Several Pairs of Differential Operators and Their Applications in Variational Calculus of a General Third Order Energy *

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Abstract

Gradient, divergence and Laplace-Beltrami operators over surfaces have played important roles in many fields related to geometry and analysis, such as differential geometry, computational geometry and geometry modeling etc. In this paper, several pairs of new differential operators on 2-manifolds are introduced and some of their properties are given. We establish several elegant relationships among the proposed operators and apply them to the deriving of the complete-variation of a general third order energy of surfaces. A vector-valued Euler-Lagrange equation as well as its weak form formulation is obtained.

Key words: Differential operator; Euler-Lagrange equation; Complete-variation; Geometric energy functional; Curvature.

1 Introduction

Gradient operator, divergence operator and Laplace-Beltrami operator(LBO) over surfaces play important roles in many fields related to geometry and analysis, such as differential geometry, Riemannian manifolds, nonlinear analysis, computational geometry and partial differential equations (PDEs), etc. These differential operators are also widely used in several other fields, such as physics, chemistry, biology, mechanism, shape modeling, image processing and so on. For example, one important problem in these fields is to characterize and trace the interface motion, such as grain growth and phase transition([16, 17]), melting and combustion, solidification([9]), biomembrane-vesicle ([18]), surface modeling ([10, 23]), image restoration and denoising ([19]) any and all. Such an interface motion problem is

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often described by a geometric partial differential equation in terms of various geometric differential operators mentioned above.

New Differential Operators. During our study of the construction of higher order geometric partial differential equations, some new differential operators are required to be introduced to solve more complicated variation problems. These new operators allow us to formulate the derived geometric partial differential equations in a neat and concise way. After these operators are introduced, their elementary properties are important for further research. For example, these differential operators are geometry essential, that is, they do not depend on the specific parametrization. Green formulae for these operators also exist. In this paper, all these properties are established.

Energy Functional Minimization. The interface motion problem aforementioned can be translated into such a problem that is to minimize a kind of given energy on the interface with prescribed conditions. Usually this minimizing problem is highly nonlinear. One approach to solve this problem is using the optimization technique. A lot of studies have been carried out along this line (e.g., [8, 15]). Another approach is the gradient descent technique, that is, to find a road, which is called flow, to reach the minimized energy. Both of these two approaches have been extensively employed. It is difficult to say which one is superior to the other. For optimization technique, it is well-known that to pursue a global optimized result is not an easy task. For gradient descent technique, the first task is to calculate the Euler-Lagrange equations by variation calculus, and then often the deduced parabolic type differential equations must be solved. To obtain analytical solutions to these highly nonlinear equations is difficult and becoming a daydream although much unfailing endeavor has been on-going. Therefore, seeking numerical solutions becomes a feasible alternative with the help of the modern computers and advanced computation methods, such as finite element method, finite difference method and finite volume method etc. Even though, to gain a global stationary solution is still challenging.

Higher Order Flows. As is well-known, mean curvature flow ([2, 16]) and surface diffusion flow ([14]) are the L^2 and H^{-1} gradient descent flows of area functional, respectively. Willmore flow ([24]) minimizes the squared mean curvature functional. But second-order flows, such as mean curvature flow and averaged mean curvature flow ([11]), can merely achieve G^0 continuity on the boundary. The fourth-order flows, for instance, the surface diffusion flow and Willmore flow, can produce G^1 continuous surfaces. However, in many application areas, such as the the design of streamlined surfaces of aircraft, ships and cars ([28]), higher order continuity is prerequisite. In this paper, we minimize a higher order geometric energy, that is the general curvature energy functional proposed in [13]

$$\mathscr{F}_1(\mathcal{M}) = \int_{\mathcal{M}} \|\nabla f(H,K)\|^2 dA, \tag{1.1}$$

where f is assumed to be a C^4 function depending on two variables and the other notations are referred to Section 2. A special case of (1.1), the mean-curvature-variation energy functional

$$\mathscr{F}_2(\mathcal{M}) = \int_{\mathcal{M}} \|\nabla H\|^2 dA, \tag{1.2}$$

has been studied in [27], where the normal-variation for functional (1.2) is carried out and corresponding curvature flow is termed as minimal mean-curvature-variation flow. The applications in surface modeling and designing show that this flow leads to very desirable results. [13] also takes the normal-variation into account for functional (1.1) and some applications in surface modeling were carried out. The numerical solutions to these geometric PDEs are based on a biquadratic fitting approach proposed in [25]. For the convenience of using finite element method to solve the PDEs, we consider the complete-variation for these functionals in this paper. Complete-variation has also been used in [26] for a lower order geometric energy functional.

Main Contributions. The contributions of this paper is summarized as follows. Several new differential operators are introduced and their elementary properties are given. Some theoretical results on these operators are developed. Complete-variation for the third order geometric energy functional (1.1) is considered. A vector-valued Euler-Lagrange equation as well as its weak form formulation is obtained.

The rest of this paper is organized as follows. In Section 2, some used notations and preliminaries are introduced, including several differential operators and their properties. The vector-valued Euler-Lagrange equation from complete-variation with respect to functional (1.1) is derived in Section 3. For the numerical solving of the obtained PDEs by the finite element method, a weak form formulation for the Euler-Lagrange equation is presented as well.

2 Notations and Preliminaries

In this section, we introduce some notations and several differential operators defined on surfaces used throughout this paper. Their elementary properties and Green's formulae are also provided.

Let $\mathcal{M}$ be a regular parametric surface represented by $\mathbf{x}(u,v) \in \mathbb{R}^3, \quad (u,v) \in \Omega \subset \mathbb{R}^2$, whose unit normal vector is $\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$, where the subscript of $\mathbf{x}$ denotes the partial derivative and $\|\mathbf{x}\| := \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}} := (\mathbf{x}^T\mathbf{x})^{\frac{1}{2}}$ is the usual Euclidean norm. To simplify notation we sometimes write $w = (u,v)$ and $u^1 = u, u^2 = v$. Superscript T stands for the transpose operation. Throughout the paper, we use $\langle \cdot, \cdot \rangle$ to denote Euclidean inner product, $[\ \ ]$ to denote matrix. All the column vectors are written in bold faced characters. Sometimes to emphasize matrices, bold faced characters are employed too. We assume at least $\mathbf{x} \in C^6(\overline{\Omega}, \mathbb{R}^3)$. The coefficients of the first fundamental form I, the second fundamental form II, and the third fundamental form III are

$$g_{\alpha\beta} = \langle \mathbf{x}_{u^\alpha}, \mathbf{x}_{u^\beta} \rangle, \quad b_{\alpha\beta} = \langle \mathbf{n}, \mathbf{x}_{u^\alpha u^\beta} \rangle = -\langle \mathbf{n}_{u^\alpha}, \mathbf{x}_{u^\beta} \rangle, \quad l_{\alpha\beta} = \langle \mathbf{n}_{u^\alpha}, \mathbf{n}_{u^\beta} \rangle, \quad \alpha, \beta = 1, 2.$$

For later use, we introduce $g_{\alpha\beta\delta} = \langle \mathbf{x}_{u^\alpha}, \mathbf{x}_{u^\beta u^\delta} \rangle, \alpha, \beta, \delta = 1, 2$, and

$$[\, g^{\alpha\beta} \,] = [\, g_{\alpha\beta} \,]^{-1}, \quad g = \det[\, g_{\alpha\beta} \,], \quad b = \det[\, b_{\alpha\beta} \,].$$

To define the mean curvature and Gaussian curvature, let us first introduce the concept of *shape operator* or *Weingarten map*. The shape operator of surface $\mathcal{M}$ is a self-adjoint linear map on the tangent space $T_{\mathbf{x}}\mathcal{M} := \text{span}\{\mathbf{x}_u, \mathbf{x}_v\}$ defined by

$$\mathcal{W} : T_{\mathbf{x}}\mathcal{M} \to T_{\mathbf{x}}\mathcal{M},$$

such that

$$\mathcal{W}(\mathbf{x}_u) = -\mathbf{n}_u, \quad \mathcal{W}(\mathbf{x}_v) = -\mathbf{n}_v.$$

We can represent this linear map by a matrix $S = [\, b_{\alpha\beta}\,][\, g^{\alpha\beta}\,]$. In particular,

$$[\, \mathbf{n}_u,\, \mathbf{n}_v\,] = -[\, \mathbf{x}_u,\, \mathbf{x}_v\,]S^T \tag{2.1}$$

is valid. The two eigenvalues k_1, k_2 of S are *principal curvatures* of $\mathcal{M}$ and their corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2$ are *principal directions.* The average and product of k_1 and k_2 are *mean curvature* and *Gaussian curvature*, respectively. That is

$$H = \frac{k_1 + k_2}{2} = \frac{\text{tr}(S)}{2}, \qquad K = k_1 k_2 = \det(S).$$

Let $\mathbf{H} = H\mathbf{n}$ be the *mean curvature normal* and $\mathbf{K} = K\mathbf{n}$ be the *Gaussian curvature normal.*

It follows from (2.1) that $[\, l_{\alpha\beta}\,] = [\, b_{\alpha\beta}\,][\, g^{\alpha\beta}\,][\, b_{\alpha\beta}\,]$. Hence the coefficients of the third fundamental form could be represented by the coefficients of the first and second fundamental forms.

2.1 Differential Operators Definitions

In this subsection, some used differential operators defined on surface $\mathcal{M}$ will be introduced.

Tangent gradient operator. Let $f \in C^1(\mathcal{M})$. Then the *tangent gradient operator* ∇ acting on f is given by (see [5], page 102)

$$\nabla f = [\, \mathbf{x}_u,\, \mathbf{x}_v\,]\,[\, g^{\alpha\beta}\,]\,(f_u,\, f_v)^T \in \mathbb{R}^3. \tag{2.2}$$

From (2.2), we can derive that

$$\begin{aligned} \langle \nabla h, \nabla f\rangle &= (h_u,\, h_v)\,[\, g^{\alpha\beta}\,]\,(f_u,\, f_v)^T \\ &= \frac{1}{g}\big((g_{22}h_u - g_{12}h_v)f_u + (g_{11}h_v - g_{12}h_u)f_v\big), \quad f,\ h \in C^1(\mathcal{M}). \end{aligned} \tag{2.3}$$

Second tangent operator. Let $f \in C^1(\mathcal{M})$. Then the *second tangent operator* $\Diamond$ acting on f is given by

$$\Diamond f = [\, \mathbf{x}_u,\, \mathbf{x}_v\,]\,[\, h^{\alpha\beta}\,]\,(f_u,\, f_v)^T \in \mathbb{R}^3, \tag{2.4}$$

where

$$[\, h^{\alpha\beta}\,] := \frac{1}{g}\begin{bmatrix} b_{22} & -b_{12} \\ -b_{12} & b_{11} \end{bmatrix}.$$

From (2.2) and (2.4), we can derive that

$$\begin{aligned}\langle \nabla h, \Diamond f\rangle &= (h_u,\, h_v)\,[\,h^{\alpha\beta}\,]\,(f_u,\, f_v)^T \\ &= \frac{1}{g}\big((b_{22}h_u - b_{12}h_v)f_u + (b_{11}h_v - b_{12}h_u)f_v\big) \\ &= \langle \Diamond h, \nabla f\rangle, \quad f,\; h \in C^1(\mathcal{M}).\end{aligned} \tag{2.5}$$

This operator always involves the second order derivatives of surfaces considered.

