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Parametric Well-posedness for Weak Vector Equilibrium Problems*

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Abstract The paper studied two kinds of parametric well-posedness for weak vector equilibrium problems in real Banach spaces. It established some metric characterizations of unique well-posedness and well-posedness for the problems. It proved that under suitable conditions, the unique well-posedness is equivalent to the existence and uniqueness of solutions. Finally, it gave sufficient conditions to well-posedness in finite dimensional space.

Keywords weak vector equilibrium problems, parametric well-posedness, metric characterizations

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弱向量均衡问题的含参适定性

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摘要 在实 Banach 空间中研究了弱向量均衡问题的两种适定性. 给出了该问题唯一适定与适定的距离刻划. 在适当条件下证明了弱向量均衡问题的唯一适定性等价于解的存在性与唯一性. 最后，在有限维空间给出了弱向量均衡问题适定的充分性条件.

关键词 弱向量均衡问题，含参适定性，距离刻划

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0 Introduction

The concept of well-posedness is closely related to stability, approximation and numerical analysis. An initial concept of well-posedness introduced by Tykhonov, was given in scalar optimization (see [1]). Since then many kinds of well-posedness concepts and applications in game theory and vector optimization problems were studied (see [2]). Extensions

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of well-posedness to other related problems, such as fixed point problems[3], variational inequality problems[4] and believel optimization problems[5], were also considered. Fang et al.[6] discussed the well-posedness for equilibrium problems and for optimization problems with equilibrium constrains in real Banach spaces. Kimura et al.[7] studied the parametric well-posedness for vector equilibriums and proposed a generalized well-posed concept for equilibrium problems with equilibrium constrains in topological vector spaces setting. Huang et al.[8] investigated Levitin-Polyak well-posedness of variational inequality problems with functional constraints. Li and Li[9] studied two types of Levitin-Polyak well-posedness of vector equilibrium problems with variable domination structures in metricable locally convex Hausdorff topological vector spaces. Bianchi et al.[10] introduced and studied two notions of well-posedness for vector equilibrium problems in topological vector spaces and provided sufficient conditions for well-posedness in finite dimensional spaces.

Motivated and inspired by the above works, in this paper we shall investigate parametric well-posedness for weak vector equilibrium problems in real Banach space. We establish some metric characterizations of unique well-posedness and well-posedness for the problems. We also obtain that under suitable conditions, the unique well-posedness is equivalent to the existence and uniqueness of solutions. Finally, we derive sufficient conditions for well-posedness in finite dimensional space.

1 Preliminaries and definitions

Throughout this paper, let X be a real Banach space, Y be a real Hausdorff topological vector space, P be a normed space; and let C be a closed convex pointed cone in Y with $\mathrm{int}C \neq \emptyset$, where $\mathrm{int}C$ denotes the interior of C. We denote the class of the neighborhoods of 0 in X and Y by $N_X(0)$ and $N_Y(0)$, respectively, and denote the closed unit ball of X by B.

Let A be a nonempty subset of X, and $F : A \times A \to Y$ be a bifunction. The weak vector equilibrium problem is: find $x \in A$ such that

$$F(x, y) \notin -\mathrm{int}C, \quad \forall y \in A.$$

Let E be a nonempty subset of X. When the set A and the function F are perturbed by a parameter p, which varies over the space P, we can define the parameterized weak vector equilibrium problem $(\mathrm{WVEP})_p$: find $x \in A(p)$ such that

$$F(p, x, y) \notin -\mathrm{int}C, \quad \forall y \in A(p),$$

where A is a set-valued mapping from P to X with $A(p) \subset E$, for all $p \in P$, and $F : P \times E \times E \to Y$ is a trifunction.

For each $p \in P$, let $S(p)$ denote the solution set of $(\mathrm{WVEP})_p$:

$$S(p) = \{x \in A(p) : F(p, x, y) \notin -\mathrm{int}C, \quad \forall y \in A(p)\}.$$

Definition 1.1 Let W be a topological space, and Q be a topological vector space with a partial ordering cone K. Suppose that h is a vector-valued mapping from W to Q.

We say that h is K-continuous at $w_0 \in W$ if, for any neighborhood V of $0 \in Q$, there exists a neighborhood $U(w_0)$ of w_0 such that

$$h(w) \in h(w_0) + V + K, \quad \forall w \in U(w_0).$$

Moreover, h is said to be K-continuous on W if h is K-continuous at every point of W

Definition 1.2 Let W and Q be two topological spaces, H be a set-valued mapping from W to Q.

We say that H is upper semicontinuous at $w_0 \in W$ if, for any neighborhood $U(H(w_0))$ of $H(w_0)$, there exists a neighborhood $U(w_0)$ of w_0 such that

$$H(w) \subset U(H(w_0)), \quad \forall w \in U(w_0).$$

H is said to be upper semicontinuous on W if H is upper semicontinuous at every point of W.

We say that H is lower semicontinuous at $w_0 \in W$ if, for any net $\{w_\alpha : \alpha \in I\}$ converging to w_0 and for any $y_0 \in H(w_0)$, there exists a net $y_\alpha \in H(w_\alpha)$ such that converges to y_0. H is said to be lower semicontinuous on W if H is lower semicontinuous at every point of W

H is said to be continuous on W if H is both upper semicontinuous and lower semicontinuous on W.

Definition 1.3[11−12] Let E be a nonempty bounded subset of X. The measure of noncompactness μ of set E is defined by

$$\mu(E) = \inf\left\{t > 0 : \text{there exists } n \in N \text{ such that } E = \bigcup_{i=1}^{n} E_i,\ \text{diam} E_i < t, i = 1, \cdots, n\right\},$$

where the diam means the diameter of a set.

