Consistent Approximations of Some Geometric Differential Operators

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Abstract

The numerical integration of many geometric partial differential equations involve discrete approximations of some differential geometric operators. In this paper, we consider consistent discretized approximations of these operators based on a quadratic fitting scheme. Asymptotic error analysis on the quadratic fitting are conducted. The experiments show that the proposed approach is effective.

Key words: Geometric Differential Operators; Consistent discretized approximations; Triangular Surface Mesh; Error Analysis.

1 Introduction

To solve geometric partial differential equations (PDE) using a divided-difference-like method, discrete approximations of several differential operators (such as surface normal, mean curvature, Gaussian curvature and Laplace-Beltrami operators) are required (see [18]). Many discrete schemes have been proposed for these differential operators from different point of views (see [9], [10], [12], [11], [15] for references). Taubin [13] discussed the discretization of the Laplacian and related approaches in the context of generalized frequencies on meshes. Kobbelt [8] considered discrete approximations of the Laplacian in the construction of fair interpolatory subdivision schemes. Asymptotic error analysis for some of these schemes have been conducted under various conditions (see [9], [10], [14], [15]). An elegant asymptotic estimation of Gaussian curvature by the angular deficit has been given by Borrelli et al [2] under regular assumptions. Except for the schemes based on the interpolation or fitting, non of these schemes converges without any restriction on the regularity of the meshes considered. In [15], it has been claimed that the mean curvature computed from a parametric quadratic fitting surface converges. However, this fact has never been formally proved. In this paper, several commonly used differential operators are approximated based on a parametric quadratic fitting. Hence, the convergence problem of the quadratic fitting does need to be further addressed.

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Main Contributions. PDEs are often solved using a divided-difference-like method, where discretizations of several differential geometric operators are required. Previous work on the discrete approximations of these differential geometric operators are based on several different theorems from differential geometry. Therefore, they are not consistent in general, meaning they do not come from a single surface. We give a consistent estimation of a set of differential geometric operators. Furthermore, we present a convergence analysis of these approximations.

The rest of the paper is organized as follows: Section 2 introduces some basic matterial on differential geometry. Section 3 gives the dicretization scheme for the involved differential operators. Convergence analysis of these discrete differential operators are conducted in section 4. Section 5 concludes the paper.

2 Geometric Differential Operators

In this section, we introduce a set of geometric differential operators, including mean curvature, Gaussian curvature, Laplace-Beltrami operator and Giaquinta-Hildebrandt operator.

Curvatures. Let $M(u,v)$, $(u,v) \in B \subset I\!R^2$ be a regular smooth parametric surface in $I\!R^3$. Let $g_{\alpha\beta} = \langle t_\alpha, t_\beta \rangle$ be the coefficients of the first fundamental form of M with $t_1 = \frac{\partial M}{\partial u}, t_2 = \frac{\partial M}{\partial v}$, set

$$g = \det(G), \quad G = (g_{\alpha\beta}), \quad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}.$$

Let

$$b_{\alpha\beta} = \langle N, t_{\alpha\beta} \rangle, \quad b = \det(b_{\alpha\beta}), \quad (h^{\alpha\beta}) = (b_{\alpha\beta})^{-1},$$

where $t_{11} = \frac{\partial^2 M}{\partial u^2}, t_{12} = \frac{\partial^2 M}{\partial u \partial v}, t_{22} = \frac{\partial^2 M}{\partial v^2}$ and $N = g^{-1/2} t_1 \times t_2$. Then the *mean curvature* H and the *Gaussian curvature* K are

$$H = \frac{1}{2} b_{\alpha\beta} g^{\alpha\beta}, \quad K = \frac{b}{g}.$$

Let p be a surface point. Then the *mean curvature normal* and *Gaussian curvature* can be expressed as (see [17])

$$\mathbf{H}(p) = \frac{Q(g_{22}t_{11} + g_{11}t_{22} - 2g_{12}t_{12})}{2g} \in I\!R^3, \tag{2.1}$$

$$K(p) = \frac{t_{11}^T Q t_{22} - t_{12}^T Q t_{12}}{g}, \tag{2.2}$$

where $Q = I - [t_1, t_2] G^{-1} [t_1, t_2]^T \in I\!R^{3\times 3}$. The advantage of using the expression (2.1) for mean curvature normal is that it does not involve the orientation of the surface normal. Let

$$H(p) = \langle \mathbf{H}(p), n(p) \rangle = n(p)^T \mathbf{H}(p).$$

Then $H(p)$ is the mean curvature, which depends on the orientation of the surface normal $n(p)$.

