A Non-overlapping Domain Decomposition Method with Inexact Solvers[1]

Qiya Hu and Dudu Sun

LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences,

Abstract

In this paper we are concerned with non-overlapping domain decomposition methods for the second-order elliptic problems in three-dimensional domains. We develop a new kind of substructuring method, in which one can use an inexact solver both in each subdomain and on each local interface. The main ideas are to decompose the interface space into the sum of small local subspaces and a series of coarse subspaces spanned by *constant-like* basis functions, and to design a cheap approximate harmonic extension of the *constant-like* function. It will be shown that the condition number of the preconditioned system grows only as the logarithm of the dimension of the local problem associated with an individual substructure, and is independent of possible jumps of the coefficient in the elliptic equation.

Key words. domain decomposition, interface space, multilevel decomposition, constant-like function, nearly harmonic extension, preconditioner, inexact solvers, condition number

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1 Introduction

Non-overlapping domain decomposition methods (DDMs) have been shown to be powerful techniques for solving large-scale partial differential equations, especially for solving partial differential equations with large jump coefficients, and for solving coupled partial differential equations. One's main task in non-overlapping DDMs is the construction of an efficient substructuring preconditioner for discretization system associated with the partial differential equations. The construction of this preconditioner has been investigated from various ways and to various models in literature, see, for example, [1]-[9], [11]-[15], [17]-[21], [23]-[24], [26].

Most non-overlapping DDMs studied so far require exact subdomain solvers; we refer [25] and [29] (and the references cited therein). Such a requirement severely degrade the efficiency of the methods. There are only a few works studying substructuring methods with inexact subdomain solvers [3], [4] and [9]. In [3], analysis and numerical experiments with inexact algorithms of Neumann-Dirichlet type was done under the additional assumption of high accuracy of the inexact solvers. The essential difficulty is that discrete harmonic extensions on each subdomain are used in non-overlapping domain decomposition methods. In [4], the harmonic extension on a subdomain was replaced by a simple *average* extension, and substructuring preconditioners with the average extension are constructed. Because of such *average* extension, nearly optimal convergence can not be gotten for these substructuring preconditioners. To avoid harmonic extensions, [9] considered so called *approximate* harmonic basis functions, which still involve high accuracy of the inexact solvers. It seems difficult to construct a nearly optimal substructuring preconditioner with inexact solvers without any additional assumption.

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In the present paper, inspired by [9] and [4], we make a new attempt for the design of substructuring preconditioner with inexact solvers. The core work is the construction of a kind of multilevel *nearly* harmonic basis on general quasi-uniform meshes. The main ingredients in our construction are: (i) develop a special multilevel space decomposition to the interface space; (ii) design a cheap *nearly* harmonic extension for each basis function of the subspaces involved in the decomposition. It will be shown that the new substructuring preconditioner possesses nearly optimal convergence, which is independent of possible large jumps of the coefficient across the interface. For the new method, no additional assumption is required , and the total computational complexity is optimal.

The outline of the remainder of the paper is as follows. In Section 2, we introduce some notation and our motive. In Section 3, we present a multilevel decomposition for the interface space. The main results on the substructuring preconditioner are described in Section 4. In Section 5, we give an analysis of the convergence of the preconditioner. In section 6, we prove the stability of the extension of *constant-like* function, which is used in Section 5. Some numerical results are reported in Section 7.

2 Preliminaries

2.1 Domain decomposition

Let Ω be a bounded polyhedron in $\mathcal{R}^3$. Consider the model problem

$$\begin{cases} -div(\omega\nabla u) = f, & in \;\; \Omega, \\ \quad u = 0, & on \;\; \partial\Omega, \end{cases} \tag{2.1}$$

where $\omega \in L^\infty(\Omega)$ is a positive function.

Let $H_0^1(\Omega)$ denote the standard Sobolev space, and let $(\cdot,\cdot)$ denote the $L^2(\Omega)$-inner product. The weak formulation of (2.1) in $H_0^1(\Omega)$ is then given by the following.

Find $u \in H_0^1(\Omega)$ such that

$$\mathcal{A}(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$, and

$$\mathcal{A}(u,v) = \int_\Omega \omega\nabla u \cdot \nabla v dp.$$

We will apply a kind of non-overlapping domain decomposition method to solving (2.2). For simplicity of exposition, we consider only the case with matching grids in this paper.

Let $\mathcal{T}_h = \{\tau_i\}$ be a regular and quasi-uniform triangulation of Ω with $\tau_i's$ being non-overlapping simplexes of size h ($\in (0,1]$). The set of notes of $\mathcal{T}_h$ is denoted by $\mathcal{N}_h$. We then define $V_h(\Omega)$ to be the piecewise linear finite element subspace of $H_0^1(\Omega)$ associated with $\mathcal{T}_h$:

$$V_h(\Omega) = \{v \in H_0^1(\Omega) : \; v|_\tau \in \mathcal{P}_1 \;\; \forall \tau \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ is the space of linear polynomials. Then the finite element approximation for (2.2) is to find $u_h \in V_h(\Omega)$ such that

$$\mathcal{A}(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h(\Omega). \tag{2.3}$$

Let Ω be decomposed into the union of N polyhedrons $\Omega_1, \cdots, \Omega_{\mathrm{N}}$, which satisfy $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$. We assume that each $\partial\Omega_k$ can be written as a union of boundaries of elements in $\mathcal{T}_h$, and all Ω_k are of size H in the usual sense (see [5] and [29]). Without loss of

generality, we assume that the coefficient $\omega(p)$ is piecewise constant, then each subdomain Ω_k is chosen such that $\omega(p)$ equals to a constant ω_k in Ω_k. Note that $\{\Omega_k\}$ may not constitute a triangulation of Ω.

The common part of two neighboring subdomains Ω_i and Ω_j may be a vertex, an edge or a face. In particular, we denote by Γ_{ij} the common face of two neighboring subdomains Ω_i and Ω_j (i.e., $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$). The union of all Γ_{ij} is denoted by Γ, which is called the interface. In this paper, we choose Dirichlet data as the interface unknown.

Define the operator $A_h : V_h(\Omega) \to V_h(\Omega)$ by

$$(A_h v,\ w) = \mathcal{A}(v, w) = \sum_{k=1}^{N} \omega_k \int_{\Omega_k} \nabla v \cdot \nabla w dx, \quad v \in V_h(\Omega),\ \forall w \in V_h(\Omega).$$

The equation (2.3) can be written as

$$A_h u_h = f_h, \quad u_h \in V_h(\Omega). \tag{2.4}$$

The goal of this paper is to construct a substructuring preconditioner for A_h based on the domain decomposition described above.

2.2 Notations

To introduce the new method, we need some more notations. Throughout this paper, a subset G of Ω are always understood as an open set. The closure of G is denoted by $\bar{G}$.

• subdomain spaces

For subdomain Ω_k, define

$$V_h(\Omega_k) = \{v|_{\Omega_k} :\ \forall v \in V_h(\Omega)\},$$

and

$$V_h^p(\Omega_k) = \{v_h \in V_h(\Omega) :\ supp\ v_h \subset \Omega_k\}.$$

Set

$$\Omega_{ij} = \Omega_i \cup \Gamma_{ij} \cup \Omega_j,$$

and define

$$V_h^p(\Omega_{ij}) = \{v_h \in V_h(\Omega) :\ supp\ v_h \subset \Omega_{ij}\}.$$

• interface space and face spaces

As usual, we define the (global) interface space by

$$W_h(\Gamma) = \{v|_\Gamma :\ \forall v \in V_h(\Omega)\}.$$

For each $\partial\Omega_k$, set

$$W_h(\partial\Omega_k) = \{v|_{\partial\Omega_k} :\ \forall v \in W_h(\Gamma)\}.$$

For a subset G of Γ, define

$$\tilde{W}_h(G) = \{\phi_h \in W_h(\Gamma) :\ supp\ \phi_h \subset G\}.$$

In particular, for $G = \Gamma_{ij}$, the face space $\tilde{W}_h(\Gamma_{ij})$ will be used repeatedly.

• interpolation-type operator and *constant-like* basis

For a subset G of Γ, define the interpolation-type operator $I_G^0 : W_h(\Gamma) \to W_h(\Gamma)$ as

$$(I_G^0 \phi_h)(p) = \begin{cases} \phi_h(p), & \text{if } p \in \mathcal{N}_h \cap G, \\ 0, & \text{if } p \in \mathcal{N}_h \cap (\Gamma \backslash G). \end{cases}$$

In particular, we have

$$(I_G^0 1)(p) = \begin{cases} 1, & \text{if } p \in \mathcal{N}_h \cap G, \\ 0, & \text{if } p \in \mathcal{N}_h \cap (\Gamma \backslash G). \end{cases}$$

If G is just the union of some elements on Γ, we call $\phi_G = I_G^0 1$ to be *constant-like* basis function on G, which will be used repeatedly.

• integration average and algebraic average

For a function $\varphi_h \in W_h(\Gamma)$, let $\gamma_G(\varphi_h)$ denote the integration average of φ_h on G, and let $\gamma_{h,G}(\varphi)$ denote the algebraic average of the values of φ on the nodes in G.

• sets of faces, edges, vertices and subdomains

For convergence, let $\mathcal{F}_\Gamma$ denote the set of all the faces Γ_{ij}. Besides, let $\mathcal{E}_\Gamma$ and $\mathcal{V}_\Gamma$ denote the set of the interior edges and the set of interior vertices generated by the decomposition

$$\bar{\Omega} = \bigcup \bar{\Omega}_k,$$

respectively. For an edge $\mathrm{E} \in \mathcal{E}_\Gamma$, let $\mathcal{Q}_\mathrm{E}$ denote the set of the indices k of the subdomains Ω_k which contain E as an edge. Namely,

$$\mathcal{Q}_\mathrm{E} = \{k : \ \mathrm{E} \subset \partial\Omega_k\}.$$

Define

$$\Omega_\mathrm{E} = \bigcup_{k \in \mathcal{Q}_\mathrm{E}} \Omega_k, \quad \mathrm{E} \in \mathcal{E}_\Gamma.$$

• face inner-products, scaling norm and interface norm

For a subset G of Γ, let $\langle \cdot, \cdot \rangle_G$ denote the L^2 inner product on G. In particular, the $\langle \cdot, \cdot \rangle_\Gamma$ is abbreviated as $\langle \cdot, \cdot \rangle$. Let $\| \cdot \|_{0,\ G}$ denote the norm induced from $\langle \cdot,\ \cdot \rangle_G$.

For a sub-faces G of Γ, let H_G denote the "size" of G. Define the scaling norm

$$\|\phi\|_{\frac{1}{2},\ G} = (|\phi|^2_{\frac{1}{2},\ G} + H_G^{-1} \|\phi\|^2_{0,\ G})^{\frac{1}{2}}, \quad \forall \phi \in H^{\frac{1}{2}}(G).$$

For convenience, define

$$\|\phi_h\|_{*,\Gamma} = (\sum_{k=1}^{\mathrm{N}} \omega_k |\phi_h|^2_{\frac{1}{2},\partial\Omega_k})^{\frac{1}{2}}, \quad \forall \phi_h \in W_h(\Gamma).$$

• discrete norms and discrete inner-product

Discrete norms (or semi-norms) of finite element functions will be used repeatedly in this paper, since the discrete norms are defined on a set of nodes only, and do not depend on the geometric shape of the underlying domain.

We first give definitions of three well known discrete norms (refer to [29]), which are equivalent to their respective continuous norms. For $v_h \in V_h(\Omega_k)$, the discrete H^1 semi-norm is defined by

$$|v_h|^2_{1,\ h,\ \Omega_k} = h^3 \sum_{p_i,\ p_j \in \mathcal{N}_h \cap \Omega_k} |v_h(p_i) - v_h(p_j)|^2,$$

where p_i and p_j denote two neighboring nodes. Similarly, the discrete L^2 norm on an edge E of Ω_k is defined by

$$\|v_h\|^2_{0,\ h,\ \mathrm{E}} = h \sum_{p \in \mathcal{N}_h \cap \mathrm{E}} |v_h(p)|^2.$$

Let $G \subset \Gamma_{ij}$ be the union of some elements. Define the discrete $H^{\frac{1}{2}}$ semi-norm on $W_h(G)$ by

$$|\varphi_h|^2_{h,\ \frac{1}{2},\ G} = \sum_{p \in \mathcal{N}_h \cap G} \sum_{q \in \mathcal{N}_h \cap G} \frac{|\varphi_h(p) - \varphi_h(q)|^2}{|p-q|^3}, \quad \varphi_h \in W_h(G),$$

where p and q denote two different nodes on G.

Then, we define a discrete inner-product. For each $\partial\Omega_k$, set

$$\langle \phi, \psi \rangle_{h,\ \partial\Omega_k} = \sum_{p\in\mathcal{N}_h\cap\partial\Omega_k} \sum_{q\in\mathcal{N}_h\cap\partial\Omega_k} \frac{(\phi(p)-\phi(q))(\psi(p)-\psi(q))}{|p-q|^3}, \quad \phi,\ \psi \in W_h(\partial\Omega_k),$$

where p and q denote two different nodes on $\partial\Omega_k$. Define discrete $H^{\frac{1}{2}}$ inner-product on $W_h(\Gamma)$ by

$$\langle \phi, \psi \rangle_{h,\Gamma} = \sum_{k=1}^{\mathrm{N}} \omega_k \langle \phi, \psi \rangle_{h,\partial\Omega_k}, \quad \phi,\ \psi \in W_h(\Gamma).$$

• $H^{\frac{1}{2}}_{00}$ norms

Let $G \subset \Gamma_{ij}$ be the union of some elements. For a function φ_h satisfying *supp* $\varphi_h \subset G$, define

$$|\varphi_h|^2_{H^{\frac{1}{2}}_{00}(G)} = |\varphi_h|^2_{\frac{1}{2},\ G} + \int_G \frac{|\varphi_h(x)|^2}{dist(x,\ \partial G)} ds(x).$$

It is known that

$$|\varphi_h|^2_{H^{\frac{1}{2}}_{00}(G)} \eqsim |\tilde{\varphi}_h|^2_{\frac{1}{2},\ \Omega_i} \eqsim |\tilde{\varphi}_h|^2_{\frac{1}{2},\ \Omega_j},$$

where $\tilde{\varphi}_h \in W_h(\Gamma)$ denotes the zero extension of φ_h. Moreover, we have

$$|\varphi_h|^2_{H^{\frac{1}{2}}_{00}(G)} \eqsim \int_G \frac{|\varphi_h(x)|^2}{dis(x,\ \partial G)} ds(x).$$

The corresponding discrete semi-norm is defined by

$$|\varphi_h|^2_{h,\ H^{\frac{1}{2}}_{00}(G)} = \sum_{p\in\mathcal{N}_h\cap G} \frac{|\varphi_h(p)|^2}{dis(p,\ \partial G)}.$$

• spectrally equivalences

For simplicity, we will frequently use the notations $\lesssim$ and $\eqsim$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \le Cy$ for some constant C independent of mesh size h, subdomain size d and the related parameters. $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

2.3 Motivation

We first recall the main ideas of the existing substucturing preconditions.

Let $E_k : W_h(\partial\Omega_k) \to V_h(\Omega_k)$ be the discrete harmonic extension. Define the *harmonic* subspace

$$V_h^{\Gamma}(\Omega) = \{v_h \in V_h(\Omega) :\ v_h|_{\Omega_k} = E_k(\phi_h|_{\partial\Omega_k})\ (k = 1, \cdots, \mathrm{N})\ \text{ for some } \phi_h \in V_h(\Gamma)\}.$$

Then, we have the initial space decomposition

$$V_h(\Omega) = \sum_{k=1}^{\mathrm{N}} V_h^p(\Omega_k) + V_h^{\Gamma}(\Omega).$$

Let $A_{h,k} : V_h^p(\Omega_k) \to V_h^p(\Omega_k)$ be the restriction of the operator A_h on the local space $V_h^p(\Omega_k)$, and let $B_{h,\Gamma} : V_h^{\Gamma}(\Omega) \to V_h^{\Gamma}(\Omega)$ be a symmetric and positive definite operator which is spectrally equivalent to the restriction of the operator A_h on the harmonic subspace

$V_h^\Gamma(\Omega)$. Then, the classical substructuring preconditioner (refer to [5]) can be defined in the rough form

$$B_h^{-1} = \sum_{k=1}^{N} A_{h,k}^{-1} Q_k + B_{h,\Gamma}^{-1} Q_\Gamma, \tag{2.5}$$

where Q_k and Q_Γ denote the standard L^2 projectors into their respective subspaces. For the preconditioner B_h, we have (see [5])

$$cond(B_h^{-1} A_h) \lesssim \log^2(H/h). \tag{2.6}$$

In many applications, the subspaces $V_h^p(\Omega_k)$ still have high dimensions, so it is expensive to use the exact solvers $A_{h,k}^{-1}$.

