2011 年 9 月 运 筹 学 学 报 第 15 卷 第 3 期
Sept., 2011 Operations Research Transactions Vol.15 No.3

Properly Colored Paths and Cycles in Complete Graphs*

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Abstract Let K_n^c denote a complete graph on n vertices whose edges are colored in an arbitrary way. Let $\Delta^{mon}(K_n^c)$ denote the maximum number of edges of the same color incident with a vertex of K_n^c. A properly colored cycle (path) in K_n^c is a cycle (path) in which adjacent edges have distinct colors. B. Bollobás and P. Erdös (1976) proposed the following conjecture: If $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, then K_n^c contains a properly colored Hamiltonian cycle. This conjecture is still open. In this paper, we study properly colored paths and cycles under the condition mentioned in the above conjecture.

Keywords properly colored cycle, complete graph

Chinese Library Classification O157.5

2010 Mathematics Subject Classification 05C38

完全图中的正常染色的路和圈

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摘要 令 K_n^c 表示 n 个顶点的边染色完全图. 令 $\Delta^{mon}(K_n^c)$ 表示 K_n^c 的顶点上关联的同种颜色的边的最大数目. 如果 K_n^c 中的一个圈（路）上相邻的边染不同颜色，则称它为正常染色的. B. Bollobás 和 P. Erdös (1976) 提出了如下猜想：若 $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, 则 K_n^c 中含有一个正常染色的 Hamilton 圈. 这个猜想至今还未被证明. 我们研究了上述条件下的正常染色的路和圈.

关键词 正常染色圈，完全图

中图分类号 O157.5

数学分类号 05C38

0 Introduction and notation

We use [5] for terminology and notations not defined here. Let $G = (V, E)$ be a graph. An *edge coloring* of G is a function $c : E \to \mathbb{N}$ ($\mathbb{N}$ is the set of nonnegative integers). If G is assigned such a coloring c, then we say that G is an *edge colored graph*. Let $c(e)$ denote the color of the edge e. A properly colored cycle (path) in an edge colored graph is a cycle

收稿日期： 2011 年 8 月 13 日.

* Supported by National Natural Science Foundation of China (61070230, 11026184, 11101243), Independent Innovation Foundation of Shandong University (2009hw001), Research Fund for the Doctoral Program of Higher Education of China (20100131120017) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars.

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(path) in which adjacent edges have distinct colors. A good resource on properly colored paths and cycles is the survey paper [2] by Bang-Jensen and Gutin. The the color degree of a vertex v in an edge colored graph is defined as the maximum number of edges adjacent to v, that have distinct colors. For study of properly colored paths and cycles under color degree conditions, we refer to [7, 14, 12].

Let K_n^c be an edge colored complete graph. Bollobás and Erdős [4] proposed the following interesting conjecture.

Conjecture 1 [4] *If* $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$, *then* K_n^c *contains a properly colored Hamiltonian cycle.*

Bollobás and Erdős [4] managed to prove that $\Delta^{mon}(K_n^c) < \frac{n}{69}$ implies the existence of a properly colored Hamiltonian cycle in K_n^c. This result was improved by Chen and Daykin [6] to $\Delta^{mon}(K_n^c) < \frac{n}{17}$ and by Shearer [13] to $\Delta^{mon}(K_n^c) < \frac{n}{7}$. So far the best asymptotic estimate was obtained by Alon and Gutin [1]. They proved that: For every $\epsilon > 0$ there exists an $n_o = n_0(\epsilon)$ so that for every $n > n_o$, K_n^c satisfying $\Delta^{mon}(K_n^c) \leqslant (1 - \frac{1}{\sqrt{2}} - \epsilon)n$ has a properly colored Hamiltonian cycle; furthermore, there exist all properly colored cycles of length from 3 to n.

First, we show that, under the condition mentioned in the conjecture K_n^c is not pancyclic. Note that there exist edge colored complete graphs with $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$ such that K_n^c contains no tricolored triangle. Consider the red-blue coloring of K_5 where both color classes form pentagons. Each vertex in this base graph is substituted by green complete graphs, then we can get a edge colored K_n^c with no rainbow triangle. Clearly $\Delta^{mon}(K_n^c) \leqslant \frac{2n}{5} < \lfloor \frac{n}{2} \rfloor$, when $n \geqslant 10$. Nevertheless, we prove the following results.

