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On the Spectral Radii of m-Starlike Tree*

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Abstract A tree is said to be starlike if exactly one of its vertices has degree larger than 2. A m-starlike tree is obtained by appending a starlike tree to every one terminus of a starlike tree $S_0 = S(m_{01}, m_{02}, ..., m_{0\Delta_0})$. Gutman and L. Shi give a bound of the spectral radii of starlike tree. In this paper, we give an another short proof and further discussions about this result. Sometime, we give a new upper bound of the spectral radii of m-starlike tree.

Keywords m-Starlike tree, Spectra of graphs, Spectral radius
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m 重似星树的谱半径

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摘要 仅有一个顶点的度大于 2 的树称为似星树．在一棵似星树的每个一度点粘接一棵似星树构成的图称为 m 重似星树． Gutman 和 L. Shi 给出了似星树谱半径的一个界．在本文中我们给出了另外一个更简洁的证明方法并做了深入的讨论，同时给出了 m 重似星树谱半径的一个最好界．

关键词 m 重似星树，图的谱，谱半径
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0 Introduction

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. All graphs considered here are simple and undirected. For $v \in V(G)$, we use $N(v)$ to denote the neighbors of v. Let $d(v_i)$ denote the vertex degree of v_i and let Δ denote the maximum vertex degree of G. Let $A(G)$ be the adjacency matrix of G. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\lambda_1(G) \geqslant \lambda_2(G) \geqslant ... \geqslant \lambda_n(G)$. The sequence of n eigenvalues is called the *spectrum* of G, the largest eigenvalue $\lambda_1(G)$ is often called the *spectral radius* of G. The characteristic polynomial of $A(G)$ is called the characteristic polynomial of the graph G and is denoted by $\phi(G, \lambda)$.

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A tree is said to be *starlike* if exactly one of its vertices has degree larger than two. Let P_n denote the path on n vertices. By $S = S(n_1, n_2, ..., n_\Delta)$ we denote the starlike tree which has a vertex v of degree $\Delta \geqslant 3$ and which has the property

$$S(n_1, n_2, ..., n_\Delta) - v = P_{n_1} \cup P_{n_2} \cup ... \cup P_{n_\Delta},$$

where $n_1 + n_2 + \ldots + n_\Delta + 1 = n$. The starlike tree with maximal degree $\Delta = 3$ is called *T-shape*.

A *m-starlike tree* is obtained by appending a starlike tree to every terminus of a starlike tree $S_0 = S(m_{01}, m_{02}, ..., m_{0\Delta_0})$ (see Fig.1). We will denoted by $G(S_0, S_1, S_2, ..., S_{\Delta_0})$, a m-starlike tree such that

$$G(S_0, S_1, ..., m_{0\Delta_0}) - u - \sum_{i=1}^{\Delta_0} v_i = P_{m_{01}} \cup \ldots \cup P_{m_{\Delta_0 \Delta_{\Delta_0 - 1}}},$$

where

$u + \sum_{i=1}^{\Delta_0} v_i + m_{01} + m_{02} + ... + m_{\Delta_1 - 1} + m_{11} + ... + m_{1\Delta_1} + ... + m_{\Delta_0 \Delta_{\Delta_0 - 1}} = n$

and

$$S_i - v_i = \sum_{j=1}^{\Delta_i} m_{ij}.$$

If $\Delta_0 = \Delta_1 = \Delta_2 = ... = \Delta_{\Delta_0} = \Delta \geqslant 3$, $m_{0i} (i = 1, 2, ..., \Delta_0)$ and $m_{ij} = m (i = 1, 2, ..., \Delta_0, j = 1, 2, ..., \Delta_{\Delta_0})$, then we will denoted by $G(\Delta m, (\Delta - 1)m, ..., (\Delta - 1)m)$ the m-starlike tree.

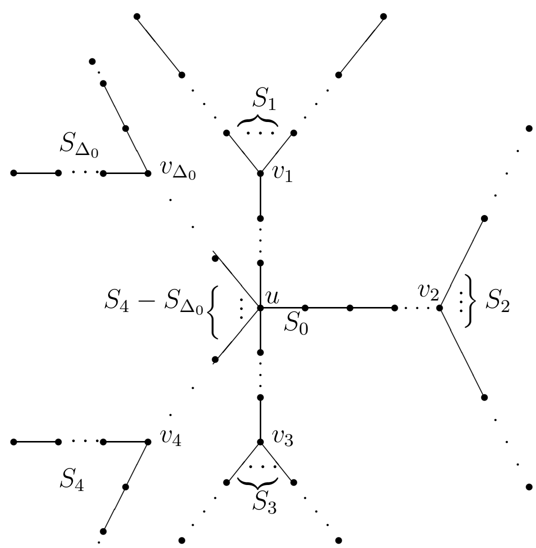

Fig. 1 m-starlike tree

In this paper, we give a new upper bound of the spectral radius of m-starlike tree.