It was shown in [3] that substructuring preconditioners with inexact solvers $B_{h,k}^{-1}$ still possess nearly optimal convergence, if each $B_{h,k}$ has some spectrally *approximation* to $A_{h,k}$ (the usual spectrally equivalence is not enough). Hereafter, "inexact" means that $B_{h,k}$ is only spectrally equivalent to $A_{h,k}$, for example, $B_{h,k}$ is a multigrid preconditioner for $A_{h,k}$. It seems difficult to design an efficient substructuring preconditioner with completely inexact solvers $B_{h,k}^{-1}$, instead of $A_{h,k}^{-1}$ itself or its approximation. In essence, one has to modify the harmonic subspace $V_h^\Gamma(\Omega)$ by replacing each harmonic extension E_k with another extension.

In [4], a substructuring preconditioner $\hat{B}_h^{-1}$ with inexact solvers was been designed by replacing each harmonic extension E_k with a simple *average* extension. It has been shown that the condition number of the resulting preconditioned system can be estimated by

$$cond(\hat{B}_h^{-1} A_h) \lesssim H/h. \tag{2.7}$$

In [9], another substructuring preconditioner $\tilde{B}_h^{-1}$ with inexact solvers was been designed by replacing each harmonic extension E_k with an *approximate* harmonic extension. If the approximate harmonic extension is exact enough, then

$$cond(\tilde{B}_h^{-1} A_h) \lesssim \log^2(H/h). \tag{2.8}$$

The approximate harmonic extension can be defined by *approximate* harmonic basis functions. It is not practical to compute all the *approximate* harmonic basis functions. Because of this, an alternative method, which still require high accuracy of $B_{h,k}$, was considered in [9].

The main idea of this paper is to avoid *approximate* harmonic basis functions by constructing multilevel *nearly* harmonic basis functions on general quasi-uniform triangulations $\mathcal{T}_h$. The new method contains three main ingredients (refer to [12] and [13] for the original versions):

• decompose each face space $\tilde{W}_h(\Gamma_{ij})$ into the sum of multilevel coarse subspaces and small local subspaces, such that each coarse subspace is spanned by several *constant-like* basis functions as $I_G^0 1$, and each local subspace contains only a few nodal basis functions;

• define an *explicit* extension for each *constant-like* basis function $I_G^0 1$ of coarse subspaces, and use the *average* extension for each nodal basis function of local subspaces.

• build a multilevel space decomposition to the global space $V_h(\Omega)$ by using the decomposition of the interface space and the basis extensions.

Based on these, we construct a multilevel substructuring preconditioner with *completely inexact* solvers. As we will see, the new substructuring preconditioner not only possesses a nearly optimal convergence, but also possesses optimal computational complexity.

3 A multilevel space decomposition for $W_h(\Gamma)$

In this section we establish a new kind of multilevel decomposition for $W_h(\Gamma)$. To this end, we first give several basic auxiliary results.

3.1 Basic tools

The following results can be found in [29].

Lemma 3.1 *Let* E *and* F *be an edge and a face of* Ω_k*. Then,*

$$\|\phi_h\|_{0,\partial\mathrm{F}} \lesssim \log^{\frac{1}{2}}(H/h)\|\phi_h\|_{\frac{1}{2},\partial\Omega_k}, \quad \forall \phi_h \in W_h(\partial\Omega_k) \tag{3.1}$$

and

$$\|I^0_{\mathrm{E}}\phi_h\|_{\frac{1}{2},\partial\Omega_k} \lesssim \log^{\frac{1}{2}}(H/h)\|\phi_h\|_{\frac{1}{2},\partial\Omega_k}, \quad \forall \phi_h \in W_h(\partial\Omega_k). \tag{3.2}$$

□

The following result can be proved as in Lemma 6.2 in [15], together with the standard technique.

Lemma 3.2 *Let* E *be an edge of* Ω_k*. Then,*

$$\|\phi_h - \gamma_{\mathrm{E}}(\phi_h)\|_{\frac{1}{2},\partial\Omega_k} \lesssim \log^{\frac{1}{2}}(H/h)|\phi_h|_{\frac{1}{2},\partial\Omega_k}, \quad \forall \phi_h \in W_h(\partial\Omega_k). \tag{3.3}$$

□

Let F denote a face Γ_{ij} itself or a sub-face of Γ_{ij}. We assume that F is just the union of some elements τ on the face Γ_{ij}, and possesses the size d in the usual sense. Here, we do not require that F is a usual polygon.

Lemma 3.3 *Let* F *be defined above. Then,*

$$|I^0_{\mathrm{F}}\phi_h|_{H^{\frac{1}{2}}_{00}(\mathrm{F})} \lesssim \log(d/h)\|\phi_h\|_{\frac{1}{2},\mathrm{F}}, \quad \forall \phi_h \in W_h(\mathrm{F}). \tag{3.4}$$

Proof. We can prove the result directly by using the discrete norms described in Subsection 2.2 (note that F may be a nonstandard polygon). But, the proof will involve some complicated formulas. Because of this, we try to use another simpler proof. Our idea is to map F into a rectangle on the plane of F, and to use an existing result on rectangles.

Choose four nodes p_1, p_2, p_3 and p_4 on $\partial\mathrm{F}$ in order. Assume that $|p_1p_2|$, $|p_2p_3|$, $|p_3p_4|$ and $|p_4p_1|$ has almost the same length. Let ℓ denote the straight line through p_1 and p_3, and let ℓ' denote the straight line through p_2 and p_4. The unique intersection of ℓ and ℓ' is denoted by o, which is almost the barycenter of F by the assumption. Make a sufficiently small rectangle D containing F, so that ℓ and ℓ' are just two diagonal lines of D. Moreover, we require that o is the center of D. It is clear that the size of D is also d. For each node $p \in \bar{\mathrm{F}}$ $(p \neq o)$, draw a line ℓ_p through o and p. Let p' and p'' denote the intersection of ℓ_p with $\partial\mathrm{F}$ and with ∂D, respectively. Define the well known *projection*-type mapping F by

$$F(p) = \frac{|op''|}{|op'|}p, \quad \forall p \in \mathcal{N}_h \cap \bar{\mathrm{F}}\ (p \neq o).$$

If o is just a node, then define $F(o) = o$. It is easy to see that F maps $\partial\mathrm{F}$ onto ∂D, and maps the meshes on F onto a quasi-uniform and regular meshes on D. Moreover, the four vertices of D equal to $F(p_i)$ $(i = 1, 2, 3, 4)$, which are just four nodes of such resulting triangulation $\mathcal{T}_h^D$ (with the diameter h).

Let $W_h(\bar{D})$ denote the linear finite element space associated with $\mathcal{T}_h^D$. For $\phi_h \in W_h(\bar{\mathrm{F}})$, define $F\phi_h \in W_h(\bar{D})$ by

$$(F\phi_h)(F(p)) = \phi_h(p), \quad \forall p \in \mathcal{N}_h \cap \mathrm{F}.$$

By the discrete norm, one can verify that

$$|I_{\mathrm{F}}^0 \phi_h|_{H_{00}^{\frac{1}{2}}(\mathrm{F})} \overset{=}{\sim} |I_D^0(F\phi_h)|_{H_{00}^{\frac{1}{2}}(D)}. \tag{3.5}$$

It is known that (see [5] and [29])

$$|I_D^0(F\phi_h)|_{H_{00}^{\frac{1}{2}}(D)} \lesssim \log(d/h)\|F\phi_h\|_{\frac{1}{2},D}.$$

Plugging this in (3.5), and using the discrete norm again, gives (3.4).

□

Similar mapping with F has been used to construct an explicit harmonic extension by [16].

3.2 Initial space decomposition for $W_h(\Gamma)$

This subsection is devoted to introduction of an initial stable space decomposition for $W_h(\Gamma)$. There are various ways to design this kind of decomposition (refer to [5], [6] and [29]). How to design this decomposition is not the main interest of this paper, since our *multilevel* method has no essential dependence on such a concrete decomposition. In the subsection we consider only an example of this kind of decomposition.

For ease of notation, let $F \in \mathcal{F}_\Gamma$ denote a generic interface Γ_{ij} throughout this Section. Let φ_p denote the nodal basis function on the node $p \in \mathcal{N}_h \cap \Omega$. Define the global coarse subspace

$$W_h^0(\Gamma) = span\{I_F^0 1,\ I_{\mathrm{E}}^0 1,\ \varphi_p :\ F \in \mathcal{F}_\Gamma,\ \mathrm{E} \in \mathcal{E}_\Gamma,\ p \in \mathcal{V}_\Gamma\}.$$

Note that $I_F^0 1$ and $I_{\mathrm{E}}^0 1$ denotes the *constant-like* basis functions on the face F and the coarse edge E, respectively.

Let $\tilde{W}_h(\mathrm{E})$ and $\tilde{W}_h(F)$ be the local spaces defined as in Subsection 2.2. Then, we have the space decomposition

$$W_h(\Gamma) = W_h^0(\Gamma) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \tilde{W}_h(\mathrm{E}) + \sum_{F\in\mathcal{F}_\Gamma} \tilde{W}_h(F). \tag{3.6}$$

The following result gives a stability of the above space decomposition of $W_h(\Gamma)$.

Theorem 3.1 *For any $\phi_h \in W_h(\Gamma)$, there exists a decomposition*

$$\phi_h = \phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_{\mathrm{E}} + \sum_{F\in\mathcal{F}_\Gamma} \phi_F \tag{3.7}$$

with

$$\phi_0 \in W_h^0(\Gamma);\ \phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E}) \quad and \quad \phi_F \in \tilde{W}_h(F),$$

such that

$$\|\phi_0\|_{*,\Gamma}^2 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \|\phi_{\mathrm{E}}\|_{*,\Gamma}^2 + \sum_{F\in\mathcal{F}_\Gamma} \|\phi_F\|_{*,\Gamma}^2 \le \log^2(H/h)\|\phi_h\|_{*,\Gamma}^2. \tag{3.8}$$

Proof. The idea of the proof is standard. But, for readers' convenience, we still give a complete proof of this theorem below. For $\mathrm{E} \in \mathcal{E}_\Gamma$ and $F \in \mathcal{F}_\Gamma$, set

$$\phi_{\mathrm{E}} = I^0_{\mathrm{E}}[\phi_h - \gamma_{\mathrm{E}}(\phi_h)], \text{ and } \phi_F = I^0_F[\phi_h - \gamma_F(\phi_h)].$$

It is clear that $\phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E})$ and $\phi_F \in \tilde{W}_h(F)$. Define $\phi_0 \in W^0_h(\Gamma)$ by

$$\phi_0(p) = \begin{cases} \phi_h(p), & if\ p \in \mathcal{N}_h \cap \mathcal{V}_\Gamma, \\ \gamma_{\mathrm{E}}(\phi_h), & if\ p \in \mathcal{N}_h \cap \mathrm{E}\ (\mathrm{E} \in \mathcal{E}_\Gamma), \\ \gamma_F(\phi_h), & if\ p \in \mathcal{N}_h \cap F\ (F \in \mathcal{F}_\Gamma). \end{cases}$$

It is easy to see that

$$\phi_h = \phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_{\mathrm{E}} + \sum_{F\in\mathcal{F}_\Gamma} \phi_F. \tag{3.9}$$

Then, we have by (3.2)

$$\begin{aligned} \|\phi_{\mathrm{E}}\|^2_{*,\Gamma} &= \sum_{k\in\mathcal{Q}_{\mathrm{E}}} \omega_k |\phi_{\mathrm{E}}|^2_{\frac{1}{2},\partial\Omega_k} \\ &\lesssim \log(H/h) \sum_{k\in\mathcal{Q}_{\mathrm{E}}} \omega_k \|\phi_h - \gamma_{\mathrm{E}}(\phi_h)\|^2_{\frac{1}{2},\partial\Omega_k}. \end{aligned}$$

This, together with (3.3), yields

$$\begin{aligned} \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \|\phi_{\mathrm{E}}\|^2_{*,\Gamma} &\lesssim \log^2(H/h) \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \sum_{k\in\mathcal{Q}_{\mathrm{E}}} \omega_k |\phi_h|^2_{\frac{1}{2},\partial\Omega_k} \\ &\lesssim \log^2(H/h) \sum_{k=1}^{\mathrm{N}} \omega_k |\phi_h|^2_{\frac{1}{2},\partial\Omega_k} \\ &= \log^2(H/h) \|\phi_h\|^2_{*,\Gamma}. \end{aligned} \tag{3.10}$$

On the other hand, one gets by (3.4) (with $\mathrm{F} = F$) and Poincare inequality

$$\begin{aligned} |\phi_F|_{\frac{1}{2},\partial\Omega_i} &\lesssim \log(H/h) \|\phi_h - \gamma_F(\phi_h)\|_{\frac{1}{2},\partial\Omega_i} \\ &\lesssim \log(H/h) |\phi_h|_{\frac{1}{2},\partial\Omega_i}. \end{aligned} \tag{3.11}$$

Similarly, we have

$$|\phi_F|_{\frac{1}{2},\Omega_j} \lesssim \log(H/h) |\phi_h|_{\frac{1}{2},\partial\Omega_j}. \tag{3.12}$$

Combining (3.11) and (3.12), leads to

$$\|\phi_F\|^2_{*,\Gamma} \lesssim \log^2(H/h)[\omega_i |\phi_h|^2_{\frac{1}{2},\partial\Omega_i} + \omega_j |\phi_h|^2_{\frac{1}{2},\partial\Omega_j}].$$

Thus,

$$\sum_{F\in\mathcal{F}_\Gamma} \|\phi_F\|^2_{*,\Gamma} \lesssim \log^2(H/h) \sum_{k=1}^{\mathrm{N}} \omega_k |\phi_h|^2_{\frac{1}{2},\partial\Omega_k} = \|\phi_h\|^2_{*,\Gamma}. \tag{3.13}$$

It follows by (3.9) that

$$\begin{aligned} \|\phi_0\|^2_{*,\Gamma} &\lesssim \|\phi_h\|^2_{*,\Gamma} + \|\sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \phi_{\mathrm{E}}\|^2_{*,\Gamma} + \|\sum_{F\in\mathcal{F}_\Gamma} \phi_F\|^2_{*,\Gamma} \\ &\lesssim \|\phi_h\|^2_{*,\Gamma} + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \|\phi_{\mathrm{E}}\|^2_{*,\Gamma} + \sum_{F\in\mathcal{F}_\Gamma} \|\phi_{ij}\|^2_{*,\Gamma}. \end{aligned}$$

This, together with (4.4) and (3.13), leads to

$$\|\phi_0\|^2_{*,\Gamma} \lesssim \log^2(H/h) \|\phi_h\|^2_{*,\Gamma}. \tag{3.14}$$

Now, the inequality (3.8) is a direct consequence of (4.4), (3.13) and the above inequality. □

In the rest of this section, we construct a multilevel space decomposition to each $\tilde{W}_h(\Gamma_{ij})$.

3.3 Multilevel decomposition for each Γ_{ij}

In this subsection, we describe a multilevel decomposition for each face $F = \Gamma_{ij}$. In short words, we make a multilevel decomposition for F, such that each sub-face generated by the final level contains a few nodes.

For a positive integer m_1, we first decompose F into the union of non-overlapping polygons $\tilde{F}_1^{(1)}, \cdots, \tilde{F}_{m_1}^{(1)}$ in the standard way. As usual, we assume that the decomposition is conforming, and all the polygons $\tilde{F}_r^{(1)}$ have the same "size" $d_1 \in (0,\ H)$. Note that $\tilde{F}_r^{(1)}$ may not be the union of some elements on F.

Let J and m_k $(k = 1, \cdots, J)$ be given positive integers. Set $\mathrm{M}_k = m_1 \cdots m_k$ for $k = 1, \cdots, J$. As we will see, M_k denotes the total number of all the sub-faces generated from the k-level decomposition for F. Successively continuing the above procedure, we get the following hierarchical decompositions of F (see Figure 1).