Tangential gradient operator. Let f be smooth function on $\mathcal{M}$. Then the *tangential gradient operator* ∇ acting on f is given by (see [4], page 102)

$$\begin{aligned}
\nabla f &= [\ \mathbf{r}_u,\ \mathbf{r}_v\ ][g^{\alpha\beta}][\ f_u, f_v\ ]^T \in \mathbb{R}^3 \\
&= \frac{1}{g}(g_{22}f_u\mathbf{r}_u + g_{11}f_v\mathbf{r}_v - g_{12}f_u\mathbf{r}_v - g_{12}f_v\mathbf{r}_u) \\
&= g_u^{\nabla} f_u + g_v^{\nabla} f_v, \qquad (2.3)
\end{aligned}$$

where $g_u^{\nabla} = \frac{1}{g}(g_{22}\mathbf{r}_u - g_{12}\mathbf{r}_v)$ and $g_v^{\nabla} = \frac{1}{g}(g_{11}\mathbf{r}_v - g_{12}\mathbf{r}_u)$.

Second tangential gradient operator. Let f be a smooth function on $\mathcal{M}$. Then we introduce the *second tangential gradient operator* $\diamond$ acting on f, which is defined as

$$\begin{aligned}
\diamond f &= [\ \mathbf{r}_u,\ \mathbf{r}_v\ ][K\, b^{\alpha\beta}][\ f_u, f_v\ ]^T \in \mathbb{R}^3 \\
&= \frac{1}{g}(b_{22}f_u\mathbf{r}_u + b_{11}f_v\mathbf{r}_v - b_{12}f_u\mathbf{r}_v - b_{12}f_v\mathbf{r}_u) \\
&= g_u^{\diamond} f_u + g_v^{\diamond} f_v, \qquad (2.4)
\end{aligned}$$

where $g_u^{\diamond} = \frac{1}{g}(b_{22}\mathbf{r}_u - b_{12}\mathbf{r}_v)$ and $g_v^{\diamond} = \frac{1}{g}(b_{11}\mathbf{r}_v - b_{12}\mathbf{r}_u)$.

Divergence operator. Let $\mathbf{v}$ be a C^1 smooth vector field on $\mathcal{M}$. Then the *divergence* of $\mathbf{v}$ is defined by

$$\operatorname{div}(\mathbf{v}) = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\sqrt{g}\,[g^{\alpha\beta}]\,[\mathbf{r}_u, \mathbf{r}_v]^T\mathbf{v}\right].$$

Note that if $\mathbf{v}$ is the normal vector field of $\mathcal{M}$, $\operatorname{div}(\mathbf{v}) = 0$.

Laplace-Beltrami operator. Let $f \in C^2(\mathcal{M})$. Then ∇f is a smooth vector field on $\mathcal{M}$. The Laplace-Beltrami operator (LBO) Δ applying to f is defined by (see [5])

$$\Delta f = \operatorname{div}(\nabla f).$$

From the definition of ∇ and div, it is easy to derive that

$$\begin{aligned}
\Delta f &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right]\left[\sqrt{g}\,[g^{\alpha\beta}]\,[\ f_u, f_v\ ]^T\right] \\
&= \frac{1}{g}(g_{22}f_{11} + g_{11}f_{22} - 2g_{12}f_{12}) \\
&= g_u^{\triangle} f_u + g_v^{\triangle} f_v + g_{uu}^{\triangle} f_{uu} + g_{uv}^{\triangle} f_{uv} + g_{vv}^{\triangle} f_{vv}, \qquad (2.5)
\end{aligned}$$

where

$$\begin{aligned}
&f_{\alpha\beta} = f_{u^\alpha u^\beta} - (\nabla f)^T \mathbf{r}_{u^\alpha u^\beta}, \quad \alpha, \beta = 1, 2, \\
&g_u^{\triangle} = -[g_{11}(g_{22}g_{122} - g_{12}g_{222}) + 2g_{12}(g_{12}g_{212} - g_{22}g_{112}) + g_{22}(g_{22}g_{111} - g_{12}g_{211})]/g^2, \\
&g_v^{\triangle} = -[g_{11}(g_{11}g_{222} - g_{12}g_{122}) + 2g_{12}(g_{12}g_{112} - g_{11}g_{212}) + g_{22}(g_{11}g_{211} - g_{12}g_{111})]/g^2, \\
&g_{uu}^{\triangle} = g_{22}/g, \qquad g_{uv}^{\triangle} = -2g_{12}/g, \qquad g_{vv}^{\triangle} = g_{11}/g,
\end{aligned}$$

and $g_{\alpha\beta\delta} = \langle \mathbf{r}_{u^\alpha}, \mathbf{r}_{u^\beta u^\delta} \rangle$. It is easy to see that Δ is a second-order differential operator.