Definition 1.4 Let E_1, E_2 be two nonempty subsets of X. The Hausdorff metric $H(\cdot,\cdot)$ between E_1 and E_2 is defined by

$$H(E_1,E_2) = \max\{e(E_1,E_2), e(E_2, E_1)\},$$

where

$$e(E_1, E_2) = \sup_{a\in E_1} d(a, E_2), \qquad d(a, E_2) = \inf_{b\in E_2} \|a - b\|.$$

Let $\{E_n\}$ be a sequence of nonempty subsets of X. We say that E_n converges to E in the sense of Hausdorff metric if $H(E_n, E) \to 0$.

Lemma 1.1[13] If H is an upper semicontinuous set-valued mapping from topological vector space W to another topological vector space Q with closed values, then H is closed.

2 Well-posedness for weak vector equilibrium problems

Let $c \in \text{int}C$ be a given point. Fang et al.[6] discussed the well-posedness for numerical equilibrium problems and for optimization problems with numerical equilibrium constrains in real Banach spaces. Based on the definition of well-posedness in [6], we extend the definition to the case of weak vector equilibrium problems.

Definition 2.1 Let $p \in P$, and $\{p_n\} \subset P$ be a sequence converging to p. A sequence $\{x_n\}$ $(x_n \in A(p_n))$ is said to be an approximating sequence for $(\mathrm{WVEP})_p$ corresponding to $\{p_n\}$ if and only if there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$F(p_n, x_n, y) + \varepsilon_n c \notin -\mathrm{int}C, \quad \forall\, y \in A(p_n).$$

Definition 2.2 The family $\{(\mathrm{WVEP})_p : p \in P\}$ is said to be parametric uniquely well-posed if and only if,

(i) there exists a unique solution x_p to $(\mathrm{WVEP})_p$ for each $p \in P$;

(ii) for any given $p \in P$ and with $p_n \to p$, every approximating sequence for $(\mathrm{WVEP})_p$ corresponding to $\{p_n\}$ converging to x_p.

Definition 2.3 The family $\{(\mathrm{WVEP})_p : p \in P\}$ is said to be parametric well-posed if and only if,

(i) the solution set $S(p)$ of $(\mathrm{WVEP})_p$ is nonempty for all $p \in P$;

(ii) for any given $p \in P$ and $\{p_n\} \subset P$ with $p_n \to p$, every approximating sequence for $(\mathrm{WVEP})_p$ corresponding to $\{p_n\}$ has a subsequence converging to some point of $S(p)$.

The approximating solution set of $(\mathrm{WVEP})_p$ is defined by

$$\Omega_p(\delta, \varepsilon) = \bigcup_{p' \in B(p,\delta)} \{x \in A(p') : F(p', x, y) + \varepsilon c \notin -\mathrm{int}C,\, \forall y \in A(p')\}, \quad \delta \geqslant 0, \quad \varepsilon > 0.$$

where $B(p, \delta)$ denotes the closed ball centered at p with radius δ. When $\delta = 0$, $B(p, \delta)$ reduces to the point p.

The following theorem shows that the parametric uniquely well-posedness for $(\mathrm{WVEP})_p$ can be characterized by considering the behavior of the diameter of the approximating solution set.

Theorem 2.1 Let E be a nonempty subset of X, A be a continuous set-valued mapping from P to X with nonempty closed values and $A(p) \subset E$ for all $p \in P$. Let $F : P \times E \times E \to Y$ be a C-continuous mapping. Then the family $\{(\mathrm{WVEP})_p : p \in P\}$ is parametric uniquely well-posed if and only if $\forall p \in P$,

$$\Omega_p(\delta, \varepsilon) \neq \emptyset, \quad \forall \delta \geqslant 0, \varepsilon > 0, \quad \text{and } \mathrm{diam}\Omega_p(\delta, \varepsilon) \to 0, \quad \text{as}(\delta, \varepsilon) \to (0, 0). \tag{1}$$

Proof Suppose that the family $\{(\mathrm{WVEP})_p : p \in P\}$ is parametric uniquely well-posed. Then $(\mathrm{WVEP})_p$ has a unique solution x_p for each $p \in P$. Clearly $\Omega_p(\delta, \varepsilon) \neq \emptyset$ since $x_p \in \Omega_p(\delta, \varepsilon)$ for all $\delta \geqslant 0, \varepsilon > 0$. If $\mathrm{diam}\Omega_p(\delta, \varepsilon) \nrightarrow 0$ as $(\delta, \varepsilon) \to (0, 0)$, then there exist $l > 0$, $\delta_n \geqslant 0$, $\varepsilon_n > 0$ with $\delta_n \to 0$, $\varepsilon_n \to 0$, and $x_n, y_n \in \Omega_p(\delta_n, \varepsilon_n)$ such that

$$\|x_n - y_n\| > l, \quad \forall n \in N. \tag{2}$$

Since $x_n, y_n \in \Omega_p(\delta_n, \varepsilon_n)$ for each $n \in N$, there exist $p_n, q_n \in B(p, \delta_n)$ such that

$$F(p_n, x_n, y) + \varepsilon_n c \notin -\mathrm{int}C, \quad \forall y \in A(p_n)$$

and

$$F(q_n, y_n, y) + \varepsilon_n c \notin -\text{int}C, \quad \forall y \in A(q_n).$$

It is clear that $p_n \to p$, $q_n \to p$, so $\{x_n\}$ and $\{y_n\}$ are approximating sequences for $(\text{WVEP})_p$ corresponding to $\{p_n\}$ and $\{q_n\}$, respectively. By the parametric uniquely well-posedness of family $\{(\text{WVEP})_p : p \in P\}$, both $\{x_n\}$ and $\{y_n\}$ have to converge to x_p, which contradicts (15). Therefore (1) holds.