- the first level decomposition

$$F = \bigcup_{r=1}^{m_1} \tilde{F}_r^{(1)}.$$

- the second level decomposition

Let each $\tilde{F}_r^{(1)}$ be further decomposed into the union of m_2 sub-polygons of $\tilde{F}_r^{(1)}$

$$\tilde{F}_r^{(1)} = \bigcup_{l=1}^{m_2} \tilde{F}_{m_1(r-1)+l}^{(2)} \quad (r = 1, \cdots, m_1).$$

Thus,

$$F = \bigcup_{r=1}^{m_1} \bigcup_{l=1}^{m_2} \tilde{F}_{m_2(r-1)+l}^{(2)} = \bigcup_{r=1}^{\mathrm{M}_2} \tilde{F}_r^{(2)}.$$

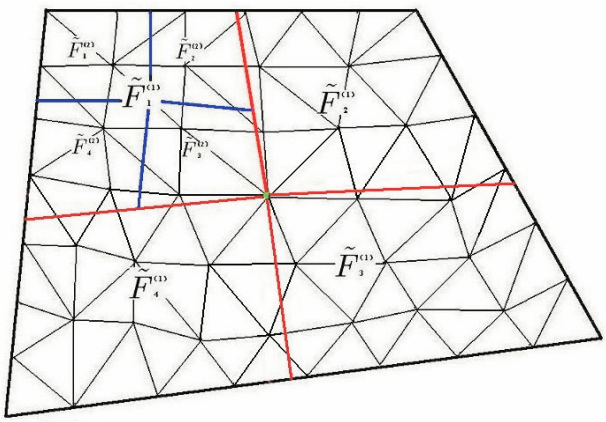

Figure 1: A simple decomposition of F

- the k level decomposition for $2 \le k \le J$

After generating $\tilde{F}_r^{(k-1)}$ from the $k-1$ level decomposition, we decompose each $\tilde{F}_r^{(k-1)}$ into the union of m_k sub-polygons of $\tilde{F}_r^{(k-1)}$

$$\tilde{F}_r^{(k-1)} = \bigcup_{l=1}^{m_k} \tilde{F}_{m_k(r-1)+l}^{(k)} \quad (r = 1 \cdots \mathrm{M}_{k-1}).$$

Then, we get a multilevel decomposition of F

$$F = \bigcup_{r=1}^{m_1} \tilde{F}_r^{(1)} = \cdots = \bigcup_{r=1}^{\mathrm{M}_{J-1}} \bigcup_{l=1}^{m_J} \tilde{F}_{m_J(r-1)+l}^{(J)} = \bigcup_{r=1}^{\mathrm{M}_J} \tilde{F}_r^{(J)}.$$

Sine the sub-polygons $\tilde{F}_r^{(k)}$ above may not be the union of some elements on F, it is inconvenient to make a detailed theoretical analysis for the multilevel method introduced later. Then, for the purpose of analysis, we define a perturbation $F_r^{(k)}$ for each polygon $\tilde{F}_r^{(k)}$ as follows (see Figure 2)

$$F_r^{(k)} = \bigcup_{p \in \mathcal{N}_h \cap \tilde{F}_r^{(k)}} supp\ \varphi_p, \quad 1 \le r \le m_k; 1 \le k \le J,$$

where φ_p denotes the nodal basis function on the node p. It is clear that $\tilde{F}_r^{(1)} \subset F_r^{(1)}$. In particular, if $\tilde{F}_r^{(k)}$ is just the union of some elements on F, then $F_r^{(k)} = \tilde{F}_r^{(k)}$. It is clear that these $F_r^{(k)}$ still constitute a decomposition of $F_r^{(k-1)}$:

$$F_r^{(k-1)} = \bigcup_{l=1}^{m_k} F_{m_k(r-1)+l}^{(k)} \quad (r = 1 \cdots \mathrm{M}_{k-1}; k = 1, \cdots, J). \tag{3.15}$$

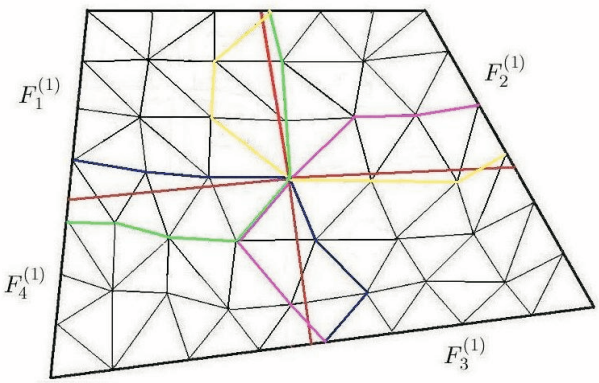

Figure 2: the decomposition of F for exposition

Finally, we get another multilevel decomposition for F

$$\bar{F} = \bigcup_{r=1}^{m_1} \bar{F}_r^{(1)} = \cdots = \bigcup_{r=1}^{\mathrm{M}_{J-1}} \bigcup_{l=1}^{m_J} \bar{F}_{m_J(r-1)+l}^{(J)} = \bigcup_{r=1}^{\mathrm{M}_J} \bar{F}_r^{(J)}.$$

For a fixed k, the sub-faces $F_r^{(k)}$ $(r = 1, \cdots, \mathrm{M}_k)$ satisfy the following conditions:
(a) each $F_r^{(k)}$ is just the union of some elements on F;
(b) each $F_r^{(k)}$ has the same size d_k for some $d_k \in (h,\ d_{k-1})$ (set $d_0 = H$);
(c) the union of all $\bar{F}_r^{(k)}$ $(r = 1, \cdots, \mathrm{M}_k)$ constitute a decomposition of $\bar{F}$.

Remark 3.1 *In general the sub-face $F_r^{(k)}$ is not a usual polygon yet, except for some particular situations. It is clear that the sub-faces $F_r^{(k)}$ satisfying the conditions (a)-(c) can be generated directly if the grids on F have some particular structure.*

Remark 3.2 *In applications, we use only basis functions on the nodes in $F_r^{(k)}$, instead of $F_r^{(k)}$ itself. Since $\tilde{F}_r^{(k)}$ contains the same nodes with $F_r^{(k)}$, one needs not to actually generate $F_r^{(k)}$ when implementing the method introduced later (i.e., $\tilde{F}_r^{(k)}$ is ok.).*

3.4 Multilevel decomposition for $\tilde{W}_h(\Gamma_{ij})$

In this subsection, we define a multilevel space decomposition for each $\tilde{W}_h(\Gamma_{ij})$ based on the decomposition given in the last subsection.

For convenience, define $\mathrm{M}_0 = 1$ and $F_1^{(0)} = F$. For $k \geq 0$, let $\mathcal{F}_r^{(k)}$ denote the set of all the sub-faces generated by the decomposition

$$F_r^{(k)} = \bigcup_{l=1}^{m_{k+1}} F_{m_{k+1}(r-1)+l}^{(k+1)} \quad (r = 1 \cdots \mathrm{M}_k). \tag{3.16}$$

Namely,

$$\mathcal{F}_r^{(k)} = \{F_{m_{k+1}(r-1)+1}^{(k+1)},\ F_{m_{k+1}(r-1)+2}^{(k+1)}, \cdots, F_{m_{k+1}r}^{(k)}\}.$$

If the common edge of two sub-polygons generated by the decomposition

$$\tilde{F}_r^{(k)} = \bigcup_{l=1}^{m_{k+1}} \tilde{F}_{m_{k+1}(r-1)+l}^{(k+1)} \quad (r = 1 \cdots \mathrm{M}_k)$$

contains a node at least, then we call the common edge to be a "real" edge of the decomposition (3.16). Similarly, if the common vertex of several sub-polygons generated by the decomposition

$$\tilde{F}_r^{(k)} = \bigcup_{l=1}^{m_{k+1}} \tilde{F}_{m_{k+1}(r-1)+l}^{(k+1)} \quad (r = 1 \cdots \mathrm{M}_k)$$

is just a node, then we call the common vertex to be a "real" vertex of the decomposition (3.16). Let $\mathcal{E}_r^{(k)}$ and $\mathcal{V}_r^{(k)}$ denote the sets of "real" edges and "real" vertices of the decomposition (3.16).

• the local subspaces

As before, let φ_p denote the nodal basis function on the node $p \in \mathcal{N}_h \cap \Gamma$. Set

$$\tilde{W}_h(F_r^{(k)}) = span\{\varphi_p :\ p \in \mathcal{N}_h \cap (\bar{F}_r^{(k)} \backslash \partial F)\} \quad (r = 1, \cdots, \mathrm{M}_k;\ k = 0, \cdots, J).$$

It is clear that $\tilde{W}_h(F_1^{(0)}) = \tilde{W}_h(F)$ ($\mathrm{M}_0 = 1$). Moreover, for each $1 \leq k \leq J$, the sum of all the local subspaces $\tilde{W}_h(F_r^{(k)})$ $(r = 1, \cdots, \mathrm{M}_k)$ gives a decomposition of $\tilde{W}_h(F)$.

• local coarse subspaces

For a sub-face $F_r^{(k)}$, define

$$W_h^0(F_r^{(k)}) = span\{I_{\mathrm{F}}^0 1,\ I_{\mathrm{E}}^0 1,\ \phi_p :\ \mathrm{F} \in \mathcal{F}_r^{(k)},\ \mathrm{E} \in \mathcal{E}_r^{(k)},\ p \in \mathcal{V}_r^{(k)}\}$$

$$(r = 1, \cdots, \mathrm{M}_k;\ k = 0, \cdots, J-1).$$

Note that $W_h^0(F_r^{(k)})$ is a subspace of $\tilde{W}_h(F_r^{(k)})$. If both $\mathcal{E}_r^{(k)}$ and $\mathcal{V}_r^{(k)}$ are empty set, then we have

$$W_h^0(F_r^{(k)}) = span\{I_{\mathrm{F}}^0 1 :\ \mathrm{F} \in \mathcal{F}_r^{(k)}\}.$$

In this case, the dimensions of the subspace $W_h^0(F_r^{(k)})$ equal the number m_{k+1} of the sub-faces generated by the first decomposition of $F_r^{(k)}$.

• the final decomposition

It is easy to see that

$$\tilde{W}_h(F) = \sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} W_h^0(F_r^{(k)}) + \sum_{r=1}^{\mathrm{M}_J} \tilde{W}_h(F_r^{(J)}). \tag{3.17}$$

Remark 3.3 *In the above space decomposition, the spaces $\tilde{W}_h(F_r^{(J)})$ $(r = 1, \cdots, \mathrm{M}_J)$ are the finest local subspaces. The spaces $W_h^0(F_r^{(k)})$ $(r = 1, \cdots, \mathrm{M}_k)$, which can be viewed as extensions of $W_h^0(\Gamma)$ defined in subsection 3.2, are the coarse subspaces associated with the first level decomposition of $F_r^{(k)}$ $(k = 0, \cdots, J-1)$. Note that each function in $W_h^0(F_r^{(k)})$ vanishes at all nodes in $F \backslash \bar{F}_r^{(k)}$.*

Remark 3.4 *In applications, the multilevel decomposition would be generated in a suitable manner such that both each local subspace $W_h(F_r^{(J)})$ and each coarse subspace $W_h^0(F_r^{(k)})$ has a low dimension. As will see, we usually set $m_k = m$ (a fixed positive integer).*

3.5 Stability of the multilevel space decomposition

For convenience, set

$$W_h^{00}(\Gamma) = W_h^0(\Gamma) + \sum_{\mathrm{E} \in \mathcal{E}_\Gamma} W_h(\mathrm{E}).$$

From (3.6) and (3.17), we get a multilevel space decomposition of $W_h(\Gamma)$

$$W_h(\Gamma) = W_h^{00}(\Gamma) + \sum_{F \in \mathcal{F}_\Gamma} [\sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} W_h^0(F_r^{(k)}) + \sum_{r=1}^{\mathrm{M}_J} \tilde{W}_h(F_r^{(J)})].$$

This subsection is devoted to analysis of the stability of the above multilevel space decomposition.

Before giving the main result, we prove two auxiliary results for a generic face $F \in \mathcal{F}_\Gamma$. For simplicity of exposition, we assume that both $\mathcal{E}_r^{(k)}$ and $\mathcal{V}_r^{(k)}$ are empty set for each $F_r^{(k)}$, where $\mathcal{E}_r^{(k)}$ and $\mathcal{V}_r^{(k)}$ were defined in the last subsection. Then, we need only to consider sub-face interpolations in the construction of a stable multilevel decomposition. If the assumption does not hold, we need also to consider edge interpolations as in Subsection 3.2, and to make some obvious modifications in the following construction. But, this change will not increase any essential difficulty of the analysis.

We first construct a decomposition for the generic face F.

For k satisfying $1 \le k \le J$, let $I^0_{F_r^{(k)}}$ denote the interpolation-type operator defined in Subsection 2.2 with $G = F_r^{(k)}$ $(r = 1, \cdots, \mathrm{M}_k)$. For ease of notation, set

$$\theta_r^{(k)} = I^0_{F_r^{(k)}} \quad (r = 1, \cdots, \mathrm{M}_k).$$

Under the assumption mentioned above, the union of $F^{(k)}_{m_k(r-1)+1}, \cdots, F^{(k)}_{m_k(r-1)+m_k}$ gives an open cover of $F_r^{(k-1)}$. Then, the operators

$$\theta^{(k)}_{m_k(r-1)+l} : W_h(\Gamma) \to \tilde{W}_h(F^{(k)}_{m_k(r-1)+l}) \subset \tilde{W}_h(F_r^{(k-1)}) \quad (l = 1, \cdots, m_k)$$

satisfy

$$\sum_{l=1}^{m_k} \theta^{(k)}_{m_k(r-1)+l} = \mathrm{I} \quad \text{on} \quad \tilde{W}_h(F_r^{(k-1)}) \quad (r = 1, \cdots, \mathrm{M}_{k-1}; k = 1, \cdots, J). \tag{3.18}$$

For $\phi_h \in W_h(\Gamma)$, let $\phi_F \in \tilde{W}_h(F)$ be defined in the proof of Theorem 3.1. For convenience, set $\phi_1^{(0)} = \phi_F$. Define

$$\phi_r^{(k)} = \theta_r^{(k)}[\phi_h - \gamma_{F_r^{(k)}}(\phi_h)] \in \tilde{W}_h(F_r^{(k)}) \quad (1 \le k \le J;\ r = 1, \cdots, \mathrm{M}_k) \tag{3.19}$$

and

$$\phi_{r,\ 0}^{(k-1)} = \phi_r^{(k-1)} - \sum_{l=1}^{m_k} \phi_{m_k(r-1)+l}^{(k)} \quad (1 \le k \le J;\ r = 1, \cdots, \mathrm{M}_{k-1}). \tag{3.20}$$

By the definition of ϕ_F, the equality (3.19) is also valid for $k = 0$ ($r = 1$). The following result indicates that they constitute a decomposition of ϕ_F.

Lemma 3.4 *Let $\phi_r^{(J)}$ and $\phi_{r,\ 0}^{(k-1)}$ ($k = 1, \cdots, J$) be defined above. Then,*

$$\phi_{r,\ 0}^{(k-1)} \in W_h^0(F_r^{(k-1)}) \quad (1 \le k \le J;\ r = 1, \cdots, \mathrm{M}_{k-1}). \tag{3.21}$$

Moreover, we have

$$\phi_F = \sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} \phi_{r,\ 0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J} \phi_r^{(J)}. \tag{3.22}$$

Proof. By (3.19) and (3.18), we have

$$\begin{aligned} \phi_r^{(k-1)} &= \theta_r^{(k-1)}[\phi_h - \gamma_{F_r^{(k-1)}}(\phi_h)] \\ &= \sum_{l=1}^{m_k} \theta_{m_k(r-1)+l}^{(k)} \theta_r^{(k-1)}[\phi_h - \gamma_{F_r^{(k-1)}}(\phi_h)] \\ &= \sum_{l=1}^{m_k} \theta_{m_k(r-1)+l}^{(k)} [\phi_h - \gamma_{F_r^{(k-1)}}(\phi_h)]. \end{aligned}$$

Substituting this into (3.20), and using (3.19) for $\phi_{m_k(r-1)+l}^{(k)}$, yields

$$\begin{aligned} \phi_{r,\ 0}^{(k-1)} &= \sum_{l=1}^{m_k} \theta_{m_k(r-1)+l}^{(k)} [\phi_h - \gamma_{F_r^{(k-1)}}(\phi_h)] - \sum_{l=1}^{m_k} \theta_{m_k(r-1)+l}^{(k)} [\phi_h - \gamma_{F_{m_k(r-1)+l}^{(k)}}(\phi_h)] \\ &= \sum_{l=1}^{m_k} \theta_{m_k(r-1)+l}^{(k)} [\gamma_{F_{m_k(r-1)+l}^{(k)}}(\phi_h) - \gamma_{F_r^{(k-1)}}(\phi_h)]. \end{aligned}$$

Since $\gamma_{F_{m_k(r-1)+l}^{(k)}}(\phi_h) - \gamma_{F_r^{(k-1)}}(\phi_h)$ is a constant, we further get

$$\phi_{r,\ 0}^{(k-1)} = \sum_{l=1}^{m_k} [\gamma_{F_{m_k(r-1)+l}^{(k)}}(\phi_h) - \gamma_{F_r^{(k-1)}}(\phi_h)] \theta_{m_k(r-1)+l}^{(k)} 1.$$

This gives (3.21).

Using (3.20) for $k = 1$ and $k = 2$, we deduce

$$\phi_F = \phi_1^{(0)} = \phi_{1,\ 0}^{(0)} + \sum_{r=1}^{m_1} \phi_r^{(1)} = \phi_{1,\ 0}^{(0)} + \sum_{r=1}^{m_1} \phi_{r,\ 0}^{(1)} + \sum_{r=1}^{m_1} \sum_{l=1}^{m_2} \phi_{m_2(r-1)+l}^{(2)}. \tag{3.23}$$

It follows by (3.15) that

$$\sum_{r=1}^{m_1} \sum_{l=1}^{m_2} \phi_{m_2(r-1)+l}^{(2)} = \sum_{r=1}^{\mathrm{M}_2} \phi_r^{(2)}.$$

Substituting this into (3.23), and using (3.20) repeatedly, yields

$$\phi_F = \phi_{1,\ 0}^{(0)} + \sum_{r=1}^{m_1} \phi_{r,\ 0}^{(1)} + \sum_{r=1}^{\mathrm{M}_2} \phi_{r,\ 0}^{(2)} + \sum_{r=1}^{\mathrm{M}_3} \phi_r^{(3)} = \cdots$$

$$= \quad \phi^{(0)}_{1,\,0} + \sum_{r=1}^{m_1} \phi^{(1)}_{r,\,0} + \cdots + \sum_{r=1}^{\mathrm{M}_{J-1}} \phi^{(J-1)}_{r,\,0} + \sum_{r=1}^{\mathrm{M}_J} \phi^{(J)}_r.$$

This implies (3.22).