Giaquinta-Hildebrandt Operator. Let f be a smooth function on $\mathcal{M}$. Then the Giaquinta-Hildebrandt operator (GHO) acting on f is given by

$$\Box f = \mathrm{div}(\Diamond f).$$

From the definition of $\Diamond$ and div, it is easy to derive that (see [6], page 84)

$$\begin{aligned} \Box f &= \frac{1}{\sqrt{g}} \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] \left[\sqrt{g} \left[K b^{\alpha\beta} \right] \left[\, f_u, f_v \, \right]^T \right] && (2.6) \\ &= \frac{1}{g}(b_{22} f_{11} + b_{11} f_{22} - 2 b_{12} f_{12}) \\ &= g_u^\Box f_u + g_v^\Box f_v + g_{uu}^\Box f_{uu} + g_{uv}^\Box f_{uv} + g_{vv}^\Box f_{vv}, && (2.7) \end{aligned}$$

where

$$\begin{aligned} g_u^\Box &= -[b_{11}(g_{22}g_{122} - g_{12}g_{222}) + 2b_{12}(g_{12}g_{212} - g_{22}g_{112}) + b_{22}(g_{22}g_{111} - g_{12}g_{211})]/g^2, \\ g_v^\Box &= -[b_{11}(g_{11}g_{222} - g_{12}g_{122}) + 2b_{12}(g_{12}g_{112} - g_{11}g_{212}) + b_{22}(g_{11}g_{211} - g_{12}g_{111})]/g^2, \\ g_{uu}^\Box &= b_{22}/g, \qquad g_{uv}^\Box = -2b_{12}/g, \qquad g_{vv}^\Box = b_{11}/g. \end{aligned}$$

Differential operator $\Box$ is introduced by Giaquinta and Hildebrandt (see [6], pages 82–85), we therefore call it as Giaquinta-Hildebrandt operator. Since b_{ij} involves the second order derivatives of the surface considered, equation (2.6) implies that $\Box f$ is a third order differential operator at first glance. However, (2.7) shows that it is a second order differential operator, since the terms involving the third order derivatives are canceled fortunately. Similar to the relation $\Delta \mathbf{r} = 2H\mathbf{n}$, we have $\Box \mathbf{r} = 2K\mathbf{n}$ (see [19]).

3 Discretizations of Geometric Differential Operators

Discrete Surface. Let T be a triangulation of surface M. Let $\{p_i\}_{i=1}^N$ be the vertex set of M. For vertex p_i with valence n, denote by $N(i) = \{i_1, i_2, \cdots, i_n\}$ the set of the vertex indices of one-ring neighbors of p_i. We assume in the following that these $i_1, \cdots, i_n$ are arranged such that the triangles $[p_i p_{i_k} p_{i_{k-1}}]$ and $[p_i p_{i_k} p_{i_{k+1}}]$ are in M, and $p_{i_{k-1}}$, $p_{i_{k+1}}$ opposite to the edge $[p_i p_{i_k}]$. For $j = i_k \in N(i)$, we use j_+ and j_- to denote i_{k+1} and i_{k-1}, respectively, for simplifying the notation. Furthermore, we use the following convention throughout the paper: $i_{n+1} = i_1, \quad i_0 = i_n$.

To solve the geometric PDEs using a divided-difference-like method, discrete approximations of the mean curvature, Gaussian curvature and Laplace-Beltrami operator and Giaquinta-Hildebrandt operator are required. In order to use the semi-implicit scheme, the approximations of the above mentioned differential operators require to have the following form

$$H(p_i)(\text{or } K(p_i)) = \sum_j w_{ij} p_j, \quad \Delta f(p_i)(\text{or } \Box f(p_i)) = \sum_j \omega_{ij} f(p_j), \tag{3.1}$$

where w_{ij}, $\omega_{ij} \in I\!R$. There are several discretization schemes of Laplace-Beltrami operator and Gaussian curvature (see [15, 16] for a review). However, the discretizations of Gaussian curvature are not in the required form and may not consistent in the following sense.

Definition 3.1 *A set of approximations of differential geometric operators is said consistent if there exists a C^2 smooth surface S, such that the approximate operators coincide with the exact counterparts of S.*

Here we use a biquadratic fitting of the surface data and function data to calculate the approximate differential operators. The algorithm we adopted is from [15]. Let p_i be a vertex of T with valence n, p_j be its neighbor vertices for $j \in N(i)$.

Algorithm 3.1. *Quadratic Fit*

1. Compute angles $\alpha_k = cos^{-1}[(p_{i_k} - p_i, p_{i_{k+1}} - p_i)/(\|p_{i_k} - p_i\|\|p_{i_{k+1}} - p_i\|)]$, and then compute the angles $\beta_k = 2\pi\alpha_k / \sum_{j=1}^{n} \alpha_j$ for $k = 1, \cdots, n$. Set $q_0 = (0, 0)$, $\theta_1 = 0$ and $q_k = \|p_{i_k} - p_i\|(cos\theta_k, sin\theta_k)$, $\theta_k = \beta_1 + \cdots + \beta_{k-1}$, for $k = 1, \cdots, n$.