Conversely, suppose that condition (1) holds. Note that $\forall \delta \geqslant 0$, $\varepsilon > 0$, $\Omega_p(\delta, \varepsilon) \neq \emptyset$, let $p \in P$, $\{p_n\} \subset P$ with $p_n \to p$ and $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$, thus there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$F(p_n, x_n, y) + \varepsilon_n c \notin -\text{int}C, \quad \forall\, y \in A(p_n). \tag{3}$$

This yields that $x_n \in \Omega_p(\delta_n, \varepsilon_n)$ with $\delta_n = \|p_n - p\| \to 0$. It follows from (1) that $\{x_n\}$ is a Cauchy sequence and since X is complete, we can suppose that $\{x_n\}$ converges to $\bar{x} \in X$. Since A is upper semicontinuous with closed values, by Lemma 1.1, A is a closed mapping. Thus $\bar{x} \in A(p)$. Now we show that $\bar{x} \in S(p)$. If $\bar{x} \notin S(p)$, then there exists $\bar{y} \in A(p)$such that

$$F(p, \bar{x}, \bar{y}) \in -\text{int}C.$$

Therefore, there exists balanced $V \in N_Y(0)$ such that

$$F(p, \bar{x}, \bar{y}) + V + V \subset -\text{int}C. \tag{4}$$

By the lower semicontinuity of A, there exists $y_n \in A(p_n)$ such that $y_n \to \bar{y}$. Since F is $-C$-continuous on $P \times E \times E$ and $(p_n, x_n, y_n) \to (p, \bar{x}, \bar{y})$, there exists $n_1 \in N$, such that

$$F(p_n, x_n, y_n) \in F(p, \bar{x}, \bar{y}) + V - C, \quad \forall n \geqslant n_1. \tag{5}$$

For $\varepsilon_n \to 0$, there exists $n_2 \in N$, satisfying

$$\varepsilon_n c \in V, \quad \forall n \geqslant n_2. \tag{6}$$

By (4)-(6), there exists $n_0 = \max\{n_1, n_2\}$, $\forall n \geqslant n_0$, we have

$$F(p_n, x_n, y_n) + \varepsilon_n c \in F(p, \bar{x}, \bar{y}) + V + V - C \subset -\text{int}C - C = -\text{int}C. \tag{7}$$

This contradicts (3). Thus, $\bar{x} \in S(p)$.

To complete the proof, it is sufficient to prove that $(\text{WVEP})_p$ has a unique solution. If $(\text{WVEP})_p$ has two distinct solutions x_1 and x_2, it is easy to see that x_1, $x_2 \in \Omega_p(\delta, \varepsilon)$ for all $\delta \geqslant 0$, $\varepsilon > 0$. It follows that

$$0 < \|x_1 - x_2\| \leqslant \text{diam}\Omega_p(\delta, \varepsilon) \nrightarrow 0,$$

which contradicts with (1).

For the parametric well-posedness, we give the following characterization by considering the noncompactness of approximate solution set.

Theorem 2.2 Let E be a nonempty subset of X, P be a finite dimensional normed space. Let A be a continuous set-valued mapping from P to X with nonempty bounded closed values and $A(p)\subset E$ for all $p\in P$. Let $F:P\times E\times E\to Y$ be a $-C$-continuous mapping. Then the family $\{(\mathrm{WVEP})_p:p\in P\}$ is parametric well-posed if and only if $\forall p\in P$,

$$\Omega_p(\delta,\varepsilon)\neq\emptyset,\quad \forall\delta>0,\varepsilon>0,\ \text{ and }\ \mu(\Omega_p(\delta,\varepsilon))\to 0,\ \text{ as }\ (\delta,\varepsilon)\to(0,0). \tag{8}$$

Proof Suppose that the family $\{(\mathrm{WVEP})_p:p\in P\}$ is parametric well-posed. Then $S(p)\neq\emptyset$ for all $p\in P$. Since for any $\delta>0$, $\varepsilon>0$, $S(p)\subset\Omega_p(\delta,\varepsilon)$, we have $\Omega_p(\delta,\varepsilon)\neq\emptyset$.

We first show that $S(p)$ is compact. Let $\{x_n\}\subset S(p)$ be any sequence in $S(p)$, thus

$$F(p,x_n,y)\notin-\mathrm{int}C,\quad \forall y\in A(p),\quad \forall n\in N.$$

Therefore, $\{x_n\}$ is an approximating sequence corresponding to $\{p_n\}(\forall n\in N,p_n=p)$. Since the family $\{(\mathrm{WVEP})_p:p\in P\}$ is parametric well-posed, $\{x_n\}$ has a subsequence which converges to some point of $S(p)$. Thus, $S(p)$ is compact.