□

The following result shows the stability of the decomposition (3.22).

Lemma 3.5 *Let $\phi^{(k)}_{r,\,0}$ and $\phi^{(J)}_r$ be defined in (3.19) and (3.20). Then,*

$$\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k} |\phi^{(k)}_{r,\,0}|^2_{H^{\frac12}_{00}(F^{(k)}_r)} + \sum_{r=1}^{\mathrm{M}_J} |\phi^{(J)}_r|^2_{H^{\frac12}_{00}(F^{(k)}_r)} \lesssim J[1+\log(H/h)]^2 |\phi_h|^2_{\frac12,\,F} \quad (J \geq 1). \tag{3.24}$$

Proof. It follows by (3.20) that

$$\begin{aligned} |\phi^{(k-1)}_{r,\,0}|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)} &\leq 2(|\phi^{(k-1)}_r|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)} + |\sum_{l=1}^{m_k} \phi^{(k)}_{m_k(r-1)+l}|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)}) \\ &\lesssim |\phi^{(k-1)}_r|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)} + \sum_{l=1}^{m_k} |\phi^{(k)}_{m_k(r-1)+l}|^2_{H^{\frac12}_{00}(F^{(k)}_{m_k(r-1)+l})}. \end{aligned} \tag{3.25}$$

One needs to estimate each semi-norm in the right side of the above inequality.

Using (3.4) with $\mathrm{F} = F^{(k-1)}_r$, yields ($d_0 = H$)

$$\begin{aligned} |\phi^{(k-1)}_r|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)} &\lesssim \log^2(d_{k-1}/h)\|\phi_h - \gamma_{F^{(k-1)}_r}(\phi_h)\|^2_{\frac12,\,F^{(k)}_r} \\ &\lesssim \log^2(d_{k-1}/h)|\phi_h|^2_{\frac12,\,F^{(k)}_r}, \quad r = 1,\cdots,\mathrm{M}_{k-1}. \end{aligned}$$

Then,

$$\sum_{r=1}^{\mathrm{M}_{k-1}} |\phi^{(k-1)}_r|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)} \lesssim \log(d_{k-1}/h)|\phi_h|^2_{\frac12,\,F} \quad (1 \leq k \leq J). \tag{3.26}$$

Let $r = 1,\cdots,\mathrm{M}_{k-1};\ l = 1,\cdots,m_k$. By (3.4) with $\mathrm{F} = F^{(k)}_{m_k(r-1)+l}$, one can verify that

$$\begin{aligned} |\phi^{(k)}_{m_k(r-1)+l}|^2_{H^{\frac12}_{00}(F^{(k)}_{m_k(r-1)+l})} &\lesssim \log^2(d_k/h)\|\phi_h - \gamma_{F^{(k)}_{m_k(r-1)+l}}(\phi_h)\|^2_{\frac12,\,F^{(k)}_{m_k(r-1)+l}} \\ &\lesssim \log^2(d_k/h)|\phi_h|^2_{\frac12,\,F^{(k)}_{m_k(r-1)+l}}. \end{aligned}$$

This, together with (3.15) and the discrete $H^{\frac12}$ semi-norm, leads to

$$\sum_{l=1}^{m_k} |\phi^{(k)}_{m_k(r-1)+l}|^2_{H^{\frac12}_{00}(F^{(k)}_{m_k(r-1)+l})} \lesssim \log^2(d_k/h)|\phi_h|^2_{\frac12,\,F^{(k-1)}_r}.$$

We further get

$$\sum_{r=1}^{\mathrm{M}_{k-1}}\sum_{l=1}^{m_k} |\phi^{(k)}_{m_k(r-1)+l}|^2_{H^{\frac12}_{00}(F^{(k)}_{m_k(r-1)+l})} \lesssim \log^2(d_k/h)|\phi_h|^2_{\frac12,\,F} \quad (1 \leq k \leq J). \tag{3.27}$$

Combining (3.25) with (3.26)-(3.27), leads to

$$\sum_{r=1}^{\mathrm{M}_{k-1}} |\phi^{(k-1)}_{r,\,0}|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)} \leq \sum_{r=1}^{\mathrm{M}_{k-1}} |\phi^{(k-1)}_r|^2_{H^{\frac12}_{00}(F^{(k-1)}_r)} + \sum_{r=1}^{\mathrm{M}_{k-1}}\sum_{l=1}^{m_k} |\phi^{(k)}_{m_k(r-1)+l}|^2_{H^{\frac12}_{00}(F^{(k)}_{m_k(r-1)+l})}$$

$$\lesssim \quad \log^2(d_{k-1}/h)|\phi_h|^2_{\frac{1}{2},\ F} \quad (k=1,\cdots,J).$$

Thus,

$$\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}|\phi^{(k)}_{r,\ 0}|^2_{H^{\frac{1}{2}}_{00}(F^{(k)}_r)} \lesssim \sum_{k=0}^{J-1}\log^2(d_k/h)|\phi_h|^2_{\frac{1}{2},\ F} \lesssim J\log^2(H/h)|\phi_h|^2_{\frac{1}{2},\ F}. \tag{3.28}$$

On the other hand, using (3.19) (with $k=J$) and (3.4), we deduce

$$|\phi^{(J)}_r|^2_{H^{\frac{1}{2}}_{00}(F^{(J)}_r)} \lesssim \log^2(d_J/h)|\phi_h|^2_{\frac{1}{2},\ F^{(J)}_r}.$$

Then,

$$\sum_{r=1}^{\mathrm{M}_J}|\phi^{(J)}_r|^2_{H^{\frac{1}{2}}_{00}(F^{(J)}_r)} \lesssim \log^2(d_J/h)|\phi_h|^2_{\frac{1}{2},\ F}.$$

Now, the inequality (3.24) follows by (3.28) and and the above inequality.

□

Now, we give the stability of the multilevel space decomposition of $W_h(\Gamma)$.

For $\phi_h \in W_h(\Gamma)$, let ϕ_0 and ϕ_{E} be defined by Subsection 3.2. For simplicity of exposition, we define

$$\phi_{00} = \phi_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\phi_{\mathrm{E}}.$$

The decomposition (3.7) can be written as

$$\phi_h = \phi_{00} + \sum_{F\in\mathcal{F}_\Gamma}\phi_F. \tag{3.29}$$

Then, the following result is a direct consequence of Lemma 3.4 and Lemma 3.5.

Theorem 3.2 *For any $\phi_h \in W_h(\Gamma)$, let $\phi_{00} \in W^{00}_h(\Gamma)$ be defined above. Then, there exist functions*

$$\phi^{(k)}_{ij,\ r,\ 0} \in W^0_h(\Gamma^{(k)}_{ij,\ r})\ \ (0\le k\le J-1;\ r=1,\cdots,\mathrm{M}_k)\ \ \textit{and}\ \ \phi^{(J)}_{ij,\ r} \in \tilde{W}_h(\Gamma^{(J)}_{ij,\ r})\ \ (r=1,\cdots,\mathrm{M}_J),$$

such that

$$\phi_h = \phi_{00} + \sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}\phi^{(k)}_{ij,\ r\ ,0} + \sum_{r=1}^{\mathrm{M}_J}\phi^{(J)}_{ij,\ r}]. \tag{3.30}$$

Moreover, we have

$$\sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k}\|\phi^{(k)}_{ij,\ r\ ,0}\|^2_{*,\Gamma} + \sum_{r=1}^{\mathrm{M}_J}\|\phi^{(J)}_{ij,\ r}\|^2_{*,\Gamma}] \lesssim J[1+\log(H/h)]^2\|\phi_h\|^2_{*,\Gamma} \quad (J\ge 1). \tag{3.31}$$

□

Remark 3.5 *We conjecture that the factor J in (3.31) (and (3.24)) can be dropped. Unfortunately, we fail to prove this conjecture.*

4 A substructuring method with inexact subdomain solvers

This section is devoted to construction of a new substructuring preconditioner with inexact subdomain solvers. The new preconditioner is based on a multilevel decomposition of $V_h(\Omega)$. This multilevel decomposition of $V_h(\Omega)$ depends on the multilevel decomposition of $W_h(\Gamma)$ developed in the last section and various nearly harmonic extension operators.

4.1 Nearly harmonic extensions

In this subsection, we define various extension operators, which will play a key role in the new preconditioner.

As in the last section, let F denote a face Γ_{ij}, and let $F_r^{(k)}$ be defined in Subsection 3.3. For convenience, we use F to denote the face F or a sub-face $F_r^{(k)}$ ($r = 1, \cdots, \mathrm{M}_k$; $k = 0, \cdots, J$). Throughout this section, we always use $\phi_{\mathrm{F}} = I_{\mathrm{F}}^0 1 \in \tilde{W}_h(F)$ to denote the *constant-like* basis function on F.

• an explicit extension for the *constant-like* basis function ϕ_{F}

Since F may not be a polygon, a stable extension of the *constant-like* basis ϕ_{F} is trick to design.

Let o_{F} be a central node on F, and let d_{F} denote the "size" of F in the standard sense ($d_{\mathrm{F}} = d_k$ if $\mathrm{F} = F_r^{(k)}$). For a node p in Ω_{ij}, let p' denote the projection of p on F. Besides, we use l_p to denote the straight line drawing from o_{F} to p' (if $p' \neq o_{\mathrm{F}}$), and use $q \in \partial\mathrm{F}$ to denote the intersection point of l_p with $\partial\mathrm{F}$ (see Figure 3).

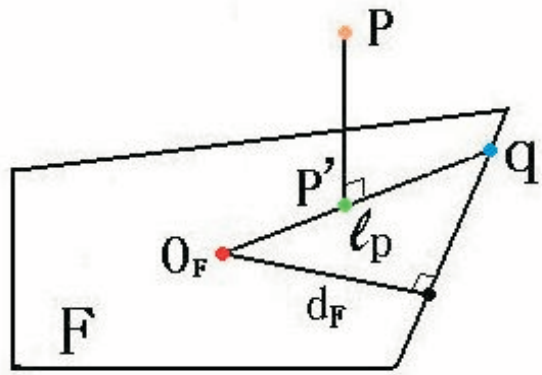

Figure 3: illustration for the notations

In short words, we define the extension $E_{\mathrm{F}}\phi_{\mathrm{F}}$ such that the values $(E_{\mathrm{F}}\phi_{\mathrm{F}})(p)$ decrease gradually when the length $|pp'|$ or $|o_{\mathrm{F}}p'|$ increases. To give the exact definition of $E_{\mathrm{F}}\phi_{\mathrm{F}}$, we set

$$\Lambda_{\mathrm{F}}^{(1)} = \{p \in \mathcal{N}_h \cap \Omega_{ij} : \ p' = o_{\mathrm{F}}, \ |pp'| \leq d_{\mathrm{F}}\}$$

and

$$\Lambda_{\mathrm{F}}^{(2)} = \{p \in \mathcal{N}_h \cap \Omega_{ij} : \ p' \in \mathrm{F}, \ p' \neq o_{\mathrm{F}}, \ |pp'| \leq |p'q| \cdot \frac{d_{\mathrm{F}}}{|o_{\mathrm{F}}q|}\}.$$

For a face or a sub-face F, define the extension operator E_{F} as

$$(E_{\mathrm{F}}\phi_{\mathrm{F}})(p) = \begin{cases} \phi_{\mathrm{F}}(p), & \text{if } \ p \in \bar{\mathrm{F}}, \\ 1 - \frac{|pp'|}{d_{\mathrm{F}}}, & \text{if } p \in \Lambda_{\mathrm{F}}^{(1)}, \\ 1 - \frac{|o_{\mathrm{F}}q|}{d_{\mathrm{F}}} \cdot \frac{|pp'|}{|p'q|}, & \text{if } p \in \Lambda_{\mathrm{F}}^{(2)}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

For rectangular face F with uniform triangulation, the values of $E_{\mathrm{F}}\phi_{\mathrm{F}}$ at some nodes are given in Figure 4.

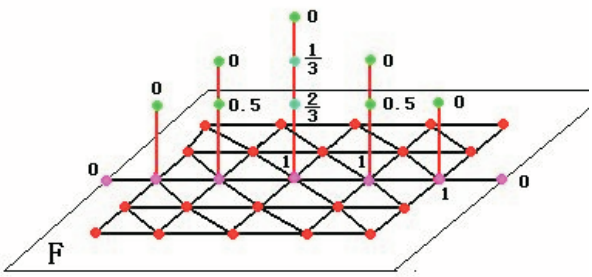

Figure 4: extension of ϕ_{F} for particular case

It is easy to see that the extension operator E_{F} possesses the properties:

(i) the support set Ω_{F} of $E_{\mathrm{F}}\phi_{\mathrm{F}}$ is a simply connected domain with the size d_{F};

(ii) the implementation of E_{F} possesses the optimal computational complexity $O(n_{\mathrm{F}})$ with $n_{\mathrm{F}} = (d_{\mathrm{F}}/h)^3$ being the number of the nodes in Ω_{F}.

Besides, the extension E_{F} is nearly harmonic in the following sense

Theorem 4.1 *The extension E_{F} defined by (4.1) satisfies the stability condition*

$$|E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_i},\ |E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_j} \lesssim |\phi_{\mathrm{F}}|^2_{H^{\frac{1}{2}}_{00}(\mathrm{F})}. \tag{4.2}$$

The proof of this theorem will be given in Section 6.

• extensions associated with the finest sub-faces

For $F = \Gamma_{ij}$, define $E^{(J)}_{ij,\ r} : \tilde{W}_h(F^{(J)}_r) \to V_h(\Omega_{ij})$ as follows ($\mathrm{F} = F^{(J)}_r$)

$$(E^{(J)}_{ij,\ r}\varphi)(p) = \begin{cases} \varphi(p), & \text{if } \ p \in \bar{\mathrm{F}}, \\ \gamma_{h,\mathrm{F}}(\varphi), & \text{if } p \in \Lambda^{(1)}_{\mathrm{F}} \cup \Lambda^{(2)}_{\mathrm{F}}, \\ 0, & \text{otherwise.} \end{cases} \quad (p \in \mathcal{N}_h \cap \Omega_{ij},\ \varphi \in \tilde{W}_h(F^{(J)}_r)) \tag{4.3}$$

• an extension on the global coarse space

Let $R^0_k : W_h(\partial\Omega_k) \to V_h(\Omega_k)$ denote the zero extension operator in the sense that $R^0_k\phi = \phi$ on $\partial\Omega_k$, and $R^0_k\phi$ vanishes at all internal nodes of Ω_k for $\phi \in W_h(\partial\Omega_k)$. For $\psi \in W^0_h(\Gamma)$, we have

$$\psi = \gamma_{h,\partial\Omega_k}(\psi) + I^0_{\mathcal{W}_k}(\psi - \gamma_{h,\partial\Omega_k}(\psi)) + \sum_{\mathrm{F}\subset\partial\Omega_k} \gamma_{h,\mathrm{F}}(\psi - \gamma_{h,\partial\Omega_k}(\psi))\phi_{\mathrm{F}}, \quad \text{on } \ \partial\Omega_k.$$

Hereafter, $\mathcal{W}_k$ denotes the *wire-basket* set of Ω_k, i.e., the union of all the edges of Ω_k. Then, we define on each Ω_k

$$E_0\psi = \gamma_{h,\partial\Omega_k}(\psi) + R^0_k[I^0_{\mathcal{W}_k}(\psi - \gamma_{h,\partial\Omega_k}(\psi))] + \sum_{\mathrm{F}\subset\partial\Omega_k} \gamma_{h,\mathrm{F}}(\psi - \gamma_{h,\partial\Omega_k}(\psi))E_{\mathrm{F}}\phi_{\mathrm{F}}. \tag{4.4}$$

It is easy to see that $E_0\psi = \psi$ on Γ.

• extensions on the local coarse spaces

Let $G^{(k)}_r$ denote the union of the "real" edges and the "real" vertices generated by the first decomposition of $F^{(k)}_r$ (see Subsection 3.4). For $\psi \in W^0_h(F^{(k)}_r)$, let $a_{\mathrm{F}}(\psi)$ denote the (constant) value of ψ at the interior nodes in F. It is easy to see that

$$\psi = I^0_{G^{(k)}_r}\psi + \sum_{\mathrm{F}\in\mathcal{F}^{(k)}_r} a_{\mathrm{F}}(\psi)\phi_{\mathrm{F}}, \quad \text{on } \ F^{(k)}_r.$$

Note that $I^0_{G_r^{(k)}}\psi \equiv 0$ if $G_r^{(k)}$ is an empty set. Then, we define on Ω_l $(l = i, j)$

$$E^{(k)}_{ij,\ r,\ 0}\psi = R^0_l(I^0_{G_r^{(k)}}\psi) + \sum_{\mathrm{F}\in\mathcal{F}_r^{(k)}} a_{\mathrm{F}}(\psi)E_{\mathrm{F}}\phi_{\mathrm{F}}. \tag{4.5}$$

As we will see in Section 5, all the extension operators defined above keep the stability of the exact harmonic extension, and so they are called *nearly* harmonic extensions.

