Optimization Methods for a Class of Integer Polynomial Programming Problems*

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Abstract In this paper, a class of integer polynomial programming problems is considered. This class of integer polynomial programming problems has a wide range of practical applications and is NP hard. For these problems, necessary global optimality conditions and sufficient global optimality conditions have been presented recently. We will design some optimization methods to this class of integer polynomial programming problems by using these global optimality conditions. Firstly, a local optimization method is designed according to the necessary global optimality conditions for these integer polynomial programming problems. Moreover, a new global optimization method for this class of integer polynomial programming problems is presented by combining the sufficient global optimality conditions, the local optimization method and an auxiliary function. Some numerical examples are presented to illustrate the efficiency and reliability of these optimization methods.

Keywords polynomial integer programming problem, local optimization method, global optimization method

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一类特殊多项式整数规划问题的最优化算法

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摘要 考虑一类特殊的多项式整数规划问题．此类问题有很广泛的实际应用，并且是 NP 难问题．对于这类问题，最优性必要条件和最优性充分条件已经给出，利用这些最优性条件设计最优化算法．首先，利用最优性必要条件，给出一种新的局部优化算法．进而结合最优性充分条件、新的局部优化算法和辅助函数，设计新的全局最优化算法．给出的算例展示算法是有效的和可靠的．

关键词 多项式整数规划，局部最优化算法，全局最优化算法

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0 Introduction

Polynomial programming problems (PP) have a wide range of applications, such as in engineering design, network distribution and location-allocation context[1−3]. In (PP) formulations, variables are usually assumed to take real continuous values. However, variables that can take only integer values occur naturally and frequently in engineering design models. Examples are the number of teeth in a gear, the number of bars in a truss, and the size of components available only in standard sizes[4]. Finding the global optimal solution and how to characterize it for polynomial programming problems are very difficult tasks except for some special cases. Recently, [5] presented optimal conditions for quadratic integer problem with general box integer constraints and [6] discussed some global optimality conditions for a special kind of cubic polynomial optimization problems where the cubic objective function contains no cross terms. Furthermore, [7] considered the following class of polynomial integer programming problems:

$$(POP)_I \qquad \min \quad f(x)=\sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)} x_i^k + \frac{1}{2}x^{\mathrm{T}}Ax + a^{\mathrm{T}}x$$
$$s.t. \quad x\in U_I=\{(x_1,\dots,x_n)^{\mathrm{T}} \mid x_i\in\{0,1,\dots,J\}, i=1,2,\cdots,n\},$$

where $a\in R^n, A\in S^n$ and S^n is the set of all symmetric $n\times n$ matrices, $k\geqslant 3$ is a positive integer, J is a positive integer.

Many combinatorial optimization problems can be modeled as this class of polynomial scalar objective functions as above, such as in cubic polynomial approximation optimization[8], engineering design and presence of noise[9]. Particularly, some famous test functions belong to this class of polynomial scalar objective functions, such as Six-hump camelback function, Modified fourth De Jong function and Aluffi-Pentini's function. More examples can be found in [10]. As quadratic integer problem with general box integer constraints is NP hard[5], $(POP)_I$ is also an NP hard problem. The necessary global optimality condition and sufficient global optimality condition for this polynomial integer programming problem $(POP)_I$ have been presented in [7], which can be used to check a given point is or is not a global minimizer. The conditions given in [7] extends the results given by references [5-6], [11-12], where the necessary global optimality conditions and sufficient global optimality conditions are presented for integer quadratic or cubic programming problems.

We know that it is more important to design the methods for finding the global minimizer by using the obtained necessary global optimality conditions and sufficient global optimality conditions. Recently, [13] has given some local and global optimization methods according to the necessary global optimality conditions and sufficient global optimality conditions for mixed integer quadratic programming problems. In this paper, we will design a local optimization method according to the necessary global optimality condition given in [7] for the integer polynomial programming problem $(POP)_I$ and then we will propose an auxiliary function to improve the obtained local minimizer and finally design a global optimization method for $(POP)_I$ by combining the sufficient global optimality condition given in [7], the local optimization methods and the auxiliary function proposed in this paper.