2. Take the basis functions $\{B_l(\xi_1, \xi_2)\}_{l=0}^{5} = \{1, \xi_1, \xi_2, \frac{1}{2}\xi_1^2, \xi_1\xi_2, \frac{1}{2}\xi_2^2\}$, and determine the coefficient $c_l \in I\!R^3$ of $\sum_{l=0}^{5} c_l B_l$ so that

$$\sum_{l=0}^{5} c_l B_l(q_k) = p_{i_k}, \quad k = 0, \cdots, n$$

   in the least square sense (assume $i_0 = i$). This system is solved by solving the normal equation. Let $A = (B_l(q_k))_{k=0,l=0}^{n,5} \in I\!R^{(n+1)\times 6}$, and let

$$C = (A^T A)^{-1} A^T \in I\!R^{6\times(n+1)}, \tag{3.2}$$

   then $[c_0, \cdots, c_5] = [p_{i_0}, \cdots, p_{i_n}]C^T$.

Remark. The construction algorithm above may fail if the coefficient matrix of the normal equation is singular or nearly singular. In this case, we look for a least square solution with minimal normal. Let $A^T Ax = b$ be the linear system in the matrix form. We find a least square solution x such that $\|x\|_2 = min$. That is, we replace $(A^T A)^{-1}$ in (3.2) with $(A^T A)^+$, the Moore-Penrose inverse. It is well known that $(A^T A)^+$ could be computed by the SVD decomposition of $A^T A$ (see [7], Chapter 5). Let $V = \text{diag}[\sigma_1, \cdots, \sigma_6]$, where $\sigma_1 \geq \cdots \geq \sigma_6 \geq 0$ are the singular value of $A^T A$. If the computed singular value $\sigma_i < 10^{-8}$, we regard this singular value as zero (we use double precision arithmetic operations). In the practice, $\|p_{i_k} - p_i\|$ may be very small and the result in singular values are also small. Then the treatment of the singular values mentioned above may be misleading. To overcome this difficulty, the matrix A is normalized by multiplying a diagonal matrix $D = \text{diag}[1, h^{-1}, h^{-1}, h^{-2}, h^{-2}, h^{-2}]$ on the left, where $h = \max_k \|p_{i_k} - p_i\|$.

Partial derivatives. Let $[d_0, \cdots, d_5]^T = C[f(\mathbf{r}_{i_0}), \cdots, f(\mathbf{r}_{i_n})]^T$. Then we compute partial derivatives up to the second order. Denote the second, third, fourth, fifth and sixth rows of C as C_1, C_2, C_{11}, C_{12} and C_{22}, respectively, then we can see that

$$\begin{array}{lll} \mathbf{r}_{u^\alpha} & = [\mathbf{r}_{i_0}, \cdots, \mathbf{r}_{i_n}] C_\alpha^T, & \alpha = 1, 2, \\ \frac{\partial f}{\partial u^\alpha} & = [f(\mathbf{r}_{i_0}), \cdots, f(\mathbf{r}_{i_n})] C_\alpha^T, & \alpha = 1, 2, \\ \mathbf{r}_{u^\alpha u^\beta} & = [\mathbf{r}_{i_0}, \cdots, \mathbf{r}_{i_n}] C_{\alpha\beta}^T, & 1 \leq \alpha \leq \beta \leq 2, \\ \frac{\partial^2 f}{\partial u^\alpha \partial u^\beta} & = [f(\mathbf{r}_{i_0}), \cdots, f(\mathbf{r}_{i_n})] C_{\alpha\beta}^T, & 1 \leq \alpha \leq \beta \leq 2. \end{array} \tag{3.3}$$

Tangential gradient operator. Substituting (3.3) into (2.3), we get an approximation of tangential gradient operator as follows:

$$\nabla f(\mathbf{r}_i) \approx \sum_{j \in N_1(i)} w_{ij}^{\nabla} f(\mathbf{r}_j), \quad w_{i,i_j}^{\nabla} = g_u^{\nabla} c_1^{(j)} + g_v^{\nabla} c_2^{(j)} \in \mathbb{R}^3.$$

Here $c_\alpha^{(j)}$ are the j-th component of C_α and $c_{\alpha\beta}^{(j)}$ are the j-th component of $C_{\alpha\beta}$ by analogy in the sequel.