4.2 Multilevel decomposition for $V_h(\Omega)$

In this subsection, we define a series of "nearly harmonic" subspaces of $V_h(\Omega)$ by using the extension operators described in the last subsection.

For $\mathrm{E} \in \mathcal{E}_\Gamma$, let $E_{\mathrm{E}} : \tilde{W}_h(\mathrm{E}) \to V_h(\Omega)$ denote the standard zero extension operator. Define

$$V_h(\Omega_{\mathrm{E}}) = \{v_h \in V_h(\Omega) :\ v_h = E_{\mathrm{E}}\phi_h \text{ for some } \phi_h \in \tilde{W}_h(\mathrm{E})\}.$$

Namely,

$$V_h(\Omega_{\mathrm{E}}) = span\{\varphi_p :\ p \in \mathcal{N}_h \cap \mathrm{E}\}.$$

For the extension $E_0 : W^0_h(\Gamma) \to V_h(\Omega)$, define

$$V^0_h(\Omega) = \{v_h \in V_h(\Omega) :\ v_h = E_0\phi \text{ for some } \phi_h \in W^0_h(\Gamma)\}.$$

As before, let $F = \Gamma_{ij}$. For each $F_r^{(k)}$ $(k = 0, \cdots, J;\ r = 1, \cdots, \mathrm{M}_k)$, let $E^{(k)}_{ij,\ r,\ 0}$: $W^0_h(F_r^{(k)}) \to V_h(\Omega_{ij})$ $(k = 1, \cdots, J-1)$ and $E_r^{(J)} : \tilde{W}_h(F_r^{(J)}) \to V_h(\Omega_{ij})$ be the extension operators defined in the last subsection. Set

$$\Omega^{(k)}_{ij,\ r} = \bigcup_{\psi\in W^0_h(F_r^{(k)})} supp\ E^{(k)}_{ij,\ r,\ 0}\psi \quad (k = 0, \cdots, J-1)$$

and

$$\Omega^{(J)}_{ij,\ r} = \bigcup_{\psi\in \tilde{W}_h(F_r^{(J)})} supp\ E_r^{(J)}\psi \quad .$$

Define the subspaces

$$V^0_h(\Omega^{(k)}_{ij,\ r}) = \{v_h \in V_h(\Omega_{ij}) :\ v_h = E^{(k)}_{ij,\ r,\ 0}\phi_h \text{ for } \phi_h \in W^0_h(F_r^{(k)})\} \quad (k = 1, \cdots, J-1),$$

and

$$\tilde{V}_h(\Omega^{(J)}_{ij,\ r}) = \{v_h \in V_h(\Omega_{ij}) :\ v_h = E^{(J)}_{ij,\ r}\phi_h \text{ for } \phi_h \in \tilde{W}_h(F_r^{(J)})\}.$$

The functions in the subspaces defined in this subsection are not harmonic yet, but they still keep the energy stability (see Section 5). Thus, the basis functions of these subspaces are called *nearly* harmonic basis functions.

Since we have

$$W_h(\Gamma) = W^0_h(\Gamma) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} \tilde{W}_h(\mathrm{E}) + \sum_{F\in\mathcal{F}_\Gamma} [\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k} W^0_h(F_r^{(k)}) + \sum_{r=1}^{\mathrm{M}_J} \tilde{W}_h(F_r^{(J)})],$$

the space decomposition holds

$$V_h(\Omega) = \sum_{k=1}^{\mathrm{N}} V^p_h(\Omega_k) + V^0_h(\Omega) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} V_h(\Omega_{\mathrm{E}}) + \sum_{\Gamma_{ij}}[\sum_{k=0}^{J-1}\sum_{r=1}^{\mathrm{M}_k} V^0_h(\Omega^{(k)}_{ij,\ r}) + \sum_{r=1}^{\mathrm{M}_J} \tilde{V}_h(\Omega^{(J)}_{ij,\ r})].$$

4.3 New substructuring preconditioner

Based on the space decomposition in the last subsection, we can define a preconditioner in the standard way.

Define symmetric and positive definite operators as follows:

• the subdomain solver $B_k : V_h^p(\Omega_k) \to V_h^p(\Omega_k)$ satisfies:

$$(B_k v_h, v_h)_{\Omega_k} \overset{=}{\sim} \omega_k \int_{\Omega_k} |\nabla v_h|^2 dp, \quad \forall v_h \in V_h^p(\Omega_k); \tag{4.6}$$

• the global coarse solver $B_0 : V_h^0(\Omega) \to V_h^0(\Omega)$ satisfies:

$$(B_0 v_h, v_h) \overset{=}{\sim} (A_h v_h, v_h), \quad \forall v_h \in V_h^0(\Omega); \tag{4.7}$$

• the edge solver $B_{\mathrm{E}} : V_h(\Omega_{\mathrm{E}}) \to V_h(\Omega_{\mathrm{E}})$ satisfies:

$$(B_{\mathrm{E}} v_h, v_h)_{\Omega_{\mathrm{E}}} \overset{=}{\sim} (A_h v_h, v_h), \quad \forall v_h \in V_h(\Omega_{\mathrm{E}}); \tag{4.8}$$

• local coarse solver $B_{ij,\ r,\ 0}^{(k)} : V_h^0(\Omega_{ij,\ r}^{(k)}) \to V_h^0(\Omega_{ij,\ r}^{(k)})$ $(k = 0, \cdots, J-1)$ satisfies

$$(B_{ij,\ r,\ 0}^{(k)} v_h, v_h) \overset{=}{\sim} (A_h v_h, v_h), \quad \forall v_h \in V_h^0(\Omega_{ij,\ r}^{(k)}); \tag{4.9}$$

• the finest local solver $\tilde{B}_{ij,\ r}^{(J)} : \tilde{V}_h(\Omega_{ij,\ r}^{(J)}) \to \tilde{V}_h(\Omega_{ij,\ r}^{(J)})$ satisfies

$$(B_{ij,\ r}^{(J)} v_h, v_h) \overset{=}{\sim} (A_h v_h, v_h), \quad \forall v_h \in \tilde{V}_h(\Omega_{ij,\ r}^{(J)}). \tag{4.10}$$

In applications, the subdomain solver B_k is usually chosen as a symmetric multigrid preconditioner. Since all the subspaces $V_h^0(\Omega)$, $V_h(\Omega_{\mathrm{E}})$, $V_h^0(\Omega_{ij,\ r}^{(k)})$ and $\tilde{V}_h(\Omega_{ij,\ r}^{(J)})$ have low dimensions, the solvers B_0, B_{E}, $B_{ij,\ r,\ 0}^{(k)}$ and $\tilde{B}_{ij,\ r}^{(J)}$ can be simply defined as the restriction operators of A_h on their respective subspaces.

Now, the desired multilevel preconditioner for A_h is defined as

$$\begin{aligned} B_J^{-1} &= \sum_{k=1}^{\mathrm{N}} B_k^{-1} Q_k + B_0^{-1} Q_0 + \sum_{\mathrm{E} \in \mathcal{E}_\Gamma} B_{\mathrm{E}}^{-1} Q_{\mathrm{E}} \\ &+ \sum_{\Gamma_{ij}} [\sum_{k=0}^{J-1} \sum_{r=1}^{\mathrm{M}_k} (B_{ij,\ r,\ 0}^{(k)})^{-1} Q_{ij,\ r,\ 0}^{(k)} + \sum_{r=1}^{\mathrm{M}_J} (\tilde{B}_{ij,\ r}^{(J)})^{-1} Q_{ij,\ r}^{(J)}], \end{aligned}$$

where $Q_k : V_h(\Omega) \to V_h^p(\Omega_k)$, $Q_0 : V_h(\Omega) \to V_h^0(\Omega)$, $Q_{\mathrm{E}} : V_h(\Omega) \to V_h(\Omega_{\mathrm{E}})$, $Q_{ij,\ r,\ 0}^{(k)} : V_h(\Omega) \to V_h^0(\Omega_{ij,\ r}^{(k)})$ $(k = 0, \cdots, J-1)$, and $Q_{ij,\ r}^{(J)} : V_h(\Omega) \to \tilde{V}_h(\Omega_{ij,\ r}^{(J)})$ denote L^2 projectors.

4.4 Implementation

Let B_k, B_0, B_{E}, $B_{ij,\ r,\ 0}^{(k)}$ $(k = 0, \cdots, J-1)$ and $B_{ij,\ r}^{(J)}$ be defined as in the last subsection. The action of B_J^{-1} can be described by the following algorithm

Algorithm 4.1. For $g \in V_h(\Omega)$, the solution $u_g \in V_h(\Omega)$ satisfying

$$(B_J u_g, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega)$$

can be gotten as follows:

Step 1. Computing $u_k^p \in V_k^p(\Omega_k)$ in parallel by

$$(B_k u_k^p, v_h)_{\Omega_k} = (g, v_h)_{\Omega_k}, \quad \forall v_h \in V_k^p(\Omega_k);$$

Step 2. Computing $u_0 \in V_h^0(\Omega)$ by

$$(B_0 u_0, v_h) = (g, v_h), \quad \forall v_h \in V_h^0(\Omega);$$

Step 3. Computing $u_{\mathrm{E}} \in V_h(\Omega_{\mathrm{E}})$ in parallel by

$$(B_{\mathrm{E}} u_{\mathrm{E}}, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega_{\mathrm{E}});$$

Step 4. Computing $u_{ij,\ r,\ 0}^{(l)} \in V_h^0(\Omega_{ij,\ r}^{(l)})$ $(l = 0, \cdots, J-1)$ in parallel by

$$(B_{ij,\ r,\ 0}^{(l)} u_{ij,\ r,\ 0}^{(l)},\ v) = (g, v_h)_{\Omega_{ij,\ r,\ 0}^{(l)}}, \quad \forall v_h \in V_h^0(\Omega_{ij,\ r}^{(l)});$$

Step 5. Computing $\tilde{u}_{ij,\ r}^{(J)} \in \tilde{V}_h(\Gamma_{ij,\ r}^{(J)})$ in parallel by

$$(B_{ij,\ r}^{(J)} \tilde{u}_{ij,\ r}^{(J)},\ v_h) = (g, v_h)_{\Omega_{ij,\ r}^{(J)}}, \quad \forall v_h \in \tilde{V}_h(\Omega_{ij,\ r}^{(J)});$$

Step 6. Set

$$u_g \quad = \quad \sum_{k=1}^{\mathrm{N}} u_k^p + u_0 + \sum_{\mathrm{E} \in \mathcal{E}_\Gamma} u_{\mathrm{E}} + \sum_{\Gamma_{ij}} [\sum_{l=1}^{J-1} \sum_{r=1}^{\mathrm{M}_k} u_{ij,\ r,\ 0}^{(l)} + \sum_{r=1}^{\mathrm{M}_J} \tilde{u}_{ij,\ r}^{(J)}].$$

Remark 4.1 *Before implementing the above algorithm, one needs to compute the nearly harmonic basis $E_0\phi_h$, $E_{\mathrm{E}}\phi_h$, $E_{ij,\ r,\ 0}^{(k)}\phi_h$ and $E_{ij,\ r}^{(J)}\phi_h$ for each basis function ϕ_h of $W_h^0(\Gamma)$, $\tilde{W}_h(\mathrm{E})$, $W_h^0(F_{r,\ 0}^{(k)})$ or of $\tilde{W}_h(F_r^{(J)})$ $(F = \Gamma_{ij})$. As we will see in the next subsection, the complexity of computing all the extension is optimal. When all the harmonic basis functions are gotten, the cost for implementing Step 2-Step 5 is very cheap, since each subspace involved in these steps has very low dimension.*

Algorithm 4.1 can be also described in term of interface solvers. To this end, we need to define a series of solvers on interface subspaces.

- The global coarse solver

  A symmetric operator $M_0 : W_h^0(\Gamma) \to W_h^0(\Gamma)$ satisfying

$$\langle M_0 v_h, v_h \rangle \overset{=}{\sim} \|v_h\|_{*,\ \Gamma}^2, \quad \forall v_h \in W_h^0(\Omega).$$

- The edge solver

  A symmetric operator $M_{\mathrm{E}} : W_h(\mathrm{E}) :\to W_h(\mathrm{E})$ satisfying

$$\langle M_{\mathrm{E}} v_h, v_h \rangle_{\mathrm{E}} \overset{=}{\sim} \|v_h\|_{*,\Gamma}^2, \quad \forall v_h \in W_h(\mathrm{E}).$$

- The local coarse solver

  A symmetric operator $M_{ij,\ r,\ 0}^{(k)} :\ W_h^0(F_r^{(k)}) \to W_h^0(F_r^{(k)})$ $(F = \Gamma_{ij})$ satisfying

$$\langle M_{ij,\ r,\ 0}^{(k)} v_h, v_h \rangle_{F_r^{(k)}} \overset{=}{\sim} (\omega_i + \omega_j) |v_h|_{\frac{1}{2}, \Gamma_{ij}}^2, \quad \forall v_h \in W_h^0(F_r^{(k)}).$$

- The finest local solver

  A symmetric operator $M_{ij,\ r}^{(J)} : \tilde{W}_h(F_r^{(J)}) \to \tilde{W}_h(F_r^{(J)})$ $(F = \Gamma_{ij})$ satisfying

$$\langle M_{ij,\ r}^{(J)} v_h, v_h \rangle_{F_r^{(J)}} \overset{=}{\sim} (\omega_i + \omega_j) |v_h|_{\frac{1}{2}, \Gamma_{ij}}^2, \quad \forall v_h \in \tilde{W}_h(F_r^{(J)}).$$

All the above operators can be defined by the discrete $H^{\frac{1}{2}}$ inner-product (see Subsection 2.2) on their respective subspaces of $W_h(\Gamma)$ (see [14] and [13] for the details). Under suitable assumptions, the local operators can be also defined explicitly (see [12]).

The following algorithm can be viewed as a variant of Algorithm 4.1

Algorithm 4.2. For $g \in V_h(\Omega)$, the solution $u_g \in V_h(\Omega)$ satisfying

$$(B_J u_g, v_h) = (g, v_h), \quad \forall v_h \in V_h(\Omega)$$

can be gotten as follows:

Step 1. Computing $u_k^p \in V_k^p(\Omega_k)$ in parallel by

$$(B_k u_k^p, v_h)_{\Omega_k} = (g, v_h)_{\Omega_k}, \quad \forall v_h \in V_k^p(\Omega_k);$$

Step 2′. Computing $\phi_0 \in W_h^0(\Gamma)$ by

$$\langle M_0 \phi_0, \psi_h \rangle_\Gamma = (g, v_h), \quad v_h = E_0 \psi_h, \quad \forall \psi_h \in W_h^0(\Gamma);$$

Step 3′. Computing $\phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E})$ in parallel by

$$\langle M_{\mathrm{E}} \phi_{\mathrm{E}}, \psi_h \rangle_{\mathrm{E}} = (g, v_h)_{\Omega_{\mathrm{E}}}, \quad v_h = E_{\mathrm{E}} \psi_h, \quad \forall \psi_h \in \tilde{W}_h(\mathrm{E});$$

Step 4′. Computing $\phi_{ij,\ r,\ 0}^{(l)} \in W_h^0(F_r^{(l)})$ $(l = 0, \cdots, J-1)$ in parallel by $(F = \Gamma_{ij})$

$$\langle M_{ij,\ r,\ 0}^{(l)} \phi_{ij,\ r,\ 0}^{(l)},\ \psi_h \rangle_{F_r^{(l)}} = (g, v_h)_{\Omega_{ij,\ r,\ 0}^{(l)}}, \quad v_h = E_{ij,\ r,\ 0}^{(l)} \psi_h, \quad \forall \psi_h \in W_h^0(F_r^{(l)});$$

Step 5′. Computing $\tilde{\phi}_{ij,\ r}^{(J)} \in \tilde{W}_h(F_r^{(J)})$ in parallel by $(F = \Gamma_{ij})$

$$\langle M_{ij,\ r}^{(J)} \tilde{\phi}_{ij,\ r}^{(J)}, \psi_h \rangle_{F_r^{(J)}} = (g, v_h)_{\Omega_{ij,\ r}^{(J)}}, \quad v_h = E_{ij,\ r}^{(J)} \psi_h, \quad \forall \psi_h \in \tilde{W}_h(F_r^{(J)});$$

Step 6′. Computing

$$u_0 = E_0 \phi_0, \quad u_{\mathrm{E}} = E_{\mathrm{E}} \phi_{\mathrm{E}}, \quad u_{ij,\ r,\ 0}^{(l)} = E_{ij,\ r,\ 0}^{(l)} \phi_{ij,\ r,\ 0}^{(l)} \quad (l = 0, \cdots, J-1)$$

and

$$\tilde{u}_{ij,\ r}^{(J)} = E_{ij,\ r}^{(J)} \phi_{ij,\ r}^{(J)}.$$

Step 7. Set

$$u_g \quad = \quad \sum_{k=1}^{\mathrm{N}} u_k^p + v_0 + \sum_{\mathrm{E} \in \mathcal{E}_\Gamma} u_{\mathrm{E}} + \sum_{\Gamma_{ij}} [\sum_{l=1}^{J-1} \sum_{r=1}^{\mathrm{M}_k} u_{ij,\ r,\ 0}^{(l)} + \sum_{r=1}^{\mathrm{M}_J} \tilde{u}_{ij,\ r}^{(J)}].$$

4.5 Computational complexity

When implementing an existing substructuring method with *exact* subdomain solvers, one needs to compute harmonic extension $E_k \varphi_h \in V_h(\Omega_k)$ for some $\varphi_h \in W_h(\partial\Omega_k)$ $(k = 1, \cdots, \mathrm{N})$. Since each Ω_k has $O((H/h)^3)$ nodes, the optimal complexity for computing all such extensions is $O(\mathrm{N} \times (H/h)^3)$ (but, there seems no optimal harmonic extension in literature, except that the triangulation $\mathcal{T}_h$ possesses particular structure). For our substructuring method with *inexact* subdomain solvers, we need not to compute such harmonic extensions, but we have to compute all the *nearly* harmonic basis functions before implementing Algorithm 4.1 (or Algorithm 4.2). A natural question is: whether the complexity for computing all the *nearly* harmonic basis functions is much greater than the complexity

for computing the original harmonic extensions? If yes, the advantage of inexact subdomain solvers is not obvious. The following result gives an answer to this question.

