^{-1}B^t\hat{A}^{-1}B)$. In the following we will recall two efficient preconditioners $\hat{A}$ and $\hat{C}$ respectively for A and C developed in [22]. It was shown in [22] that the aforementioned two condition numbers are nearly optimal, i.e. nearly independent of the subdomain size d and finite element mesh size h. But it is unclear how the two condition numbers depend on the jumps of coefficients in (1.1). This will be the main target of the remaining part of this paper, to rigorously demonstrate the independence of these condition numbers of the jumps in coefficients.

Let $\tilde{A} : V_h(\Omega) \to V_h(\Omega)$ be a self-adjoint operator defined by

$$(\tilde{A}\mathbf{u}, \mathbf{v}) = (\alpha\, \mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v}) + (\alpha\, \mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V_h(\Omega)\,. \tag{4.6}$$

One may note that both coefficients above are $\alpha(x)$, instead of the coefficient $\beta(x)$ appearing in the lower-order term of (1.1). The following theorem is an important observation from [22], which indicates that the newly introduced augmented operator A in (4.4) is spectrally equivalent to the operator $\tilde{A}$ in (4.6), as long as $\hat{C}$ is spectrally equivalent to the discrete Laplacian.

Theorem 4.1 *Let $G(\cdot) \geq 1$ be some given function, and the operator $\hat{C}$ satisfy*

$$(\beta\nabla\phi, \nabla\phi) \lesssim (\hat{C}\phi, \phi) \lesssim G(d/h)(\beta\nabla\phi, \nabla\phi), \quad \forall\, \phi \in Z_h(\Omega)\,, \tag{4.7}$$

then we have

$$(A\mathbf{v}_h, \mathbf{v}_h) \lesssim (\tilde{A}\mathbf{v}_h, \mathbf{v}_h) \lesssim G(d/h)\log^{m+1}(1/h)(\tilde{A}\mathbf{v}_h, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in V_h(\Omega)\,.$$

The proof of this theorem will be given in Section 5. With this result, it suffices for us to construct a preconditioner for $\tilde{A}$, instead of A.

4.2 Construction of a preconditioner for $\tilde{A}$

In this section, we recall a substructuring preconditioner for $\tilde{A}$ proposed in [22].

For this, we first give the definition of non-overlapping domain decomposition.

Domain decomposition. We decompose Ω into N non-overlapping tetrahedral subdomains $\{\Omega_k\}_{k=1}^N$, with each Ω_k of size d (see [3] [31]) such that

1. Each Ω_k is a subdomain of some Ω_r^0, i.e., each Ω_r^0 is the union of some subdomains in $\{\Omega_k\}_{k=1}^N$;
2. Each Ω_k is the union of some elements of $\mathcal{T}_h$.

The common face of the subdomains Ω_i and Ω_j by Γ_{ij}. Also we shall often write $\Gamma = \cup\Gamma_{ij}$, and $\Gamma_i = \Gamma \cap \partial\Omega_i$, and Γ will be called *the interface.* For the definiteness, a unique unit normal direction $\mathbf{n}$ is assigned to each face F of Γ, and this normal vector is meant in any context whenever a unit normal direction is used on any face in the subsequent analysis. As in Subsection 2.2, let $V_h(\Gamma)$ denote the restriction space of the tangential components of the functions in $V_h(\Omega)$ on Γ.

We then define a suitable space decomposition.

Space decomposition. We define two subspaces of $V_h(\Omega)$, one is basically local in each subdomain, while the other is global in the entire domain Ω but discrete A_k-harmonic in each subdomain:

$$V^p(\Omega) = \Big\{\mathbf{v} \in V_h(\Omega);\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma\Big\} = \prod_{k=1}^{N} V_h^0(\Omega_k),$$

$$V^H(\Omega) = \Big\{\mathbf{v} \in V_h(\Omega);\ \mathbf{v} \text{ is the discrete } A_k\text{-extension of } (\mathbf{v} \times \mathbf{n})|_{\partial\Omega_k} \text{ in each } \Omega_k\Big\}.$$