Second tangent operator. Substituting (3.3) into (2.4), we get an approximation of second tangent operator as follows:

$$\diamond f(\mathbf{r}_i) \approx \sum_{j \in N_1(i)} w_{ij}^{\diamond} f(\mathbf{r}_j), \quad w_{i,i_j}^{\diamond} = g_u^{\diamond} c_1^{(j)} + g_v^{\diamond} c_2^{(j)} \in \mathbb{R}^3.$$

Laplace-Beltrami operator. Substituting (3.3) into (2.5), we get an approximation of LBO as follows:

$$\Delta f(\mathbf{r}_i) \approx \sum_{j \in N_1(i)} w_{ij}^{\triangle} f(\mathbf{r}_j),$$

where $w_{i,i_j}^{\triangle} = g_u^{\triangle} c_1^{(j)} + g_v^{\triangle} c_2^{(j)} + g_{uu}^{\triangle} c_{11}^{(j)} + g_{uv}^{\triangle} c_{12}^{(j)} + g_{vv}^{\triangle} c_{22}^{(j)}$.

Giaquinta-Hildebrandt operator. Substituting (3.3) into (2.7), we get an approximation of $\Box$ as follows:

$$\Box f(\mathbf{r}_i) \approx \sum_{j \in N_0(i)} w_{ij}^{\Box} f(\mathbf{r}_j), \tag{3.4}$$

where $w_{i,i_j}^{\Box} = g_u^{\Box} c_1^{(j)} + g_v^{\Box} c_2^{(j)} + g_{uu}^{\Box} c_{11}^{(j)} + g_{uv}^{\Box} c_{12}^{(j)} + g_{vv}^{\Box} c_{22}^{(j)}$.

Mean curvature normal and mean curvature. Using the relation $\Delta \mathbf{r} = 2\mathbf{H}$, we have

$$\mathbf{H}(\mathbf{r}_i) \approx \frac{1}{2} \sum_{j \in N_1(i)} w_{ij}^{\triangle} \mathbf{r}_j, \quad H(\mathbf{r}_i) \approx \frac{1}{2} \sum_{j \in N_1(i)} w_{ij}^{\triangle} \mathbf{n}(\mathbf{r}_i)^T \mathbf{r}_j,$$

where $\mathbf{n}(\mathbf{r}_i)$ is the surface normal at $\mathbf{r}_i$, which is computed as $(\mathbf{r}_u \times \mathbf{r}_v)/\|\mathbf{r}_u \times \mathbf{r}_v\|$ with a proper orientation.

Gaussian curvature normal and Gaussian curvature. Using the relation $\Box\mathbf{r} = 2K\mathbf{n}$, we have

$$K(\mathbf{r}_i)\mathbf{n}(\mathbf{r}_i) \approx \frac{1}{2}\sum_{j\in N_1(i)} w_{ij}^{\Box}\mathbf{r}_j, \qquad K(\mathbf{r}_i) \approx \frac{1}{2}\sum_{j\in N_1(i)} w_{ij}^{\Box}\mathbf{n}(\mathbf{r}_i)^T\mathbf{r}_j.$$

Remark. Now we explain why we derive the used differential operators based on the parametric fitting. The first reason is that this fitting scheme yields a convergent approximation as the mesh size (the maximal edge length) $h \to 0$ (see next section). The second reason is that the computation of these operators is consistent, meaning they are computed from the same surface. The third reason is that the fitting scheme yields the required form expressions: $\sum w_{ij}^{\triangle} p_j$ and $\sum w_{ij}^{\Box} p_j$, which are ready for use in the semi-implicit discretization of the geometric PDEs, while the widely used discrete scheme based on Gauss-Bonnet theorem for Gaussian curvature (see [1]) is not the form of (3.1). The last reason is that all the differential operators considered in this paper involve the first and second order derivatives of the surface or functions on the surface. Hence, quadratic function is enough to provide these partial derivative data.

4 Convergence of Discrete Differential Operators

It is well known that (see [3]) the errors of the coefficients c_{ij} of the Lagrange interpolation polynomial of degree n versus the Taylor expansion of a function around the origin are bounded by $|c_{ij} - f_{ij}(0)/i!j!| \le Ch^{n+1-(i+j)}$, where C is a constant and h is the maximal distance of the interpolation nodes to the origin. For approximation (the least square fitting), a similar result holds (see [3]). However, these results do not imply explicitly the convergence for our quadratic fitting algorithm because of the following two reasons. First, our fitting surface is in the parametric form. Second, the nodes of the fitting are determined in a way that is different from the functional case (see Step 1 of Algorithm 3.1). In [15], it has been claimed that the mean curvature computed from the quadratic fitting converges. However, this fact has never been formally proved. Hence, the convergence property of the quadratic fitting does need to be analyzed.