Let $N(J)$ denote the total complexity for computing all the *nearly* harmonic basis functions.

Proposition 4.1. Let $m \geq 4$ be a given positive integer. If J and m_k are defined by

$$J \eqsim \log_{\sqrt{m}}(H/h) \quad \text{and} \quad m_1 = \cdots = m_J = m, \tag{4.11}$$

then

$$N(J) \lesssim \mathrm{N} \times (H/h)^3, \tag{4.12}$$

which is optimal. Moreover, the complexity for implementing Step 3-Step 5 in Algorithm 4.1 (or Step 3′-Step 5′ in Algorithm 4.2) equals $O((H/h)^2)$ for each Γ_{ij}, which is also the optimal.

Proof. We first derive a general estimate of $N(J)$. It is clear that the global coarse space $W_h^0(\Gamma)$ has the dimensions $O(\mathrm{N})$. Using (4.4), together with the property (ii) in Subsection 4.1, we know that the cost for computing $E_0\phi_h$ with all basis functions ϕ_h of $W_h^0(\Gamma)$ equals

$$N_0 = O(\mathrm{N} \times (H/h)^3). \tag{4.13}$$

Since E_{E} is chosen as the simple zero extension, the cost for computing $E_{\mathrm{E}}\phi_h$ with all basis functions ϕ_h of each $\tilde{W}_h(\mathrm{E})$ is

$$N_{\mathrm{E}} = O(H/h).$$

Note that the number of the coarse edges $\mathrm{E} \in \mathcal{E}_\Gamma$ equals to $O(\mathrm{N})$, we have

$$\sum_{\mathrm{E}\in\mathcal{E}_\Gamma} N_{\mathrm{E}} = O(\mathrm{N} \times H/h). \tag{4.14}$$

For a face $F = \Gamma_{ij}$, let n_{ij} denote the number of the nodes in Γ_{ij}. It is easy to see that the number of nodes in $F_r^{(k)}$ is about n_{ij}/M_k, and so the number of nodes in $\Omega_{ij,\,r}^{(k)}$ is about $(n_{ij}/\mathrm{M}_k)^{\frac{3}{2}}$ with $\mathrm{M}_0 = 1$ $(k = 0, \cdots, J)$. By the definition of $W_h^0(F_r^{(k)})$, the dimensions of each $W_h^0(F_r^{(k)})$ equal $O(m_{k+1})$ $(k = 0, \cdots, J-1)$. Moreover, the number of the local coarse subspaces $W_h^0(F_r^{(k)})$ equals M_k $(k = 0, \cdots, J-1)$. Besides, the dimensions of each $\tilde{W}_h(F_r^{(J)})$ equal n_{ij}/M_J. Since the cost for computing each local harmonic extension is optimal, the complexity for computing all the harmonic basis functions of $V_h^0(\Omega_{ij,\,r}^{(k)})$ and $\tilde{V}_h(\Omega_{ij,\,r}^{(J)})$ is about

$$\sum_{k=0}^{J-1} \mathrm{M}_k \cdot [(\frac{n_{ij}}{\mathrm{M}_k})^{\frac{3}{2}} \cdot m_{k+1}] + \mathrm{M}_J[(\frac{n_{ij}}{\mathrm{M}_J})^{\frac{3}{2}} \cdot \frac{n_{ij}}{\mathrm{M}_J}]. \tag{4.15}$$

This, together with (4.13) and (4.14), leads to

$$N(J) \lesssim \mathrm{N} \times (\frac{H}{h})^3 + \mathrm{N} \times \frac{H}{h} + \mathrm{N} \times \{\sum_{k=0}^{J-1} \mathrm{M}_k \cdot [(\frac{n_{ij}}{\mathrm{M}_k})^{\frac{3}{2}} \cdot m_{k+1}] + \mathrm{M}_J[(\frac{n_{ij}}{\mathrm{M}_J})^{\frac{3}{2}} \cdot \frac{n_{ij}}{\mathrm{M}_J}]\}. \tag{4.16}$$

Now, we verify the desired result by (4.16). By the choices of J and m_k in (4.11), we have

$$\mathrm{M}_k = m^k, \quad m_{k+1} = m \quad (k = 0, \cdots, J-1) \quad \text{and} \quad \mathrm{M}_J = m^J \eqsim (H/h)^2.$$

Then,

$$\mathrm{M}_k \cdot [(\frac{n_{ij}}{\mathrm{M}_k})^{\frac{3}{2}} \cdot m_{k+1}] = m \cdot n_{ij}^{\frac{3}{2}} \cdot (\frac{1}{\sqrt{m}})^k \quad \text{and} \quad \mathrm{M}_J[(\frac{n_{ij}}{\mathrm{M}_J})^{\frac{3}{2}} \cdot \frac{n_{ij}}{\mathrm{M}_J}] \eqsim n_{ij}^{\frac{5}{2}} \cdot (\frac{H}{h})^{-3}.$$

Substituting these into (4.16), and note that $n_{ij}^{\frac{1}{2}} = O(H/h)$, we deduce (4.12).

The number of subspaces $V_h^0(\Omega_{ij,\ r}^{(k)})$ and $\tilde{V}_h(\Omega_{ij,\ r}^{(J)})$ equals

$$\mathrm{M}_1 + \mathrm{M}_2 + \cdots \mathrm{M}_J \eqsim m^{J+1} = O((H/h)^2) \eqsim n_{ij}.$$

Note that each subspace has almost the fixed dimension m, we get the second conclusion.

□

4.6 Convergence

Now, we give an estimate of $cond(B_J^{-1}A_h)$.

Theorem 4.2 *Let J and m_k be defined as in Proposition 4.1, and let the extensions be defined by Subsection 4.1. For the preconditioner B_J defined in Subsection 4.3, we have*

$$cond(B_J^{-1}A_h) \leq CJ\log^3(H/h), \tag{4.17}$$

where C is a constant independent of h, H, d_k and the jumps of the coefficient ω across the faces Γ_{ij}.

A proof of the above theorem will be given in the next section.

5 Analysis

This section is devoted to a proof of Theorem 4.1. We first to prove several auxiliary results, which involve stabilities of the extensions defined in Subsection 4.1.

5.1 Auxiliary results

The following result can be verified directly by the discrete semi-norms described in Subsection 2.2.

Lemma 5.1 *The coarse edge extension E_{E} satisfies*

$$\sum_{k\in\mathcal{Q}_{\mathrm{E}}} \omega_k |E_{\mathrm{E}}\phi_{\mathrm{E}}|_{1,\Omega_k}^2 \lesssim \|\phi_{\mathrm{E}}\|_{*,\Gamma}^2, \quad \forall \phi_{\mathrm{E}} \in \tilde{W}_h(\mathrm{E}). \tag{5.1}$$

□

Lemma 5.2 *Let m_k and J be defined in Proposition 4.1. For $F = \Gamma_{ij}$, we have*

$$\omega_i |E_{ij,\ r}^{(J)}\varphi_h|_{1,\Omega_i}^2 + \omega_j |E_{ij,\ r}^{(J)}\varphi_h|_{1,\Omega_j}^2 \lesssim \|\phi_h\|_{*,\Gamma}^2, \quad \forall \varphi_h \in \tilde{W}_h(F_r^{(J)}). \tag{5.2}$$

Proof. Using the discrete H^1 semi-norm of finite element functions, together with the definition of the extension $E_{ij,\ r}^{(J)}$, yields

$$\begin{aligned} |E_{ij,\ r}^{(J)}\varphi_h|_{1,\Omega_i}^2 &\lesssim h[\sum_{p\in \mathrm{F}\cap\mathcal{N}_h} (\varphi_h(p) - \gamma_{h,\mathrm{F}}(\varphi_h))^2 + (\frac{d_J}{h})^2 |\gamma_{h,\mathrm{F}}(\varphi_h)|^2] \\ &\lesssim h^{-1}[\|\varphi_h - \gamma_{h,\mathrm{F}}(\varphi_h)\|_{0,\mathrm{F}}^2 + \|\varphi_h\|_{0,\mathrm{F}}^2]. \end{aligned} \tag{5.3}$$

Since φ_h vanishes on $\Gamma_{ij}\backslash\mathrm{F}$, we get by Friedrich's inequality

$$\|\varphi_h - \gamma_{h,\mathrm{F}}(\varphi_h)\|_{0,\mathrm{F}}^2,\ \|\varphi_h\|_{0,\mathrm{F}}^2 \lesssim d_J |\varphi_h|_{\frac{1}{2},\partial\Omega_i}^2. \tag{5.4}$$

It is easy to see that

$$\frac{d_J}{h} = (\frac{n_{ij}}{\mathrm{M}_J})^{\frac{1}{2}} = \frac{H}{h}/(m_1 \cdots m_J)^{\frac{1}{2}}.$$

Then, we deduce by the assumption $d_J/h = O(1)$. Plugging (5.4) in (5.3), leads to

$$|E^{(J)}_{ij,\ r}\varphi_h|^2_{1,\Omega_i} \lesssim |\varphi_h|^2_{\frac{1}{2},\partial\Omega_i}.$$

Similarly, we have

$$|E^{(J)}_{ij,\ r}\varphi_h|^2_{1,\Omega_j} \lesssim |\varphi_h|^2_{\frac{1}{2},\partial\Omega_j}.$$

Then, we deduce (5.2) by the definition of $\|\cdot\|_{*,\Gamma}$.

□

Lemma 5.3 *The global coarse extension E_0 satisfies*

$$\sum_{k=1}^{\mathrm{N}} \omega_k |E_0\phi_0|^2_{1,\Omega_k} \lesssim \log(H/h)\|\phi_0\|^2_{*,\Gamma}, \quad \forall \phi_0 \in W^0_h(\Gamma). \tag{5.5}$$

Proof. By the definition of E_0, we have

$$\begin{aligned} |E_0\phi_0|^2_{1,\Omega_k} &\lesssim |R^0_k[I^0_{\mathcal{W}_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0))]|^2_{1,\Omega_k} \\ &+ \sum_{\mathrm{F}\subset\partial\Omega_k} |\gamma_{h,\mathrm{F}}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0))|^2 \cdot |E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_k}. \end{aligned} \tag{5.6}$$

Using the discrete norm, together with (3.1) and Poincare inequality, yields

$$\begin{aligned} |R^0_k[I^0_{\mathcal{W}_k}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0))]|^2_{1,\Omega_k} &\lesssim \|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\mathcal{W}_k} \\ &\lesssim \log(H/h)|\phi_0|^2_{\frac{1}{2},\partial\Omega_k}. \end{aligned} \tag{5.7}$$

On the other hand, we have

$$|\gamma_{h,\mathrm{F}}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0))|^2 \lesssim H^{-3}\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\partial\Omega_k}$$

and (by (4.2))

$$|E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_k} \lesssim |\phi_{\mathrm{F}}|^2_{H^{\frac{1}{2}}_{00}(\mathrm{F})} \overset{=}{\sim} H\log(H/h).$$

Thus,

$$\begin{aligned} |\gamma_{h,\mathrm{F}}(\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0))|^2 \cdot |E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_k} &\lesssim H^{-2}\log(H/h)\|\phi_0 - \gamma_{h,\partial\Omega_k}(\phi_0)\|^2_{0,\partial\Omega_k} \\ &\lesssim \log(H/h)|\phi_0|^2_{\frac{1}{2},\partial\Omega_k}. \end{aligned}$$

Substituting (5.7) and the above inequality into (5.6), we get (5.5).

□

Lemma 5.4 *Let $F^{(k)}_r$ be associated with $F = \Gamma_{ij}$. The local coarse extensions $E^{(k)}_{ij,\ r,\ 0}$ satisfy*

$$\begin{aligned} &\omega_i|E^{(k)}_{ij,\ r,\ 0}\phi_h|^2_{1,\Omega_i} + \omega_j|E^{(k)}_{ij,\ r,\ 0}\phi_h|^2_{1,\Omega_j} \\ &\lesssim \|\phi_h\|^2_{*,\Gamma}, \quad \forall \phi_h \in W^0_h(F^{(k)}_r) \ \ (k = 1, \cdots, J-1). \end{aligned} \tag{5.8}$$

Proof. Without loss of generality, we assume that $F_r^{(k)}$ is a quadrilateral, which is just decomposed into four quadrilaterals in the standard way.

It follows by (4.5) that

$$|E_{ij,\ r,\ 0}^{(k)}\phi_h|_{1,\Omega_i}^2 \lesssim |R_l^0(I_{G_r^{(k)}}^0\phi_h)|_{1,\Omega_i}^2 + \sum_{\mathrm{F}\in\mathcal{F}_r^{(k)}} |a_{\mathrm{F}}(\phi_h)|^2 \cdot |E_{\mathrm{F}}\phi_{\mathrm{F}}|_{1,\Omega_i}^2. \tag{5.9}$$

We first estimate $|E_{\mathrm{F}}\phi_{\mathrm{F}}|_{1,\Omega_i}^2$ for each $\mathrm{F}\in\mathcal{F}_r^{(k)}$. Using (4.2) and the discrete norm given in Subsection 2.2, yields

$$|E_{\mathrm{F}}\phi_{\mathrm{F}}|_{1,\Omega_i}^2 \lesssim |\phi_{\mathrm{F}}|_{H_{00}^{\frac{1}{2}}(\mathrm{F})}^2 \eqsim |\phi_{\mathrm{F}}|_{h,\ H_{00}^{\frac{1}{2}}(\mathrm{F})}^2 \eqsim \sum_{p\in\mathrm{F}\cap\mathcal{N}_h} \frac{1}{dis(p,\partial\mathrm{F})}. \tag{5.10}$$

Set (see Figure 5)

$$L_{\mathrm{F}} = \partial\mathrm{F}\cap\partial F_r^{(k)},\quad D = \{p\in\mathrm{F}:\ dis(p,\partial\mathrm{F}) = dis(p,L_{\mathrm{F}})\}\quad\text{and}\quad D^{\partial} = \mathrm{F}\backslash D.$$

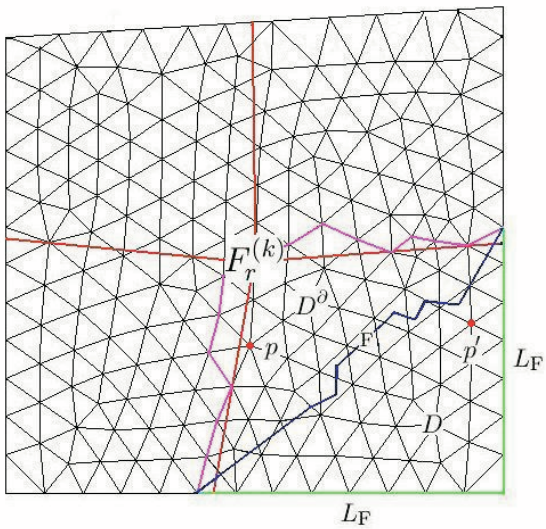

Figure 5: illustration to L_{F}, D and D^{∂}

Then,

$$\sum_{p\in\mathrm{F}\cap\mathcal{N}_h} \frac{1}{dis(p,\partial\mathrm{F})} = \sum_{p\in D\cap\mathcal{N}_h} \frac{1}{dis(p,L_{\mathrm{F}})} + \sum_{p\in D^{\partial}\cap\mathcal{N}_h} \frac{1}{dis(p,\partial\mathrm{F})}. \tag{5.11}$$

Since the triangulation is quasi-uniform, the domains D and D^{∂} have the same "size". For each $p\in D^{\partial}\cap\mathcal{N}_h$, there exist a $p'\in D\cap\mathcal{N}_h$ such that

$$dis(p,\partial\mathrm{F}) \gtrsim dis(p',\partial\mathrm{F}) = dis(p',L_{\mathrm{F}}).$$