Third tangent operator. Let $f \in C^1(\mathcal{M})$. Then the *third tangent operator* $\oslash$ acting on f is given by

$$\oslash f = [\,\mathbf{x}_u,\, \mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\,S\,(f_u,\, f_v)^T \in \mathbb{R}^3,$$

which can also be written as

$$\oslash f = -[\,\mathbf{n}_u,\, \mathbf{n}_v\,]\,[\,g^{\alpha\beta}\,]\,(f_u,\, f_v)^T = -\frac{1}{g}(g_{22}f_u\mathbf{n}_u + g_{11}f_v\mathbf{n}_v - g_{12}f_u\mathbf{n}_v - g_{12}f_v\mathbf{n}_u),$$

where the first equality is valid owing to (2.1) and the equality

$$S^T[\,g^{\alpha\beta}\,] = [\,g^{\alpha\beta}\,]\,S.$$

Analogously, this operator needs the second order derivatives information of surfaces.

Definition 2.1 (Geometry intrinsic) *If a quality on surface is determined by the first fundamental form of the surface, then the quality is called* geometry intrinsic*.*

Definition 2.2 (Geometry essential) *If a quality on surface is independent of specific parametrization of surface, we call this quality* geometry essential*.*

Although the operators introduced above are defined by the local parametrization of surfaces, they do not depend on the specific choice of parameters. So they are geometry essential. In more detail, we have the following lemma.

Lemma 2.1 *Let $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \mathbf{x}(\bar{u},\bar{v})$ be two different parametric representations of surface $\mathcal{M}$. Provided that the determinant of $J := \begin{bmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{u}} \\ \frac{\partial u}{\partial \bar{v}} & \frac{\partial v}{\partial \bar{v}} \end{bmatrix}$ is positive, then for all $f \in C^1(\mathcal{M})$,*

$$\nabla f = \bar{\nabla} f, \tag{2.6}$$

$$\Diamond f = \bar{\Diamond} f, \tag{2.7}$$

$$\oslash f = \bar{\oslash} f, \tag{2.8}$$

where $\nabla,\ \Diamond,\ \oslash$ and $\bar{\nabla},\ \bar{\Diamond},\ \bar{\oslash}$ are two groups of operators on $\mathcal{M}$ under distinct parameters choices.

Proof: We only prove equation (2.7). Similar proofs can be performed for (2.6) and (2.8). Assume that the transform of parameters is

$$\sigma : (\bar{u}, \bar{v}) \in \bar{\Omega} \to (u, v) \in \Omega.$$

Then the transformation of basis of the tangent space is

$$[\,\mathbf{x}_{\bar{u}},\, \mathbf{x}_{\bar{v}}\,]^T = J\,[\,\mathbf{x}_u,\, \mathbf{x}_v\,]^T.$$

For $\det(J) > 0$, $\mathbf{n} = \bar{\mathbf{n}}$ holds. Therefore the transformations between the coefficients of first and second fundamental forms are

$$[\,\bar{g}_{\alpha\beta}\,] = J\,[\,g_{\alpha\beta}\,]\,J^T, \quad [\,\bar{b}_{\alpha\beta}\,] = J\,[\,b_{\alpha\beta}\,]\,J^T.$$

Hence

$$\begin{aligned}\bar{\Diamond} f &= [\,\mathbf{x}_{\bar{u}},\, \mathbf{x}_{\bar{v}}\,]\,[\,\bar{h}^{\alpha\beta}\,]\,(f_{\bar{u}},\, f_{\bar{v}})^T \\ &= [\,\mathbf{x}_u,\, \mathbf{x}_v\,]\,J^T J^{-T}[\,h^{\alpha\beta}\,]J^{-1}J\,(f_u,\, f_v)^T = \Diamond f.\end{aligned}$$

Tangential divergence operator. Let $\mathbf{v}$ be a C^1 smooth vector field on $\mathcal{M}$. Then the *tangential divergence* of $\mathbf{v}$ is defined by

$$\mathrm{div}(\mathbf{v}) = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\,\sqrt{g}\,[\,g^{\alpha\beta}\,]\,[\,\mathbf{x}_u,\, \mathbf{x}_v\,]^T\mathbf{v}\,\right].$$

Noticing that if $\mathbf{v}$ is the normal vector field of $\mathcal{M}$, $\mathrm{div}(\mathbf{v}) = 0$, therefore, we refer to this operator as tangential divergence operator(see Remark 2.3).

Laplace-Beltrami operator(LBO). Let $f \in C^2(\mathcal{M})$. Then ∇f is a smooth vector field on $\mathcal{M}$. The Laplace-Beltrami Δ applying to f is defined by

$$\Delta f = \mathrm{div}(\nabla f).$$

From the definitions of ∇ and div, it is easy to derive that

$$\begin{aligned}\Delta f &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\,\sqrt{g}\,[\,g^{\alpha\beta}\,]\,(f_u,\, f_v)^T\,\right] && (2.9)\\ &= g_u^{\triangle} f_u + g_v^{\triangle} f_v + g_{uu}^{\triangle} f_{uu} + g_{uv}^{\triangle} f_{uv} + g_{vv}^{\triangle} f_{vv}, \\ &= \frac{1}{g}(g_{22}f_{11} + g_{11}f_{22} - 2g_{12}f_{12}) \\ &= [\,g^{\alpha\beta}\,]{:}[\,f_{\alpha\beta}\,], && (2.10)\end{aligned}$$

where

$$\begin{aligned}g_u^{\triangle} &= -\big(g_{11}(g_{22}g_{122} - g_{12}g_{222}) + 2g_{12}(g_{12}g_{212} - g_{22}g_{112}) + g_{22}(g_{22}g_{111} - g_{12}g_{211})\big)/g^2, \\ g_v^{\triangle} &= -\big(g_{11}(g_{11}g_{222} - g_{12}g_{122}) + 2g_{12}(g_{12}g_{112} - g_{11}g_{212}) + g_{22}(g_{11}g_{211} - g_{12}g_{111})\big)/g^2, \\ g_{uu}^{\triangle} &= g_{22}/g, \qquad g_{uv}^{\triangle} = -2g_{12}/g, \qquad g_{vv}^{\triangle} = g_{11}/g,\end{aligned}$$

and

$$f_{\alpha\beta} = f_{u^\alpha u^\beta} - (\nabla f)^T \mathbf{x}_{u^\alpha u^\beta}, \quad \alpha, \beta = 1, 2,$$

are the second covariant derivatives, and the notation $A{:}B$ stands for the trace of A^TB. One can easily see that Δ is a second order differential operator with respect to surfaces considered and functions domained on surfaces.

Giaquinta-Hildebrandt operator(GHO). Let $f \in C^2(\mathcal{M})$. Then $\Diamond f$ is a smooth vector field on $\mathcal{M}$. The Giaquinta-Hildebrandt operator $\Box$ applying to f is newly defined by

$$\Box f = \mathrm{div}(\Diamond f).$$

From the definitions of $\Diamond$ and div, it is easy to derive that

$$\begin{aligned}
\Box f &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\sqrt{g}\,[\,h^{\alpha\beta}\,]\,(f_u,\, f_v)^T\,\right] && (2.11)\\
&= g_u^{\Box} f_u + g_v^{\Box} f_v + g_{uu}^{\Box} f_{uu} + g_{uv}^{\Box} f_{uv} + g_{vv}^{\Box} f_{vv}\\
&= \frac{1}{g}(b_{22}f_{11} + b_{11}f_{22} - 2b_{12}f_{12}) && (2.12)\\
&= [\,h^{\alpha\beta}\,]{:}[\,f_{\alpha\beta}\,], && (2.13)
\end{aligned}$$

where

$$\begin{aligned}
g_u^{\Box} &= -\big(b_{11}(g_{22}g_{122} - g_{12}g_{222}) + 2b_{12}(g_{12}g_{212} - g_{22}g_{112}) + b_{22}(g_{22}g_{111} - g_{12}g_{211})\big)/g^2,\\
g_v^{\Box} &= -\big(b_{11}(g_{11}g_{222} - g_{12}g_{122}) + 2b_{12}(g_{12}g_{112} - g_{11}g_{212}) + b_{22}(g_{11}g_{211} - g_{12}g_{111})\big)/g^2,\\
g_{uu}^{\Box} &= b_{22}/g, \qquad g_{uv}^{\Box} = -2b_{12}/g, \qquad g_{vv}^{\Box} = b_{11}/g.
\end{aligned}$$

To the best of the authors' knowledge, differential operator $\Box$ is introduced by Giaquinta and Hildebrandt (see [7], p. 84), we therefore call it Giaquinta-Hildebrandt operator (GHO). Since b_{ij} involves the second order derivatives of the surfaces considered, the first equality of equation (2.11) implies that $\Box$ involves the third order derivatives at first glance, but the second equality tells us that it does not depend on the third order derivatives of surfaces. Therefore this operator is of second order with respect to surfaces and functions domained on them.

$\boxtimes$ **operator**. Let $f \in C^2(\mathcal{M})$. Then $\oslash f$ is a smooth vector field on $\mathcal{M}$. The $\boxtimes$ operator applying to f is defined by

$$\boxtimes f = \mathrm{div}(\oslash f).$$

Form the definitions of $\oslash$ and div, we can derive that

$$\begin{aligned}
\boxtimes f &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\sqrt{g}\,[\,g^{\alpha\beta}\,]\,S\,(f_u,\, f_v)^T\,\right]\\
&= \big[\,[\,g^{\alpha\beta}\,]\,S\,\big]{:}[\,f_{\alpha\beta}\,] + 2\langle\nabla f, \nabla H\rangle. && (2.14)
\end{aligned}$$

Obviously, this operator is of second order with respect to f. But it is of third order with respect to the surface. Since $\oslash f = 2H\nabla f - \Diamond f$ (see Lemma 2.6), we have

$$\boxtimes f = 2H\Delta f - \Box f + 2\langle\nabla f, \nabla H\rangle.$$

Then (2.14) is obtained using the relations (2.33), (2.10) and (2.13).

Remark 2.1 We have presented three pairs of differential operators. That is

1. Pair one (∇, Δ): tangential gradient operator and LB operator.
2. Pair two ($\Diamond$, $\Box$): second tangent operator and GH opertor.
3. Pair three ($\oslash$, $\boxtimes$): third tangent operator and $\boxtimes$ operator.

The last two pairs of these differential operators are quite new. We have noticed that in [21] these operators are also introduced to solving variation problems in biomembrane. There they used symbols $\nabla, \bar{\nabla}, \tilde{\nabla}$ to represent $\nabla, \oslash, \Diamond$, respectively.

Remark 2.2 Since tangential divergence operator is geometric essential, we can prove that these three pairs of operators are all geometry essential, as Lemma 2.1 has proved. But only the first pair of operators ∇ and Δ is geometry intrinsic. Both the second and the third pairs of differential operators are not geometry intrinsic.

Remark 2.3 There is another definition of the divergence operator in literatures (e. g. [1, 9]), that is, Div is defined by

$$\mathrm{Div}(\mathbf{v}) = \mathrm{tr}[\nabla \mathbf{v}].$$

Our tangential divergence operator div is not completely identical with Div. Specifically, if $\mathbf{v}$ is a tangential vector field to $\mathcal{M}$, $\mathrm{div}(\mathbf{v})$ coincides with $\mathrm{Div}(\mathbf{v})$. But if $\mathbf{v}$ is not on the tangential direction, e. g., the normal vector field $\mathbf{n}$, $\mathrm{div}(\mathbf{n}) = 0$ but $\mathrm{Div}(\mathbf{n}) = -2H$, which can be found in subsection 2.4.

2.2 Properties of the Proposed Differential Operators

In the following, we will list or prove some basic formulae for these differential operators.

