Some Results on the Laplacian Spectral Radii of Tricyclic Graphs*

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Abstract A tricyclic graph is a connected graph in which the number of edges equals the number of vertices plus two. Let $\Delta(G)$ and $\mu(G)$ denote the maximum degree and the Laplacian spectral radius of a graph G, respectively. Let $\mathcal{T}(n)$ be the set of tricyclic graphs on n vertices. In this paper, it is proved that, for two graphs H_1 and H_2 in $\mathcal{T}(n)$, if $\Delta(H_1) > \Delta(H_2)$ and $\Delta(H_1) \geqslant \frac{n+7}{2}$, then $\mu(H_1) > \mu(H_2)$. As an application of this result, we determine the seventh to the nineteenth largest values of the Laplacian spectral radii among all the graphs in $\mathcal{T}(n)(n \geqslant 9)$ together with the corresponding graphs.

Keywords Laplacian spectral radius, tricyclic graphs, maximum degree

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关于三圈图的拉普拉斯谱半径的一些结果

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摘要 边数等于点数加二的连通图称为三圈图. 设 $\Delta(G)$ 和 $\mu(G)$ 分别表示图 G 的最大度和其拉普拉斯谱半径，设 $\mathcal{T}(n)$ 表示所有 n 阶三圈图的集合，证明了对于 $\mathcal{T}(n)$ 的两个图 H_1 和 H_2，若 $\Delta(H_1) > \Delta(H_2)$ 且 $\Delta(H_1) \geqslant \frac{n+7}{2}$, 则 $\mu(H_1) > \mu(H_2)$. 作为该结论的应用， 确定了 $\mathcal{T}(n)(n \geqslant 9)$ 中图的第七大至第十九大的拉普拉斯谱半径及其相应的极图.

关键词 拉普拉斯谱半径，三圈图，最大度

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0 Introduction

In this paper, all the graphs are simple graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. Denote by $N_G(v)$(or simply $N(v)$) the set of all neighbors of a vertex v of G, and by $d_G(v)$(or simply $d(v)$) the degree of v. Let $D(G) = \mathrm{diag}(d(v_1), d(v_2), \cdots, d(v_n))$ be the diagonal matrix of vertex degrees. The Laplacian matrix $L(G)$ of G is defined by $L(G) = D(G) - A(G)$, where $A(G)$ is the (0,1)-adjacency matrix of G. The characteristic

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polynomial $\det(xI-L(G))$ is denoted by $\Phi(G;x)$. It is well known that $L(G)$ is positive semi-definite, symmetric and singular. We denote the ith eigenvalue of $L(G)$ by $\mu_i(L(G)))$ (or simply $\mu_i(G)$) and order them in non-increasing order, i.e., $\mu_1(G)\geqslant\mu_2(G)\geqslant\cdots\geqslant\mu_n(G)$, and $\mu_1(G)$ is called the Lapalcian spectral radius of G, denoted by $\mu(G)$ in this paper.

A tricyclic graph is a connected graph in which the number of edges equals the number of vertices plus two. Let $\mathcal{T}(n)$ be the set of tricyclic graphs on n vertices. From [1] [1] we know that for any graph $G\in\mathcal{T}(n)$, G can be obtained from some $\mathcal{T}_i$ shown in Fig.1 by attaching trees (maybe empty) to some vertices. Let $\Delta(G)$ denote the maximum degree of a graph G. Denote by $\mathcal{T}(n,\Delta)$ the graphs whose maximum degree is Δ in $\mathcal{T}(n)$.