Now we show that $\mu(\Omega_p(\delta,\varepsilon))\to 0$, as$(\delta,\varepsilon)\to(0,0)$. Since P is finite dimensional, we know that $B(p,\delta)$ is compact. Since A is upper semicontinuous with bounded values, we can know that $A(B(p,\delta))$ is bounded in X(see [13], p.123, Proposition 2 and Proposition 4). Hence, $\Omega_p(\delta,\varepsilon)\subset A(B(p,\delta))$ is bounded in X. Observe that for every $\delta>0$, $\varepsilon>0$,

$$H(\Omega_p(\delta,\varepsilon),S(p))=\max\{e(\Omega_p(\delta,\varepsilon),S(p)),e(S(p),\Omega_p(\delta,\varepsilon))\}=e(\Omega_p(\delta,\varepsilon),S(p)).$$

Taking into account the compactness of $S(p)$, we get $\mu(S(p))=0$ (see [11,12]) and

$$\mu(\Omega_p(\delta,\varepsilon))\leqslant 2H(\Omega_p(\delta,\varepsilon),S(p))+\mu(S(p))=2e(\Omega_p(\delta,\varepsilon),S(p)).$$

To prove (8), it is sufficient to show

$$e(\Omega_p(\delta,\varepsilon),S(p))\to 0,\quad \text{as}\ \ (\delta,\varepsilon)\to(0,0).$$

If $e(\Omega_p(\delta,\varepsilon),S(p))\nrightarrow 0$ as $(\delta,\varepsilon)\to(0,0)$, then there exist $l>0$, $\delta_n>0$, $\varepsilon_n>0$ with $\delta_n\to 0$, $\varepsilon_n\to 0$,and $x_n\in\Omega_p(\delta_n,\varepsilon_n)$ such that

$$x_n\notin S(p)+lB,\quad \forall n\in N. \tag{9}$$

As $x_n\in\Omega_p(\delta_n,\varepsilon_n)$, there exists $p_n\in B(p,\delta_n)$ such that $x_n\in A(p_n)$ and

$$F(p_n,x_n,y)+\varepsilon_n c\notin-\mathrm{int}C,\quad \forall y\in A(p_n).$$

So $\{x_n\}$ is an approximating sequence corresponding to $\{p_n\}$. By the parametric well-posedness of family $\{(\mathrm{WVEP})_p:p\in P\}$, $\{x_n\}$ has a subsequence converging to some point of $S(p)$. This contradicts (9) and so

$$e(\Omega_p(\delta,\varepsilon),S(p))\to 0,\ \text{ as }\ (\delta,\varepsilon)\to(0,0).$$

Conversely, suppose that condition (8) holds. We first show that $\Omega_p(\delta,\varepsilon)$ is closed for all $\delta>0$, $\varepsilon>0$. Let $x_n\in\Omega_p(\delta,\varepsilon)$ with $x_n\to x$. Then for each $n\in N$, there exists $p_n\in B(p,\delta)$, $x_n\in A(p_n)$ such that

$$F(p_n,x_n,y)+\varepsilon c\notin -\mathrm{int}C,\quad \forall y\in A(p_n). \tag{10}$$

Since $B(p,\delta)$ is compact, without loss of generality, we can suppose that $\{p_n\}$ converges to $\bar p\in B(p,\delta)$. Since A is upper semicontinuous with closed values, by Lemma 1.1, A is a closed mapping. Thus $x\in A(p)$. We claim that

$$F(\bar p,x,y)+\varepsilon c\notin -\mathrm{int}C,\quad \forall y\in A(\bar p). \tag{11}$$

Otherwise, there exists $\bar y\in A(\bar p)$ such that

$$F(\bar p,x,\bar y)+\varepsilon c\in -\mathrm{int}C.$$

Hence, there exists balanced $V\in N_Y(0)$ such that

$$F(\bar p,x,\bar y)+\varepsilon c+V\subset -\mathrm{int}C. \tag{12}$$

By the lower semicontinuity of A, there is $y_n\in A(p_n)$ with $y_n\to\bar y$. Since F is $-C$-continuous on $P\times E\times E$ and $(p_n,x_n,y_n)\to(\bar p,x,\bar y)$, there exists $n_1\in N$, satisfying

$$F(p_n,x_n,y_n)\in F(\bar p,x,\bar y)+V-C,\quad \forall n\geqslant n_1. \tag{13}$$

By (12) and (13), $\forall n\geqslant n_1$, we have

$$F(p_n,x_n,y_n)+\varepsilon c\in F(\bar p,x,\bar y)+\varepsilon c+V-C\subset -\mathrm{int}C-C=-\mathrm{int}C.$$

This contradicts (10). Therefore, (11) holds. Hence, $\Omega_p(\delta,\varepsilon)$ is closed.

Now we show that

$$S(p)=\bigcap_{n=1}^{\infty}\Omega_p(\delta_n,\varepsilon_n),$$

where $\delta_n>0$,$\varepsilon_n>0$, $0<\delta_{n+1}\leqslant\delta_n$,$0<\varepsilon_{n+1}\leqslant\varepsilon_n$, and δ_n, $\varepsilon_n\to 0$. According to the definition of $\Omega_p(\delta,\varepsilon)$ and the proof above, we know that $\Omega_p(\delta_n,\varepsilon_n)$ is nonempty bounded closed and $\forall n\in N$, $\Omega_p(\delta_n,\varepsilon_n)\supset\Omega_p(\delta_{n+1},\varepsilon_{n+1})$. With the condition $\mu(\Omega_p(\delta_n,\varepsilon_n))\to 0$, we obtain that $\bigcap\limits_{n=1}^{\infty}\Omega_p(\delta_n,\varepsilon_n)$ is a nonempty compact set (see [12], p.139 Lemma 5.2). We choose $x\in\bigcap\limits_{n=1}^{\infty}\Omega_p(\delta_n,\varepsilon_n)$, thus $x\in\Omega_p(\delta_n,\varepsilon_n)$ for all $n\in N$. Therefore, there exists $p_n\in B(p,\delta_n)$ such that $x\in A(p_n)$ and

$$F(p_n,x,y)+\varepsilon_n c\notin -\mathrm{int}C,\quad \forall y\in A(p_n).$$

Obviously, $p_n\to p$. Since A is upper semicontinuous with closed values, by Lemma 1.1, A is a closed mapping. Thus $x\in A(p)$. By the sufficient proof of Theorem 2.1, we get $x\in S(p)$. Thus $\bigcap\limits_{n=1}^{\infty}\Omega_p(\delta_n,\varepsilon_n)\subset S(p)$. It is clear that $S(p)\subset\bigcap\limits_{n=1}^{\infty}\Omega_p(\delta_n,\varepsilon_n)$,