Lemma 2.2 ([3], pp. 139–142) *For any functions $f, h \in C^2(\mathcal{M})$, the following equalities hold*

$$\begin{aligned}
\nabla(fh) &= h\nabla f + f\nabla h, \\
\Delta(af + bh) &= a\Delta f + b\Delta h, \quad (\forall\, a, b \in \mathbb{R}), \\
\mathrm{div}(f\,\nabla h) &= f\Delta h + \langle \nabla f, \nabla h \rangle, \\
\Delta(fh) &= f\Delta h + 2\langle \nabla f, \nabla h \rangle + h\Delta f.
\end{aligned}$$

For the operator pair ($\Diamond$, $\Box$), we similarly have the following newly established

Lemma 2.3 *For any functions $f, h \in C^2(\mathcal{M})$, the following equalities hold*

$$\Diamond(fh) = h\Diamond f + f\Diamond h, \tag{2.15}$$

$$\Box(af + bh) = a\,\Box f + b\,\Box h, \quad (\forall\, a, b \in \mathbb{R}) \tag{2.16}$$

$$\begin{aligned}
\mathrm{div}(f\,\Diamond h) &= f\,\Box h + \langle \nabla f, \Diamond h \rangle \\
&= f\,\Box h + \langle \nabla h, \Diamond f \rangle,
\end{aligned} \tag{2.17}$$

$$\begin{aligned}
\Box(fh) &= f\,\Box h + \langle \nabla f, \Diamond h \rangle + \langle \Diamond f, \nabla h \rangle + h\,\Box f \\
&= f\,\Box h + 2\langle \nabla f, \Diamond h \rangle + h\,\Box f \\
&= f\,\Box h + 2\langle \nabla h, \Diamond f \rangle + h\,\Box f.
\end{aligned} \tag{2.18}$$

Proof: (2.15) and (2.16) are evident by the definitions of $\Diamond$ and $\Box$. Now we prove (2.17) as

$$\begin{aligned} \operatorname{div}(f\,\Diamond h) &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u},\,\frac{\partial}{\partial v}\right]\left[f\frac{\begin{bmatrix} b_{22} & -b_{12} \\ -b_{12} & b_{11}\end{bmatrix}\begin{pmatrix} h_u \\ h_v\end{pmatrix}}{\sqrt{g}}\right] \\ &= f\,\Box h + \frac{1}{g}(f_u,\,f_v)\begin{bmatrix} b_{22} & -b_{12} \\ -b_{12} & b_{11}\end{bmatrix}\begin{pmatrix} h_u \\ h_v\end{pmatrix} \\ &= f\,\Box h + \langle\nabla f, \Diamond h\rangle = f\,\Box h + \langle \nabla h, \Diamond f\rangle. \end{aligned}$$

Taking (2.15) and (2.17) into account, we can prove (2.18) without any difficulty.

For a vector-valued function $\mathbf{f} = (f_1, \cdots, f_k)^T \in C^2(\mathcal{M})^k$, we define

$$\begin{aligned} \nabla\mathbf{f} &= [\,\nabla f_1, \cdots, \nabla f_k\,] && \in \mathbb{R}^{3\times k}, \\ \Delta\mathbf{f} &= (\Delta f_1, \cdots, \Delta f_k)^T && \in \mathbb{R}^{k}, \\ \Diamond\mathbf{f} &= [\,\Diamond f_1, \cdots, \Diamond f_k\,] && \in \mathbb{R}^{3\times k}, \\ \Box\mathbf{f} &= (\Box f_1, \cdots, \Box f_k)^T && \in \mathbb{R}^{k}, \\ \oslash\mathbf{f} &= [\,\oslash f_1, \cdots, \oslash f_k\,] && \in \mathbb{R}^{3\times k}, \\ \boxtimes\mathbf{f} &= (\boxtimes f_1, \cdots, \boxtimes f_k)^T && \in \mathbb{R}^{k}. \end{aligned}$$

Hence, it is easy to see that

$$\begin{aligned} \nabla\mathbf{x} &= [\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]^T, && (2.19) \\ \nabla\mathbf{n} &= -[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\,S\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]^T, && (2.20) \end{aligned}$$

and both $\nabla\mathbf{x}$ and $\nabla\mathbf{n}$ are symmetric 3×3 matrices.

For the compatibility between operators, we prescribe the divergence operator acting on a matrix-valued function $\mathbf{Q} = [\,\mathbf{q}_1, \cdots, \mathbf{q}_k\,] \in C^1(\mathcal{M})^{3\times k}$ as

$$\operatorname{div}[\mathbf{Q}] = (\operatorname{div}(\mathbf{q}_1), \cdots, \operatorname{div}(\mathbf{q}_k))^T \in \mathbb{R}^k.$$

Hence

$$\begin{aligned} \Delta\mathbf{f} &= \operatorname{div}[\,\nabla\mathbf{f}\,], && (2.21) \\ \Box\mathbf{f} &= \operatorname{div}[\,\Diamond\mathbf{f}\,], && (2.22) \end{aligned}$$

still hold.

Lemma 2.4 *For any scalar function $f \in C^2(\mathcal{M})$ and vector-valued functions $\mathbf{f}, \mathbf{h} \in C^2(\mathcal{M})^3$, we have*

$$\begin{aligned} \nabla(\langle\mathbf{f},\mathbf{h}\rangle) &= \nabla\mathbf{f}\,\mathbf{h} + \nabla\mathbf{h}\,\mathbf{f}, && (2.23) \\ \operatorname{div}(\nabla\mathbf{f}\;\mathbf{h}) &= \langle\Delta\mathbf{f},\mathbf{h}\rangle + \nabla\mathbf{f}{:}\nabla\mathbf{h}, && (2.24) \\ \Delta(\langle\mathbf{f},\mathbf{h}\rangle) &= \langle\mathbf{f},\Delta\mathbf{h}\rangle + 2\nabla\mathbf{f}{:}\nabla\mathbf{h} + \langle\mathbf{h},\Delta\mathbf{f}\rangle, && (2.25) \\ \operatorname{div}[f\nabla\mathbf{h}] &= f\Delta\mathbf{h} + [\,\nabla\mathbf{h}\,]^T\nabla f, && (2.26) \\ \nabla(f\mathbf{h}) &= f\nabla\mathbf{h} + \nabla f\mathbf{h}^T, && (2.27) \\ \Delta(f\mathbf{h}) &= \Delta f\mathbf{h} + 2[\,\nabla\mathbf{h}\,]^T\nabla f + f\Delta\mathbf{h} && (2.28) \\ \operatorname{div}[\mathbf{f}\,\mathbf{h}^T] &= [\,\nabla\mathbf{h}\,]^T\,\mathbf{f} + \operatorname{div}(\mathbf{f})\,\mathbf{h}. && (2.29) \end{aligned}$$

Proof: Let $\mathbf{f} = (f_1,\, f_2,\, f_3)^T$ and $\mathbf{h} = (h_1,\, h_2,\, h_3)^T$. (2.23) is obvious. For (2.24), we have

$$\begin{aligned} & \operatorname{div}\big([\nabla f_1, \nabla f_2, \nabla f_3](h_1,\, h_2,\, h_3)^T\big) \\ = \;& \operatorname{div}(\nabla f_1\, h_1 + \nabla f_2\, h_2 + \nabla f_3\, h_3) \\ = \;& h_1 \Delta f_1 + \langle \nabla f_1, \nabla h_1\rangle + h_2 \Delta f_2 + \langle \nabla f_2, \nabla h_2\rangle + h_3 \Delta f_3 + \langle \nabla f_3, \nabla h_3\rangle \\ = \;& \langle \Delta \mathbf{f}, \mathbf{h}\rangle + \nabla \mathbf{f} : \nabla \mathbf{h}. \end{aligned}$$

From (2.23) and (2.24), (2.25) is easily deduced. Let us prove (2.26) as

$$\begin{aligned} \operatorname{div}[f\nabla\mathbf{h}] &= \operatorname{div}[\, f\nabla h_1, f\nabla h_2, f\nabla h_3\,] \\ &= \begin{pmatrix} \operatorname{div}(f\nabla h_1) \\ \operatorname{div}(f\nabla h_2) \\ \operatorname{div}(f\nabla h_3) \end{pmatrix} = \begin{pmatrix} f\Delta h_1 + \langle \nabla h_1, \nabla f\rangle \\ f\Delta h_2 + \langle \nabla h_2, \nabla f\rangle \\ f\Delta h_3 + \langle \nabla h_3, \nabla f\rangle \end{pmatrix} \\ &= f\Delta\mathbf{h} + [\,\nabla\mathbf{h}\,]^T \nabla f. \end{aligned}$$

For (2.27), we have

$$\begin{aligned} \nabla(f\mathbf{h}) &= [\,\nabla(fh_1),\; \nabla(fh_2),\; \nabla(fh_3)\,] \\ &= [\,\nabla f\, h_1 + \nabla h_1\, f,\; \nabla f\, h_2 + \nabla h_2\, f,\; \nabla f\, h_3 + \nabla h_3\, f\,] \\ &= f\nabla\mathbf{h} + \nabla f\, \mathbf{h}^T. \end{aligned}$$

From (2.26) and (2.27), (2.28) follows. To prove (2.29), we have

$$\begin{aligned} \operatorname{div}[\mathbf{f}\,\mathbf{h}^T] &= (\operatorname{div}(h_1\mathbf{f}),\, \operatorname{div}(h_2\mathbf{f}),\, \operatorname{div}(h_3\mathbf{f}))^T \\ &= (h_1\operatorname{div}(\mathbf{f}) + \langle \nabla h_1, \mathbf{f}\rangle,\, h_2\operatorname{div}(\mathbf{f}) + \langle \nabla h_2, \mathbf{f}\rangle,\;\, h_3\operatorname{div}(\mathbf{f}) + \langle \nabla h_3, \mathbf{f}\rangle)^T \\ &= [\,\nabla\mathbf{h}\,]^T\mathbf{f} + \operatorname{div}(\mathbf{f})\mathbf{h}. \end{aligned}$$

Similar to Lemma 2.4, we can verify the following newly established

Lemma 2.5 *For any scalar function $f \in C^2(\mathcal{M})$ and vector-valued functions $\mathbf{f}, \mathbf{h} \in C^2(\mathcal{M})^3$, we have*

$$\begin{aligned} \Diamond(\langle \mathbf{f}, \mathbf{h}\rangle) &= \Diamond\mathbf{f}\,\mathbf{h} + \Diamond\mathbf{h}\,\mathbf{f}, \\ \operatorname{div}(\Diamond\mathbf{f}\;\mathbf{h}) &= \langle \Box\mathbf{f}, \mathbf{h}\rangle + \nabla\mathbf{f}:\Diamond\mathbf{h} \qquad (2.30) \\ &= \langle \Box\mathbf{f}, \mathbf{h}\rangle + \nabla\mathbf{h}:\Diamond\mathbf{f}, \\ \Box(\langle \mathbf{f}, \mathbf{h}\rangle) &= \langle \mathbf{f}, \Box\mathbf{h}\rangle + 2\nabla\mathbf{f}:\Diamond\mathbf{h} + \langle \mathbf{h}, \Box\mathbf{f}\rangle \\ &= \langle \mathbf{f}, \Box\mathbf{h}\rangle + 2\nabla\mathbf{h}:\Diamond\mathbf{f} + \langle \mathbf{h}, \Box\mathbf{f}\rangle, \\ \operatorname{div}[f\Diamond\mathbf{h}] &= f\,\Box\mathbf{h} + [\,\Diamond\mathbf{h}\,]^T\nabla f \\ &= f\,\Box\mathbf{h} + [\,\nabla\mathbf{h}\,]^T\Diamond f, \\ \Diamond(f\mathbf{h}) &= f\Diamond\mathbf{h} + \Diamond f\mathbf{h}^T, \\ \Box(f\mathbf{h}) &= \Box f\,\mathbf{h} + 2[\,\Diamond\mathbf{h}\,]^T\nabla f + f\,\Box\mathbf{h} \qquad (2.31) \\ &= \Box f\,\mathbf{h} + 2[\,\nabla\mathbf{h}\,]^T\Diamond f + f\,\Box\mathbf{h}. \end{aligned}$$

Remark 2.4 Because the ranges of the first three tangent operators are in the tangent space, there should be some relations between them. That means, they may be not independent to each other. Interestingly enough, there is a nice relationship between them characterized by the following

Lemma 2.6 *For any function $f \in C^1(\mathcal{M})$, the following equality*

$$2H\nabla f - \oslash f - \Diamond f = \mathbf{0} \tag{2.32}$$

is valid, where H is the mean curvature of surface $\mathcal{M}$.