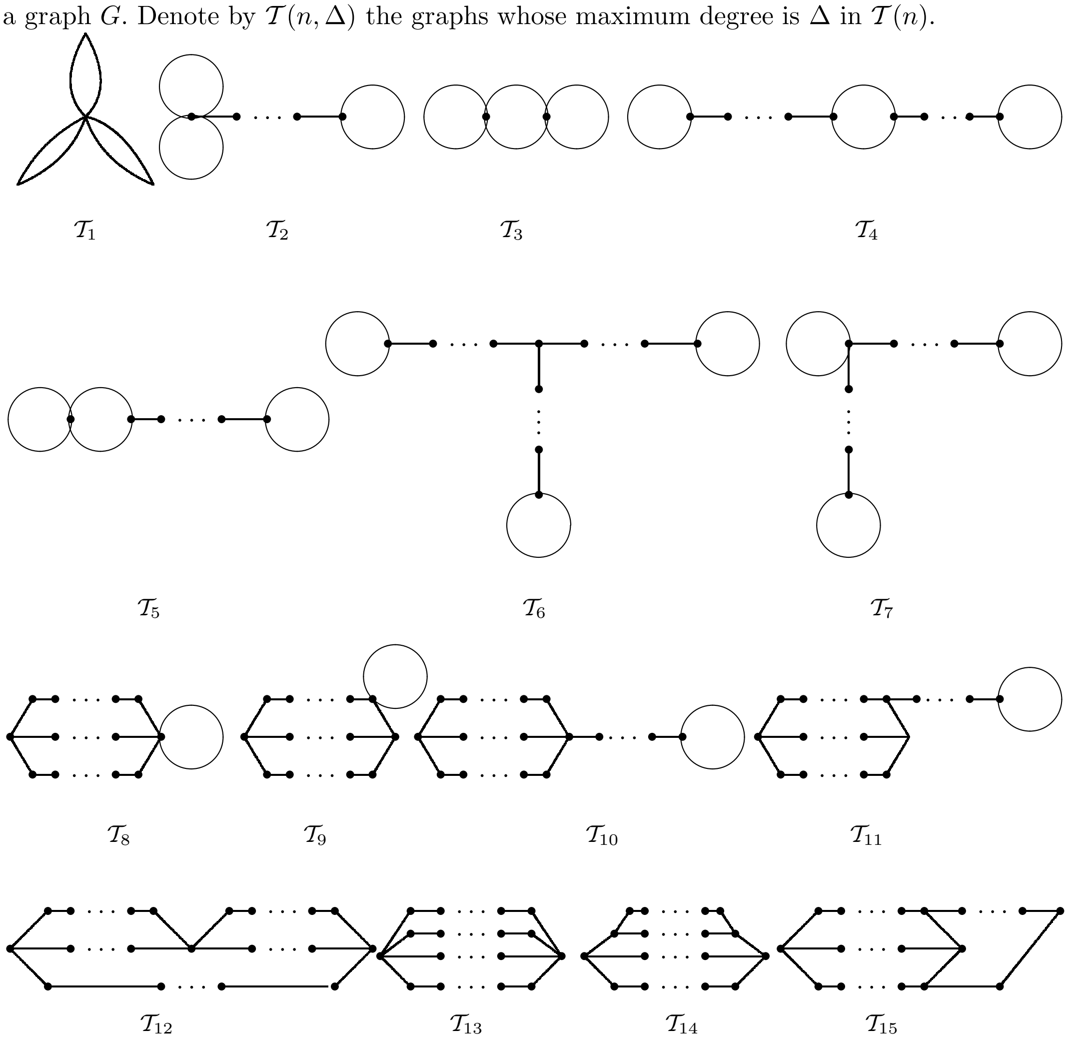

Fig. 1 $\mathcal{T}_1$ - $\mathcal{T}_{15}$

1 A relation between $\mu(G)$ and $\Delta(G)$ of a graph G in $\mathcal{T}(n)$

Ren in [2] determined the first six largest Laplacian spectral radii among all the graphs in $\mathcal{T}(n)(n \geqslant 9)$ together with the corresponding graphs(see Lemma 1).

Lemma 1 [2] Let $G_1, G_2, \cdots, G_{11} (n \geqslant 9)$ be the graphs in $\mathcal{T}(n)$ as shown in Figure.2, and G be any graph in $\mathcal{T}(n) \backslash \{G_1, G_2, \cdots, G_{11}\}$. Then we have $\mu(G_1) = \mu(G_2) = \mu(G_3) = \mu(G_4) = \mu(G_5) > \mu(G_6) > \mu(G_7) > \mu(G_8) = \mu(G_9) > \mu(G_{10}) > \mu(G_{11}) > \mu(G)$.

Lemma 2 [4] If G has at least one edge, then $\mu(G) \geqslant \Delta(G)+1$. For G being a connected graph on $n > 1$ vertices, equality is attained if and only if $\Delta(G) = n-1$.