1 Some lemmas

In the section, we will present some lemmas which are required in the proof of the main results.

Lemma 1.1[1] Let G be a connected graph, and let H be a proper subgraph G. Then

$$\lambda_1(H) < \lambda_1(G).$$

Lemma 1.2[1] Let u be a vertex of G, and let $C(u)$ be the set of all cycles containing u. Then

$$\phi(G,\lambda) = \lambda\phi(G-u,\lambda) - \sum_{v\in N(u)} \phi(G-u-v,\lambda) - 2\sum_{Z\in C(u)} \phi(G-V(Z),\lambda).$$

In particular, if G is a tree, then

$$\phi(G,\lambda) = \lambda\phi(G-u,\lambda) - \sum_{v\in N(u)} \phi(G-u-v,\lambda).$$

Lemma 1.3[1] The characteristic polynomial of a graph satisfies the following identities:

(a) $\phi(G\cup H,\lambda) = \phi(G,\lambda)\phi(H,\lambda)$,

(b) $\phi(G,\lambda) = \phi(G-e,\lambda) - \phi(G-v_1-v_2,\lambda)$ if $e = v_1v_2$ is a cut-edge of G,

where $G-e$ denotes the graph obtained from G by deleting the edge e and $G-v_1v_2$ denotes the graph obtained from G by deleting the vertices v_1, v_2 and the edges incident to it.

Lemma 1.4[1] Let C_n, P_n denote the cycle and the path on n vertices respectively. Then

$$\phi(C_n,\lambda) = \prod_{j=1}^{n}\left(\lambda - 2\cos\frac{2\pi j}{n}\right) = 2\cos(n\arccos\lambda/2) - 2,$$

$$\phi(P_n,\lambda) = \prod_{j=1}^{n}\left(\lambda - 2\cos\frac{\pi j}{n+1}\right) = \frac{\sin((n+1)\arccos\lambda/2)}{\sin(\arccos\lambda/2)}.$$

Let $\lambda = 2\cos\theta$, set $t^{1/2} = e^{i\theta}$, it is useful to write the characteristic polynomial of C_n, P_n in the following form:

$$\phi(C_n, t^{1/2}+t^{-1/2}) = t^{n/2} + t^{-n/2} - 2, \tag{1}$$

$$\phi(P_n, t^{1/2}+t^{-1/2}) = t^{-n/2}(t^{n+1}-1)/(t-1). \tag{2}$$

Hoffman and Smith[3] define an *internal path* of a graph G as a walk $v_0, v_1, ..., v_k$ $(k \geqslant 1)$ such that the vertices $v_1, v_2, ..., v_k$ are distinct (v_0, v_k need not be distinct), $d(v_0) > 2, d(v_k) > 2$ and $d(v_i) = 2$ for $0 < i < k$.

Lemma 1.5[3] Let G be a connected graph that is not isomorphic to W_n, where W_n is a graph obtained from the path P_{n-2} (indexed in natural order $1, 2, ..., n-2$) by adding two pendant edges at vertices 2 and $n-3$. Let G_{uv} be the graph obtained from G by subdividing the edge uv of G. If uv lies on internal path of G, then $\lambda_1(G_{uv}) < \lambda_1(G)$.

We now give some known bounds for the spectral radius $\lambda_1(G)$ in terms of the largest vertex degree Δ.

It is well-known that

Lemma 1.6[1] For a connected graph G

$$\sqrt{\Delta} \leqslant \lambda_1(G) \leqslant \Delta. \tag{3}$$

The lower bound in (3) follows from the fact that the star $K_{1,\Delta}$ is a subgraph (not necessarily induced) of any graph G with $d_1 = \Delta$ and therefore $\lambda_1(G) \geqslant \lambda_1(K_{1,\Delta}) = \sqrt{\Delta}$.

The upper bound in [3] is also well-known (see [1]).

Lemma 1.7[2,5] Let T be a tree. Then

$$\lambda_1(T) < 2\sqrt{\Delta - 1}. \tag{4}$$

2 Main results

The following well-known result was proposed by Gutman[4] and L. Shi[7]. We will give a short proof and further discussions.