One can easily find that $V_h(\Omega)$ has the orthogonal decomposition with respect to the inner product $(\tilde{A}\cdot, \cdot)$:

$$V_h(\Omega) = V^p(\Omega) \oplus V^H(\Omega). \tag{4.8}$$

Furthermore, we define two subspaces of $V^H(\Omega)$:

$$V^{ij}(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \mathrm{supp}(\mathbf{v}) \subset \Omega_{ij} = \Omega_i \cup \Omega_j \cup \Gamma_{ij}\Big\},$$

$$V^0(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \lambda_e(\mathbf{v}) = 0 \text{ for each } \ e \in \mathrm{F}_\partial \ \text{ with } \mathrm{F} \subset \Gamma\Big\}.$$

The space $V^0(\Omega)$ is called the *coarse* subspace. It is easy to see that the space $V_h(\Omega)$ has the following decomposition (not a direct sum):

$$V_h(\Omega) = V^p(\Omega) \oplus (V^0(\Omega) + \sum_{\Gamma_{ij}} V^{ij}(\Omega)). \tag{4.9}$$

Next, we define the desired substructuring preconditioner.

A substructuring preconditioner. We define the corresponding local and global coarse solvers on the subspaces $V^p(\Omega)$, $V^0(\Omega)$ and $V^{ij}(\Omega)$ adopted in the decomposition (4.9).

Let $\hat{A}_p : V^p(\Omega) \to V^p(\Omega)$ and $\hat{A}_{ij} : V^{ij}(\Omega) \to V^{ij}(\Omega)$ be symmetric and positive definite local solvers such that

$$(\hat{A}_p\mathbf{v}, \mathbf{v}) \eqsim \sum_{k=1}^{N} (A_k\mathbf{v}_k, \mathbf{v}_k)_{\Omega_k}, \quad \forall \mathbf{v} \in V^p(\Omega),$$

where $\mathbf{v}_k = \mathbf{v}|_{\Omega_k}$ for $k = 1, 2, \cdots, N$ and

$$(\hat{A}_{ij}\mathbf{v}, \mathbf{v}) \eqsim (A_i\mathbf{v}_i, \mathbf{v}_i)_{\Omega_i} + (A_j\mathbf{v}_j, \mathbf{v}_j)_{\Omega_j}, \quad \forall \mathbf{v} \in V^{ij}(\Omega).$$

The global coarse solvers should be solvable in an efficient way on $V^0(\Omega)$. For the ease of notation, we assume that the coefficient $\alpha(\mathbf{x})$ in (1.1) is piecewise constant, namely $\alpha(x) = \alpha_i$ for $x \in \Omega_i$, with α_i's being constants. For any face F of Ω_i, we shall use F_b to denote the union of all $\mathcal{T}_h$-induced (closed) triangles on F, which have either one single vertex or one edge lying on $\partial\mathrm{F}$, and F_∂ to denote the open set $\mathrm{F}\backslash\mathrm{F}_b$. For any subdomain Ω_i, define

$$\Delta_i = \bigcup_{\mathrm{F}\subset\Gamma_i} \mathrm{F}_b, \quad i = 1, \cdots, N.$$

Then the following coarse solver $\hat{A}_0 : V^0(\Omega) \to V^0(\Omega)$ was proposed in [22]: for any $\mathbf{v}$, $\mathbf{w} \in V^0(\Omega)$,

$$(\hat{A}_0\mathbf{v}, \mathbf{w}) = h[1 + \log(d/h)] \sum_{i=1}^{N} \alpha_i \Big\{\langle \mathrm{div}_\tau(\mathbf{v} \times \mathbf{n})|_{\Gamma_i}, \mathrm{div}_\tau(\mathbf{w} \times \mathbf{n})|_{\Gamma_i}\rangle_{\Delta_i} + \langle \mathbf{v} \times \mathbf{n}, \mathbf{w} \times \mathbf{n}\rangle_{\Delta_i}\Big\}.$$

Let $Q_p : V^p(\Omega) \to V^p(\Omega)$, $Q_0 : V_h(\Omega) \to V^0(\Omega)$ and $Q_{ij} : V_h(\Omega) \to V^{ij}(\Omega)$ be the L^2-projections. Then the following preconditioner $\hat{A}$ for A:

$$\hat{A}^{-1} = \hat{A}_p^{-1}Q_p + \hat{A}_0^{-1}Q_0 + \sum_{\Gamma_{ij}} \hat{A}_{ij}^{-1}Q_{ij}. \tag{4.10}$$