By Lemma 2 and the fact that $\mathcal{T}(n, n-1) = \{G_1, G_2, G_3, G_4, G_5\}$, we may obtain the following result.

Corollary 1 If G is a graph in $\mathcal{T}(n) \backslash \{G_1, G_2, G_3, G_4, G_5\}$, then $\mu(G) > \Delta(G)+1$.

In the following we will give a relation between $\mu(G)$ and $\Delta(G)$ of a graph G in $\mathcal{T}(n)$(see Theorem 1).

Lemma 3 [6] Let G be a simple graph, then

$$\mu(G) \leqslant \max\{d(v) + m(v) | v \in V(G)\},$$

where $m(v) = \frac{\sum\limits_{u \in N(v)} d(u)}{d(v)}$.

Theorem 1 Let H_1, H_2 be graphs in $\mathcal{T}(n)$. If $\Delta(H_1) > \Delta(H_2)$ and $\Delta(H_1) \geqslant \frac{n+7}{2}$, then $\mu(H_1) > \mu(H_2)$.

Proof In order to prove Theorem 1, we first give two claims in the following.

Claim 1. Let G be graphs in $\mathcal{T}(n)$. If $\Delta(G) \geqslant \frac{n+5}{2}$, then $\mu(G) \leqslant \Delta(G)+2$.

Proof of Claim 1. Let v be a vertex of G and write $d(v) = t$. Set

$$N(v) = \{v_1, v_2, \cdots, v_t\}, \quad A(v_i) = N(v_i) \backslash \{v\}, i = 1, 2, \cdots, t.$$

Since G is a graph in $\mathcal{T}(n)$, we have

$$| A(v_1) | + | A(v_2) | + \cdots + | A(v_t) | \leqslant n - t + 5.$$

Hence

$$\sum_{v_i \in N(v)} d(v_i) = \sum_{i=1}^{t} (| A(v_i) | + 1) \leqslant n + 5.$$

and

$$d(v) + m(v) \leqslant t + \frac{n+5}{t}.$$

Let $g(t) = t + \frac{n+5}{t}$, then $g(t)$ is convex when $t > 0$. Hence when $2 \leqslant t \leqslant \Delta(G)$, we have

$$d(v) + m(v) \leqslant g(t) \leqslant \max\{g(2), g(\Delta(G))\}. \tag{1}$$

Let v be any vertex of G. If $d(v) = 1$, then

$$d(v) + m(v) \leqslant 1 + \Delta(G) < \Delta(G) + 2. \tag{2}$$

If $2 \leqslant d(v) \leqslant \Delta(G)$ and $\Delta(G) \geqslant \frac{n+5}{2}$, then from Ineq.(1) we have

$$d(v) + m(v) \leqslant \max\{2 + \frac{n+5}{2}, \Delta(G) + \frac{n+5}{\Delta(G)}\} \leqslant \Delta(G) + 2. \tag{3}$$

Hence $\mu(G) \leqslant \Delta(G) + 2$ follows from Ineqs.(2) and (3) and Lemma 3.

Claim 2. Let G be graphs in $\mathcal{T}(n)$. If $\Delta(G) \leqslant \frac{n+5}{2}$, then $\mu(G) \leqslant \frac{n+9}{2}$.

Proof of Claim 2. Similarly to the proof of Claim 1, let v be any vertex of G. If $d(v) = 1$, and $\Delta(G) \leqslant \frac{n+5}{2}$, then

$$d(v) + m(v) \leqslant 1 + \Delta(G) < \frac{n+9}{2}. \tag{4}$$

If $2 \leqslant d(v) \leqslant \Delta(G) \leqslant \frac{n+5}{2}$, and noting that $g(2) = g(\frac{n+5}{2}) = \frac{n+9}{2}$, then from Ineq.(1) we have

$$d(v) + m(v) \leqslant \frac{n+9}{2}. \tag{5}$$

Hence $\mu(G) \leqslant \frac{n+9}{2}$ follows from Ineqs.(4) and (5) and Lemma 3.