Thus, we have by (5.11)

$$\sum_{p\in\mathrm{F}\cap\mathcal{N}_h} \frac{1}{dis(p,\partial\mathrm{F})} \lesssim \sum_{p\in D\cap\mathcal{N}_h} \frac{1}{dis(p,L_{\mathrm{F}})} \eqsim \sum_{p\in D\cap\mathcal{N}_h} \frac{1}{dis(p,\partial F_r^{(k)})}.$$

Plugging this in (5.10), leads to

$$|a_{\mathrm{F}}(\phi_h)|^2 \cdot |E_{\mathrm{F}}\phi_{\mathrm{F}}|_{1,\Omega_i}^2 \lesssim \sum_{p\in D\cap\mathcal{N}_h} \frac{|a_{\mathrm{F}}(\phi_h)|^2}{dis(p,\partial F_r^{(k)})} \le \sum_{p\in F_r^{(k)}\cap\mathcal{N}_h} \frac{\phi_h^2(p)}{dis(p,\partial F_r^{(k)})}.$$

We further get by the discrete norm

$$|a_{\mathrm{F}}(\phi_h)|^2 \cdot |E_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_i} \lesssim \int_{F_r^{(k)}} \frac{\phi_h^2(p)}{dis(p, \partial F_r^{(k)})} ds(p) \lesssim |\phi_h|^2_{H^{\frac{1}{2}}_{00}(F_r^{(k)})}. \tag{5.12}$$

Now, we estimate $|R_l^0(I^0_{G_r^{(k)}}\phi_h)|^2_{1,\Omega_i}$. It can be verified, by the discrete norm again, that

$$|R_l^0(I^0_{G_r^{(k)}}\phi_h)|^2_{1,\Omega_i} \lesssim |I^0_{G_r^{(k)}}\phi_h|^2_{H^{\frac{1}{2}}_{00}(F_r^{(k)})}.$$

This, together with the relation

$$I^0_{G_r^{(k)}}\phi_h = \phi_h - \sum_{\mathrm{F}\in\mathcal{F}_r^{(k)}} a_{\mathrm{F}}(\phi_h)\phi_{\mathrm{F}},$$

yields

$$|R_l^0(I^0_{G_r^{(k)}}\phi_h)|^2_{1,\Omega_i} \lesssim |\phi_h|^2_{H^{\frac{1}{2}}_{00}(F_r^{(k)})} + \sum_{\mathrm{F}\subset\mathcal{F}_r^{(k)}} |a_{\mathrm{F}}(\phi_h)|^2 \cdot |\phi_{\mathrm{F}}|^2_{H^{\frac{1}{2}}_{00}(F_r^{(k)})}.$$

Then, we obtain by the above proof

$$|R_l^0(I^0_{G_r^{(k)}}\phi_h)|^2_{1,\Omega_i} \lesssim |\phi_h|^2_{H^{\frac{1}{2}}_{00}(F_r^{(k)})}.$$

Plugging this and (5.12) in (5.9), leads to

$$|E^{(k)}_{ij,\ r,\ 0}\phi_h|^2_{1,\Omega_i} \lesssim |\phi_h|^2_{H^{\frac{1}{2}}_{00}(F_r^{(k)})}.$$

Similarly, we have

$$|E^{(k)}_{ij,\ r,\ 0}\phi_h|^2_{1,\Omega_j} \lesssim |\phi_h|^2_{H^{\frac{1}{2}}_{00}(F_r^{(k)})}.$$

Combining the two inequalities, gives (5.8).

□

5.2 Proof of Theorem 4.2

One needs to establish a suitable decomposition for $v_h \in V_h(\Omega)$

$$v_h = \sum_{k=1}^{\mathrm{N}} v_k + v_0 + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma} v_{\mathrm{E}} + \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l} v^{(l)}_{ij,\ r\ ,0} + \sum_{r=1}^{\mathrm{M}_J} \tilde{v}^{(J)}_{ij,\ r}], \tag{5.13}$$

with

$$v_k \in V_h^p(\Omega_k) \quad (k = 1, \cdots, \mathrm{N}), \quad v_0 \in V_h^0(\Omega), \quad v_{\mathrm{E}} \in V_h(\Omega_{\mathrm{E}}),$$

and

$$v^{(l)}_{ij,\ r,\ 0} \in V_h^0(\Omega^{(l)}_{ij,\ r}) \quad (l = 0, \cdots, J-1) \quad \text{and} \quad v^{(J)}_{ij,\ r} \in \tilde{V}_h(\Omega^{(J)}_{ij,\ r}.$$

This decomposition should satisfy the stability condition

$$\sum_{k=1}^{\mathrm{N}}(B_k v_k,\ v_k)_{\Omega_k} + (B_0 v_0,\ v_0) + \sum_{\mathrm{E}\in\mathcal{E}_\Gamma}(B_{\mathrm{E}} v_{\mathrm{E}},\ v_{\mathrm{E}})_{\Omega_{\mathrm{E}}}$$
$$+ \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}(B^{(l)}_{ij,\ r,\ 0} v^{(l)}_{ij,\ r,\ 0},\ v^{(l)}_{ij,\ r,\ 0}) + \sum_{r=1}^{\mathrm{M}_J}(B^{(J)}_{ij,\ r}\tilde{v}^{(J)}_{ij,\ r}, \tilde{v}^{(J)}_{ij,\ r})]$$

$$\lesssim J\log^3(H/h)(A_hv_h,v_h). \tag{5.14}$$

Set $\phi_h=v_h|_\Gamma$, and decompose ϕ_h into

$$\phi_h=\phi_0+\sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\phi_{\mathrm{E}}+\sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}\phi^{(l)}_{ij,\ r,\ 0}+\sum_{r=1}^{\mathrm{M}_J}\tilde{\phi}^{(J)}_{ij,\ r}]. \tag{5.15}$$

Hereafter, ϕ_0, ϕ_{E}, $\phi^{(l)}_{ij,\ r,\ 0}$ $(l=0,\cdots,J-1)$ and $\tilde{\phi}^{(J)}_{ij,\ r}$ are defined by Theorem 3.2. Define $v_0=E_0\phi_0$, $v_{\mathrm{E}}=E_{\mathrm{E}}\phi_{\mathrm{E}}$, $v^{(l)}_{ij,\ r,\ 0}=E^{(l)}_{ij,\ r,\ 0}\phi^{(l)}_{ij,\ r,\ 0}$ $(l=0,\cdots,J-1)$ and $\tilde{v}^{(J)}_{ij,\ r}=E^{(J)}_{ij,\ r}\tilde{\phi}^{(J)}_{ij,\ r}$. Moreover, we set for each Ω_k

$$v_k=\{v_h-v_0-\sum_{\mathrm{E}\in\mathcal{E}_\Gamma}v_{\mathrm{E}}-\sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}v^{(l)}_{ij,\ r\ ,0}+\sum_{r=1}^{\mathrm{M}_J}\tilde{v}^{(J)}_{ij,\ r}]\}|_{\Omega_k}. \tag{5.16}$$

Then, we have $v_k\in V_h^p(\Omega_k)$ by (5.15). It suffices to verify (5.14) for the functions defined above.

For convenience, set

$$\begin{aligned}\mathcal{G}(\phi_h) &= (B_0v_0,v_0)+\sum_{\mathrm{E}\in\mathcal{E}_\Gamma}(B_{\mathrm{E}}v_{\mathrm{E}},\ v_{\mathrm{E}})_{\Omega_{\mathrm{E}}}\\ &+ \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}(B^{(l)}_{ij,\ r,\ 0}v^{(l)}_{ij,\ r,\ 0},\ v^{(l)}_{ij,\ r,\ 0})+\sum_{r=1}^{\mathrm{M}_J}(B^{(J)}_{ij,\ r}\tilde{v}^{(J)}_{ij,\ r},\tilde{v}^{(J)}_{ij,\ r})].\end{aligned}$$

By (4.7)-(4.10) and Lemma 5.1-Lemma 5.4, we deduce

$$\begin{aligned}\mathcal{G}(\phi_h) &\lesssim \log(H/h)\|\phi_0\|^2_{*,\Gamma}+\sum_{\mathrm{E}\in\mathcal{E}_\Gamma}\|\phi_{\mathrm{E}}\|^2_{*,\Gamma}\\ &+ \sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}\|\phi^{(l)}_{ij,\ r,\ 0}\|^2_{*,\ \Gamma}+\sum_{r=1}^{\mathrm{M}_J}\|\tilde{\phi}^{(J)}_{ij,\ r}\|^2_{*,\ \Gamma}].\end{aligned} \tag{5.17}$$

Here, we have used the fact that $(F=\Gamma_{ij})$

$$|\phi_h|_{H^{\frac12}_{00}(F^{(l)}_r)}\overset{=}{\sim}|\phi_h|_{\frac12,\partial\Omega_i}\overset{=}{\sim}|\phi_h|_{\frac12,\partial\Omega_j},\quad \forall\phi_h\in H^{\frac12}_{00}(F^{(l)}_r)\ (l=0,\cdots,J;\ r=1,\cdots,\mathrm{M}_l).$$

Substituting (3.8) and (3.31) into (5.17), and note that $J\lesssim\log(H/h)$, yields

$$\mathcal{G}(\phi_h)\lesssim\log^3(H/h)\|\phi_h\|^2_{*,\Gamma}$$

Using the trace theorem, we further get

$$\mathcal{G}(\phi_h)\lesssim\log^3(H/h)(A_hv_h,v_h). \tag{5.18}$$

Besides, it follows by (5.16) that

$$\begin{aligned}|v_k|^2_{1,\Omega_k} &\lesssim |v_h|^2_{1,\Omega_k}+|v_0|^2_{1,\Omega_k}+\sum_{\mathrm{E}\in\mathcal{E}_\Gamma}|v_{\mathrm{E}}|^2_{1,\Omega_k}\\ &+ J\sum_{\Gamma_{ij}}[\sum_{l=0}^{J-1}\sum_{r=1}^{\mathrm{M}_l}|v^{(l)}_{ij,\ r,\ 0}|^2_{1,\ \Omega_k}+\sum_{r=1}^{\mathrm{M}_J}|\tilde{v}^{(J)}_{ij,\ r}|^2_{1,\ \Omega_k}].\end{aligned}$$

This, together with (4.6)-(4.10), leads to

$$\sum_{k=1}^{\mathrm{N}}(B_k v_k, v_k)_{\Omega_k} \lesssim (A_h v_h, v_h) + J\ \mathcal{G}(\phi_h).$$

Now, (5.14) is a direct consequence of the above inequality, together with (5.18).

On the other hand, one can derive the strength Cauchy-Schwarz inequality by using (4.6)-(4.10).

Finally, we deduce (4.17) by the convergence theory [28].

□

6 Analysis for the stability of the extension E_{F}

This section is devoted to verification of the stability condition (4.2), which has been used in Lemma 4.2-Lemma 4.3. Since general sub-face F and general quasi-uniform meshes are considered, the analysis is a bit technical. Our basic idea is to introduce a suitable "approximate" extension of E_{F}. This auxiliary extension is defined by positive integers, so that its stability can be verified more easily by estimating two finite sums associated with the discrete H^1 semi-norm.

6.1 An auxiliary extension

For a sub-face $\mathrm{F} \subset \Gamma_{ij}$, let o_{F}, d_{F}, p' and q be defined as in Subsection 4.1. Without loss of generality, we assume that $h \leq \min\{|pp'| :\ p \in \mathcal{N}_h \cap \Omega_{ij}\}$. For a positive number x, let $[x]$ denote the integer part of x. Set $n_{\mathrm{F}} = [\frac{d_{\mathrm{F}}}{h}]$. For any node p in $\Omega_{ij}\backslash\mathrm{F}$, define

$$m_p = [\frac{|pp'|}{h}] \qquad \text{and} \quad n_p = [\frac{d_{\mathrm{F}}}{|o_{\mathrm{F}}q|} \cdot \frac{|p'q|}{h}] \quad (\text{if } \ p' \neq o_{\mathrm{F}}).$$

For convenience, set $n_p = n_{\mathrm{F}}$ if $p' = o_{\mathrm{F}}$. To understand the meaning of the integers m_p and n_p more intuitively, we image that the segments pp' and $p'q$ are divided into some smaller segments with the size h and $h \cdot \frac{|o_{\mathrm{F}}q|}{d_{\mathrm{F}}}$ respectively. Then, m_p and n_p can be viewed roughly as the numbers of the division points on the segments pp' and $p'q$ respectively. In particular, the position of a node p can be determined by the two integers m_p and n_p roughly.

By the definition of the sets $\Lambda_{\mathrm{F}}^{(1)}$ and $\Lambda_{\mathrm{F}}^{(2)}$, it is easy to see that the positive integers m_p and n_p possess the properties:

Property A. For each node $p \in \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)}$, we have $1 \leq m_p \leq n_p \leq n_{\mathrm{F}}$;

Property B. For two integers r and k satisfying $1 \leq r \leq k \leq n_{\mathrm{F}}$, there are at most $O(n_{\mathrm{F}})$ nodes $p \in \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)}$, such that these nodes p define the same $m_p = r$ and $n_p = k$;

Property C. For an integer k satisfying $1 \leq k \leq n_{\mathrm{F}}$, there are at most $O(m_p n_{\mathrm{F}})$ nodes $p \in \Lambda_{\mathrm{F}}^{(2)}$, such that these nodes p define the same $n_p = k$.

For the *constant-like* basis function $\phi_{\mathrm{F}} = I_{\mathrm{F}}^0 1$, define the auxiliary extension

$$(E'_{\mathrm{F}}\phi_{\mathrm{F}})(p) = \begin{cases} \phi_{\mathrm{F}}(p), & \text{if } \ p \in \bar{\mathrm{F}}, \\ 1 - \frac{m_p}{n_{\mathrm{F}}}, & \text{if } p \in \Lambda_{\mathrm{F}}^{(1)}, \\ 1 - \frac{m_p}{n_p}, & \text{if } p \in \Lambda_{\mathrm{F}}^{(2)}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

For the verification of the stability (4.2), one needs to prove the following two inequalities

$$|(E_{\mathrm{F}} - E'_{\mathrm{F}})\phi_{\mathrm{F}}|^2_{1,\Omega_i} \lesssim d_{\mathrm{F}} \log(d_{\mathrm{F}}/h) \tag{6.2}$$

and

$$|E'_{\mathrm{F}}\phi_{\mathrm{F}}|^2_{1,\Omega_i} \lesssim d_{\mathrm{F}} \log(d_{\mathrm{F}}/h). \tag{6.3}$$

6.2 Auxiliary results

In this subsection, we estimate two finite sums, which will be used to verify the inequality (6.3).

In the rest of this section, p_1 and p_2 always denote two neighboring nodes. Since the triangulation $\mathcal{T}_h$ is quasi-uniform, there exists a (fixed) positive integer k_0, such that $|p_1p_2| \le k_0 h$ for any two neighboring nodes p_1 and p_2. Then, it can be verified, by the definitions of m_p and n_p, that

$$|m_{p_1} - m_{p_2}| \le k_0 + 1 \quad \text{and} \quad |n_{p_1} - n_{p_2}| \le r_0 + 1 \tag{6.4}$$

with

$$r_0 = [2k_0 \max_{q\in\partial\mathrm{F}} \frac{d_{\mathrm{F}}}{|o_{\mathrm{F}}q|}] + 1.$$

Lemma 6.1 *Let $\Lambda_{\mathrm{F}}^{(3)}$ be the set of nodes p, at which the extension $E_{\mathrm{F}}\phi_{\mathrm{F}}$ vanish. Namely,*

$$\Lambda_{\mathrm{F}}^{(3)} = \{p \in \mathcal{N}_h \cap \Omega_{ij} :\ p \notin \mathrm{F},\ p \notin \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)}\}.$$

Then,

$$h \sum_{p_1\in\Lambda_{\mathrm{F}}^{(3)}} \sum_{p_2\in\Lambda_{\mathrm{F}}^{(1)}\cup\Lambda_{\mathrm{F}}^{(2)}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 \lesssim d_{\mathrm{F}}. \tag{6.5}$$

Note that p_1 and p_2 above denote two neighboring nodes.