$S(p) = \bigcap_{n=1}^{\infty} \Omega_p(\delta_n, \varepsilon_n)$ is nonempty, compact. According to the theorem on p.412 in [11], we conclude that

$$e(\Omega_p(\delta,\varepsilon), S(p)) \to 0, \quad \text{as} \quad (\delta,\varepsilon) \to (0,0). \tag{14}$$

(To show (14) holds. If $e(\Omega_p(\delta_n,\varepsilon_n), S(p)) \nrightarrow 0$, without loss of generality, we can suppose that there exists $l > 0$ such that $e(\Omega_p(\delta_n,\varepsilon_n), S(p)) > l$ for every $n \in N$. Thus, there exists $a_n \in \Omega_p(\delta_n,\varepsilon_n)$ such that

$$d(a_n, S(p)) > l.$$

By the definition of $\mu(\Omega_p(\delta_n,\varepsilon_n))$, there exist finite sets $E_\alpha^{(n)}$ such that

$$\bigcup_\alpha E_\alpha^{(n)} = \quad \Omega_p(\delta_n,\varepsilon_n)$$

with

$$\mathrm{diam} E_\alpha^{(n)} \leqslant \mu(\Omega_p(\delta_n,\varepsilon_n)) + \frac{1}{n}.$$

Since $\Omega_p(\delta_n,\varepsilon_n) \supset \Omega_p(\delta_{n+1},\varepsilon_{n+1})$, we know that $\{a_1, a_2, \cdots\} \subset \Omega_p(\delta_1,\varepsilon_1)$. Hence there exists some $E_\alpha^{(1)}$ contains a subsequence $\{a_{i_1}, a_{i_2}, a_{i_3} \cdots\}$ of $\{a_1, a_2, \cdots\}$, where $1 < i_1$. For $\{a_{i_1}, a_{i_2}, \cdots\} \subset \Omega_p(\delta_2,\varepsilon_2)$, there exists some $E_\alpha^{(2)}$ contains a subsequence $\{a_{j_1}, a_{j_2}, \cdots\}$of $\{a_{i_1}, a_{i_2}, \cdots\}$, where $i_1 < j_1$. Continue doing this, we get infinite subsequences $\{a_{i_1}, a_{i_2}, \cdots\}$, $\{a_{j_1}, a_{j_2}, \cdots\}$,$\cdots$. Choose the first element of every subsequence, we get a new subsequence $\{a_{n_1}, a_{n_2}, \cdots\}(n_1 = i_1 < n_2 = j_1 < \cdots)$ of $\{a_1, a_2, \cdots\}$. Hence $\{a_{n_k}, a_{n_{k+1}}, a_{n_{k+2}}, \cdots\} \subset E_\alpha^{(k)}$. For $\mathrm{diam} E_\alpha^{(k)} \leqslant \mu(\Omega_p(\delta_k,\varepsilon_k)) + \frac{1}{k} \to 0$, $\{a_{n_1}, a_{n_2}, \cdots\}$ is a Cauchy sequence. Since X is complete, we can suppose that $\{a_{n_1}, a_{n_2}, \cdots\}$ converges to $\bar{a} \in X$.

Now we show that $\bar{a} \in S(p)$. For any given $n \in N$, there exists $m_0 \in N$ such that $n_{m_0} \geqslant n$. When $m \geqslant m_0$, we have $n_m \geqslant n_{m_0} \geqslant n$. Together with $\Omega_p(\delta_n,\varepsilon_n) \supset \Omega_p(\delta_{n+1},\varepsilon_{n+1})$, we have $a_{n_m} \in \Omega_p(\delta_n,\varepsilon_n)$. Since $\Omega_p(\delta_n,\varepsilon_n)$ is closed, we know that $\bar{a} \in \Omega_p(\delta_n,\varepsilon_n)$. By the arbitrary of n, we have $\bar{a} \in \bigcap_{n=1}^{\infty} \Omega_p(\delta_n,\varepsilon_n) = S(p)$. This contradicts with $d(a_n, S(p)) > l$. Hence $e(\Omega_p(\delta_n,\varepsilon_n), S(p)) \to 0$.)

Let $p_n \to p \in P$ and $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$. Thus, there exist $\delta_n > 0$, $\varepsilon_n > 0$ with $\delta_n \to 0$, $\varepsilon_n \to 0$ such that

$$F(p_n, x_n, y) + \varepsilon_n c \notin -\mathrm{int} C, \quad \forall\, y \in A(p_n).$$

This means that $x_n \in \Omega_p(\delta_n,\varepsilon_n)$ with $\delta_n = \|p_n - p\| + \frac{1}{n}$. It follows from (14) that

$$d(x_n, S(p)) \leqslant e(\Omega_p(\delta_n,\varepsilon_n), S(p)) \to 0.$$

Since $S(p)$ is compact, there exists $\bar{x}_n \in S(p)$ such that

$$\|x_n - \bar{x}_n\| = d(x_n, S(p)) \to 0.$$

Again from the compactness of $S(p)$, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to $\bar{x} \in S(p)$. Hence the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x} \in S(p)$. Thus the family $\{(\mathrm{WVEP})_p : p \in P\}$ is parametric well-posed.

The following theorem shows that under suitable conditions, the parametric uniquely well- posedness of $(\text{WVEP})_p$ is equilivalent to the existence and uniqueness of the solution.

