On the Linear Arboricity of 1-Planar Graphs*

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Abstract It is proved that the linear arboricity of every 1-planar graph with maximum degree $\Delta \geqslant 33$ is $\lceil \Delta/2 \rceil$.

Keywords 1-planar graph, 1-embedded graph, linear arboricity
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1- 平面图的线性荫度

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摘要 证明了最大度 $\Delta \geqslant 33$ 的 1- 平面图的线性荫度为 $\lceil \Delta/2 \rceil$

关键词 1- 平面图， 1- 嵌入图，线性荫度
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0 Introduction

All graphs considered here are finite, simple and undirected. Most of the notions are standard and we refer the readers to [1]. A linear forest is a forest in which every connected component is a path. The linear arboricity $la(G)$ of a graph G is the minimum number of linear forests in G, whose union is the set of all edges of G. Akiyama, Exoo and Harary[2] conjectured that $la(G) = \lceil (\Delta(G)+1)/2 \rceil$ for any regular graph G. It is obviously that $la(G) \geqslant \lceil \Delta(G)/2 \rceil$ for every graph G and $la(G) \geqslant \lceil (\Delta(G)+1)/2 \rceil$ for every regular graph G. So this conjecture is equivalent to the following conjecture.

Conjecture 1 For any graph G,

$$\left\lceil \frac{\Delta(G)}{2} \right\rceil \leqslant la(G) \leqslant \left\lceil \frac{\Delta(G)+1}{2} \right\rceil .$$

Now this conjecture was only proved for several special classes of graphs such as planar graphs[3−4] and is still widely open. Note that if this conjecture is true and G is a graph with even (resp. odd) maximum degree, then the linear arboricity of G is either $\lceil \Delta(G)/2 \rceil$ or $\lceil (\Delta(G)+1)/2 \rceil$ (resp. exactly $\lceil \Delta(G)/2 \rceil$). So the determination of $la(G)$ for a graph G

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seems interesting, although Péroche showed that this is an NP-hard problem[5]. In fact, the linear arboricity has been determined for many classes of graphs (see the introduction of [6] for detail) such as series-parallel graphs[7].

In this paper, we focus on 1-planar graphs. Given a surface S we call a graph G 1-embedded on S if G can be drawn on S so that each edge is crossed by at most one other edge. In particular, if S is a plane, then such a graph G is called 1-planar graph. The notion of 1-planar graphs was introduced by Ringel[8], who proved that the chromatic number of each 1-planar graph is at most 7; this bound was latter improved to 6 (being sharp) by Borodin[9−10]. In [11], Albertson and Mohar considered the list vertex coloring of graphs 1-embedded on a given surface. Wang and Lih proved that each 1-planar graph is list 7-colorable[12]. It is also known that each 1-planar graph G is acyclically 20-colorable[13] and is edge $\Delta(G)$-colorable if $\Delta(G) \geqslant 10$[14] or $\Delta(G) \geqslant 7$ and $g(G) \geqslant 4$[15]. Recently, Zhang et al. investigated the $(p,1)$-total labelling of 1-planar graphs[16].

In this paper we aim to investigate the linear arboricity of 1-planar graphs. One of the main results is the following Theorem 2, which implies that the linear arboricity of every 1-planar graph with maximum degree $\Delta \geqslant 33$ is exactly $\lceil \Delta/2 \rceil$. The other result, which dedicates to the linear arboricity of graphs 1-embedded on a given surface, will be shown at the end of the paper.

Theorem 2 For every 1-planar graph G with maximum degree $\Delta \leqslant M$ and $M \geqslant 34$, we have

$$la(G) \leqslant \left\lceil \frac{M}{2} \right\rceil.$$

From now on, for any 1-planar graph G, we always assume that G has been embedded on a plane such that every edge is crossed by at most one other edge and the number of crossings is as small as possible. We call such an embedding 1-plane graph. The associated plane graph $G^{\times}$ of a 1-plane graph G is the plane graph that is obtained from G by turning all crossings of G into new 4-vertices. A vertex in $G^{\times}$ is called false if it is not a vertex of G and true otherwise. Note that no two false vertices are adjacent in $G^{\times}$. By false face, we mean a face f in $G^{\times}$ that is incident with at least one false vertex; otherwise we say that f is true. For a true vertex v in $G^{\times}$, we use $\alpha(v)$ and $\tau(v)$ to denote the number of false 3-faces and 3-faces incident with v in $G^{\times}$, respectively. Throughout this paper, a k-, k^+- and k^--vertex (resp. face) is a vertex (resp. face) of degree k, at least k and at most k.