The layout of the paper is as follows. In section 1, sufficient global optimality condition and the necessary global optimality condition given in [7] are reviewed. In section 2, two local optimization methods are provided. In section 3, a global optimization methods by combining the sufficient global optimality condition and the local optimization method and the auxiliary function is proposed. In section 4, several examples are given. We conclude this paper in section 5.

1 Global optimality conditions for problem $(POP)_I$

In this section, we will review the sufficient global optimality condition and the necessary global optimality condition for problem $(POP)_I$ given in [7].
For $\bar{x} \in U_I$, let

$$\begin{aligned}\alpha_{\bar{x}_i} : &= \min\left\{\frac{(a+A\bar{x})_i + \sum_{k=3}^{m} b_i^{(k)}(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1})}{(x_i - \bar{x}_i)},\right.\\ &\qquad \left. x_i \in \{0,1,\ldots,J\}, x_i \neq \bar{x}_i\right\},\\ \alpha_{\bar{x}} : &= (\alpha_{\bar{x}_1}.\ldots,\alpha_{\bar{x}_n})^{\mathrm{T}},\\ \mathrm{diag}(\alpha_{\bar{x}}) : &= \mathrm{diag}(\alpha_{\bar{x}_1}.\ldots,\alpha_{\bar{x}_n}),\end{aligned}$$

where $\mathrm{diag}(\alpha_1,\ldots,\alpha_{\mathrm{n}})$ denotes a diagonal matrix with diagonal elements $\alpha_1,\ldots,\alpha_n$.

Theorem 1.1 (Sufficient Global Optimality Condition for $(POP)_I$)[7] Let $\bar{x} \in U_I, J \geqslant 1$. If

$$[SC1] \qquad -\mathrm{diag}(\alpha_{\bar{x}}) \preceq \frac{1}{2}A,$$

then $\bar{x}$ is a global minimizer of problem $(POP)_I$.

Theorem 1.2 (Necessary Global Optimality Condition for $(POP)_I$)[7] Let $\bar{x} \in U_I$, $e := (1,\ldots,1)^{\mathrm{T}}$ and let diag $(A) =$ diag $(a_{11},\ldots,a_{nn})$, where $a_{11},\ldots,a_{nn}$ are the diagonal elements of matrix A. If $\bar{x}$ is a global minimizer of $(POP)_I$, then the following condition holds:

$$[NC1] \qquad -\mathrm{diag}(\alpha_{\bar{x}}) \preceq \frac{1}{2}\mathrm{diag}\ (A).$$

2 Local optimization methods

In this section, we will introduce two local optimization methods. One is designed for general integer problem (IP) which will be used for looking for the local minimizer of the auxiliary function problem (AFP) proposed in Section 3; another one is designed according to the necessary global optimality condition $[NC1]$ for searching the local minimizer of the problem $(POP)_I$. Consider the following general integer problem (IP):

$$\begin{aligned}(IP) \qquad &\min \quad f(x)\\ &s.t. \quad x \in U_I,\end{aligned}$$

where $f : R^n \to R$ is a general continuous function on U_I. Firstly, we will give some definitions, such as neighborhood of a given point, local minimizer (maximizer) and strict local minimizer (maximizer) of problem (IP).

Let e_i be the ith unit vector (the n dimensional vector with the ith component equals to one and the other components equal to zero). For any $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)^{\mathrm{T}} \in U_I$, let

$$N_i(\bar{x}) := \{\bar{x} + (w_i - \bar{x}_i)e_i | w_i = 0, 1, \cdots, J\}.$$

Definition 2.1 Let $\bar{x} \in U_I$, $\bigcup\limits_{i=1}^{n} N_i(\bar{x})$ is said to be a neighborhood of $\bar{x}$ with respect to U_I.