Theorem 2.3 Let X be a finite dimensional normed space and E be a nonempty subset of X. Let A be a continuous set-valued mapping from P to X with nonempty compact values and $A(p) \subset E$ for all $p \in P$. Let $F : P \times E \times E \to Y$ be a $-C$-continuous mapping. Then the family $\{(\text{WVEP})_p : p \in P\}$ is parametric uniquely well-posed if and only if $\forall p \in P$, $(\text{WVEP})_p$ has a unique solution.

Proof The necessity is obvious. For the sufficiency, suppose that $S(p) = \{x_p\}$ for each $p \in P$. Let $p_n \to p \in P$ and $\{x_n\}(x_n \in A(p_n))$be an approximating sequence corresponding to $\{p_n\}$. Thus, there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$F(p_n, x_n, y) + \varepsilon_n c \notin -\text{int}C, \quad \forall\ y \in A(p_n).$$

We assert that $\{x_n\}$ is bounded. Indeed, if $\{x_n\}$ is unbounded, without loss of generality, we can suppose that $\|x_n\| \to +\infty$. Since $p_n \to p$, the set $\{p_n : n \in N\} \cup \{p\}$ is compact. Since A is upper semicontinuous, $A(\{p_n : n \in N\} \cup \{p\})$ is compact in X(see [13], p.121, Proposition 11). So $\{x_n\} \subset A(\{p_n : k \in N\} \cup \{p\})$ is bounded, which contradicts $\|x_n\| \to +\infty$.

Now we show that $x_n \to x_p$. Suppose to the contrary that $\{x_n\}$ dosen't converge to x_p. Then there exist $l > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k} \notin x_p + lB, \quad \forall k \in N. \tag{15}$$

Since $\{x_{n_k}\}$ is bounded, it has a convergent subsequence. Without loss of generality, we assume that $\{x_{n_k}\}$ converges to $\bar{x} \in X$. Since A is upper semicontinuous with closed values, A is a closed mapping. Hence, $\bar{x} \in A(p)$. By the sufficient proof of Theorem 3.1, we get $\bar{x} \in S(p)$. By the uniqueness of the solution, $\bar{x} = x_p$. This contradicts (15). Hence $x_n \to x_p$. Therefore, the family $\{(\text{WVEP})_p : p \in P\}$ is parametric uniquely well-posed.

For the parametric well-posedness, we have the following result.

Theorem 2.4 Let X be a finite dimensional normed space and E be a nonempty subset of X. Let A be a continuous set-valued mapping from P to X with nonempty compact values and $A(p) \subset E$ for all $p \in P$. Let $F : P \times E \times E \to Y$ be a $-C$-continuous mapping. If $\forall p \in P$, $S(p) \neq \emptyset$, then the family $\{(\text{WVEP})_p : p \in P\}$ is parametric well-posed.

Proof Let $p_n \to p \in P$ and $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$. Thus, there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$F(p_n, x_n, y) + \varepsilon_n c \notin -\text{int}C, \quad \forall\ y \in A(p_n).$$

By the sufficient proof of Theorem 2.3, we know that $\{x_n\}$ has a subsequence converging to some point of $S(p)$. Therefore, the family $\{(\text{WVEP})_p : p \in P\}$ is parametric well-posed.

Remark 2.1 Fang et al.[6] discussed the well-posedness for numerical equilibrium problems and for optimization problems with numerical equilibrium constrains in real Banach spaces. In this paper, for Theorem 2.1-2.4, we use the method which was utilized in [6] to deal with the vector equilibrium problems.

It is well known that both the assumptions of concavity and monotonicity together with their generalizations are strictly related to many results concerning equilibrium problems. The following theorem focus on concave functions, by providing results that extend a similar one in [10]. Bianchi et al.[10] studied the well-posedness of vector equilibrium problems without any parameter, next we consider the well-posedness of weak vector equilibrium problems which was perturbed by parameter.

We denote the boundary and closure of set E by $\mathrm{bd}(E)$ and $\mathrm{cl}(E)$ respectively.

Theorem 2.5 Let X be a finite dimensional normed space, E be a nonempty closed convex subset of X, and $\forall p \in P$, $A(p) = E$. Let $F : P \times E \times E \to Y$. If the following conditions

(i) the solution set $S(p)$ is nonempty and bounded for all $p \in P$;

(ii) $F(p,x,x) = 0$ for all $p \in P$, $x \in E$; and for each $p \in P$, $F(p,x,y) \notin -\mathrm{bd}(C)$, whenever $x \in S(p)$, and $y \in E \backslash S(p)$;

(iii) for any given $(x,y) \in E \times E$, $F(\cdot,x,y)$ is continuous on P; for any given $y \in E$, $F(\cdot,\cdot,y)$ is $-C$-continuous on $P \times E$;

(iv) for any given $(p,y) \in P \times E$, $F(p,\cdot,y)$ is C-concave on E;

(v) if $p_n \to p \in P$, $\{x_n\}$ is an approximating sequence corresponding to $\{p_n\}$, then for any $y \in E$, the sequence $\{F(p_n,x_n,y)\}$ is bounded in Y

hold, then the family $\{(\mathrm{WVEP})_p : p \in P\}$ is parametric well-posed.