1 Main results and their proofs

First of all, we prove Theorem 2. Let G be a minimum counterexample to Theorem 2. It is easy to see that G is 2-connected and $\delta(G) \geqslant 2$. Moreover, G also has the following properties.

Claim 1[17] For every edge uv of G,

$$d_G(u) + d_G(v) \geqslant 2\left\lceil \frac{M}{2} \right\rceil + 2.$$

Let G_2 be the subgraph of G induced by the edges incident with 2-vertices. It is proved in [6] that G_2 is a forest. So it is easy to find a matching M in G saturating all

2-vertices. If $uv \in M$ and $d_G(u) = 2$, then we call v the 2-master of u. For $3 \leqslant t \leqslant \lfloor \frac{\Delta}{2} \rfloor$, let $X_t \subseteq \{v \mid 2 \leqslant d_G(v) \leqslant t\}$, $Y_t = N(X_t)$ and B_t be the induced bipartite subgraph of G with partite sets X_t and Y_t. It follows from Claim 1 that X_t is an independent set of G. If $X_t \neq \emptyset$ and there exists a bipartite subgraph M_t of B_t such that $d_{M_t}(x) = 1$ for each $x \in X_t$ and $d_{M_t}(y) \leqslant 2t-1$ for each $y \in Y_t$, then we call y the t-master of x in G for $xy \in M_t$ and $x \in X_t$. The following claim is due to [6].

Claim 2[6] Each i-vertex in G (if exits) has one j-master, where $2 \leqslant i \leqslant j \leqslant 7$, and each M-vertex (if exits) in G can be 2-master of at most one vertex and each $(M-i)$-vertex (if exits) can be j-masters of at most $2j-1$ vertices, where $2 \leqslant \max\{i+2, 3\} \leqslant j \leqslant 7$.

We call a vertex in G small if it is of degree no more than seven and big otherwise. A false 3-face in $G^{\times}$ is called unbalanced or balanced according to whether or not it is incident with a small vertex. For a true vertex v in $G^{\times}$, let $\alpha_a(v)$ be the number of unbalanced false 3-faces that are incident with v in $G^{\times}$.

Claim 3[14] Let v be a vertex in G. If $d_G(v) = 2$, then $\alpha(v) = 0$; if $d_G(v) = 3$ and $\alpha(v) \geqslant 2$, then v is incident with a 5^+-face in $G^{\times}$; if $d_G(v) = 4$, then $\alpha(v) \leqslant 3$; and if $d_G(v) \geqslant 5$, then $\alpha(v) \leqslant 2\lfloor \frac{d_G(v)}{2} \rfloor$.

Claim 4 Let v be a big vertex in G. If $\tau(v) = d_G(v)$, then

$$\alpha_a(v) \leqslant \left\lfloor \frac{\tau(v)}{2} \right\rfloor;$$

and if $\tau(v) = d_G(v) - i \geqslant \frac{2}{3} d_G(v)$, then

$$\alpha_a(v) \leqslant \left\lceil \frac{\tau(v)}{2} \right\rceil + i - 1.$$

Proof If any of the two facts does not hold, then there must be three consecutive unbalanced false 3-faces that are incident with v in $G^{\times}$, which implies that two small vertices are adjacent in G, a contradiction to Claim 1

Now we continue the proof of Theorem 2 by the discharging method. Define an initial charge c on $V(G) \cup F(G^{\times})$ by letting $c(v) = d_G(v) - 4$ for every $v \in V(G)$ and $c(f) = d_{G^{\times}}(f) - 4$ for every $f \in F(G^{\times})$. By Euler's formula,

$$\sum_{x \in V(G) \cup F(G^{\times})} c(x) = -8.$$

Now we redistribute the charges by the following rules.

R1. If f is a true or balanced false 3-face in $G^{\times}$, then f receives $\frac{1}{2}$ from each of its incident big vertices.

R2. If f is an unbalanced false 3-face in $G^{\times}$, then f receives $\frac{1}{4}$ from its incident small vertex and $\frac{3}{4}$ from its incident big vertex.

R3. If f is a 5^+ face in $G^{\times}$, then f sends $\frac{1}{2}$ to each of its incident 3-vertices.

R4. If v is 2-vertex in G, then v receives $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ from each of its 2-masters, 3-masters and 4-masters, respectively.

R5. If v is 3-vertex in G, then v receives $\frac{1}{2}$ and $\frac{3}{4}$ from each of its 3-masters and 4-masters, respectively.

R6. If v is 4-vertex in G, then v receives $\frac{3}{4}$ from each of its 4-masters.