Theorem 2 Let K_n^c be an edge colored complete graph with $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$. Then

(i) K_n^c contains either a properly colored triangle or properly colored C_4 with two colors;

(ii) if $n \geqslant 5$, then each vertex of K_n^c is contained in a properly colored cycle of length at least 5;

(iii) there is a properly colored path of length at least 2 between any two vertices of K_n^c.

1 Proof of Theorem 2 (i)

We will use a lemma for Gallai colorings. Note that edge colorings of complete graphs in which no triangle is colored with three distinct colors are called Gallai colorings.

Lemma A [8, 9] Every Gallai coloring with at least three colors has a color which spans a disconnected graph.

Let K_n^c be a edge colored complete graph with $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$. Clearly, each vertex is incident with three edges with distinct colors. If K_n^c has no tricolored triangles, then this edge coloring is a Gallai coloring. By Lemma A, we have a color with at least two components. Then the edges between any two components are colored with the same color, otherwise we can find a tricolored triangle easily. Thus we can get a properly colored C_4 with two colors. This completes the proof.

2 Proof of Theorem 2 (ii)

Firstly, we give a notation which has been used in [7]. In an edge colored complete graph, a set of vertices $A(|A|\geqslant 2)$ is said to have the *dependence property* with respect to a vertex $v\notin A$ (denoted by DP_v) if $c(aa')\in\{c(va),c(va')\}$ for all $a,a'\in A$. For a vertex $a\in A$, let $d_A^v(a)$ denote the number of the edges incident to a and with color $c(va)$ (in the subgraph induced by A). For simplicity, let $V=V(K_n^c)$.

Lemma B If a set A has DP_v, then there exists a vertex $a\in A$ with $d_A^v(a)\geqslant\frac{|A|+1}{2}$.

Proof of the Lemma We prove it by induction on $|A|$. When $|A|=2$, it is easy to check. So assume it is true, when $|A|\leqslant p-1$ and suppose $|A|=p$. By the induction hypothesis, there is a vertex a such that

$$d_A^v(a)\geqslant\frac{p-1+1}{2}=\frac{p}{2}.$$

If there is an edge with color $c(va)$ incident with v, then we are done. So assume that there is no such edge. By induction, there exists a vertex a' in $A\backslash a$ such that

$$d_{A\backslash a}^v(a')\geqslant\frac{p}{2}.$$

By the assumption of a, we know that

$$c(aa')\neq c(va)$$

which implies that $c(aa')=c(va')$. Therefore a' is the desired vertex. This completes the proof of Lemma B.

In [11], Li, Wang and Zhou [11] studied long properly cycles in K_n^c and they proved that if $\Delta^{mon}(K_n^c)<\lfloor\frac{n}{2}\rfloor$, then K_n^c contains a properly colored cycle of length at least $\lceil\frac{n+2}{3}\rceil+1$. Recently, this bound was improved by Wang, Wang and Liu [15], we remark it as a lemma as follows.

Lemma C [15] If $\Delta^{mon}(K_n^c)<\lfloor\frac{n}{2}\rfloor$, then K_n^c contains a properly colored cycle of length at least $\lceil\frac{n}{2}\rceil+2$.

Next we will prove Theorem 2(ii). Let K_n^c be an edge colored graph with $\Delta^{mon}(K_n^c)<\lfloor\frac{n}{2}\rfloor$ and v be an arbitrary vertex in K_n^c. If $n=5$, then

$$\Delta^{mon}(K_5)<\left\lfloor\frac{5}{2}\right\rfloor=2,$$

which means that K_5 is properly colored. Then there is a properly colored Hamiltonian cycle. Hence our conclusion holds clearly. So we assume that $n\geqslant 6$. By Lemma C, there exists a longest properly colored cycle C of length

$$m\geqslant\left\lceil\frac{n}{2}\right\rceil+2\geqslant 5.$$

If $v\in C$, then we are done. So assume that $v\notin C$. In the following, all indices will be modulo m. We say a vertex $x\in V\backslash C$ *follows the color of* C *increasing* if $c(xv_i)=c(v_iv_{i+1})$ (or symmetrically we say that x follows the colors of C *decreasing* if $c(xv_i)=c(v_iv_{i-1})$) for each i. Since $m \geqslant \lceil \frac{n}{2} \rceil + 2$, v has at least two colors on all edges to C. We distinguish the following two cases.