Proof Let $p_n \to p \in P$ and $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$. Then there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$F(p_n,x_n,y) + \varepsilon_n c \notin -\mathrm{int}C, \quad \forall\, y \in E. \tag{16}$$

We first show that $\{x_n\}$ exists bounded subsequences. If not, then any subsequence of $\{x_n\}$ is unbounded. Without loss of generality, let us suppose that $\|x_n\| \to +\infty$ and $x_n \notin \mathrm{cl}(S(p)+B)$ for all $n \in N$. Let $Q = \mathrm{cl}(S(p)+B)\backslash\mathrm{int}(S(p)+B)$. Since $S(p)$ is bounded and X is finite dimensional, we know that Q is compact. For any $x \in S(p)$ we have $x + B \subset S(p) + B$, therefore $S(p) \subset \mathrm{int}(S(p)+B)$, which implies that $S(p) \cap Q = \emptyset$. Fix an arbitrary $\bar{x} \in S(p)$ and let

$$\lambda_n = \sup\{\lambda \in [0,1] : \lambda\bar{x} + (1-\lambda)x_n \notin S(p) + B\}, \quad n \in N.$$

Let $x'_n := \lambda_n\bar{x} + (1-\lambda_n)x_n$. We claim that $x'_n \in Q$ for each $n \in N$ and$\lambda_n \to 1$.

(In fact, by $\forall n \in N,\ x_n \notin \mathrm{cl}(S(p)+B)$, we know $0 \in \{\lambda \in [0,1] : \lambda\bar{x} + (1-\lambda)x_n \notin S(p)+B\}$, thus λ_n is meaningful and $\lambda_n > 0$. If there exists $n_0 \in N$ such that $\lambda_{n_0} = 1$, we know that there exists $\varepsilon_m \to 0^+$ such that

$$(1-\varepsilon_m)\bar{x} + \varepsilon_m x_{n_0} = \bar{x} + \varepsilon_m(x_{n_0} - \bar{x}) \notin S(p) + B$$

for any $m \in N$. But for $\bar{x} \in S(p)$ and $\varepsilon_m \to 0^+$, there exists enough large m such that

$$\bar{x} + \varepsilon_m(x_{n_0} - \bar{x}) \in S(p) + B.$$

It is a contradiction. Hence $0 < \lambda_n < 1$ for all $n \in N$.

If there exists $n_0 \in N$ such that $x'_{n_0} = \lambda_{n_0}\bar{x} + (1-\lambda_{n_0})x_{n_0} \notin Q$, then we can divide the following two cases:

1. If $x'_{n_0} \in \mathrm{int}(S(p)+B)$, then there exists $\lambda'_{n_0} > 0$ such that

$$x'_{n_0} + \lambda'_{n_0}B \subset S(p)+B.$$

That is

$$\lambda_{n_0}\bar{x} + (1-\lambda_{n_0})x_{n_0} + \lambda'_{n_0}B \subset S(p)+B.$$

Since $\frac{\lambda'_{n_0}}{2}B$ is a balanced absorbing set, there exists $0 < \xi < \lambda_{n_0}$ such that

$$-r\bar{x} \in \frac{\lambda'_{n_0}}{2}B, \quad rx_{n_0} \in \frac{\lambda'_{n_0}}{2}B$$

and

$$(\lambda_{n_0} - r)\bar{x} + [1-(\lambda_{n_0}-r)]x_{n_0} \in S(p)+B$$

for all $0 \leqslant r \leqslant \xi$. By the definition of λ_{n_0}, there exists $\lambda_{n_0} - \xi < \lambda_0 \leqslant \lambda_{n_0}$ such that

$$\lambda_0\bar{x} + (1-\lambda_0)x_0 \notin S(p)+B.$$

That is to say that there exists $0 \leqslant r_0 = \lambda_{n_0} - \lambda_0 < \xi$ such that

$$(\lambda_{n_0} - r_0)\bar{x} + [1-(\lambda_{n_0}-r_0)]x_{n_0} \notin S(p)+B.$$

It is a contradiction.

2. If $x'_{n_0} \in X\backslash(\mathrm{cl}(S(p)+B))$, then there exists $\lambda'_{n_0} > 0$ such that

$$x'_{n_0} + \lambda'_{n_0}B \subset X\backslash(\mathrm{cl}(S(p)+B)).$$

That is

$$\lambda_{n_0}\bar{x} + (1-\lambda_{n_0})x_{n_0} + \lambda'_{n_0}B \subset X\backslash(\mathrm{cl}(S(p)+B)).$$

Since $\frac{\lambda'_{n_0}}{2}B$ is a balanced absorbing set, there is $0 < \xi < 1-\lambda_{n_0}$ such that

$$\xi\bar{x} \in \frac{\lambda'_{n_0}}{2}B \quad \text{and} \quad -\xi x_{n_0} \in \frac{\lambda'_{n_0}}{2}B.$$

Hence

$$(\lambda_{n_0} + \xi)\bar{x} + [1-(\lambda_{n_0}+\xi)]x_{n_0} \in X\backslash(\mathrm{cl}(S(p)+B)),$$
$$(\lambda_{n_0} + \xi)\bar{x} + [1-(\lambda_{n_0}+\xi)]x_{n_0} \notin S(p)+B.$$

It contradicts the definition of λ_{n_0}.

Therefore, $x'_n = \lambda_n\bar{x} + (1-\lambda_n)x_n \in Q$ for all $n \in N$. By the compactness of Q, $\{\|x'_n\|\}$ is a bounded sequence of real numbers.

If $\lambda_n \nrightarrow 1$, then there exist some $\delta < 1$ and a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_k} \leqslant \delta$ for all $k \in N$. Thus we may write

$$(1-\lambda_{n_k})x_{n_k} = x'_{n_k} - \lambda_{n_k}\bar{x},$$

from which

$$\|x_{n_k}\| \leqslant \frac{1}{(1-\lambda_{n_k})}\left\|x'_{n_k}\right\| + \frac{\lambda_{n_k}}{(1-\lambda_{n_k})}\|\bar{x}\|$$
$$\leqslant \frac{1}{1-\delta}(\left\|x'_{n_k}\right\| + \|\bar{x}\|).$$

But this contradicts the fact that $\|x_n\| \to +\infty$.)