For this preconditioner, we have

Theorem 4.2 *The condition number of the preconditioned system can be estimated by*

$$\mathrm{cond}(\hat{A}^{-1}A) \lesssim G(d/h)[1 + \log(d/h)]^2. \tag{4.11}$$

Theorem 4.2 was proved in [22] in the sense that the constant appearing in the upper bound of estimate (4.11) is independent of the subdomain size d and the finite element mesh size h, but possibly depending on the jumps of the coefficients in (1.1). Following the proof of Theorem 3.2 in [22], one can come to the estimate

$$\mathrm{cond}(\hat{A}^{-1}\tilde{A}) \lesssim [1+\log(d/h)]^2, \tag{4.12}$$

where the constant appearing in its upper bound is independent of the jumps of the coefficient $\alpha(\mathbf{x})$. One can easily see that Theorem 4.2 is a consequence of Theorem 4.1 and (4.12). Hence it remains only to prove Theorem 4.1, which will be carried out in Section 5.

Remark 4.2 *The entries of the stiffness matrix associated with $\hat{A}_0$ can be expressed as*

$$h[1+\log(d/h)]\sum_{i=1}^{N}\alpha_i\Big\{\langle(\mathbf{curl}\ L_e)\cdot\mathbf{n},(\mathbf{curl}\ L_{e'})\cdot\mathbf{n}\rangle_{\Delta_i}+\langle L_e\times\mathbf{n},L_{e'}\times\mathbf{n}\rangle_{\Delta_i}\Big\},\quad e,\ e'\in\bigcup_{\mathrm{F}\subset\Gamma}\mathrm{F}_b\,,$$

see [22]. Here L_e and $L_{e'}$ are the basis functions of $V_h(\Omega)$ associated with the edges e and e' respectively. So the coarse solver $\hat{A}_0$ involves computations only on Δ_i, a very small fraction of the interface Γ. And it is much simpler compared with the coarse solvers in many existing substructuring preconditioners for standard elliptic problems, where some optimization systems need to be solved [3] [31]. Preconditioner $\hat{A}$ in (4.10) can be implemented as in [3] and [31].

An alternative preconditioner. In some applications, especially when the lower order term in (1.1) is present, i.e., $\gamma_0 = 1$, it may be more convenient to precondition directly the operator A^* given by

$$(A^*\mathbf{u},\mathbf{v}) = (\alpha\mathbf{curl}\ \mathbf{u},\mathbf{curl}\ \mathbf{v}) + (\beta\mathbf{u},\mathbf{v}),\quad \mathbf{u},\mathbf{v}\in V_h(\Omega)\,,$$

instead of A in (4.4). Set $\beta_k^* = \beta|_{\Omega_k}$ (compare Subsection 2.1). It is clear that $\beta_k^* = \beta_l$ when $\Omega_k\subset\Omega_l^0$. Assume that the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ satisfy that

$$1\lesssim \frac{\beta_i^*}{\alpha_i}\Big/\frac{\beta_j^*}{\alpha_j}\lesssim 1\quad \text{for each face}\quad \Gamma_{ij}, \tag{4.13}$$

and

$$1\lesssim \beta_k^*\lesssim d^{-2}\alpha_k\quad k=1,\cdots,N. \tag{4.14}$$

In this case, we may make some obvious modifications to the preconditioner $\hat{A}$ in (4.10). Namely, replacing A_k by $A^*|_{V_h(\Omega_k)}$, and replacing $\hat{A}_0$ by

$$(\hat{A}_0^*\mathbf{v},\mathbf{w}) = h[1+\log(d/h)]\sum_{k=1}^{N}\Big\{\alpha_k\langle\mathrm{div}_\tau(\mathbf{v}\times\mathbf{n})|_{\Gamma_k},\mathrm{div}_\tau(\mathbf{w}\times\mathbf{n})|_{\Gamma_i}\rangle_{\Delta_i}+\beta_k^*\langle\mathbf{v}\times\mathbf{n},\mathbf{w}\times\mathbf{n}\rangle_{\Delta_i}\Big\}.$$