Proof: Proving equality (2.32) is equivalent to proving

$$2H[\, g^{\alpha\beta}\,] - [\, g^{\alpha\beta}\,]S - [\, h^{\alpha\beta}\,] = \mathbf{0}, \tag{2.33}$$

which is valid by a straightforward calculation.

From this Lemma, we can see that three pairs of differential operators can be reduced to two pairs. That is, the third pair differential operators can be represented by the other two pairs. Thus we did not write out the analogous lemmas for $\oslash$ and $\boxtimes$ as in Lemma 2.3 and Lemma 2.5. In particular, we own

Lemma 2.7 *For surface $\mathcal{M}$, we have*

$$\nabla\mathbf{n} + \oslash\mathbf{x} = \mathbf{0}, \tag{2.34}$$

$$\Diamond\mathbf{n} + K\nabla\mathbf{x} = \mathbf{0}, \tag{2.35}$$

$$2H\nabla\mathbf{x} + \nabla\mathbf{n} - \Diamond\mathbf{x} = \mathbf{0}, \tag{2.36}$$

$$2H\nabla\mathbf{n} - \oslash\mathbf{n} - \Diamond\mathbf{n} = \mathbf{0}. \tag{2.37}$$

Proof: (2.34) can be proved by the definitions of operators ∇ and $\oslash$. We can prove (2.35) by the definitions of $\Diamond$ and ∇ as well as (2.1). (2.36) is the outcome of (2.34) and Lemma 2.6. If we replace f in Lemma 2.6 with $\mathbf{n}$, equality (2.37) is obvious.

Remark 2.5 Equation (2.33) is of fundamental importance because we can draw the well-known relationship between the first, second and third fundamental forms, that is

$$\mathrm{III} - 2H\mathrm{II} + K\mathrm{I} = 0. \tag{2.38}$$

Proof: Multiplying equality (2.33) with matrix $[\, b_{\alpha\beta}\,]$ and $[g_{\alpha\beta}\,]$ from the left and the right, respectively, we have

$$2H[\, b_{\alpha\beta}\,] - K[\, g_{\alpha\beta}\,] - [\, b_{\alpha\beta}\,]\,[\, g^{\alpha\beta}\,]\,[\, b_{\alpha\beta}\,] = \mathbf{0}, \tag{2.39}$$

where the entities of $[\, b_{\alpha\beta}\,]\,[\, g^{\alpha\beta}\,]\,[\, b_{\alpha\beta}\,]$ are the coefficients of the third fundamental form. Multiplying $[\, du,\, dv\,]$ from the left and $[\, du,\, dv\,]^T$ from right sides of (2.39), we obtain the famous (2.38). From this proof, we can regard equality (2.32) as a dual of relationship (2.38) in some sense. We believe, this approach of proof is really new.

In what follows, we shall prove several important formulae with the help of these operators, which will be widely utilized in Section 3.

Theorem 2.1 *Let $\mathcal{M} := \{\mathbf{x}(u, v) \in \mathbb{R}^3; (u, v) \in \Omega \subset \mathbb{R}^2\}$ be a regular parametric surface with normal vector field $\mathbf{n}$, and operators $\Delta, \Box, \mathrm{div}, \nabla, \Diamond$ defined as above. Then we have*

$$\langle \Delta\mathbf{x}, \mathbf{n}\rangle = 2H, \tag{2.40}$$

$$\langle \Box\mathbf{x}, \mathbf{n}\rangle = 2K, \tag{2.41}$$

$$\langle \Delta\mathbf{n}, \mathbf{n}\rangle = -(4H^2 - 2K), \tag{2.42}$$

$$\langle \Box\mathbf{n}, \mathbf{n}\rangle = -2HK. \tag{2.43}$$

Proof: (2.40) is well-known, but we would like to reprove it quickly with the help of the operators introduced above.

$$\begin{aligned}
\langle \Delta \mathbf{x}, \mathbf{n}\rangle &\overset{(2.24)}{=} \operatorname{div}(\nabla \mathbf{x}\,\mathbf{n}) - \nabla \mathbf{x}{:}\nabla \mathbf{n} = -\operatorname{tr}\big[[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\,[-S\,]\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]^T\big] = \operatorname{tr}[\,S\,] = 2H,\\
\langle \Box \mathbf{x}, \mathbf{n}\rangle &\overset{(2.30)}{=} -\nabla \mathbf{x}{:}\Diamond \mathbf{n} = \operatorname{tr}\big[\,[\,h^{\alpha\beta}\,]S[\,g_{\alpha\beta}\,]\big] = 2K,\\
\langle \Delta \mathbf{n}, \mathbf{n}\rangle &\overset{(2.24)}{=} -\nabla \mathbf{n}{:}\nabla \mathbf{n} = -\operatorname{tr}[\,SS\,] = -(k_1^2 + k_2^2) = -(4H^2 - 2K),\\
\langle \Box \mathbf{n}, \mathbf{n}\rangle &\overset{(2.30)}{=} -\nabla \mathbf{n}{:}\Diamond \mathbf{n} = -K\operatorname{tr}[\,S\,] = -2HK.
\end{aligned}$$

Hence (2.40)–(2.43) are true. Here we point out that (2.42) is the negative of total curvature ([20], p. 1011, [12]) and the deduction can also be found in ([4], p. 216).

From this theorem, we can derive a stronger conclusion.

Theorem 2.2 *Given $\mathcal{M}, \mathbf{n}$ and operators as in Theorem* 2.1*, we have*

$$\begin{aligned}
\Delta \mathbf{x} &= 2\mathbf{H}, && (2.44)\\
\Box \mathbf{x} &= 2\mathbf{K}, && (2.45)\\
\Delta \mathbf{n} &= \Box \mathbf{x} - 2\nabla H - 2H\Delta \mathbf{x} = 2(\mathbf{K} - 2H\mathbf{H} - \nabla H), && (2.46)\\
\Box \mathbf{n} &= -\nabla K - K\Delta \mathbf{x} = -\nabla K - H\Box \mathbf{x} = -\nabla K - 2K\mathbf{H}. && (2.47)
\end{aligned}$$

Proof: To prove (2.44), we need to verify that

$$\begin{aligned}
\langle \Delta \mathbf{x}, \mathbf{x}_u\rangle &= 0, && (2.48)\\
\langle \Delta \mathbf{x}, \mathbf{x}_v\rangle &= 0. && (2.49)
\end{aligned}$$

Frankly speaking, this result is not trivial at all if we do not know (2.44) as a prerequisite. From (2.24), we have

$$\operatorname{div}(\mathbf{x}_u) = \operatorname{div}(\nabla \mathbf{x}\,\mathbf{x}_u) = \langle \Delta \mathbf{x}, \mathbf{x}_u\rangle + \nabla \mathbf{x}{:}\nabla \mathbf{x}_u.$$

We can calculate that

$$\begin{aligned}
\operatorname{div}(\mathbf{x}_u) &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\frac{1}{\sqrt{g}}\begin{bmatrix} g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{bmatrix}\begin{pmatrix} g_{11}\\ g_{12}\end{pmatrix}\right]\\
&= \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g})}{\partial u} = \frac{g_u}{2g}, && (2.50)\\
\nabla \mathbf{x}{:}\nabla \mathbf{x}_u &= \operatorname{tr}\big[[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\,[\,\mathbf{x}_{uu},\,\mathbf{x}_{uv}\,]^T\big]\\
&= \operatorname{tr}\left[[\,g^{\alpha\beta}\,]\begin{bmatrix}\langle \mathbf{x}_{uu}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uu}, \mathbf{x}_v\rangle\\ \langle \mathbf{x}_{uv}, \mathbf{x}_u\rangle & \langle \mathbf{x}_{uv}, \mathbf{x}_v\rangle\end{bmatrix}\right]\\
&= \operatorname{tr}\left[[\,g^{\alpha\beta}\,]\begin{bmatrix}\Gamma_{11}^1 g_{11} + \Gamma_{11}^2 g_{12} & \Gamma_{11}^1 g_{12} + \Gamma_{11}^2 g_{22}\\ \Gamma_{12}^1 g_{11} + \Gamma_{12}^2 g_{12} & \Gamma_{12}^1 g_{12} + \Gamma_{12}^2 g_{22}\end{bmatrix}\right]\\
&= \Gamma_{11}^1 + \Gamma_{12}^2 = \frac{g_u}{2g}, && (2.51)
\end{aligned}$$

where

$$\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2}g^{\gamma\xi}\Big\{\frac{\partial g_{\alpha\xi}}{\partial u^{\beta}} + \frac{\partial g_{\beta\xi}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\xi}}\Big\}, \quad (\alpha, \beta, \gamma, \xi = 1, 2)$$

are Christoffel symbols. Thus (2.48) is proved and similar proof for (2.49) give rise to (2.44) with the help of (2.40).

Similarly, proving (2.45) is equivalent to proving

$$\begin{aligned} \langle \Box \mathbf{x}, \mathbf{x}_u \rangle &= 0, && (2.52) \\ \langle \Box \mathbf{x}, \mathbf{x}_v \rangle &= 0. && (2.53) \end{aligned}$$

Using (2.30), (2.52) is the same as

$$\operatorname{div}(\Diamond \mathbf{x}\, \mathbf{x}_u) = \nabla \mathbf{x} : \Diamond \mathbf{x}_u. \tag{2.54}$$