If $H_1 \in \{G_1, G_2, G_3, G_4, G_5\}$, then the hypothesis that $\Delta(H_1) > \Delta(H_2)$ insures that $H_2 \notin \{G_1, G_2, G_3, G_4, G_5\}$. Then $\mu(H_1) > \mu(H_2)$ follows from Lemma 1. Now we suppose that $H_1 \notin \{G_1, G_2, G_3, G_4, G_5\}$, then $\mu(H_1) > \Delta(H_1) + 1$ holds from Corollary 1.

If $\Delta(H_2) \geqslant \frac{n+5}{2}$, by Claim 1 and Corollary 1 we have

$$\mu(H_2) \leqslant \Delta(H_2) + 2 \leqslant \Delta(H_1) + 1 < \mu(H_1).$$

If $\Delta(H_2) \leqslant \frac{n+5}{2}$, by Claim 2 and Corollary 1 we have

$$\mu(H_2) \leqslant \frac{n+9}{2} \leqslant \Delta(H_1) + 1 < \mu(H_1).$$

The proof is completed.

2 Ordering the graphs in $\mathcal{T}(n)$ by their Laplacian spectral radii

In this section we first cite a formula for the characteristic polynomial of the matrix $L(H)$ when H is a coalescence of some two graphs. Suppose we have two graphs H_1 and H_2 with $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$; the coalescence of H_1 and H_2 with respect to v_1 and v_2 is formed by identifying v_1 and v_2 and is denoted by $H_1 \cdot H_2$. In other words, $V(H_1 \cdot H_2) = V(H_1) \cup V(H_2) \cup \{v^*\} - \{v_1, v_2\}$, with two vertices in $H_1 \cdot H_2$ adjacent if they are adjacent in H_1 or H_2, or if one is v^* and the other is adjacent to v_1 or v_2(see [3]). Let $L_v(H)$ be the principal sub-matrix of $L(H)$ obtained by deleting the row and column corresponding to the vertex v of H. Write $det(xI - L(H)) = \Phi(H; x)$, and $det(xI - L_v(H)) = \Phi(L_v(H); x)$.

Lemma 4 [7] Let $H_1 \cdot H_2$ be the coalescence of H_1 and H_2 with respect to v_1 and v_2, we have

$$\begin{aligned}\Phi(H_1 \cdot H_2; x) =&\Phi(H_1; x)\Phi(L_{v_2}(H_2); x) + \Phi(H_2; x)\Phi(L_{v_1}(H_1); x)\\ &- x\Phi(L_{v_1}(H_1); x)\Phi(L_{v_2}(H_2); x).\end{aligned}$$

Let $g_1(x) = x^5 - (n+10)x^4 + (11n+29)x^3 - (40n+16)x^2 + (54n-19)x - 21n$;
$g_2(x) = x^4 - (n+7)x^3 + (8n+5)x^2 - (13n-7)x + 5n$;
$g_3(x) = x^4 - (n+6)x^3 + (7n+6)x^2 - (13n-5)x + 6n$;
$g_4(x) = x^6 - (n+11)x^5 + (12n+41)x^4 - (53n+55)x^3 + (106n+4)x^2 - (94n-26)x + 29n$;
$g_5(x) = x^5 - (n+8)x^4 + (9n+18)x^3 - (27n+6)x^2 + (31n-10)x - 11n$;
$g_6(x) = x^3 - (n+3)x^2 + (4n-2)x - 2n$;
$g_7(x) = x^6 - (n+11)x^5 + (12n+39)x^4 - (51n+45)x^3 + (95n-9)x^2 - (77n-31)x + 21n$;
$g_8(x) = x^4 - (n+6)x^3 + (7n+4)x^2 - (11n-6)x + 4n$;
$g_9(x) = x^5 - (n+9)x^4 + (10n+21)x^3 - (31n+3)x^2 + (33n-16)x - 10n$;
$g_{10}(x) = x^6 - (n+11)x^5 + (12n+40)x^4 - (52n+48)x^3 + (99n-10)x^2 - (80n-34)x + 21n$;
$g_{11}(x) = x^5 - (n+8)x^4 + (9n+17)x^3 - (26n+2)x^2 + (27n-13)x - 8n$;
$g_{12}(x) = x^4 - (n+5)x^3 + (6n+3)x^2 - (9n-5)x + 3n$;
$g_{13}(x) = x^3 - (n+2)x^2 + (3n-2)x - n$.