We consider the final charge c' of the vertices in G and faces in $G^\times$. Note that if f is a true or balanced false 3-face in $G^\times$, then f is incident with at least two big vertices by Claim 1, and if f is an unbalanced false 3-face in $G^\times$, then f is incident with exactly one small vertex and one big vertex. So $c'(f) = 0$ for every 3-face in $G^\times$ by R1 and R2. Since 4-faces are involved in none of the rules, their final charges remain zero. For a 5^+-face f in $G^\times$, f can be incident with at most $\lfloor \frac{d_{G^\times}(f)}{2} \rfloor$ 3-vertices by Claim 1. So by R3,

$$c'(f) \geqslant d_{G^\times}(f) - 4 - \frac{1}{2}\left\lfloor \frac{d_{G^\times}(f)}{2} \right\rfloor \geqslant 0$$

for $d_{G^\times}(f) \geqslant 5$.

Let v be a 2-vertex. Then by Claim 3, v is incident with no false 3-faces and by Claim 2, v has a 2-master, a 3-master and a 4-master. So by R4,

$$c'(v) = -2 + \frac{3}{4} + \frac{1}{2} + \frac{3}{4} = 0.$$

Let v be a 3-vertex. Then v has a 3-master and a 4-master. If $\alpha(v) \leqslant 1$, then

$$c'(v) \geqslant -1 - \frac{1}{4} + \frac{1}{2} + \frac{3}{4} = 0$$

by R2 and R5, and if $\alpha(v) \geqslant 2$, then by Claim 3, v is also incident with a 5^+-face, which implies that

$$c'(v) \geqslant -1 - 2 \times \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + \frac{1}{2} > 0$$

by R2, R3 and R5. Let v be a 3-vertex. Then v has a 4-master by Claim 2 and $\alpha(v) \leqslant 3$ by Claim 3. This implies that

$$c'(v) \geqslant 0 - 3 \times \frac{1}{4} + \frac{3}{4} = 0$$

by R2 and R6. Let v be a vertex of degree between 5 and 7. Then v only sends at most $\frac{1}{4}$ to each of its incident false 3-faces by R1 and R2. So

$$c'(v) \geqslant d_G(v) - 4 - \frac{1}{4}\alpha(v) \geqslant d_G(v) - 4 - \frac{1}{2}\left\lfloor \frac{d_G(v)}{2} \right\rfloor \geqslant 0$$

for $d_G(v) \geqslant 5$ by Claim 3. Let v be a vertex of degree between 8 and $M-6$. Then by Claim 1, v is adjacent to no small vertices and thus v sends out no charges by R2 and R4–R6. This implies that

$$c'(v) \geqslant d_G(v) - 4 - \frac{1}{2}d_G(v) \geqslant 0$$

by R1 for $d_G(v) \geqslant 8$. Let v be a vertex of degree between $M-5$ and $M-3$. Then by Claim 1, v is adjacent to no 4^--vertices and thus v sends out no charges by R4–R6. This implies that

$$c'(v) \geqslant d_G(v) - 4 - \frac{3}{4}d_G(v) \geqslant 0$$

by R2 for $d_G(v) \geqslant M - 5 > 16$. Finally, let v be a vertex of degree between $M-2$ and M. If $d_G(v) = M$, then by Claim 2, v sends at most

$$\frac{3}{4} + 5 \times \frac{1}{2} + 7 \times \frac{3}{4} = \frac{17}{2}$$

to its neighbors by R4–R6. If $\tau(v) \leqslant M-6$, then by R1 and R2,

$$c'(v) \geqslant M-4-\frac{17}{2}-\frac{3}{4}(M-6)=\frac{1}{8}(2M-64)>0$$

for $M \geqslant 34$. If $M-5 \leqslant \tau(v) \leqslant M-1$, then by R1, R2 and Claim 4,

$$\begin{aligned} c'(c) &\geqslant M-4-\frac{17}{2}-\frac{3}{4}\alpha_a(v)-\frac{1}{2}(\tau(v)-\alpha_a(v)) \\ &= M-\frac{25}{2}-\frac{1}{4}\alpha_a(v)-\frac{1}{2}\tau(v) \\ &\geqslant M-\frac{25}{2}-\frac{1}{4}\left(\left\lceil\frac{\tau(v)}{2}\right\rceil+M-\tau(v)-1\right)-\frac{1}{2}\tau(v) \\ &\geqslant \frac{1}{8}(3M-96)>0 \end{aligned}$$