With this new preconditioner $\hat{A}$, we have

$$\mathrm{cond}(\hat{A}^{-1}A^*)\lesssim G(d/h)[1+\log(d/h)]^2. \tag{4.15}$$

To demonstrate the estimate (4.15), we shall use the following norm defined on the boundary Γ_k of a subdomain Ω_k:

$$\|\Phi\|_{\mathcal{X}_{\Gamma_k}} = \Big(\alpha_k\|\mathrm{div}_\tau\Phi\|^2_{-1/2,\ \Gamma_k}+\beta_k^*\|\Phi\|^2_{-\frac{1}{2},\ \Gamma_k}\Big)^{\frac{1}{2}}\quad\forall\ \ \Phi\in V_h(\Gamma_k).$$

Then the estimate (4.15) can be carried out in nearly the same manner as it was done in the proof of Theorem 3.2 of [22]. The major changes are to use the relation $\alpha_j\beta_i^* \eqsim \alpha_i\beta_j^*$ in the estimate of the terms containing $|p_h^i|_{1,\Omega_i}^2$ and $|p_h^j|_{1,\Omega_j}^2$ on pages 54 and 56 of [22], and to apply the following Lemmas 4.1 and 4.2 on page 55 of [22].

Lemma 4.1 *For any $\Phi \in V_h(\Gamma_k)$, there exists an extension $\mathbf{R}_k\Phi \in V_h(\Omega_k)$, such that*

$$\alpha_k\|\mathbf{curl}(\mathbf{R}_k\Phi)\|_{0,\Omega_k}^2 + \beta_k^*\|\mathbf{R}_k\Phi\|_{0,\Omega_k}^2 \lesssim \|\Phi\|_{\mathcal{X}_{\Gamma_k}}^2. \tag{4.16}$$

Lemma 4.2 *For any $\mathbf{v} \in V_h(\Omega_k)$, we have*

$$\|\mathbf{v}\times\mathbf{n}\|_{\mathcal{X}_{\Gamma_k}}^2 \lesssim \alpha_k\|\mathbf{curl}\ \mathbf{v}\|_{0,\ \Omega_k}^2 + \beta_k^*\|\mathbf{v}\|_{0,\ \Omega_k}^2. \tag{4.17}$$

The proof of the above Lemma 4.1 follows the one of Lemma 4.9 in [22] by means of assumption (4.14) and the inverse estimates for $\|\mathrm{div}_\tau\Phi\|_{0,\Gamma_k}^2$ and $\|\Phi\|_{0,\Gamma_k}^2$, while the proof of Lemma 4.2 above comes readily from the Green formulae and assumption (4.14) (e.g., see [1]).

A preconditioner for Schur complement. Assume that $\hat{C}$ is a preconditioner for the discrete Laplacian and satisfies the condition (4.7), then we have

Theorem 4.3 *The condition number of the preconditioned Schur complement system can be estimated by*

$$\mathrm{cond}(\hat{C}^{-1}B^t\hat{A}^{-1}B) \lesssim G(d/h)[1+\log(d/h)]^2. \tag{4.18}$$

Remark 4.3 *When $\hat{C}$ is chosen as the usual multigrid preconditioner, we have $G(d/h) = 1$; when $\hat{C}$ is chosen as the substructuring preconditioner (see [3], [31]), we have $G(d/h) = [1+\log(d/h)]^2$.*

Remark 4.4 *Theorem 4.1 and Theorem 4.3 still hold when β is replaced by β^2/α in (4.7) and r_0 is chosen to be 1.*

5 Proof of Theorem 4.1

As we pointed out right after Theorem 4.2, the major result of this paper, the theorem is a immediate consequence of Theorem 4.1 and (4.12). We are now going to prove Theorem 4.1.