In the following, we will give the characteristic polynomial of the graphs in Fig.2 by using Lemma 4 and determine the seventh to the nineteenth largest values of the Laplacian spectral radii among all the graphs in $\mathcal{T}(n)$.

It is not difficult to see that any graph in $\mathcal{T}(n, n-2)$ is obtained from some graph in $\mathcal{T}_i (i = 1, 8, 9, 12, 13, 15)$ by attaching some trees at some vertices. Furthermore, we may check that $\mathcal{T}(n, n-2) = \{G_6, G_7, \cdots, G_{32}\}$, and $G_i (i = 6, 7, \cdots, 32)$ are shown in Fig.2.

Theorem 2 Let $G_i (i = 12, 13, \cdots, 32)$ be graphs as shown in Fig.2. When $n \geqslant 9$, we have

(1) $\Phi(G_{12}; x) = x(x-1)^{n-6} g_1(x)$;
(2) $\Phi(G_{13}; x) = x(x-1)^{n-7}(x-2)^2 g_2(x)$;
(3) $\Phi(G_{14}; x) = x(x-1)^{n-7}(x^2-5x+5) g_3(x)$;
(4) $\Phi(G_{15}; x) = x(x-1)^{n-7} g_4(x)$;
(5) $\Phi(G_{16}; x) = x(x-1)^{n-7}(x-3) g_5(x)$;
(6) $\Phi(G_{17}; x) = x(x-1)^{n-7}(x-3)^2(x-2) g_6(x)$;
(7) $\Phi(G_{18}; x) = x(x-1)^{n-7}(x-2)^2(x-4) g_6(x)$;
(8) $\Phi(G_{19}; x) = x(x-1)^{n-6}(x-3)(x-4) g_6(x)$;
(9) $\Phi(G_{20}; x) = x(x-1)^{n-7} g_7(x)$;
(10) $\Phi(G_{21}; x) = x(x-1)^{n-7}(x-2)(x-3) g_8(x)$;
(11) $\Phi(G_{22}; x) = x(x-1)^{n-6}(x-4) g_8(x)$;
(12) $\Phi(G_{23}; x) = x(x-1)^{n-7}(x-2) g_9(x)$;
(13) $\Phi(G_{24}; x) = x(x-1)^{n-7} g_{10}(x)$;
(14) $\Phi(G_{25}; x) = x(x-1)^{n-7}(x-3) g_{11}(x)$;
(15) $\Phi(G_{26}; x) = x(x-1)^{n-7}(x-3)^2 g_{12}(x)$;
(16) $\Phi(G_{27}; x) = x(x-1)^{n-7}(x-2)(x-3) g_{12}(x)$;

(17) $\Phi(G_{28};x) = x(x-1)^{n-7}(x-3)^3 g_{13}(x)$;

(18) $\Phi(G_{29};x) = x(x-1)^{n-7}(x-2)(x-3)(x-4)g_{13}(x)$;

(19) $\Phi(G_{30};x) = x(x-1)^{n-7}(x-3)(x^2-6x+7)g_{13}(x)$;

(20) $\Phi(G_{31};x) = x(x-1)^{n-7}(x-2)^2(x-5)g_{13}(x)$;

(21) $\Phi(G_{32};x) = x(x-1)^{n-6}(x-4)^2 g_{13}(x)$;

(22) $\mu(G_{12}) > \mu(G_{13}) > \mu(G_{14}) > \mu(G_{15}) > \mu(G_{16}) > \mu(G_{17}) = \mu(G_{18}) = \mu(G_{19}) > \mu(G_{20}) > \mu(G_{21}) = \mu(G_{22}) > \mu(G_{23}) > \mu(G_{24}) > \mu(G_{25}) > \mu(G_{26}) = \mu(G_{27}) > \mu(G_{28}) = \mu(G_{29}) = \mu(G_{30}) = \mu(G_{31}) = \mu(G_{32})$.