Proof. For a node $p_2 \in \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)}$, there are at most finite nodes $p_1 \in \Lambda_{\mathrm{F}}^{(3)}$, such that p_1 and p_2 are neighboring each other. Then,

$$h \sum_{p_1\in\Lambda_{\mathrm{F}}^{(3)}} \sum_{p_2\in\Lambda_{\mathrm{F}}^{(1)}\cup\Lambda_{\mathrm{F}}^{(2)}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 \lesssim h \sum_{p_2\in\Lambda_{\mathrm{F}}^{*}} (1 - \frac{m_{p_2}}{n_{p_2}})^2, \tag{6.6}$$

where

$$\Lambda_{\mathrm{F}}^{*} = \{p \in \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)} :\ \text{there is } p^* \in \Lambda_{\mathrm{F}}^{(3)} \text{ such that } p \text{ and } p^* \text{ are neighboring}\}.$$

For $p_2 \in \Lambda_{\mathrm{F}}^{*}$, let $p_1 \in \Lambda_{\mathrm{F}}^{(3)}$ be a neighboring node with p_2. Then, we deduce, by (6.4), that

$$m_{p_2} \ge m_{p_1} - (k_0 + 1) \ge n_{p_1} - (k_0 + 1) \ge n_{p_2} - (r_0 + k_0 + 2).$$

This implies that

$$\max\{1,\ n_{p_2} - k^*\} \le m_{p_2} \le n_{p_2}, \quad \forall p_2 \in \Lambda_{\mathrm{F}}^{*},$$

with $k^* = r_0 + k_0 + 2$. Thus, we have from **Property A** and **Property B**

$$\begin{aligned}\sum_{p_2\in\Lambda_{\mathrm{F}}^{*}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 &\lesssim n_{\mathrm{F}} \left\{ \sum_{n_{p_2}=1}^{k^*} \sum_{m_{p_2}=1}^{k^*} (1 - \frac{m_{p_2}}{n_{p_2}})^2 + \sum_{n_{p_2}=k^*+1}^{n_{\mathrm{F}}} \sum_{m_{p_2}=n_{p_2}-k^*}^{n_{p_2}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 \right\} \\ &\lesssim n_{\mathrm{F}} \left\{ (k^*)^2 + \sum_{n_{p_2}=k^*+1}^{n_{\mathrm{F}}} \sum_{m_{p_2}=n_{p_2}-k^*}^{n_{p_2}} (1 - \frac{m_{p_2}}{n_{p_2}})^2 \right\}. \end{aligned} \tag{6.7}$$

It is easy to see that

$$\sum_{m_{p_2}=n_{p_2}-k^*}^{n_{p_2}} (1-\frac{m_{p_2}}{n_{p_2}})^2 \le \sum_{m_{p_2}=n_{p_2}-k^*}^{n_{p_2}} (1-\frac{n_{p_2}-k^*}{n_{p_2}})^2 \le (k^*)^3 \cdot \frac{1}{n_{p_2}^2}.$$

Plugging this in (6.7), and note that k^* is a constant, leads to

$$h \sum_{p_2\in\Lambda_{\mathrm{F}}^*} (1-\frac{m_{p_2}}{n_{p_2}})^2 \lesssim d_{\mathrm{F}} \left((k^*)^2 + (k^*)^3 \sum_{n_{p_2}=k^*+1}^{n_{\mathrm{F}}} \frac{1}{n_{p_2}^2} \right) \lesssim d_{\mathrm{F}}.$$

Combining (6.6) with the above inequality, gives the desired result.
□

Lemma 6.2 *The following inequality holds for two neighboring nodes p_1 and p_2*

$$h \sum_{p_1,p_2\in\Lambda_{\mathrm{F}}^{(1)}\cup\Lambda_{\mathrm{F}}^{(2)}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2 \lesssim d_{\mathrm{F}} \log(d_{\mathrm{F}}/h). \tag{6.8}$$

Proof. For $p_1 \in \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)}$, define

$$\Lambda_{\mathrm{F},p_1} = \{p \in \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)} :\ p \text{ is neighboring with } p_1\}.$$

Let k_0 and r_0 be defined by (6.4). Then, we have from **Property A** and **Property B**

$$\begin{aligned}
h \sum_{p_1,p_2\in\Lambda_{\mathrm{F}}^{(1)}\cup\Lambda_{\mathrm{F}}^{(2)}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2 &\lesssim h \cdot \frac{d_{\mathrm{F}}}{h} \sum_{n_{p_1}=1}^{n_{\mathrm{F}}} \sum_{m_{p_1}=1}^{n_{p_1}} \sum_{p_2\in\Lambda_{\mathrm{F},p_1}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2 \\
&= d_{\mathrm{F}} \sum_{n_{p_1}=1}^{r_0+1} \sum_{m_{p_1}=1}^{n_{p_1}} \sum_{p_2\in\Lambda_{\mathrm{F},p_1}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2 \\
&+ d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \sum_{m_{p_1}=1}^{k_0} \sum_{p_2\in\Lambda_{\mathrm{F},p_1}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2 \\
&+ d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \sum_{m_{p_1}=k_0+1}^{n_{p_1}} \sum_{p_2\in\Lambda_{\mathrm{F},p_1}} (\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}})^2 \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{6.9}$$

It is clear that $I_1 \lesssim d_{\mathrm{F}}$. When $m_{p_1} \le k_0$, we deduce from (6.4) that $m_{p_2} \le 2k_0+1$ for $p_2 \in \Lambda_{\mathrm{F},p_1}$. Note that k_0 is a constant, we get

$$\begin{aligned}
I_2 &\lesssim d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \sum_{p_2\in\Lambda_{\mathrm{F},p_1}} (\frac{1}{n_{p_1}^2} + \frac{1}{n_{p_2}^2}) \\
&\lesssim d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \left(\frac{1}{n_{p_1}^2} + \frac{1}{(n_{p_1}-r_0-1)^2} \right) \lesssim d_{\mathrm{F}}.
\end{aligned} \tag{6.10}$$

It follows by (6.4) that

$$I_3 \lesssim d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \sum_{m_{p_1}=k_0+1}^{n_{p_1}} \sum_{k=-r_0-1}^{r_0+1} \sum_{r=-k_0-1}^{k_0+1} \left(\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_1}+r}{n_{p_1}+k} \right)^2$$

$$
\begin{aligned}
&= d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \sum_{m_{p_1}=k_0+1}^{n_{p_1}} \sum_{k=-r_0-1}^{r_0+1} \sum_{r=-k_0-1}^{k_0+1} \left(\frac{k m_{p_1} - r n_{p_1}}{n_{p_1}(n_{p_1}+k)} \right)^2 \\
&\lesssim d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \sum_{m_{p_1}=k_0+1}^{n_{p_1}} \frac{m_{p_1}^2 + n_{p_1}^2}{n_{p_1}^2 (n_{p_1} - r_0 - 1)^2} \\
&\lesssim d_{\mathrm{F}} \sum_{n_{p_1}=r_0+2}^{n_{\mathrm{F}}} \frac{n_{p_1}}{(n_{p_1} - r_0 - 1)^2} \lesssim d_{\mathrm{F}} \log n_{\mathrm{F}}.
\end{aligned}
$$

Plugging (6.10) and the above inequality in (6.9), leads to

$$
h \sum_{p_1, p_2 \in \Lambda_{\mathrm{F}}^{(1)} \cup \Lambda_{\mathrm{F}}^{(2)}} \left(\frac{m_{p_1}}{n_{p_1}} - \frac{m_{p_2}}{n_{p_2}} \right)^2 \lesssim d_{\mathrm{F}} \log(d_{\mathrm{F}}/h).
$$

□

6.3 Proof of Theorem 4.1

It suffices to verify that

$$
|E_{\mathrm{F}} \phi_{\mathrm{F}}|_{1,\Omega_i}^2, \quad |E_{\mathrm{F}} \phi_{\mathrm{F}}|_{1,\Omega_j}^2 \lesssim d_{\mathrm{F}} \log(d_{\mathrm{F}}/h) \eqsim |\phi_{\mathrm{F}}|^2_{H_{00}^{\frac{1}{2}}(\mathrm{F})}. \tag{6.11}
$$

Let $E'_{\mathrm{F}} \phi_{\mathrm{F}}$ be the auxiliary extension defined by Subsection 6.1.

Step 1. Verify the inequality (6.2)

Let $p \in \Lambda_{\mathrm{F}}^{(1)}$. Then, we have

$$
\begin{aligned}
|(E_{\mathrm{F}} \phi_{\mathrm{F}} - E'_{\mathrm{F}} \phi_{\mathrm{F}})(p)| &= \left| \frac{|pp'|}{d_{\mathrm{F}}} - \left[\frac{|pp'|}{h}\right] / \left[\frac{d_{\mathrm{F}}}{h}\right] \right| \\
&\leq \max\left\{ \frac{|pp'| + h}{d_{\mathrm{F}}} - \frac{|pp'|}{d_{\mathrm{F}}}, \ \frac{|pp'|}{d_{\mathrm{F}}} - \frac{|pp'|}{d_{\mathrm{F}} + h} \right\} \\
&\lesssim h/d_{\mathrm{F}}.
\end{aligned} \tag{6.12}
$$

For the case with $p \in \Lambda_{\mathrm{F}}^{(2)}$, one can verify similarly that

$$
|(E_{\mathrm{F}} \phi_{\mathrm{F}} - E'_{\mathrm{F}} \phi_{\mathrm{F}})(p)| \lesssim \frac{h |o_{\mathrm{F}} q|}{d_{\mathrm{F}} |p' q|} \lesssim \frac{1}{n_p}. \tag{6.13}
$$

By the inverse estimate and the discrete L^2-norm, we get

$$
\begin{aligned}
|(E_{\mathrm{F}} - E'_{\mathrm{F}}) \phi_{\mathrm{F}}|_{1,\Omega_i}^2 &\lesssim h^{-2} \|(E_{\mathrm{F}} - E'_{\mathrm{F}}) \phi_{\mathrm{F}}|_{0,\Omega_i}^2 \\
&\eqsim h \sum_{p \in \Lambda_{\mathrm{F}}^{(1)}} (E_{\mathrm{F}} \phi_{\mathrm{F}} - E'_{\mathrm{F}} \phi_{\mathrm{F}})^2(p) \\
&+ h \sum_{p \in \Lambda_{\mathrm{F}}^{(2)}} (E_{\mathrm{F}} \phi_{\mathrm{F}} - E'_{\mathrm{F}} \phi_{\mathrm{F}})^2(p).
\end{aligned}
$$

This, together with (6.12) and (6.13), yields

$$
|(E_{\mathrm{F}} - E'_{\mathrm{F}}) \phi_{\mathrm{F}}|_{1,\Omega_i}^2 \lesssim h \sum_{p \in \Lambda_{\mathrm{F}}^{(1)}} \frac{h^2}{d_{\mathrm{F}}^2} + h \sum_{p \in \Lambda_{\mathrm{F}}^{(2)}} \frac{1}{n_p^2} \lesssim \frac{h^2}{d_{\mathrm{F}}} + h \sum_{p \in \Lambda_{\mathrm{F}}^{(2)}} \frac{1}{n_p^2}. \tag{6.14}
$$

Here, we have used the fact that the set $\Lambda_{\mathrm{F}}^{(1)}$ contains $O(n_{\mathrm{F}})$ nodes only.

By **Property A** and **Property C**, we get

$$\sum_{p\in\Lambda_{\rm F}^{(2)}}\frac{1}{n_p^2}\lesssim n_{\rm F}\sum_{n_p=1}^{n_{\rm F}}\frac{m_p}{n_p^2}\le n_{\rm F}\sum_{n_p=1}^{n_{\rm F}}\frac{n_p}{n_p^2}\lesssim n_{\rm F}\log n_{\rm F}.$$

Plugging this in (6.14), gives (6.2).

Step 2. Verify the inequality (6.3)

For simplicity of exposition, let $\Lambda_{\rm F}$ denote the set of neighboring node paring (p_1,p_2), which satisfies $(E'_{\rm F}\phi_{\rm F})(p_1)-(E'_{\rm F}\phi_{\rm F})(p_2)\neq 0$. By the discrete H^1 semi-norm, we have

$$|E'_{\rm F}\phi_{\rm F}|^2_{1,\Omega_i}\eqsim h\sum_{(p_1,\ p_2)\in\Lambda_{\rm F}}|(E'_{\rm F}\phi_{\rm F})(p_1)-(E'_{\rm F}\phi_{\rm F})(p_2)|^2. \tag{6.15}$$

For ease of notation, define

$$\Lambda_{{\rm F},\ b}=\{p\in\mathcal{N}_h\cap{\rm F}:\ p\ \text{ closes }\ \partial{\rm F}\}.$$

It is easy to see that the set $\Lambda_{\rm F}$ can be decomposed into several groups: (a) $p_1\in\Lambda_{{\rm F},b}$ and $p_2\in\partial{\rm F}$; (b) $p_1\in{\rm F}$ and $p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}$; (c) $p_1\in\Lambda_{\rm F}^{(3)}$ and $p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}$; (d) $p_1,p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}$. It is certain that one can also consider the inverse situation with exchanging the positions of p_1 and p_2, but this will not affect the result.

It follows by (6.15) that

$$\begin{aligned}|E'_{\rm F}\phi_{\rm F}|^2_{1,\Omega_i}\ &\eqsim\ h\sum_{p_1\in\Lambda_{{\rm F},b}}\sum_{p_2\in\partial{\rm F}}(1-0)^2+h\sum_{p_1\in{\rm F}}\sum_{p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}}(\frac{m_{p_2}}{n_{p_2}})^2\\ &+\ h\sum_{p_1\in\Lambda_{\rm F}^{(3)}}\sum_{p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}}(1-\frac{m_{p_2}}{n_{p_2}})^2+h\sum_{p_1,p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}}(\frac{m_{p_1}}{n_{p_1}}-\frac{m_{p_2}}{n_{p_2}})^2.\end{aligned} \tag{6.16}$$

It is clear that the set $\Lambda_{{\rm F},b}$ contains only $O(n_{\rm F})$ nodes. Then, we get for two neighboring nodes p_1 and p_2

$$h\sum_{p_1\in\Lambda_{{\rm F},b}}\sum_{p_2\in\partial{\rm F}}(1-0)^2\lesssim h\cdot\frac{d_{\rm F}}{h}=d_{\rm F}. \tag{6.17}$$

When $p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}$ is neighboring with some $p_1\in{\rm F}$, we have

$$m_{p_2}\le|p_2p_2'|/h\le|p_1p_2|/h\le k_0.$$

Besides, for any $p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}$, there are at most finite nodes $p_1\in{\rm F}$, such that p_1 and p_2 are neighboring each other. Thus, we deduce by **Property A** and **Property B**

$$h\sum_{p_1\in{\rm F}}\sum_{p_2\in\Lambda_{\rm F}^{(1)}\cup\Lambda_{\rm F}^{(2)}}(\frac{m_{p_2}}{n_{p_2}})^2\lesssim h\cdot\frac{d_{\rm F}}{h}\sum_{l=1}^{k_0}\sum_{n_{p_2}=1}^{n_{\rm F}}(\frac{l}{n_{p_2}})^2\lesssim d_{\rm F}. \tag{6.18}$$

Substituting (6.17)-(6.18), (6.5) and (6.8) into (6.16), yields (6.3).

Step 3. Prove the desired result (6.11)

Combining (6.2) and (6.3), yields

$$|E_{\rm F}\phi_{\rm F}|^2_{1,\Omega_i}\lesssim d_{\rm F}\log(d_{\rm F}/h).$$

In the same way, we can prove that

$$|E_{\rm F}\phi_{\rm F}|^2_{1,\Omega_j}\lesssim d_{\rm F}\log(d_{\rm F}/h).$$

On the other hand, it can be verified, by the discrete semi-norm in Subsection 2.2, that

$$|\phi_{\rm F}|^2_{H^{\frac{1}{2}}_{00}({\rm F})}\eqsim d_{\rm F}\log(d_{\rm F}/h).$$

□

7 Numerical experiments

In this section, we give some numerical results to confirm our theoretical results described in section 4.

Consider the elliptic problem (2.1) with Ω being the cube $\Omega = [0,\ 1]^3$, and the coefficient $a(x, y, z)$ being defined by

$$\begin{aligned} a(x,y,z) &= 10^{-5}, \quad \text{if } \ x, y \leq 0.5 \ \text{ or } \ x, y \geq 0.5; \\ a(x,y,z) &= 1, \qquad\qquad\qquad\qquad \text{otherwise.} \end{aligned}$$

The source function f is chosen in a suitable manner.

Let Ω be decomposed into N cube subdomains with the edge length H. To illustrate wide practicality of the new method, we consider tetrahedron elements instead of hexahedron elements. Let each subdomain be divided into tetrahedron elements with the size h in the standard way, and use the usual P_1 finite element approximate space.

We solve the algebraic system associated with the equation (2.4) by PCG iteration with the preconditioner B_J^{-1} defined in Section 4. Here, each local solver B_k is chosen as the symmetric multigrid preconditioner for the restriction of A_h on $V_h^p(\Omega_k)$, and decompose each $\Gamma_{ij,r}^{(k)}$ $(k \leq J-1)$ into four squares with the same size (i.e., $m_k = 4$). The iteration terminates when the relative remainder is less than $1.0D-5$. The iteration counts are listed as the following table.

Table 7.1
iteration counts(Tol.=$1.0D-5$)

H/h	$H = 1/4$	$H = 1/6$	$H = 1/8$
16 $(J = 2)$	36	38	38
32 $(J = 3)$	45	47	46

These numerical results indicates that the convergence of the new preconditioner is stable with the subdomain number N, and depends slightly on the ratio H/h. This is just predicted by Theorem 4.1.

8 Conclusions

We have developed a substructuring method with *inexact* subdomain solvers by building a new multilevel space decomposition of $V_h(\Omega)$. The new substructuring method not only possesses nearly optimal convergence but also has almost optimal computational complexity. For the new method, traditional nested grids are unnecessary. Our idea can be extended to some other equations, for example, Maxwell's equations.

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