For any $\mathbf{v}_h \in V_h(\Omega)$, we first decompose it as

$$\mathbf{v}_h = \nabla q_h \oplus \mathbf{w}_h, \tag{5.1}$$

where $q_h \in Z_h(\Omega)$ satisfies

$$(\beta\nabla q_h, \nabla\psi_h) = (\beta\mathbf{v}_h, \nabla\psi_h), \quad \forall\,\psi_h \in Z_h(\Omega)$$

and $\mathbf{w}_h$ is orthogonal to ∇q_h in the scalar product $(\beta\cdot,\cdot)$. By the Cauchy-Schwarz inequality, we know

$$\|\beta^{\frac{1}{2}}\nabla q_h\|_{0,\Omega} \le \|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}. \tag{5.2}$$

Moreover, by applying Theorem 3.1 for $\mathbf{w}_h$ we obtain

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}^2 \lesssim \log^{m+1}(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\mathbf{w}_h\|_{0,\Omega}^2 = \log^{m+1}(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\mathbf{v}_h\|_{0,\Omega}^2. \tag{5.3}$$

Let $J : Z_h(\Omega) \to Z_h(\Omega)$ be the operator defined by

$$(J\phi_h, \psi_h) = (\beta\nabla\phi_h, \nabla\psi_h), \quad \forall\, \phi,\ \psi \in Z_h(\Omega). \tag{5.4}$$

Then, by the definitions of q_h, and B^t and J in Section 4.1, we have

$$\begin{aligned}(\beta\nabla q_h, \nabla q_h) &= (\beta\mathbf{v}_h, \nabla q_h) = (B^t\mathbf{v}_h, q_h) \\ &= (Jq_h, J^{-1}B^t\mathbf{v}_h) = (\beta\nabla q_h, \nabla(J^{-1}B^t\mathbf{v}_h)) \\ &= (\beta\mathbf{v}_h, \nabla(J^{-1}B^t\mathbf{v}_h)) = (B^t\mathbf{v}_h, J^{-1}B^t\mathbf{v}_h)\,,\end{aligned}$$

that implies

$$(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h) = (\beta\nabla q_h, \nabla q_h). \tag{5.5}$$

This, along with (5.2), leads to

$$(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h) \le (\beta\mathbf{v}_h, \mathbf{v}_h).$$

By **Property B** and the definition of r_0, we have $r_0\beta \eqsim \alpha$. Now it follows from (4.7) and **Property A** that

$$\begin{aligned}(A\mathbf{v}_h, \mathbf{v}_h) &\lesssim (\alpha\mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h) + \gamma_0(\beta\mathbf{v}_h, \mathbf{v}_h) + r_0(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h) \\ &\le (\alpha\mathbf{curl}\mathbf{v}_h, \mathbf{curl}\mathbf{v}_h) + (\alpha\mathbf{v}_h, \mathbf{v}_h) + r_0(\beta\mathbf{v}_h, \mathbf{v}_h) \\ &\lesssim (\tilde{A}\mathbf{v}_h, \mathbf{v}_h).\end{aligned} \tag{5.6}$$

On the other hand, by (5.5), (5.3) and (4.7) we can derive that

$$\begin{aligned}(\alpha\mathbf{v}_h, \mathbf{v}_h) &= r_0(\beta\mathbf{v}_h, \mathbf{v}_h) = r_0[(\beta\nabla q_h, \nabla q_h) + (\beta\mathbf{w}_h, \mathbf{w}_h)] \\ &\lesssim r_0\log^{m+1}(1/h)[(BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h) + (\beta\mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h)] \\ &\lesssim \max\{1, G(d/h)\}\log^{m+1}(1/h)(A\mathbf{v}_h, \mathbf{v}_h) \lesssim G(d/h)\log^{m+1}(1/h)(A\mathbf{v}_h, \mathbf{v}_h),\end{aligned}$$

which implies

$$(\tilde{A}\mathbf{v}_h, \mathbf{v}_h) \lesssim G(d/h)\log^{m+1}(1/h)(A\mathbf{v}_h, \mathbf{v}_h).$$

This, along with (5.6), gives the desired estimates in Theorem 4.1. ♯

References

[1] A. Alonso and A. Valli, *Some remarks on the characterization of the space of tangential traces of $H(\mathbf{curl};\Omega)$ and the construction of an extension operator*, Manuscr. Math., **89**(1996), 159-178.

[2] A. Alonso and A. Valli, An optimal domain decomposition preconditioner for low-frequency time-harmonic Maxwell equations. Math. Comp., **68**(1999), 607-631.

[3] J. Bramble, J. Pasciak and A. Schatz, *The construction of preconditioner for elliptic problems by substructuring*, IV. Math. Comp., **53**(1989), 1-24

[4] J. Bramble, J. Pasciak and A. Vassilev, *Analysis of the inexact Uzawa algorithm for saddle-point problems*, SIAM J. Numer. Anal., **34**(1997), 1072-1092

[5] J. Bramble and J. Pasciak, *A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems*, Math. Comp., **50**(1988), 1-18

[6] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer-Verlag, New York, 1991.