for $M \geqslant 34$. If $\tau(v)=M$, then by R1, R2 and Claim 4,

$$\begin{aligned} c'(v) &\geqslant M-4-\frac{17}{2}-\frac{3}{4}\alpha_a(v)-\frac{1}{2}(\tau(v)-\alpha_a(v)) \\ &= M-\frac{25}{2}-\frac{1}{4}\alpha_a(v)-\frac{1}{2}\tau(v) \\ &\geqslant M-\frac{25}{2}-\frac{1}{8}\tau(v)-\frac{1}{2}\tau(v) \\ &\geqslant \frac{1}{8}(3M-100)>0 \end{aligned}$$

for $M \geqslant 34$. By similar arguments, one can also check that the final charges of the $(M-2)$-vertices and $(M-1)$-vertices are nonnegative. Hence, the proof of Theorem 2 completes, since

$$-8=\sum_{x\in V(G)\cup F(G^\times)} c(x)=\sum_{x\in V(G)\cup F(G^\times)} c'(x)>0,$$

a contradiction.

In the following, we focus on graphs 1-embedded on surfaces and prove the following theorem.

Theorem 3 Let G be graph 1-embedded on a surface with Euler characteristic ε. If

$$\Delta(G) \geqslant 25+\sqrt{841-72\varepsilon},$$

then

$$la(G)=\left\lceil\frac{\Delta(G)}{2}\right\rceil.$$

Proof The proof of Theorem 2 implies that the linear arboricity of every graph 1-embedded on a surface with nonnegative Euler characteristic is $\lceil\frac{\Delta(G)}{2}\rceil$ if $\Delta(G) \geqslant 33$. So we assume $\varepsilon < 0$ below. Similarly, choose a minimum counterexample G to the theorem and then G is 2-connected with $\delta(G) \geqslant 2$. Moreover, Claims 1 and 2 in Section 1 are also valid for this proof. But we need one additional claim here.

Claim 5[6] There are at least $\lfloor\frac{\Delta}{3}\rfloor+2$ vertices of degree greater than $\lfloor\frac{\Delta}{3}\rfloor$ in G.

Now we assign an initial charge $c(v) = d_G(v) - 8$ to every vertex $v \in V(G)$. Since $|E(G)| \leqslant 4(|V(G)| - \varepsilon)$ (see Lemma 2.2 of [18]),

$$\sum_{v\in V(G)} c(v) = 2|E(G)| - 8|V(G)| \leqslant -8\varepsilon.$$

In the following, we will redistribute the charges by the following discharging rules.

$\tilde{\mathbf{R}}$**1.** Each i-vertex receives 1 from its j-master, where $2 \leqslant i \leqslant 7$ and $i \leqslant j \leqslant 7$.

Let $c'(v)$ denote the final charge of a vertex $v \in V(G)$. By Claims 1, 2 and $\tilde{\mathrm{R}}1$, $c'(v) = 0$ for each 7^--vertices and $c'(v) = c(v) = d_G(v) - 8 \geqslant 0$ for each vertex of degree between 8 and $\Delta - 6$. Let v be a Δ-vertex. By Claim 2, v may be 7-masters, 6-masters, 5-masters, 4-masters, 3-masters and 2-master of at most thirteen, eleven, nine, seven, five and one vertices, respectively. This implies that

$$c'(v) \geqslant \Delta - 8 - 13 - 11 - 9 - 7 - 5 - 1 = \Delta - 54$$

by $\tilde{\mathrm{R}}1$. Similarly, we can prove that $c'(v) \geqslant \Delta - 54$ for every vertex of degree between $\Delta - 5$ and $\Delta - 1$. Therefore, $c'(v) > 0$ for every vertex v in G and $c'(v) > \frac{\Delta}{3} - 18$ for every vertex of degree greater than $\lfloor \frac{\Delta}{3} \rfloor$, since

$$\Delta(G) \geqslant 25 + \sqrt{841 - 72\varepsilon} > 55.$$

So by Claim 5,

$$\begin{aligned}\sum_{v\in V(G)} c'(v) &> \left(\left\lfloor \frac{\Delta}{3} \right\rfloor + 2\right)\left(\frac{\Delta}{3} - 18\right)\\ &\geqslant \left(\frac{\Delta + 4}{3}\right)\left(\frac{\Delta - 54}{3}\right)\\ &\geqslant \frac{1}{9}\left(\sqrt{841 - 72\varepsilon} + 29\right)\left(\sqrt{841 - 72\varepsilon} - 29\right)\\ &= -8\varepsilon\\ &= \sum_{v\in V(G)} c(v),\end{aligned}$$

a contradiction.

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