We can derive that

$$\begin{aligned} & \operatorname{div}(\Diamond \mathbf{x}\, \mathbf{x}_u) \\ = \; & \operatorname{div}\big([\, \mathbf{x}_u,\, \mathbf{x}_v \,] \, [h^{\alpha\beta}] \, [\, \mathbf{x}_u,\, \mathbf{x}_v \,]^T \mathbf{x}_u \big) \\ = \; & \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] \left[\frac{1}{\sqrt{g}} \begin{bmatrix} b_{22} & -b_{12} \\ -b_{12} & b_{11} \end{bmatrix} \begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix} \right] \\ = \; & \frac{1}{g} \left(\frac{\partial}{\partial u} (g_{11} b_{22} - g_{12} b_{12}) + \frac{\partial}{\partial v} (g_{12} b_{11} - g_{11} b_{12}) \right) \\ & - \frac{1}{g} (\, g_{11} b_{22} - g_{12} b_{12},\, g_{12} b_{11} - g_{11} b_{12} \,) \begin{pmatrix} \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial u} \\ \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial v} \end{pmatrix} \\ \overset{(a)}{=} \; & \frac{1}{g} (g_{11_u} b_{22} + g_{11} b_{22u} - g_{12_u} b_{12} - g_{12} b_{12u} - g_{11_v} b_{12} - g_{11} b_{12_v} + g_{12} b_{11_v} + b_{11} g_{12_v}) \\ & - \frac{1}{g} (\, g_{11} b_{22} - g_{12} b_{12},\, g_{12} b_{11} - g_{11} b_{12} \,) \begin{pmatrix} \Gamma_{11}^1 + \Gamma_{12}^2 \\ \Gamma_{12}^2 + \Gamma_{22}^2 \end{pmatrix} \\ \overset{(b)}{=} \; & \frac{1}{g} \Big((g_{11} b_{22} - g_{12} b_{12}) \Gamma_{11}^1 + (g_{12} b_{22} - g_{22} b_{12}) \Gamma_{11}^2 \\ & + (g_{12} b_{11} - g_{11} b_{12}) \Gamma_{12}^1 + (g_{22} b_{11} - g_{12} b_{12}) \Gamma_{22}^2 \Big), \end{aligned}$$

where (a) is valid because of (2.50) and (2.51). (b) is valid because of the relationship between Christoffel symbols with the first fundamental form ([5], p. 232)

$$\begin{cases} \Gamma_{11}^1 g_{11} + \Gamma_{11}^2 g_{12} = \langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \frac{1}{2} g_{11_u}, \\ \Gamma_{11}^1 g_{12} + \Gamma_{11}^2 g_{22} = \langle \mathbf{x}_{uu}, \mathbf{x}_v \rangle = g_{12_u} - \frac{1}{2} g_{11_v}, \end{cases}$$
$$\begin{cases} \Gamma_{12}^1 g_{11} + \Gamma_{12}^2 g_{12} = \langle \mathbf{x}_{uv}, \mathbf{x}_u \rangle = \frac{1}{2} g_{11_v}, \\ \Gamma_{12}^1 g_{12} + \Gamma_{12}^2 g_{22} = \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle = \frac{1}{2} g_{22_u}, \end{cases}$$
$$\begin{cases} \Gamma_{22}^1 g_{11} + \Gamma_{22}^2 g_{12} = \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle = g_{12_v} - \frac{1}{2} g_{22_u}, \\ \Gamma_{22}^1 g_{12} + \Gamma_{22}^2 g_{22} = \langle \mathbf{x}_{vv}, \mathbf{x}_v \rangle = \frac{1}{2} g_{22_v}, \end{cases}$$

and *Mainardi-Codazzi equations* ([5], p. 235)

$$\begin{aligned} b_{11_v} - b_{12_u} &= b_{11} \Gamma_{12}^1 + b_{12} (\Gamma_{12}^2 - \Gamma_{11}^1) - b_{22} \Gamma_{11}^2, \\ b_{12_v} - b_{22_u} &= b_{11} \Gamma_{22}^1 + b_{12} (\Gamma_{22}^2 - \Gamma_{12}^1) - b_{22} \Gamma_{12}^2. \end{aligned}$$

On the other hand

$$\begin{aligned}
\nabla\mathbf{x}{:}\Diamond\mathbf{x}_u &= \mathrm{tr}\Big[\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]^T[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,[h^{\alpha\beta}\,]\,[\,\mathbf{x}_{uu},\,\mathbf{x}_{uv}\,]^T\,\Big]\\
&= \mathrm{tr}\left[\,[h^{\alpha\beta}\,]\Big[\begin{array}{ll}\Gamma_{11}^1 g_{11}+\Gamma_{11}^2 g_{12} & \Gamma_{11}^1 g_{12}+\Gamma_{11}^2 g_{22}\\ \Gamma_{12}^1 g_{11}+\Gamma_{12}^2 g_{12} & \Gamma_{12}^1 g_{12}+\Gamma_{12}^2 g_{22}\end{array}\Big]\right]\\
&= \frac{1}{g}\Big((g_{11}b_{22}-g_{12}b_{12})\Gamma_{11}^1+(g_{12}b_{22}-g_{22}b_{12})\Gamma_{11}^2\\
&\quad +(g_{12}b_{11}-g_{11}b_{12})\Gamma_{12}^1+(g_{22}b_{11}-g_{12}b_{12})\Gamma_{12}^2\Big).
\end{aligned}$$

Thus (2.54) holds and similar proof can be carried out for (2.53). Therefore we complete the proof of (2.45) with the help of (2.41).

To prove (2.46), noticing (2.36), we have that

$$\nabla\mathbf{n} = \Diamond\mathbf{x} - 2H\nabla\mathbf{x}. \tag{2.55}$$

Therefore

$$\Delta\mathbf{n} \overset{(2.21)}{=} \mathrm{div}(\nabla\mathbf{n}) = \mathrm{div}(\Diamond\mathbf{x} - 2H\nabla\mathbf{x}) \overset{(a)}{=} \Box\mathbf{x} - 2\nabla H - 2H\Delta\mathbf{x},$$

where we have owned the validity of (a) to (2.26), (2.22) and the symmetry (2.19) of $\nabla\mathbf{x}$. Thus the first equality of (2.46) holds and (2.44) and (2.45) guarantee the validity of the second equality. Similarly, noticing (2.35), we have

$$\Diamond\mathbf{n} = -K\nabla\mathbf{x}.$$

Therefore

$$\Box\mathbf{n} \overset{(2.22)}{=} \mathrm{div}(\Diamond\mathbf{n}) \overset{(2.31)}{=} -\nabla K - K\Delta\mathbf{x}.$$

Taking (2.44) and (2.45) into consideration, we can show the last two equalities of (2.47) are correct. Thus we complete the proof.

Theorem 2.3 *For operator $\boxtimes$ on parametric surface $\mathcal{M}$, we have*

$$\begin{aligned}
\boxtimes\mathbf{x} &= -\Delta\mathbf{n},\\
\boxtimes\mathbf{n} &= 2\nabla\mathbf{n}\nabla H + 2H\Delta\mathbf{n} - \Box\mathbf{n}.
\end{aligned}$$

Proof: From (2.34), (2.37), (2.26) and (2.20), we can verify the theorem easily.

Remark 2.6 On the results we obtained in Theorem 2.1–2.3, we would like to point out that except that equalities (2.40), (2.42) and (2.44) are well-known, the other equalities, to the best of the authors' knowledge, are relatively new.

2.3 Divergence Theorems and Green's Formulae

In this subsection, Green's formulae for GHO will be given. First let us introduce two lemmas.

Lemma 2.8 (Riemannian divergence theorem I) ([3], p. 142) *If* $\mathbf{v}$ *is a* C^1 *vector field on* $\mathcal{M}$ *with compact support, then*

$$\int_{\mathcal{M}} \mathrm{div}(\mathbf{v}) dA = 0.$$

Lemma 2.9 (Riemannian divergence theorem II) ([3], p. 143) *Let* $\mathcal{M}$ *be oriented,* $\mathcal{U}$ *a sub-region of* $\mathcal{M}$ *with smooth boundary* $\partial\mathcal{U}$*,* ν *the outward unit vector field along* $\partial\mathcal{U}$ *which is pointwise orthogonal to* $\partial\mathcal{U}$*. Then for any* C^1 *vector filed* $\mathbf{v}$ *on* $\mathcal{M}$ *we have*

$$\int_{\mathcal{U}} \mathrm{div}(\mathbf{v}) dA = \int_{\partial\mathcal{U}} \langle \mathbf{v}, \nu \rangle d\sigma.$$

From these two divergence theorems, we can state the following two Green's formulae.

Theorem 2.4 (Green's formula I) *Let* $\mathbf{v}$ *be a smooth three dimensional vector field on* $\mathcal{M}$ *and* $f \in C^1(\mathcal{M})$ *with compact support, then*

$$\int_{\mathcal{M}} \big(\langle \mathbf{v}, \nabla f \rangle + f \,\mathrm{div}(\mathbf{v})\big) dA = 0. \tag{2.56}$$

Proof: Taking $\mathbf{v}$ as $f\mathbf{v}$, then using Riemannian divergence theorem I, we can prove the theorem with no difficulty.

Theorem 2.5 (Green's formula II) *Let* $\mathcal{M}$ *be oriented,* $\mathcal{U}$ *a sub-region of* $\mathcal{M}$ *with smooth boundary* $\partial\mathcal{U}$*,* ν *the outward unit vector field along* $\partial\mathcal{U}$ *which is pointwise orthogonal to* $\partial\mathcal{U}$*. Then for any* C^1 *vector filed* $\mathbf{v}$ *on* $\mathcal{M}$ *we have*

$$\int_{\mathcal{U}} \big(\langle \mathbf{v}, \nabla f \rangle + f \,\mathrm{div}(\mathbf{v})\big) dA = \int_{\partial\mathcal{U}} f \langle \mathbf{v}, \nu \rangle d\sigma.$$

Proof: Taking $\mathbf{v}$ as $f\mathbf{v}$, then using Riemannian divergence theorem II, we can verify this theorem.

From these two Green's formulae, we can state the following Green's formulae for LB operator.

Theorem 2.6 (Green's formula I for LB operator) ([3], p. 142) *Let* $f \in C^2(\mathcal{M}), h \in C^1(\mathcal{M})$*, with at least one of them compactly supported. Then*

$$\int_{\mathcal{M}} \big(h\Delta f + \langle \nabla f, \nabla h \rangle\big) dA = 0.$$

If both f *and* h *are* C^2*, then*

$$\int_{\mathcal{M}} \big(h \,\Delta f - f \,\Delta h\big) dA = 0. \tag{2.57}$$

Theorem 2.7 (Green's formula II for LB operator) ([3], p. 144) *Given $\mathcal{M}$,$\mathcal{U}$ and ν as in the Lemma* 2.9*, and given $f \in C^2(\mathcal{M}), h \in C^1(\mathcal{M})$. Then*

$$\int_{\mathcal{U}} \big(h\,\Delta f + \langle \nabla f, \nabla h\rangle\big) dA = \int_{\partial\mathcal{U}} h\langle \nu, \nabla f\rangle d\sigma.$$

If both f and h are C^2, then

$$\int_{\mathcal{U}} \big(h\,\Delta f - f\,\Delta h\big) dA = \int_{\partial\mathcal{U}} \big(h\langle \nu, \nabla f\rangle - f\langle \nu, \nabla h\rangle\big) d\sigma.$$

For GH operator, we can prove the following two new conclusions.

Theorem 2.8 (Green's formula I for GH operator) *Let $f \in C^2(\mathcal{M})$, $h \in C^1(\mathcal{M})$, with at least one of them compactly supported. Then*

$$\int_{\mathcal{M}} \big(h\Box f + \langle \nabla f, \Diamond h\rangle\big) dA = 0.$$

If both f and h are C^2, then

$$\int_{\mathcal{M}} \big(h\,\Box f - f\,\Box h\big) dA = 0. \tag{2.58}$$

Proof: Let $\mathbf{v} = h\Diamond f$. Then by Green's formula I, we can confirm this theorem by noticing (2.17).

There is a note here, in [22], similar results with (2.58) are obtained.

Theorem 2.9 (Green's formula II for GH operator) *Given $\mathcal{M}$,$\mathcal{U}$ and ν as in the Lemma* 2.9*, and given $f \in C^2(\mathcal{M})$, $h \in C^1(\mathcal{M})$. Then*

$$\int_{\mathcal{U}} \big(h\,\Box f + \langle \nabla f, \Diamond h\rangle\big) dA = \int_{\partial\mathcal{U}} h\langle \nu, \Diamond f\rangle d\sigma.$$

If both f and h are C^2, then

$$\int_{\mathcal{U}} \big(h\,\Box f - f\,\Box h\big) dA = \int_{\partial\mathcal{U}} \big(h\langle \nu, \Diamond f\rangle - f\langle \nu, \Diamond h\rangle\big) d\sigma.$$

Proof: Let $\mathbf{v} = h\Diamond f$. Then by Green's formula II and (2.17), we obtain the first equality. And the second equality follows from the first one and Green's formula II again.