Definition 4.1 *Let $N := \{q_i = (x_i, y_i)^T \in I\!R^2\}_{i=0}^n$, $n \ge 5$. If the matrix*

$$A(N) := \begin{bmatrix} 1 & x_0 & y_0 & x_0^2/2 & x_0y_0 & y_0^2/2 \\ 1 & x_1 & y_1 & x_1^2/2 & x_1y_1 & y_1^2/2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & y_n & x_n^2/2 & x_ny_n & y_n^2/2 \end{bmatrix}$$

is full rank in row, then we say the node set N is well-posed for the problem of the quadratic fitting: Determining a bivariate polynomial $G(x,y) = \sum_{i+j\le 2} a_{ij}x^iy^j/i!j!$ such that

$$G(q_i) = f(q_i), \quad i = 0, 1, \cdots, n \tag{4.1}$$

in the least square sense, where f is a given function.

If the node set N is well-posed, then the quadratic fitting problem has unique solution. In the following, we assume that $q_0, q_1, \cdots, q_n$ are mutual distinct and further assume that $q_0 = (0,0)^T$ for simplicity.

Lemma 4.1 *Let $N = \{q_i\}_{i=0}^n$ be a well-posed node set for the quadratic fitting problem. Then the node set $N^{(h)} := hN := \{hq_i\}_{i=0}^n$ is also well-posed for any $h > 0$. More general, let $L \in I\!R^{2\times 2}$ be a nonsingular matrix. Then the node set $N^{(L)} := \{Lq_i\}_{i=0}^n$ is well-posed.*

Proof. Since $h > 0$ and

$$A(N^{(h)}) = A(N)\Lambda, \quad \Lambda = \mathrm{diag}[1, h, h, h^2, h^2, h^2], \tag{4.2}$$

the first conclusion of the Lemma follows. To prove the second conclusion. Let $L = (a_{ij})_{i,j=1}^2$. Then under the transform L,

$$A(N^{(L)}) = A(N)\mathrm{diag}\left\{1, L, \begin{bmatrix} a_{11}^2 & a_{11}a_{21} & a_{21}^2 \\ 2a_{11}a_{12} & a_{11}a_{22} + a_{12}a_{21} & 2a_{21}a_{22} \\ a_{12}^2 & a_{12}a_{22} & a_{22}^2 \end{bmatrix}\right\}.$$

It is not difficult to calculate that the determinant of the last 3×3 block matrix above is $(a_{11}a_{22} - a_{12}a_{21})^3 \neq 0$, since $\det(L) = a_{11}a_{22} - a_{12}a_{21} \neq 0$. Hence the lemma is proved.

Lemma 4.2 *Suppose f is a sufficiently smooth bivariate function in the neighborhood of the origin. Let $N = \{q_i\}_{i=0}^n$ be a well-posed node set, and $G^{(h)}(x,y) = \sum_{i+j\leq 2} a_{ij}^{(h)} x^i y^j / i!j!$ the quadratic fitting function of f on the node set $N^{(h)} = hN$, $h > 0$. Then*

$$\left| a_{ij}^{(h)} - f_{ij}(q_0) \right| \leq c_{ij} \|[A(N)^T A(N)]^{-1} A(N)\| h^{3-i-j}, \tag{4.3}$$

where c_{ij} are constants depending on f, but independent of $N^{(h)}$.

Proof. Let

$$X = [a_{00}^{(h)}, a_{10}^{(h)}, a_{01}^{(h)}, a_{20}^{(h)}, a_{11}^{(h)}, a_{02}^{(h)}]^T, \quad F = [f(q_0), f(hq_1), \cdots, f(hq_n)]^T.$$

Then from the fitting problem (4.1), we have

$$A(N^{(h)})^T A(N^{(h)}) X = A(N^{(h)})^T F,$$

and, by (4.2),

$$A(N)^T A(N) \Lambda X = A(N)^T F. \tag{4.4}$$

Since

$$f(hq_i) = f(q_0) + hq_i^T \nabla f(q_0) + \frac{1}{2} h^2 q_i^T \nabla^2 f(q_0) q_i + O(h^3),$$

we have

$$F = A(N)\Lambda F_0 + O(h^3),$$

where $F_0 = [f(q_0), f_{10}(q_0), f_{01}(q_0), f_{20}(q_0), f_{11}(q_0), f_{02}(q_0)]^T$. Substituting this into (4.4), we have

$$A(N)^T A(N)\Lambda X = A(N)^T A(N)\Lambda F_0 + A(N)^T O(h^3),$$

and

$$X = F_0 + \Lambda^{-1}[A(N)^T A(N)]^{-1} A(N)^T O(h^3).$$

Then (4.3) follows.

Corollary 4.1 *Suppose f is a sufficiently smooth bivariate function in the neighborhood of the origin. Let $N = \{q_i\}_{i=0}^n$ be a well-posed node set. Let $G^{(h)}$ be the quadratic fitting function of f on the node set $N^{(h)}$. Then we have*

$$\|n(G^{(h)}) - n(f)\| \leq C_0 h^2, \quad |k_i(G^{(h)}) - k_i(f)| \leq C_i h, \quad i = 1, 2,$$

where $n(f)$ and $k_i(f)$ denote the normal and the principal curvatures of f at q_0, respectively.