Case 1 v does not follow the colors of C.

Note that v has at least two colors to C, so we assume that v has color c_1 to v_1 and color c_2 to v_2. If $c_1 \neq c(v_m v_1)$ and $c_2 \neq c(v_2 v_3)$, then $C' = v_1 v v_2 C v_1$ is a proper colored cycle containing v of length $m+1$. So without loss of generality, we suppose that $c_2 = c(v_2 v_3)$. If $c(vv_3) \notin \{c_2, c(v_3 v_4)\}$, then $C' = v_2 v v_3 C v_2$ is a properly colored cycle of length $m+1$ containing v. Generally, it implies that whenever $c(vv_i) = c(v_i v_{i+1})$, then $c(vv_{i+1}) \in \{c(vv_i), c(v_{i+1} v_{i+2})\}$ for $i \geqslant 3$. Since v does not follow the colors of C, there exists a vertex v_j such that $c(vv_{j+1}) = c(vv_j)$. Suppose j is the smallest such index. If $j \geqslant 3$, then $c(vv_{j-1}) = c(v_{j-1} v_j)$ and $c(v_j v_{j+1}) = c(vv_j) = c(vv_{j+1}$, thus $C' = v_{j-1} v v_{j+1} C v_{j-1}$ is a properly colored cycle containing v of length m. Hence we assume that $j = 2$, then $c(vv_2) = c(vv_3) = c(v_2 v_3)$.

We claim that $c(vv_1) = c(v_m v_1)$. Otherwise, $C' = v_1 v v_3 C v_1$ is a properly colored cycle containing v of length m. From another point of view, if $c(vv_m) \notin \{c(v_m v_{m-1}\}$, then $C' = v_m v v_1 C v_m$ is a properly colored cycle containing v of length m. Similarly, there exists a largest index j such that $c(vv_j) \neq c(v_{j-1} v_j)$, which means that $c(vv_j) = c(vv_{j+1})$. If $j < m$, then $C' = v_j v v_{j+1} C v_j$ is a properly colored cycle containing v of length m. So we can deduce that $j = m$. Now $c(vv_m) = c(vv_1) = c(v_m v_1)$. Hence $C' = v_m v v_3 C v_m$ is a properly colored cycle. Moreover, if $m \geqslant 6$, then $|C'| \geqslant 5$. So now assume that $m = 5$. Since $5 = m \geqslant \lceil \frac{n}{2} \rceil + 2$, then $n = 6$.

Recall that $c(vv_1) = c(vv_5) = c(v_5 v_1) = c_1$ and $c(vv_2) = c(vv_3) = c(v_2 v_3) = c_2$. Since v is incident with at most two edges having the same color, $c(vv_4) \notin \{c_1, c_2\}$. Suppose $c(vv_4) = c_3$. Then we claim that $c(v_3 v_4) = c_3$, otherwise $vv_1 v_2 v_3 v_4 v$ forms a properly colored cycle of length 5 containing v. Since each vertex is incident with at most two edges with the same color, $c(v_2 v_4) \notin \{c_2, c_3\}$ and $c(v_1 v_3) \neq \{c_1, c_2\}$. Then $vv_1 v_3 v_2 v_4 v$ is a properly colored cycle of length 5 containing v. This complete the proof of Case 1.

Case 2 v follows the colors of C.

By symmetry, we suppose v follows the color of C increasing. First, we show that C has DP_v. Otherwise, there exists a chord $v_i v_j$ of C for which

$$c(v_i v_j) \notin \{c(vv_i), c(vv_j)\} = \{c(v_i v_{i+1}), c(v_j v_{j+1})\}.$$

Hence either

$$c' = vv_i v_j v_{j+1} C v_{i-1} v \quad \text{or} \quad C^{''} = vv_j v_i v_{i+1} C v_{j-1} v$$

is a properly colored cycle of length at least 5 (since $|C| \geqslant 5$), which is a contradiction.