2.4 Eigeninformation

In this subsection, we list some basic eigeninformation for these three first-order differential operators acting on $\mathbf{x}$ and $\mathbf{n}$. Obviously, the 3×3 matrices obtained are all symmetric.

1. Matrix $\nabla\mathbf{x}$ is a projection operator onto tangent space to surface $\mathcal{M}$, then we have $(\nabla\mathbf{x})^T = (\nabla\mathbf{x})^2 = \nabla\mathbf{x}$. The three eigenpairs are $(1, \mathbf{e}_1), (1, \mathbf{e}_2), (0, \mathbf{n})$, respectively. So we have $\mathrm{tr}(\nabla\mathbf{x}) = 2$, as we used before.

2. Matrix $\nabla\mathbf{n}$ has three eigenpairs as $(-k_1,\mathbf{e}_1),(-k_2,\mathbf{e}_2),(0,\mathbf{n})$, respectively. Thus $\mathrm{tr}(\nabla\mathbf{n}) = -(k_1+k_2) = -2H$, the familiar result.

3. Matrix $\lozenge\mathbf{x}$ has three eigenpairs as $(k_2,\mathbf{e}_1),(k_1,\mathbf{e}_2),(0,\mathbf{n})$ by taking (2.36) into consideration. Therefore $\mathrm{tr}(\lozenge\mathbf{x}) = k_1+k_2 = 2H$.

4. Matrix $\lozenge\mathbf{n}$ has three eigenpairs as $(-K,\mathbf{e}_1),(-K,\mathbf{e}_2),(0,\mathbf{n})$, separately, by taking (2.35) into account. Therefore $\mathrm{tr}(\lozenge\mathbf{n}) = -2K$.

5. Matrix $\oslash\mathbf{x}$ has three eigenpairs as $(k_1,\mathbf{e}_1),(k_2,\mathbf{e}_2),(0,\mathbf{n})$ by taking (2.34) into consideration. Therefore $\mathrm{tr}(\oslash\mathbf{x}) = k_1+k_2 = 2H$.

6. Matrix $\oslash\mathbf{n}$ has three eigenpairs as $(-k_1^2,\mathbf{e}_1),(-k_2^2,\mathbf{e}_2),(0,\mathbf{n})$, separately, by taking (2.37) into account. Therefore $\mathrm{tr}(\oslash\mathbf{n}) = -(k_1^2+k_2^2)$.

3 Euler-Lagrange Equation Derivation

In this section, we derive the Euler-Lagrange equation for the functional (1.1) from complete-variation. We summarize the obtained result as the following theorem

Theorem 3.1 *Let $f \in C^4(\mathbb{R}\times\mathbb{R})$. Then the Euler-Lagrange equation of the geometric energy functional* (1.1) *for the complete-variation is*

$$\begin{aligned}&\Delta(f_H\Delta f\mathbf{n}) + 2\Box(f_K\Delta f\mathbf{n}) - \mathrm{div}[\|\nabla f\|^2\nabla\mathbf{x}] - 2\mathrm{div}[f_H\Delta f\nabla\mathbf{n} - 2Kf_K\Delta f\nabla\mathbf{x}]\\ &+2\mathrm{div}[\,R\nabla f(R\nabla f)^T\,] = \mathbf{0}\in\mathbb{R}^3. \end{aligned}\tag{3.1}$$

Proof: First we can rewrite (1.1) as

$$\mathscr{F}_1(\mathcal{M}) = \iint_\Omega \|\nabla f(H,K)\|^2\sqrt{g}dudv,$$

Consider a family of variation $\underline{\mathbf{x}}(w,\varepsilon)$ of $\mathcal{M}$ defined by

$$\underline{\mathbf{x}}(w,\epsilon) = \mathbf{x}(w) + \varepsilon\Phi(w),\quad w\in\overline{\Omega},\quad |\varepsilon|\ll 1,$$

where $\Phi\in C_c^\infty(\Omega)^3$.

Suppose $\mathcal{M}$ is the extremal of functional (1.1). Then we obtain

$$0 = \frac{d}{d\varepsilon}\mathscr{F}_1(\underline{\mathcal{M}}(\cdot,\varepsilon))\big|_{\varepsilon=0} =: \delta\mathscr{F}_1(\mathcal{M},\Phi),$$

where

$$\delta\mathscr{F}_1(\mathcal{M},\Phi) = \iint_\Omega \big(\delta(\|\nabla f(H,K)\|^2) + \|\nabla f\|^2(\delta\sqrt{g})/\sqrt{g}\big)\sqrt{g}dudv. \tag{3.2}$$

Noticing that

$$\begin{aligned}\underline{\mathbf{x}}_{u^\alpha} &= \mathbf{x}_{u^\alpha} + \varepsilon\Phi_{u^\alpha},\\ \underline{\mathbf{x}}_{u^\alpha u^\beta} &= \mathbf{x}_{u^\alpha u^\beta} + \varepsilon\Phi_{u^\alpha u^\beta},\end{aligned}$$

then we first obtain

$$
\begin{aligned}
\delta(g_{11}) &= 2\langle \Phi_u, \mathbf{x}_u \rangle, &(3.3)\\
\delta(g_{12}) &= \langle \Phi_u, \mathbf{x}_v \rangle + \langle \Phi_v, \mathbf{x}_u \rangle, &(3.4)\\
\delta(g_{22}) &= 2\langle \Phi_v, \mathbf{x}_v \rangle, &(3.5)\\
\delta(g) &= 2g_{22}\langle \Phi_u, \mathbf{x}_u \rangle + 2g_{11}\langle \Phi_v, \mathbf{x}_v \rangle - 2g_{12}(\langle \Phi_u, \mathbf{x}_v \rangle + \langle \Phi_v, \mathbf{x}_u \rangle) &\\
&= 2g\nabla \mathbf{x}{:}\nabla \Phi, &(3.6)\\
(\delta\sqrt{g})/\sqrt{g} &= \nabla \mathbf{x}{:}\nabla \Phi. &
\end{aligned}
$$

Let us compute the variation of normal vector $\mathbf{n}$ as follows. First, $\delta(\mathbf{n})$ is a tangent vector to surface by noticing that

$$\delta(\langle \mathbf{n}, \mathbf{n} \rangle) = 2\langle \delta(\mathbf{n}), \mathbf{n} \rangle = 0.$$

Then from

$$
\begin{aligned}
\delta(\langle \mathbf{n}, \mathbf{x}_u \rangle) &= \langle \delta(\mathbf{n}), \mathbf{x}_u \rangle + \langle \mathbf{n}, \delta(\mathbf{x}_u) \rangle = 0,\\
\delta(\langle \mathbf{n}, \mathbf{x}_v \rangle) &= \langle \delta(\mathbf{n}), \mathbf{x}_v \rangle + \langle \mathbf{n}, \delta(\mathbf{x}_v) \rangle = 0,
\end{aligned}
$$

we can draw

$$\delta(\mathbf{n}) = -[\,\mathbf{x}_u,\ \mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\begin{pmatrix} \langle \mathbf{n}, \Phi_u \rangle \\ \langle \mathbf{n}, \Phi_v \rangle \end{pmatrix} = -\nabla \Phi\ \mathbf{n}.$$

Thus

$$
\begin{aligned}
\delta(b_{11}) &= \langle \delta(\mathbf{n}), \mathbf{x}_{uu} \rangle + \langle \mathbf{n}, \delta(\mathbf{x}_{uu}) \rangle = -\Big\langle [\,\mathbf{x}_u,\ \mathbf{x}_v\,]\,[\,g^{\alpha\beta}\,]\begin{pmatrix} \langle \mathbf{n}, \Phi_u \rangle \\ \langle \mathbf{n}, \Phi_v \rangle \end{pmatrix}, \mathbf{x}_{uu} \Big\rangle + \langle \mathbf{n}, \Phi_{uu} \rangle\\
&= -(\langle \mathbf{n}, \Phi_u \rangle,\ \langle \mathbf{n}, \Phi_v \rangle)\,[\,g^{\alpha\beta}\,]\begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_{uu} \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_{uu} \rangle \end{pmatrix} + \langle \mathbf{n}, \Phi_{uu} \rangle\\
&= -(\langle \mathbf{n}, \Phi_u \rangle,\ \langle \mathbf{n}, \Phi_v \rangle)\,[\,g^{\alpha\beta}\,]\begin{pmatrix} \Gamma^1_{11} g_{11} + \Gamma^2_{11} g_{12} \\ \Gamma^1_{11} g_{12} + \Gamma^2_{11} g_{22} \end{pmatrix} + \langle \mathbf{n}, \Phi_{uu} \rangle\\
&= \langle \mathbf{n}, -\Gamma^1_{11}\Phi_u - \Gamma^2_{11}\Phi_v + \Phi_{uu} \rangle =: \langle \mathbf{n}, \Phi_{11} \rangle,\\
\delta(b_{12}) &= \langle \mathbf{n}, \Phi_{12} \rangle,\\
\delta(b_{22}) &= \langle \mathbf{n}, \Phi_{22} \rangle,
\end{aligned}
$$

where $\Phi_{\alpha\beta}$ are the second covariant derivatives. Therefore

$$
\begin{aligned}
\delta(H) &= \frac{1}{2}(b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12})\frac{-\delta(g)}{g^2} + \frac{1}{2g}\big(\langle \mathbf{n}, \Phi_{11}g_{22} + \Phi_{22}g_{11} - 2\Phi_{12}g_{12} \rangle\\
&\quad +2b_{11}\langle \mathbf{x}_v, \Phi_v \rangle + 2b_{22}\langle \mathbf{x}_u, \Phi_u \rangle - 2b_{12}(\langle \mathbf{x}_v, \Phi_u \rangle + \langle \mathbf{x}_u, \Phi_v \rangle)\big)\\
&= -2H\nabla \mathbf{x}{:}\nabla \Phi + \frac{1}{2}\langle \mathbf{n}, \Delta \Phi \rangle + \nabla \mathbf{x}{:}\Diamond \Phi\\
&\overset{(2.32)}{=} -\nabla \mathbf{x}{:}\oslash \Phi + \frac{1}{2}\langle \mathbf{n}, \Delta \Phi \rangle\\
&= -\mathrm{tr}\Big[[\,\mathbf{x}_u,\ \mathbf{x}_v\,]S^T[\,g^{\alpha\beta}\,]\,[\,\Phi_u,\ \Phi_v\,]^T\Big] + \frac{1}{2}\langle \mathbf{n}, \Delta \Phi \rangle\\
&\underset{(2.3)}{\overset{(2.1)}{=}} \nabla \mathbf{n}{:}\nabla \Phi + \frac{1}{2}\langle \mathbf{n}, \Delta \Phi \rangle, \qquad (3.7)
\end{aligned}
$$

and

$$\begin{aligned}\delta(K) &= \delta(\frac{1}{g})(b_{11}b_{22}-b_{12}^2)+\frac{1}{g}\delta(b_{11}b_{22}-b_{12}^2)\\ &\overset{(3.6)}{=} -2K\nabla\mathbf{x}{:}\nabla\Phi+\frac{1}{g}\langle\mathbf{n},\Phi_{11}b_{22}+\Phi_{22}b_{11}-2\Phi_{12}b_{12}\rangle\\ &\overset{(2.12)}{=} -2K\nabla\mathbf{x}{:}\nabla\Phi+\langle\mathbf{n},\Box\Phi\rangle. \end{aligned}\tag{3.8}$$