Remark 8.1. The corollary says that the normal $n(G^{(h)})$ has quadratic convergence rate, curvatures have linear convergence rate. These results mach Meek and Walton's results (see [10], Lemma 4.1).

Lemma 4.3 *Let $N = \{q_i\}_{i=0}^n$ be a well-posed node set. Then for any $B > \|[A(N)^T A(N)]^{-1} A(N)\|$, there exists an $\epsilon > 0$, such that*

(i). $q_0 \notin D_i$, $D_i \cap D_j = \emptyset$, $i \neq j$, $i, j \geq 1$, where $D_i = \{q \in I\!R^2 : \|q - q_i\| < \epsilon\}$.

(ii). For any node set $R := \{r_i \in D_i\}_{i=0}^n$, we have $\|[A(R)^T A(R)]^{-1} A(R)\| < B$.

(iii). For this ϵ, let $D_i^{(h)} = \{q \in I\!R^2 : \|q - hq_i\| < \epsilon h\}$ $(h > 0)$. Then the quadratic fitting problem on the node set $R^{(h)} := \{r_i \in D_i^{(h)}\}_{i=0}^n$ has unique solution $G^{(h)}(x, y) = \sum_{i+j\leq 2} a_{ij}^{(h)} x^i y^j / i! j!$ and

$$\left| a_{ij}^{(h)} - f_{ij}(q_0) \right| \leq c_{ij} B h^{3-i-j}, \quad i + j \leq 2 \tag{4.5}$$

where c_{ij} are constants depending on f, but independent of $R^{(h)}$.

Proof. Since N is a well-posed node set. q_i are distinct. It is obvious that there exists an $\epsilon_1 > 0$ such that (i) holds. Notice that the elements of the matrix $A(R)$ are continuous function of $r_0, \cdots, r_n$. Hence the inverse of $A(R)^T A(R)$ exists in the neighborhood of $q_0, \cdots, q_n$. Then there exists an $\epsilon \leq \epsilon_1$ such that (ii) holds.

Similar to the proof of Lemma 4.2, (4.5) can be derived.

Lemma 4.4 *Suppose $f(x, y)$ is smooth function such that $f(q_0) = 0$, $\nabla f(q_0) = 0$. Let $q \in I\!R^2$ be a point in the neighborhood of the origin. Then*

$$\sqrt{\|q - q_0\|^2 + f(q)^2} - \|q - q_0\| \le C\|q - q_0\|^3.$$

Proof. Since

$$f(q) = f(q_0) + (q - q_0)^T \nabla f(q_0) + O(\|q - q_0\|^2) = O(\|q - q_0\|^2),$$

we have

$$\sqrt{\|q - q_0\|^2 + f(q)^2} = \sqrt{\|q - q_0\|^2 + O(\|q - q_0\|^4)} = \|q - q_0\| + O(\|q - q_0\|^3).$$

Lemma 4.5 *Let $f(x, y)$ be a smooth function around $q_0 \in I\!R^2$. Let $T_f(q_0)$ be the tangent plane of f at q_0. Let q be a neighbor point of q_0 and $\mathbf{x}'$, $\mathbf{x}''$ the intersection point of the line $([q^T, 0]^T, [q^T, f(q)]^T)$ with $T_f(q_0)$ and the project point of $[q^T, f(q)]^T$ onto the tangent plane $T_f(q_0)$, respectively. Then there exists a constant C such that*

$$\|\mathbf{x}' - \mathbf{x}''\| \le C\|q - q_0\|^2.$$

Proof. Since

$$T_f(q_0) = \{\mathbf{x} \in I\!R^3 : ([q_0^T, f(q_0)] - \mathbf{x}^T)[(\nabla f(q_0)^T, -1]^T = 0\}.$$

or $z = -q_0^T \nabla f(q_0) + f(q_0) + (x, y)\nabla f(q_0)$. The z-value of the intersection is

$$z(q) = -q_0^T \nabla f(q_0) + f(q_0) + q^T \nabla f(q_0) = f(q_0) + (q - q_0)^T \nabla f(q_0)$$