Let U denote the set of vertices in $V \backslash C$ which do not follow C increasing. Note that $v \notin U$. Then for each $u \in U$, there exists an edge uv_i with $c(uv_i) \neq c(v_i v_{i+1})$. We conclude that $c(uv_i) = c(vu)$, since otherwise either

$$C' = vuv_i v_{i+1} C v_{i-1} v \quad \text{or} \quad C^{''} = vuv_i v_{i+1} C v_{i-2} v$$

is a properly colored cycle of length at least $m+1$ containing v.

Next, we claim that if $|U| \geqslant 2$, then U also has DP_v. Suppose to the contrary, $c(u_1u_2) \notin \{c(vu_1, vu_2\})$, where $u_1, u_2 \in U$. Then either

$$C' = vu_1u_2v_iv_{i+1}Cv_{i-1}v \quad \text{or} \quad C'' = vu_1u_2v_iv_{i+1}Cv_{i-2}v$$

is a properly colored cycle containing v of length at least $m+2$. So U has DP_v if $|U| \geqslant 2$.

Recall that for each $u \in U$, if $c(uv_i) \neq c(v_iv_{i+1})$, then $c(uv_i) = c(uv)$. It follows that $C \cup U$ has DP_v. By Lemma B, there exists a vertex

$$w \in C \cup U \quad \text{with} \quad d^v_{C\cup U}(w) \geqslant \frac{|C \cup U| + 1}{2}.$$

So w must have a neighbor $v' \in V \backslash (C \cup U \cup \{v\})$, such that $c(wv') \neq c(wv)$, otherwise this contradicts $\Delta^{mon}(K_n^c) < \lfloor \frac{n}{2} \rfloor$. By the definition of U, v' follows the color of C increasing, so if $w \in C$, $c(wv) = c(wv')$, which is a contradiction. So now we can assume that $w \in U$. It follows that there exists a vertex $v_i \in C$ such that $c(wv_i) \neq c(v_iv_{i+1})$. Now either

$$C' = v'wv_iCv_{i-1}v' \quad \text{or} \quad C'' = v'wv_iCv_{i-2}v'$$

is a properly colored cycle of length at least $m+1$, which contradicts that C is the longest properly cycle. This completes the whole proof of Theorem 2 (ii).

3 Proof of Theorem 2 (iii)

If $n \leqslant 5$, it is trivial. So we assume that $n \geqslant 6$. We prove it by contradiction. Let u, v be two vertices such that there is no properly colored path of length at least 2. Let G be the induced subgraph by $V(K_n^c)\backslash\{u, v\}$. Then for each vertex $w \in V(G)$, $c(uw) = c(vw)$. Now assume that w_1, w_2 are any two vertices in $V(G)$. If $c(w_1u) = c(w_2u)$, then $c(w_1w_2) \neq c(w_1u)$, otherwise, uw_1w_2v is a properly colored path of length 2 from u to v, a contradiction. If $c(w_1u) \neq c(w_2u)$, then $c(w_1w_2) \in \{c(uw_1), c(uw_2)\}$, otherwise, we will get a contradiction too. Now we construct a directed graph as follows: For any two $w_i, w_j \in V(G)$, if $c(w_iw_j) = c(uw_i)$, then we orient the edge w_iw_j from w_i to w_j; otherwise, we orient w_iw_j from w_j to w_i. Then we obtain a directed graph D of order $n-2$. So there must be a vertex, say w such that

$$d^+(w) \geqslant \frac{|V(G)| - 1}{2}.$$

Recall that if w_i is an outneighbor of w, then $c(ww_i) = c(uw)$. Hence in K_n^c, there are at least

$$\frac{|V(G)| - 1 + 2}{2} = \frac{n-1}{2}$$

edges incident with w, which have the same color $c(uw)$. This contradicts with

$$\Delta^{mon}(K_n^c) < \left\lfloor \frac{n}{2} \right\rfloor,$$

which completes the proof.

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