We are now in the position to compute the variation of $\|\nabla f(H,K)\|^2$.

$$\begin{aligned}&\delta(\|\nabla f(H,K)\|^2)\\ =\;& \delta(\frac{1}{g})(f_u,\,f_v)\begin{bmatrix} g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{bmatrix}\begin{pmatrix} f_u\\ f_v\end{pmatrix}+2\delta(f_u,\,f_v)[\,g^{\alpha\beta}\,]\begin{pmatrix} f_u\\ f_v\end{pmatrix}\\ &+\frac{1}{g}(f_u,\,f_v)\delta\begin{bmatrix} g_{22} & -g_{12}\\ -g_{12} & g_{11}\end{bmatrix}\begin{pmatrix} f_u\\ f_v\end{pmatrix} \qquad (3.9)\\ =\;& -2\nabla\mathbf{x}{:}\nabla\Phi\|\nabla f\|^2+2\Big(\frac{\partial}{\partial u}(f_H\delta H+f_K\delta K),\,\frac{\partial}{\partial v}(f_H\delta H+f_K\delta K)\Big)\,[\,g^{\alpha\beta}\,]\begin{pmatrix} f_u\\ f_v\end{pmatrix}\\ &+\frac{1}{g}(f_u,\,f_v)\begin{bmatrix} 2\langle\mathbf{x}_v,\Phi_v\rangle & -\langle\mathbf{x}_v,\Phi_u\rangle-\langle\mathbf{x}_u,\Phi_v\rangle\\ -\langle\mathbf{x}_v,\Phi_u\rangle-\langle\mathbf{x}_u,\Phi_v\rangle & 2\langle\mathbf{x}_u,\Phi_u\rangle\end{bmatrix}\begin{pmatrix} f_u\\ f_v\end{pmatrix} \qquad (3.10)\\ =\;& \frac{2}{g}(g_{22}f_u-g_{12}f_v)\frac{\partial}{\partial u}\big(f_H(\nabla\mathbf{n}{:}\nabla\Phi+\frac{1}{2}\langle\mathbf{n},\Delta\Phi\rangle)+f_K(-2K\nabla\mathbf{x}{:}\nabla\Phi+\langle\mathbf{n},\Box\Phi\rangle)\big)\\ &+\frac{2}{g}(-g_{12}f_u+g_{11}f_v)\frac{\partial}{\partial v}\big(f_H(\nabla\mathbf{n}{:}\nabla\Phi+\frac{1}{2}\langle\mathbf{n},\Delta\Phi\rangle)+f_K(-2K\nabla\mathbf{x}{:}\nabla\Phi+\langle\mathbf{n},\Box\Phi\rangle)\big)\\ &+\frac{2}{g}(f_v,\,-f_u)\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]^T\nabla\Phi\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,(f_v,\,-f_u)^T-2\nabla\mathbf{x}{:}\nabla\Phi\|\nabla f\|^2. \qquad (3.11)\end{aligned}$$

In the derivation from (3.9) to (3.10), equalities (2.3), (3.3)–(3.6) and the commutative property of δ with $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$ are used. From (3.10) to (3.11), the equalities (3.7) and (3.8) are used. One thing that should be emphasized here is that $f_u = f_H H_u + f_K K_u$ and similar equation for f_v.

Hence, we have

$$\begin{aligned}&\delta\mathscr{F}_1(\mathcal{M},\Phi)\\ =\;& \iint_\Omega\Big(-2\nabla\mathbf{x}{:}\nabla\Phi\|\nabla f\|^2+\frac{2}{g}(g_{22}f_u-g_{12}f_v)\frac{\partial}{\partial u}\big(f_H(\nabla\mathbf{n}{:}\nabla\Phi+\frac{1}{2}\langle\mathbf{n},\Delta\Phi\rangle)\\ &+f_K(\langle\mathbf{n},\Box\Phi\rangle-2K\nabla\mathbf{x}{:}\nabla\Phi)\big)\\ &+\frac{2}{g}(-g_{12}f_u+g_{11}f_v)\frac{\partial}{\partial v}\big(f_H(\nabla\mathbf{n}{:}\nabla\Phi+\frac{1}{2}\langle\mathbf{n},\Delta\Phi\rangle)+f_K(-2K\nabla\mathbf{x}{:}\nabla\Phi+\langle\mathbf{n},\Box\Phi\rangle)\big)\\ &+\frac{2}{g}(f_v,\,-f_u)\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]^T\nabla\Phi\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,(f_v,\,-f_u)T+\nabla\mathbf{x}{:}\nabla\Phi\|\nabla f\|^2\Big)\sqrt{g}dudv \qquad (3.12)\\ =\;& \iint_\Omega\Big(-\|\nabla f(H,K)\|^2\nabla\mathbf{x}{:}\nabla\Phi-2\Delta f\big(f_H(\nabla\mathbf{n}{:}\nabla\Phi+\frac{1}{2}\langle\mathbf{n},\Delta\Phi\rangle)\\ &+f_K(-2K\nabla\mathbf{x}{:}\nabla\Phi+\langle\mathbf{n},\Box\Phi\rangle)\big)\\ &+\frac{2}{g}\nabla\Phi:\big[[\,\mathbf{x}_u,\,\mathbf{x}_v\,]\,(f_v,\,-f_u)^T(f_v,\,-f_u)\,[\,\mathbf{x}_u,\,\mathbf{x}_v\,]^T\big]\Big)\sqrt{g}dudv \qquad (3.13)\end{aligned}$$

$$= \iint_\Omega \Big(-\|\nabla f(H,K)\|^2 \nabla \mathbf{x}{:}\nabla\Phi - 2\Delta f\big(f_H(\nabla\mathbf{n}{:}\nabla\Phi + \frac{1}{2}\langle \mathbf{n}, \Delta\Phi\rangle)$$

$$+f_K(-2K\nabla\mathbf{x}{:}\nabla\Phi + \langle \mathbf{n}, \Box\Phi\rangle)\big) + 2\nabla\Phi : [\, R\nabla f(R\nabla f)^T\,]\Big)\sqrt{g}dudv \tag{3.14}$$

$$= \iint_\Omega \Big\langle \mathrm{div}[\|\nabla f\|^2\nabla\mathbf{x}] + 2\mathrm{div}[f_H\Delta f\nabla\mathbf{n} - 2Kf_K\Delta f\nabla\mathbf{x}] - \Delta(f_H\Delta f\mathbf{n}) - 2\Box(f_K\Delta f\mathbf{n})$$

$$-2\mathrm{div}[\, R\nabla f(R\nabla f)^T\,], \Phi\Big\rangle\sqrt{g}dudv. \tag{3.15}$$

In the derivation from (3.12) to (3.13), Green's formula (2.56), (2.9), the compact support property of Φ and the following well-known equality

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad \forall A \in \mathbb{R}^{m\times n},\ \forall B \in \mathbb{R}^{n\times m}$$

are used. In deriving (3.14) from (3.13), we have utilized a newly defined rotation matrix R on tangent plane

$$R = \frac{1}{\sqrt{g}}[\, -\mathbf{x}_v,\, \mathbf{x}_u\,]\,[\,\mathbf{x}_u,\, \mathbf{x}_v\,]^T.$$

Green's formula I (2.56), Green's formulae I for LB operator (2.57) and GH operator (2.58) are used to derive (3.15) from (3.14) as well as the compact support property of Φ. Since (3.15) vanishes for any $\Phi \in C_c^\infty(\Omega)^3$, Euler-Lagrange equation (3.1) for functional (1.1) is therefore obtained. We thus complete the proof.

Corollary 3.1 *Taking scalar product of* (3.1) *with normal vector* $\mathbf{n}$ *yields*

$$\begin{aligned}
&\big\langle \Delta(f_H\Delta f\mathbf{n}) + 2\Box(f_K\Delta f\mathbf{n}) - \mathrm{div}[\|\nabla f\|^2\nabla\mathbf{x}] - 2\mathrm{div}[f_H\Delta f\nabla\mathbf{n} - 2Kf_K\Delta f\nabla\mathbf{x}] \\
&+2\mathrm{div}[\, R\nabla f(R\nabla f)^T\,], \mathbf{n}\big\rangle \\
=\ & \Delta(f_H\Delta f) + 2\Box(f_K\Delta f) + 2(2H^2-K)f_H\Delta f + 4HKf_K\Delta f - 2H\|\nabla f\|^2 + 2\langle\nabla f, \Diamond f\rangle \\
=\ & 0.
\end{aligned} \tag{3.16}$$

The corollary coincides with the result obtained in [13]. In deriving the corollary, we have utilized Lemma 2.4 and

$$\begin{aligned}
\langle \mathrm{div}\big([\, R\nabla f(R\nabla f)^T\,]\big), \mathbf{n}\rangle &\overset{(2.29)}{=} \langle [\,\nabla(R\nabla f)\,]^T R\nabla f, \mathbf{n}\rangle + \langle \mathrm{div}(R\nabla f)R\nabla f, \mathbf{n}\rangle \\
&\overset{(a)}{=} [\,\nabla(R\nabla f)\mathbf{n}\,]^T\, R\nabla f \\
&\overset{(2.23)}{=} -(R\nabla f)^T\nabla\mathbf{n}\, R\nabla f \\
&\overset{(2.1)}{=} [\, f_u,\, f_v\,]\,[\, h^{\alpha\beta}\,]\,[\, f_u,\, f_v\,] \\
&\overset{(2.5)}{=} \langle \nabla f, \Diamond f\rangle,
\end{aligned} \tag{3.17}$$

where (a) is valid because the vector $R\nabla f$ is in the tangent plane.

In fact, we have a stronger conclusion that is the following theorem.

Theorem 3.2

$$\begin{aligned}
& \Delta(f_H\Delta f\mathbf{n}) + 2\Box(f_K\Delta f\mathbf{n}) - \mathrm{div}[\|\nabla f\|^2\nabla\mathbf{x}] - 2\mathrm{div}[f_H\Delta f\nabla\mathbf{n} - 2Kf_K\Delta f\nabla\mathbf{x}] \\
& +2\mathrm{div}[\,R\nabla f(R\nabla f)^T\,] \\
= \;& \mathbf{n}\big(\Delta(f_H\Delta f) + 2\Box(f_K\Delta f) + 4KHf_K\Delta f + 4H^2 f_H\Delta f - 2Kf_H\Delta f - 2H\|\nabla f\|^2 \\
& +2\langle\nabla f, \Diamond f\rangle\big). \qquad (3.18)
\end{aligned}$$

Proof: We first have the following equalities.