Hence $\mathbf{x}' = [q^T, z(q)]^T = [q^T, f(q_0) + (q - q_0)^T \nabla f(q_0)]^T$. Since the line passing $[q^T, f(q)]^T$ and perpendicular to $T_f(q_0)$ is

$$x = [q^T, f(q)]^T + tn(f), \quad n(f) = [\nabla f(q_0)^T, -1]^T.$$

Substituting x into the equation of $T_f(q_0)$, we obtain

$$t = \frac{(q_0 - q)^T \nabla f(q_0) + f(q) - f(q_0)}{\|\nabla f(q_0)\|^2 + 1} = O(\|q - q_0\|^2),$$

$$\mathbf{x}'' = [q^T, f(q)]^T + \frac{(q_0 - q)^T \nabla f(q_0) + f(q) - f(q_0)}{\|\nabla f(q_0)\|^2 + 1}[\nabla f(q_0)^T, -1]^T.$$

Therefore,

$$\|\mathbf{x}' - \mathbf{x}''\| = |f(q) - f(q_0) - (q - q_0)^T \nabla f(q_0) + O(\|q - q_0\|^2)| = O(\|q - q_0\|^2).$$

Lemma 4.6 *Let $f(x, y)$ be a smooth function such that $f(q_0) = 0$, $\nabla f(q_0) = 0$. Let $q_1, q_2 \in I\!R^2$ be two points in the neighborhood of the origin. Let*

$$\theta = cos^{-1}\frac{\langle q_1, q_2\rangle}{\|q_1\|\|q_2\|}, \quad \theta_f^{(h)} = cos^{-1}\frac{\langle p_1, p_2\rangle}{\|p_1\|\|p_2\|},$$

where $p_i = [hq_i^T, f(hq_i)]^T$, $i = 1, 2$. Then

$$|\theta_f^{(h)} - \theta| \le Ch^2.$$

Proof. Since

$$\begin{aligned} f(hq_i) &= f(q_0) + hq_i^T \nabla f(q_0) + \frac{1}{2}h^2 q_i^T \nabla^2 f(q_0) q_i + O(h^3) \\ &= \frac{1}{2}h^2 q_i^T \nabla^2 f(q_0) q_i + O(h^3), \\ \|p_i\| &= \|q_i\|h + O(h^3), \quad \langle p_1, p_2\rangle = \langle q_1, q_2\rangle h^2 + O(h^4), \end{aligned}$$

we have

$$\begin{aligned} \theta_f^{(h)} &= cos^{-1}\frac{\langle p_1, p_2\rangle}{\|p_1\|\|p_2\|} = cos^{-1}\frac{\langle q_1, q_2\rangle + O(h^2)}{\|q_1\|\|q_2\| + O(h^2)} \\ &= cos^{-1}\frac{\langle q_1, q_2\rangle}{\|q_1\|\|q_2\|} + O(h^2) \\ &= \theta + O(h^2). \end{aligned}$$

Theorem 4.1 *Let f be a smooth function around the origin, and let $N = \{q_i\}_{i=0}^n$ be a well-posed node set. Let $G^{(h)}$ be the quadratic fitting function generated by the Algorithm 3.1 for the sampling data $[hq_i^T, f(hq_i)]^T \in I\!R^3$ of the function f. Then (4.5) holds.*

Proof. Let $T_f(q_0)$ be the tangent plane of surface f at q_0, and let $\mathbf{x}_i'$ be the intersection point of the line $([q_i^T, 0]^T, [q_i^T, f(q_i)]^T)$ with $T_f(q_0)$. Let $N' = \{\mathbf{x}_i'\}_{i=0}^n$. Then N' can be regarded as the result of a linear transform of N, and therefore is well-posed, from Lemma 4.1 .

Let $\mathbf{x}_i''$ be the project point of the point $[q_i^T, f(q_i)]^T$ on the tangent plane $T_f(q_0)$. Then by Lemma 4.5, we have

$$\|\mathbf{x}_i''^{(h)} - \mathbf{x}_i'^{(h)}\| \le Ch^2.$$

Let $\mathbf{x}_i'''^{(h)}$ be the nodes determined by Algorithm 3.1 and denote

$$\mathbf{x}_i''^{(h)} = r_i''^{(h)}(\cos\theta_i''^{(h)}, \sin\theta_i''^{(h)})^T, \quad \mathbf{x}_i'''^{(h)} = r_i'''^{(h)}(\cos\theta_i'''^{(h)}, \sin\theta_i'''^{(h)})^T.$$

Then by Lemma 4.4 and Lemma 4.6, we know that

$$|r_i''^{(h)} - r_i'''^{(h)}| < Ch^3, \quad |\theta_i''^{(h)} - \theta_i'''^{(h)}| < Ch^2.$$

Therefore,

$$\|\mathbf{x}_i'''^{(h)} - \mathbf{x}_i''^{(h)}\| \le Ch^2,$$

and

$$\|\mathbf{x}_i'''^{(h)} - \mathbf{x}_i'^{(h)}\| \le Ch^2.$$

Hence, when h is small enough, $\mathbf{x}_i'''^{(h)} \in D_i^{(h)}$. Then by Lemma 4.3, (4.5) holds.

5 Conclusions

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