Since the set Q is compact, there exists a subsequence $\{\lambda_{n_k}\bar{x} + (1-\lambda_{n_k})x_{n_k}\}$ of $\{\lambda_n\bar{x} + (1-\lambda_n)x_n\}$, which converges to $x' \in Q$. Since E is a closed convex set, we have

$$\lambda_n\bar{x} + (1-\lambda_n)x_n \in E$$

for any $n \in N$ and $x' \in E$. By condition (iv), we obtain that $\forall k \in N$:

$$\lambda_{n_k}F(p_{n_k}, \bar{x}, x') + (1-\lambda_{n_k})F(p_{n_k}, x_{n_k}, x') \in F(p_{n_k}, \lambda_{n_k}\bar{x} + (1-\lambda_{n_k})x_{n_k}, x') - C. \quad (17)$$

We claim that $F(p, \bar{x}, x') \in -C$. If not, then there exists $V \in N_Y(0)$ such that

$$(V - C) \cap (F(p, \bar{x}, x') + V) = \emptyset. \quad (18)$$

It follows from condition (iii) that there exists k_1 such that

$$F(p_{n_k}, \lambda_{n_k}\bar{x} + (1-\lambda_{n_k})x_{n_k}, x') \in F(p, x', x') + V - C, \quad \forall k \geqslant k_1.$$

By condition (ii), $F(p, x', x') = 0$, we have

$$F(p_{n_k}, \lambda_{n_k}\bar{x} + (1-\lambda_{n_k})x_{n_k}, x') - C \subset V - C, \quad \forall k \geqslant k_1. \quad (19)$$

By condition (iii), (v) and $\lambda_{n_k} \to 1$, $\lambda_{n_k}F(p_{n_k}, \bar{x}, x') + (1-\lambda_{n_k})F(p_{n_k}, x_{n_k}, x') \to F(p, \bar{x}, x')$.

Therefore, there exists k_2 such that

$$\lambda_{n_k}F(p_{n_k}, \bar{x}, x') + (1-\lambda_{n_k})F(p_{n_k}, x_{n_k}, x') \in F(p, \bar{x}, x') + V, \quad \forall k \geqslant k_2. \quad (20)$$

According to (17), (19) and (20), we get

$$(V - C) \cap (F(p, \bar{x}, x') + V) \neq \emptyset.$$

This contradicts (18). Hence $F(p, \bar{x}, x') \in -C$. Since $\bar{x} \in S(p)$ and $x' \in Q \cap E$, $F(p, \bar{x}, x') \notin -\mathrm{int}C$. Those mean that $F(p, \bar{x}, x') \in -\mathrm{bd}(C)$. Observe that $x' \in Q$, $S(p) \cap Q = \emptyset$, so $x' \in E \backslash S(p)$. It follows from the condition (ii) that$F(p, \bar{x}, x') \notin -\mathrm{bd}(C)$. This is a contradiction. Thus $\{x_n\}$ exists bounded subsequence.

Now we show that $\{x_n\}$ has a subsequence converging to some point of $S(p)$. Since X is finite dimensional and $\{x_n\}$ exists bounded subsequence, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to $\bar{x} \in A(p) = E$. We assert that $\bar{x} \in S(p)$. If $\bar{x} \notin S(p)$, then there exists $\bar{y} \in E$ such that

$$F(p, \bar{x}, \bar{y}) \in -\mathrm{int}C.$$

Therefore, there exists balanced $V \in N_Y(0)$ such that

$$F(p, \bar{x}, \bar{y}) + V + V \subset -\text{int}C. \tag{21}$$

Again from (iii), and together with $(p_{n_k}, x_{n_k}) \to (p, \bar{x})$, $\varepsilon_{n_k} \to 0$, there exists k_3 such that

$$F(p_{n_k}, x_{n_k}, \bar{y}) \in F(p, \bar{x}, \bar{y}) + V - C \quad \text{and} \quad \varepsilon_{n_k} c \in V. \ \ \forall k \geqslant k_3. \tag{22}$$

It follows from (21) and (22) that

$$\begin{aligned} F(p_{n_k}, x_{n_k}, \bar{y}) + \varepsilon_{n_k} c &\in F(p, \bar{x}, \bar{y}) + V + V - C \\ &\subset -\text{int}C - C \\ &= -\text{int}C, \quad \forall k \geqslant k_3. \end{aligned}$$

This contradicts (16). Hence $\bar{x} \in S(p)$. Hence, the family $\{(\text{WVEP})_p : p \in P\}$ is parametric well-posed.

Corollary 2.1 Let E be a nonempty compact subset of X, and $\forall p \in P$, $A(p) = E$. Let $F: \ P \times E \times E \to Y$. Assume that

(i) the solution set $S(p)$ is nonempty for all $p \in P$;

(ii) for any given $y \in E$, $F(\cdot, \cdot, y)$ is $-C$-continuous on $P \times E$.

Then the family $\{(\text{WVEP})_p : p \in P\}$ is parametric well-posed.

Proof Let $p_n \to p \in P$ and $\{x_n\}$ be an approximating sequence corresponding to $\{p_n\}$. Thus there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$F(p_n, x_n, y) + \varepsilon_n c \notin -\text{int}C, \quad \forall \ y \in E.$$

Observe that E is compact, thus $\{x_n\}$ has a convergent subsequence. By the proof of Theorem 2.5, it is easy to see that the family $\{(\text{WVEP})_p : p \in P\}$ is parametric well-posed.

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