$$\begin{aligned}
\Delta(f_H\Delta f\mathbf{n}) &\overset{(2.28)}{=} \Delta(f_H\Delta f)\mathbf{n} + 2[\,\nabla\mathbf{n}\,]^T\nabla(f_H\Delta f) + (f_H\Delta f)\Delta\mathbf{n}, \\
\Box(f_K\Delta f\mathbf{n}) &\underset{(2.35)}{\overset{(2.31)}{=}} \Box(f_K\Delta f)\mathbf{n} - 2K\nabla(f_K\Delta f) + f_K\Delta f\Box\mathbf{n} \\
&\overset{(2.47)}{=} \Box(f_K\Delta f)\mathbf{n} - 2K\nabla(f_K\Delta f) + f_K\Delta f(-\nabla K - 2KH\mathbf{n}), \\
\mathrm{div}[\|\nabla f\|^2\nabla\mathbf{x}] &\overset{(2.26)}{=} 2\|\nabla f\|^2 H\mathbf{n} + \nabla(\|\nabla f\|^2), \\
\mathrm{div}[f_H\Delta f\nabla\mathbf{n}] &\overset{(2.26)}{=} \nabla\mathbf{n}\nabla(f_H\Delta f) + f_H\Delta f\Delta\mathbf{n}, \\
\mathrm{div}[Kf_K\Delta f\nabla\mathbf{x}] &\overset{(2.26)}{=} \nabla(Kf_K\Delta f) + Kf_K\Delta f(2H\mathbf{n}), \\
\mathrm{div}[\,R\nabla f(R\nabla f)^T\,] &\overset{(2.29)}{=} [\,\nabla(R\nabla f)\,]^T R\nabla f + \mathrm{div}(R\nabla f)R\nabla f \\
&= [\,\nabla(R\nabla f)\,]^T R\nabla f,
\end{aligned}$$

where the last equality is valid because of $\mathrm{div}(R\nabla f) = 0$. After substituting these equalities into the left hand side of (3.18), we obtain

$$\begin{aligned}
& \Delta(f_H\Delta f\mathbf{n}) + 2\Box(f_K\Delta f\mathbf{n}) - \mathrm{div}[\|\nabla f\|^2\nabla\mathbf{x}] - 2\mathrm{div}[f_H\Delta f\nabla\mathbf{n} - 2Kf_K\Delta f\nabla\mathbf{x}] \\
& +2\mathrm{div}[\,R\nabla f(R\nabla f)^T\,] \\
= \;& \Delta(f_H\Delta f)\mathbf{n} - (f_H\Delta f)\Delta\mathbf{n} + 2\Box(f_K\Delta f)\mathbf{n} - 4K\nabla(f_K\Delta f) + 2f_K\Delta f(-\nabla K - 2KH\mathbf{n}) \\
& -2\|\nabla f\|^2 H\mathbf{n} - \nabla(\|\nabla f\|^2) + 4\nabla(Kf_K\Delta f) + 8Kf_K\Delta fH\mathbf{n} + 2[\,\nabla(R\nabla f)\,]^T R\nabla f \\
= \;& \Delta(f_H\Delta f)\mathbf{n} - 2(f_H\Delta f)(K\mathbf{n} - 2H^2\mathbf{n} - \nabla H) + 2\Box(f_K\Delta f)\mathbf{n} + 2f_K\Delta f\nabla K \\
& -2H\|\nabla f\|^2\mathbf{n} - \nabla(\|\nabla f\|^2) + 4KHf_K\Delta f\mathbf{n} + 2[\,\nabla(R\nabla f)\,]^T R\nabla f \\
= \;& \mathbf{n}\big(\Delta(f_H\Delta f) + 2\Box(f_K\Delta f) + 4KHf_K\Delta f + 4H^2 f_H\Delta f - 2Kf_H\Delta f - 2H\|\nabla f\|^2 \\
& +2\langle\nabla f, \Diamond f\rangle\big) + 2f_H\Delta f\nabla H + 2f_K\Delta f\nabla K - \nabla(\|\nabla f\|^2) + 2[\,\nabla(R\nabla f)\,]^T R\nabla f \\
& -2\langle\nabla f, \Diamond f\rangle\mathbf{n} \\
= \;& \mathbf{n}\big(\Delta(f_H\Delta f) + 2\Box(f_K\Delta f) + 4KHf_K\Delta f + 4H^2 f_H\Delta f - 2Kf_H\Delta f - 2H\|\nabla f\|^2 \\
& +2\langle\nabla f, \Diamond f\rangle\big) + 2\Delta f\nabla f - \nabla(\|\nabla f\|^2) + 2[\,\nabla(R\nabla f)\,]^T R\nabla f - 2\langle\nabla f, \Diamond f\rangle\mathbf{n}.
\end{aligned}$$

All left is to show that

$$2\Delta f\nabla f - \nabla(\|\nabla f\|^2) + 2[\,\nabla(R\nabla f)\,]^T R\nabla f - 2\langle\nabla f, \Diamond f\rangle\mathbf{n} = \mathbf{0}. \qquad (3.19)$$

This equality can be verified by the following approach. Let *LHS* be the left hand side of (3.19). We only need to prove that

$$\begin{aligned}
\langle LHS, \mathbf{n}\rangle &= 0, & (3.20)\\
\langle LHS, \mathbf{x}_u\rangle &= 0, & (3.21)\\
\langle LHS, \mathbf{x}_v\rangle &= 0. & (3.22)
\end{aligned}$$

Equality (3.20) is obvious by taking (3.17) into consideration. To confirm equality (3.21) is by now the unique task we should do, since (3.22) can be confirmed in a similar manner. In view of the definitions of ∇, Δ and rotation matrix R, we can prove the second equality by a straightforward calculation. In more detail,

$$
\begin{aligned}
& \langle \mathbf{x}_u, \Delta f \nabla f \rangle \\
= \; & f_u \Delta f \\
= \; & \frac{f_u}{g}(g_{22u} f_u - g_{12u} f_v + g_{22} f_{uu} - g_{12} f_{uv} - g_{12v} f_u + g_{11v} f_v - g_{12} f_{uv} + g_{11} f_{vv}) \\
& - \frac{f_u g_u}{2g^2}(g_{22} f_u - g_{12} f_v) + \frac{f_u g_v}{2g^2}(g_{11} f_v - g_{12} f_u), \qquad (3.23)
\end{aligned}
$$

$$
\begin{aligned}
& \langle \mathbf{x}_u, \nabla(\|\nabla f\|^2) \rangle \\
= \; & \Big(\frac{1}{g}(g_{22} f_u^2 - 2 g_{12} f_u f_v + g_{11} f_v^2)\Big)_u \\
= \; & \frac{1}{g}(g_{22u} f_u^2 - 2 g_{12u} f_u f_v + g_{11u} f_v^2 + 2 g_{22} f_u f_{uu} - 2 g_{12}(f_v f_{uu} + f_u f_{uv}) + 2 g_{11} f_v f_{uv}) \\
& - \frac{g_u}{g^2}(g_{22} f_u^2 - 2 g_{12} f_u f_v + g_{11} f_v^2), \qquad (3.24)
\end{aligned}
$$

$$
\begin{aligned}
& \big\langle \mathbf{x}_u, [\nabla(R\nabla f)]^T R \nabla f \big\rangle \\
= \; & \Big\langle \mathbf{x}_u, \frac{1}{\sqrt{g}}\big[\nabla(\frac{1}{\sqrt{g}}(f_v \mathbf{x}_u - f_u \mathbf{x}_v)\big]^T (f_v \mathbf{x}_u - f_u \mathbf{x}_v) \Big\rangle \\
= \; & \frac{f_v}{\sqrt{g}}\Big(\frac{1}{\sqrt{g}}(f_v \mathbf{x}_u - f_u \mathbf{x}_v)\Big)_u - \frac{f_u}{\sqrt{g}}\Big(\frac{1}{\sqrt{g}}(f_v \mathbf{x}_u - f_u \mathbf{x}_v)\Big)_v \\
= \; & \frac{1}{g}(g_{11} f_v f_{uv} - g_{12} f_v f_{uu} + \frac{g_{11u}}{2} f_v^2 - g_{11v} f_u f_v - g_{11} f_u f_{vv} + g_{12} f_u f_{uv} + (g_{12v} - \frac{1}{2} g_{22u}) f_u^2) \\
& - \frac{g_u}{2g^2}(g_{11} f_v^2 - g_{12} f_u f_v) + \frac{g_v}{2g^2}(g_{11} f_u f_v - g_{12} f_u^2). \qquad (3.25)
\end{aligned}
$$

Substituting (3.23)–(3.25) into (3.21) gives the equality. Therefore we complete the proof of the theorem.

Remark 3.1 In particular, if we take $f(H, K) = H$, then from (3.1), the Euler-Lagrange vector equation for functional (1.2) is

$$\Delta(\Delta H \mathbf{n}) - \mathrm{div}[\|\nabla H\|^2 \nabla \mathbf{x}] - 2\mathrm{div}[\Delta H \nabla \mathbf{n}] + 2\mathrm{div}[\, R\nabla H (R \nabla H)^T \,] = \mathbf{0}. \qquad (3.26)$$

Furthermore, if we take scalar product of the (3.26) with normal vector $\mathbf{n}$, (3.16) turns out to be

$$\Delta^2 H + 2(2H^2 - K)\Delta H - 2H\|\nabla H\|^2 + 2\langle \nabla H, \Diamond H \rangle = 0,$$

which has a wonderful consistence with the result we obtained in [27].

For the sake of numerical solving of the geometric PDEs using the finite element method, we propose the following corollary.

Corollary 3.2 *The weak form of* (3.1) *can be written as*

$$\int_{\mathcal{M}}\Big((4Hf_H\Delta f+4Kf_K\Delta f-\|\nabla f\|^2)\nabla\mathbf{x}\nabla\phi-2f_H\Delta f\nabla\mathbf{x}\Diamond\phi$$
$$+\nabla(f_H\Delta f\mathbf{n})\nabla\phi+2\nabla(f_K\Delta f\mathbf{n})\Diamond\phi+2[\,R\nabla f(R\nabla f)^T\,]\nabla\phi\Big)dA=\mathbf{0},\quad\forall\phi\in C_c^\infty(\Omega).\tag{3.27}$$

Proof: Rewritten (3.14) with the help of (2.36) as

$$\iint_\Omega\Big(-\|\nabla f\|^2\nabla\mathbf{x}{:}\nabla\Phi-2f_H\Delta f[\,\Diamond\mathbf{x}-2H\nabla\mathbf{x}\,]{:}\nabla\Phi-f_H\Delta f\langle\mathbf{n},\Delta\Phi\rangle$$
$$+4Kf_K\Delta f\nabla\mathbf{x}{:}\nabla\Phi-2f_K\Delta f\langle\mathbf{n},\Box\Phi\rangle+2\big[\,R\nabla f(R\nabla f)^T\,\big]{:}\nabla\Phi\Big)\sqrt{g}dudv$$
$$=\iint_\Omega\Big(\big(4Hf_H\Delta f+4Kf_K\Delta f-\|\nabla f\|^2\big)\nabla\mathbf{x}{:}\nabla\Phi-2f_H\Delta f\Diamond\mathbf{x}{:}\nabla\Phi$$
$$+[\,\nabla(f_H\Delta f\mathbf{n})\,]{:}\nabla\Phi+2[\,\nabla(f_K\Delta f\mathbf{n})\,]{:}\Diamond\Phi+2\big[\,R\nabla f(R\nabla f)^T\,\big]{:}\nabla\Phi\Big)\sqrt{g}dudv,\tag{3.28}$$

where the last equality is valid by virtue of Green's formula I for LB operator (2.57) and GH operator (2.58). If we take $\Phi=(\phi,\,0,\,0)^T$, $(0,\,\phi,\,0)^T$ and $(0,\,0,\,\phi)^T$, separately, with $\phi\in C_c^\infty(\Omega)$, then we can get three equations. After combining them together, we obtain weak form (3.27).

Remark 3.2 For convenient numerical computation, we give an easily used form as follows, that is project the last three terms of the left hand side of (3.1) to normal direction, yielding the flow equation

$$\frac{\partial\mathbf{x}}{\partial t}=\Delta(f_H\Delta f\mathbf{n})+2\Box(f_K\Delta f\mathbf{n})-\|\nabla f\|^2\Delta\mathbf{x}+2f_H\Delta f(\Box\mathbf{x}-2H\Delta\mathbf{x})+4Kf_K\Delta f\Delta\mathbf{x}$$
$$+2\langle\nabla f,\Diamond f\rangle\mathbf{n}.$$

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