Theorem 2.1 Let $S = S(n_1, n_2, ..., n_\Delta)$ be a starlike tree with maximum vertex degree Δ. Then

$$\lambda_1(S) < \frac{\Delta}{\Delta - 1}\sqrt{\Delta - 1}. \tag{5}$$

Proof Let m be a positive integer. Denote $S(\Delta m) = S(\overbrace{m, m, ..., m}^{\Delta})$. Using Lemma 1.3 (b), we have

$$\phi(S(\Delta m), \lambda) = \phi(P_m, \lambda)^{\Delta-1}(\lambda\phi(P_m, \lambda) - \Delta\phi(P_{m-1}, \lambda)).$$

By (2) we have

$$\frac{\phi(S(\Delta m), t^{1/2} + t^{-1/2})}{\phi(P_m, \lambda)^{\Delta-1}} = \frac{t^{-(m+1)/2}}{t-1}(t^{m+2} - (\Delta - 1)t^{m+1} + (\Delta - 1)t - 1) =: \psi(t).$$

Let t_1 be the largest root of $\psi(t)$, then $t_1 < \Delta - 1$ since $\psi(t) > 0$ for $t \geqslant \Delta - 1$. Let$f(t) = t^{1/2} + t^{-1/2}$, then $f'(t) = t^{-3/2}(t - 1)/2 \geqslant 0$ for $t \geqslant 1$, so $f(t)$ strictly increases in $[1, \infty)$. Thus

$$\lambda_1(S(\Delta m)) = t_1^{1/2} + t_1^{-1/2} < (\Delta - 1)^{\frac{1}{2}} + (\Delta - 1)^{-\frac{1}{2}} = \frac{\Delta}{\Delta - 1}\sqrt{\Delta - 1}.$$

Since m is an any positive integer, hence $S(n_1, n_2, ..., n_\Delta)$ can be embedded in $S(\overbrace{m, m, ..., m}^{\Delta})$ as a proper subgraph, by Lemma 1.1 we have

$$\lambda_1(S(n_1, n_2, ..., n_\Delta)) < \lambda_1(S(\overbrace{m, m, ..., m}^{\Delta})) < \frac{\Delta}{\Delta - 1}\sqrt{\Delta - 1}.$$

This proof is complete.

Corollary 2.1[6] Let λ_1 be the spectral radius of T-shape trees, then $\lambda_1 < \frac{3}{\sqrt{2}}$.

Specially, if we think the path is also a starlike tree with $\Delta = 2$, then the proof of theorem is also right. So we have

Corollary 2.2 $\lambda_1(P_n) < 2$.

By (3) we have

Corollary 2.3

$$\sqrt{\Delta} \leqslant \lambda_1(S(n_1, n_2, ..., n_\Delta)) < \frac{\Delta}{\Delta - 1}\sqrt{\Delta - 1}. \tag{6}$$

The equality holds if and only if $n_1 = n_2 = ... = n_\Delta = 1$.

Corollary 2.4

$$\lambda_1(S(n_1, n_2, ..., n_\Delta)) \sim (\sqrt{\Delta}), \Delta \to \infty.$$

Proof Since $\lim\limits_{\Delta\to\infty} \frac{\sqrt{\Delta}}{\sqrt{\Delta}} = 1$ and $\lim\limits_{\Delta\to\infty} \frac{\Delta}{\Delta-1}\frac{\sqrt{\Delta-1}}{\sqrt{\Delta}} = 1$, hence by (6) we have

$$\lim_{\Delta\to\infty} \frac{\lambda_1(S(n_1, n_2, ..., n_\Delta))}{\sqrt{\Delta}} = 1.$$

So

$$\lambda_1(S(n_1, n_2, ..., n_\Delta)) \sim (\sqrt{\Delta}), \Delta \to \infty.$$

Since $\frac{\Delta}{\Delta-1}\sqrt{\Delta-1} < 2\sqrt{\Delta-1} < \Delta$ for $\Delta \geqslant 3$. Clearly, the upper bound (5) is better than the upper bound (4) and the upper bound (3).

Theorem 2.2 Let $G(S_0, S_1, S_2, ..., S_{\Delta_0})$ be a m-starlike tree with maximum vertex degree $\Delta(\Delta \geqslant 3)$. Then

$$\lambda_1(G(S_0, S_1, S_2, ..., S_{\Delta_0})) < \frac{(2\Delta - 1)\sqrt{2\Delta - 2}}{2\Delta - 2}. \tag{7}$$

Proof Let S_0 is a *star*, that is, $m_{01}, m_{02}, ..., m_{0\Delta_0} = 1$. Let m be a positive integer such that $m_{ij} < m$ $(i = 1, 2, ..., \Delta_0, j = 1, 2, ..., \Delta_{\Delta_0})$. Now we calculate the characteristic polynomial of $G(S_0, (\Delta-1)m, ..., (\Delta-1)m)$, where $\Delta = \max\{\Delta_0, \Delta_1, ..., \Delta_{\Delta_0}\}$. By Lemma 1.2 we know,

$$\phi(S(\overbrace{m, m, ..., m}^{\Delta-1}), \lambda) = \phi^{\Delta-2}(P_m, \lambda)(\lambda\phi(P_m, \lambda) - (\Delta - 1)\phi(P_{m-1}, \lambda)).$$