[7] A. Buffa and P. Ciarlet, Jr., *On traces for functional spaces related to Maxwell's equations. II. Hodge decompositions on the boundary of Lipschitz polyhedra and applications*, Math. Methods Appl. Sci., **24**(2001), 31-48

[8] M. Cessenat, *Mathematical methods in electromagnetism*, World Scientific, River Edge, NJ, 1998

[9] Z. Chen, Q. Du and J. Zou, *Finite element methods with matching and non-matching meshes for Maxwell equations with discontinuous coefficients*, SIAM J. Numer. Anal., **37**(1999), 1542-1570.

[10] P. Ciarlet, Jr. and J. Zou, *Fully discrete finite element approaches for time-dependent Maxwell's equations*, Numer. Math., **82**(1999), 193-219.

[11] V. Girault and P. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer–Verlag, Berlin, 1986.

[12] J. Gopalakrishnan and J. Pasciak, *Overlapping Schwarz preconditioners for indefinite time harmonic Maxwell's equations*, Math. Comp. **72**(2003), 1-16.

[13] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and studies in Mathematics, 24, Pitman, London, 1985

[14] R. Hiptmair, *Multigrid method for Maxwell's equations*, SIAM J. Numer. Anal., **36**(1998), 204-225.

[15] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numerica, **11**(2002), 237-339.

[16] Q. Hu, Preconditioning Poincare-Steklov operators arising from domain decompositions with mortar multipliers, IMA J. Numer. Anal., 2004, Vol. 24, 643-669

[17] Q. Hu, G. Liang and J. Lui, Construction of a preconditioner for domain decomposition methods with polynomial multipliers. J. Comput. Math., **19** (2001), 213-224.

[18] Q. Hu, Z. Shi and D. Yu, Efficient solvers for saddle-point problems arising from domain decompositions with Lagrange multipliers, *SIAM J. Numer. Anal.*, **42**(2004), No. 3, 905-933

[19] Q. Hu and J. Zou, *An iterative method with variable parameters for saddle-point problems*, SIAM J. Matrix Anal. Appl., **23**(2001), 317-338.

[20] Q. Hu and J. Zou, *Two new variants of nonlinear inexact Uzawa algorithms for saddle-point problems*, Numer. Math., **93**(2002), 333-359.

[21] Q. Hu and J. Zou, *A non-overlapping domain decomposition method for Maxwell's equations in three dimensions*. SIAM J. Numer. Anal., **41**(2003), 1682-1708.

[22] Q. Hu and J. Zou, *Substructuring preconditioners for saddle-point problems arising from Maxwell's equations in three dimensions*. Math. Comput.,**73**(2004), No. 245, 35-61

[23] R. Dautray and J.-L. Lions, *Mathematical analysis and numerical methods for science and technology*, Springer–Verlag, New York

[24] P. Monk, *Analysis of a finite element method for Maxwell's equations*, SIAM J. Numer. Anal., **29**(1992), 32-56.

[25] Peter Monk, Finite Element Methods for Maxwell's Equations, Oxford, 2003

[26] J. Nédélec, *Mixed finite elements in* R^3, Numer. Math., **35**(1980), 315-341.

[27] T. Rusten and R. Winther, *A preconditioned iterative method for saddlepoint problems*, SIAM J. Matrix Anal. Appl., **13**(1992), 887-904

[28] J. Saranen, *On electric and magnetic problems for vector fields in anisotropic nonhomogeneous media*, J. Math. Anal. Appl. **91**(1983), 254-275

[29] A. Toselli, *Overlapping Schwarz methods for Maxwell's equations in three dimensions*, Numer. Math., **86**(2000), 733-752.

[30] A. Toselli, *Dual-primal FETI algorithms for edge element approximations: three-dimensional h finite elements on shape-regular meshes*, IMA J. Numer. Anal., **26**(2006), 96-130

[31] J. Xu and J. Zou, *Some non-overlapping domain decomposition methods.* SIAM Review, **40**(1998), 857-914