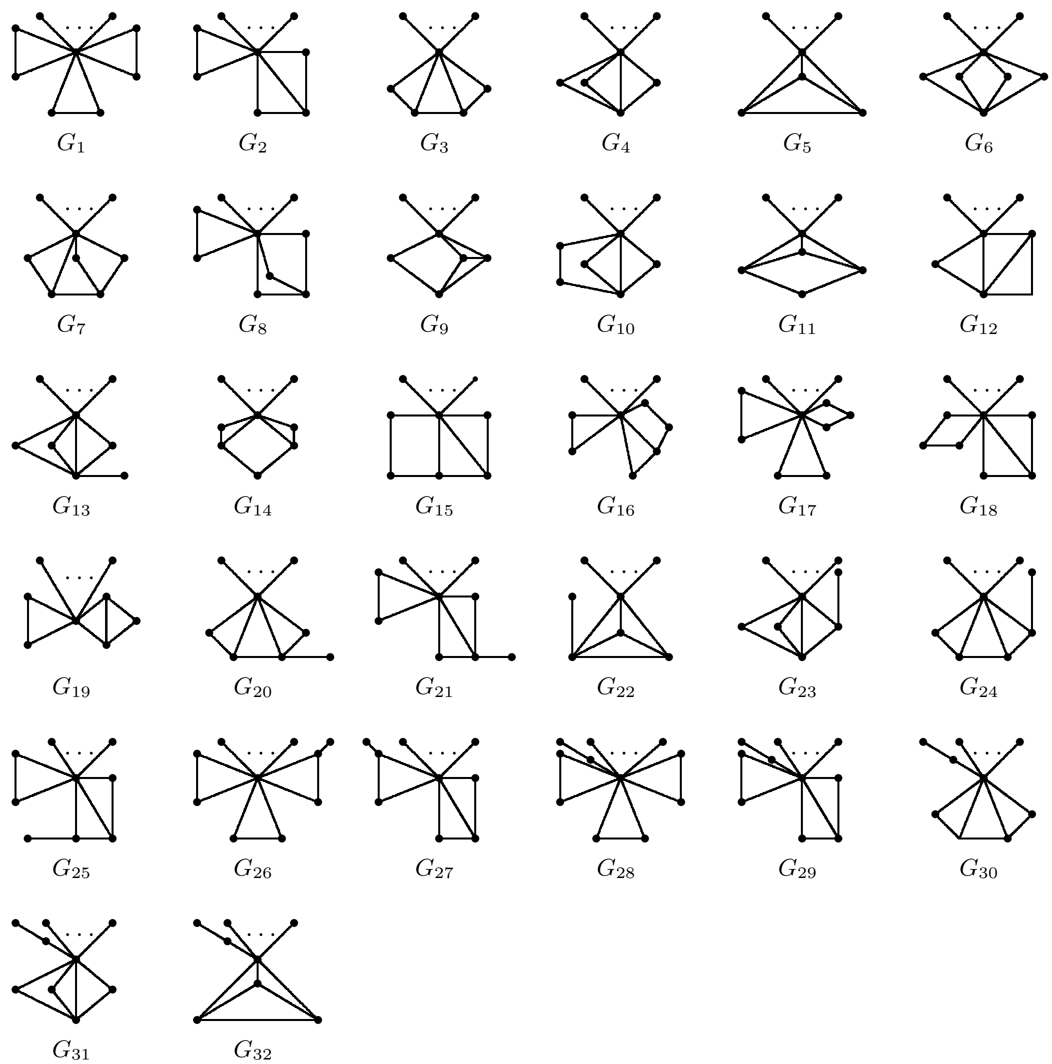

Fig. 2 The graphs $G_1, G_2, \cdots, G_{32}$ in $\mathcal{T}(n)$

Proof (1) We first use Lemma 4 to determine the characteristic polynomial $\Phi(G_{12};x)$. Let v be the vertex of G_{12} with degree $n-2$, and $G^{'}$ be the graph obtained from G_{12} by deleting all the pendant vertices in $N_{G_{12}}(v)$. Then G_{12} is the coalescence of $G^{'}$ and the star $K_{1,n-5}(i.e., G_{12}=G^{'}\cdot K_{1,n-5})$ with respect to v and u, where u is the center of $K_{1,n-5}$. It is easy to obtain that

$$\Phi(G^{'};x)=x(x-3)(x-5)(x^2-6x+7), \tag{6}$$

$$\Phi(L_v(G^{'});x)=x^4-11x^3+40x^2-54x+21, \tag{7}$$

$$\Phi(K_{1,n-5};x)=x[x-(n-4)](x-1)^{n-6}, \tag{8}$$

$$\Phi(L_u(K_{1,n-5});x)=(x-1)^{n-5}. \tag{9}$$

By using Lemma 4 and Eqs.(6)-(9) we have

$$\begin{aligned}\Phi(G_{12};x)=&\Phi(G^{'};x)\Phi(L_u(K_{1,n-5});x)+\Phi(K_{1,n-5};x)\Phi(L_v(G^{'});x)\\&-x\Phi(L_v(G^{'});x)\Phi(L_u(K_{1,n-5});x)\\=&x(x-1)^{n-6}g_1(x).\end{aligned}$$