By Lemma 1.2, we have

$$\begin{aligned}&\phi(G(\Delta m, (\Delta-1)m, ..., (\Delta-1)m), \lambda)\\ =&\phi^{\Delta-1}(S)[\lambda\phi(S) - \Delta\phi^{\Delta-1}(P_m)]\\ =&\phi^{\Delta-1}(S)\phi^{\Delta-2}(P_m)[\phi(P_m)(\lambda^2 - \Delta) - \lambda(\Delta-1)\phi(P_{m-1})].\end{aligned}$$

By (2) we have

$$\begin{aligned}&\phi(G(\Delta m, (\Delta-1)m, ..., (\Delta-1)m), t^{1/2} + t^{-1/2})/\phi^{\Delta-1}(S)\phi^{\Delta-2}(P_m)\\ =&\phi(P_m)(\lambda^2 - \Delta) - \lambda(\Delta-1)\phi(P_{m-1})\\ =&\frac{t^{-\frac{m}{2}}}{(t-1)}((t^{m+1} - 1)(t + t^{-1} + 2 - \Delta) - (t\Delta - t + \Delta - 1)(t^m - 1))\\ =&\frac{t^{-\frac{m}{2}}}{(t-1)}(t^{m+2} + (3 - 2\Delta)t^{m+1} + (2 - \Delta)t^m + (\Delta - 2)t + 2\Delta - 3 - t^{-1})\end{aligned}$$

$$\begin{aligned}&\phi(G(\Delta m,(\Delta-1)m,...,(\Delta-1)m),t^{1/2}+t^{-1/2})t^{\frac{m}{2}}(t-1)/\phi^{\Delta-1}(S)\phi^{\Delta-2}(P_m)\\ =&t^{m+2}+(3-2\Delta)t^{m+1}+(2-\Delta)t^m+(\Delta-2)t+2\Delta-3-t^{-1}=:\psi(t).\end{aligned}$$

Since

$$\begin{aligned}\psi(2\Delta-2)&=(2\Delta-2)^{m+2}+(3-2\Delta)(2\Delta-2)^{m+1}+(2-\Delta)(2\Delta-2)^m+(\Delta-2)(2\Delta-2)\\&=2\Delta-3-(2\Delta-2)^{-1}\geqslant 0.\end{aligned}$$

Because $\Delta\geqslant 3$, let t_1 be the largest root of $\psi(t)$, then $t_1<2\Delta-2$ since $\psi(t)>0$ for $t\geqslant 2\Delta-2$. Let $f(t)=t^{1/2}+t^{-1/2}$, since $f'(t)=t^{-3/2}(t-1)/2\geqslant 0$ for $t\geqslant 1$, hence $f(t)$ strictly increases in $[1,\infty)$. Thus

$$\lambda_1(G(\Delta m,(\Delta-1)m,...,(\Delta-1)m))=t_1^{1/2}+t_1^{-1/2}<(2\Delta-2)^{\frac{1}{2}}+(2\Delta-2)^{-\frac{1}{2}}=\frac{2\Delta-1}{2\Delta-2}\sqrt{2\Delta-2}.$$

Since m is an any positive integer, hence $G(S_0,S_1,S_2,...,S_{\Delta_0})$ can be embedded in $G(\Delta m,(\Delta-1)m,...,(\Delta-1)m)$ as a proper subgraph. Let u and $v_i(i=1,2,...,\Delta)$ be the maximum degree vertices of $G(\Delta m,(\Delta-1)m,...,(\Delta-1)m)$, we subdivide the edge uv_i of $G(\Delta m,(\Delta-1)m,...,(\Delta-1)m)$ for many times, to obtain the graph $G((S_0,(\Delta-1)m,...,(\Delta-1)m)$. By Lemma 1.1 and Lemma .5, we have

$$\begin{aligned}\lambda_1(G(S_0,S_1,S_2,...,S_{\Delta_0}))&<\lambda_1(G(S_0,(\Delta-1)m,...,(\Delta-1)m))\\&<\lambda_1(G(\Delta m,(\Delta-1)m,...,(\Delta-1)m))\\&<\frac{(2\Delta-1)\sqrt{2\Delta-2}}{2\Delta-2}.\end{aligned}$$

This proof is complete.

Since

$$\frac{(2\Delta-1)\sqrt{2\Delta-2}}{2\Delta-2}<2\sqrt{\Delta-1}<\Delta$$

for $\Delta\geqslant 3$. Clearly, the upper bound (7) is better than the upper bounds (4) and (3).

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