We may obtain $\Phi(G_i;x)$ for $i=13,14,\cdots,32$ by the similar argument of above.

(2) Now we'll prove $\mu(G_{12})>\mu(G_{13})$.By using Corollary 1, we have

$$\mu(G_{12})>\Delta(G_{12})+1=n-1>0.$$

By $\Phi(G_{12};x)=x(x-1)^{n-6}g_1(x)$, we get $\mu(G_{12})$ is the largest root of the equation $g_1(x)=0$. Because

$$\Phi(G_4;x)=x(x-1)^{n-7}(x-2)^2(x-3)f_4(x),$$

where $f_4=x^3-(n+4)x^2+(5n-2)x-3n$. It is easy to check that

$$(x-3)g_2(x)-g_1(x)=-3f_4(x)+q_1(x),$$

where $q_1(x)=-4x^2+(5n-8)x-3n$.

Let $\lambda^*=\mu(G_{12})$, then $\lambda^*>n-1$, and $g_1(\lambda^*)=0$. Since $f_4(\frac{1}{2})<0$, $f_4(4)>0$, $f_4(n-1)<0$, so $4<\mu_2(G_4)<n-1$. Furthermore we have $\mu_2(G_4)<n-1<\lambda^*<\mu(G_4)$ (by Lemma 1 we know $\mu(G_4)>\mu(G_{12})=\lambda^*$). So we have $f_4(\lambda^*)<0$.

If $n\geqslant 11$, then

$$q_1(n)=-4n^2+(5n-8)n-3n=n^2-11n\geqslant 0,$$

so

$$(\lambda^*-3)g_2(\lambda^*)=-3f_4(\lambda^*)+q_1(\lambda^*)+g_1(\lambda^*)>0,$$

then $\mu(G_{12})>\mu(G_{13})$.

If $n=10, \mu(G_{13})=9.02059$, $\mu(G_{12})=9.04272$, then $\mu(G_{12})>\mu(G_{13})$.

If $n=9, \mu(G_{13})=8.02991$, $\mu(G_{12})=8.05908$, then $\mu(G_{12})>\mu(G_{13})$.

Hence the largest root of the equation $g_1(x)=0$ is larger than $\mu(G_{13}), i.e., \mu(G_{12})>\mu(G_{13})$.

By the similar proof of $\mu(G_{12}) > \mu(G_{13})$, we can get $\mu(G_{13}) > \mu(G_{14})$, $\mu(G_{14}) > \mu(G_{15})$, $\mu(G_{16}) > \mu(G_{17})$, $\mu(G_{15}) > \mu(G_{16})$, $\mu(G_{17}) > \mu(G_{20})$, $\mu(G_{25}) > \mu(G_{26})$, $\mu(G_{26}) > \mu(G_{28})$, $\mu(G_{24}) > \mu(G_{25})$, $\mu(G_{24}) > \mu(G_{28})$, $\mu(G_{23}) > \mu(G_{24})$, $\mu(G_{21}) > \mu(G_{24})$, $\mu(G_{20}) > \mu(G_{21})$, $\mu(G_{21}) > \mu(G_{23})$.

By $\Phi(G_i; x)(i = 17, 18, 19)$, we know that $\mu(G_{17}) = \mu(G_{18}) = \mu(G_{19})$ i.e. the largest root of the equation $g_6(x) = 0$. Similarly, we can get $\mu(G_{26}) = \mu(G_{27})$, $\mu(G_{28}) = \mu(G_{29}) = \mu(G_{30}) = \mu(G_{31}) = \mu(G_{32})$, $\mu(G_{21}) = \mu(G_{22})$.

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