Definition 2.2 Let $\bar{x} \in U_I$. If $f(x) \geqslant f(\bar{x})$ $(f(x) \leqslant f(\bar{x}))$, $\forall x \in \bigcup\limits_{i=1}^{n} N_i(\bar{x})$, then $\bar{x}$ is said to be a local minimizer (maximizer) of problem (IP). Furthermore, if $f(x) > f(\bar{x})$ $(f(x) < f(\bar{x}))$, $\forall x \in \bigcup\limits_{i=1}^{n} N_i(\bar{x}) \setminus \{\bar{x}\}$, then $\bar{x}$ is said to be a strict local minimizer (maximizer) of problem (IP).

In the following, we will propose a local optimization method for the problem (IP).

Algorithm 2.1 (Local Optimization Method for Problem (IP))

Step 1 Take an initial point $x_0 \in U_I$. Let $\bar{x} := x_0,\ k := 1$.

Step 2 Check whether the following condition holds:

$$f(\bar{x}) \leqslant min\{f(x) | x \in \bigcup_{i=1}^{n} N_i(\bar{x})\}.$$

If this condition does not hold, go to Step 3; otherwise go to Step 4.

Step 3 Let $x^* = (x_1^*, \cdots, x_n^*)^{\mathrm{T}} := argmin\{f(x) | x \in \bigcup\limits_{i=1}^{n} N_i(\bar{x})\}$. Let k:=k+1 and $\bar{x} := x^*$, go to Step 2.

Step 4 Stop. $\bar{x}$ is a local minimizer.

Theorem 2.1 Let $\bar{x} \in U_I$. Then $\bar{x}$ is a local minimizer of problem $(POP)_I \Leftrightarrow [NC1]$ holds.

Proof If $\bar{x}$ is a local minimizer of problem $(POP)_I$, then

$$f(\bar{x}) \leqslant \min\left\{f(x) | x \in \bigcup_{i=1}^{n} N_i(\bar{x})\right\}.$$

We can easily verify that

$$
\begin{aligned}
& f(\bar{x}) \leqslant \min\{f(x)|x \in \bigcup_{i=1}^{n} N_i(\bar{x})\} \\
\Leftrightarrow\ & f(x) - f(\bar{x}) \geqslant 0\ \forall x \in N_i(\bar{x}), i = 1, \ldots, n \\
\Leftrightarrow\ & \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}x^{\mathrm{T}}Ax + a^{\mathrm{T}}x - \frac{1}{2}\bar{x}^{\mathrm{T}}A\bar{x} + a^{\mathrm{T}}\bar{x} \geqslant 0, \forall x \in N_i(\bar{x}), i = 1, \ldots, n \\
\Leftrightarrow\ & \sum_{i=1}^{n}\sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}(x-\bar{x})^{\mathrm{T}}A(x-\bar{x}) + (x-\bar{x})^{\mathrm{T}}(a + A\bar{x}) \geqslant 0, \forall x \in N_i(\bar{x}), \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, n \\
\Leftrightarrow\ & \sum_{k=3}^{m} b_i^{(k)}(x_i^k - \bar{x}_i^k) + \frac{1}{2}(x_i - \bar{x}_i)^2 a_{ii} + (x_i - \bar{x}_i)(a + A\bar{x})_i \geqslant 0, \forall x_i \in \{0, 1, \ldots, J\}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, n \\
\Leftrightarrow\ & -\frac{1}{x_i - \bar{x}_i}(\sum_{k=3}^{m} b_i^{(k)}(x_i^{k-1} + x_i^{k-2}\bar{x}_i + \ldots + x_i\bar{x}_i^{k-2} + \bar{x}_i^{k-1}) + (a + A\bar{x})_i) \leqslant \frac{a_{ii}}{2}, \\
& \forall x_i \in \{0, 1, \ldots, J\}, x_i \neq \bar{x}_i, i = 1, \ldots, n \\
\Leftrightarrow\ & -\alpha_{\bar{x}_i} \leqslant \frac{a_{ii}}{2}, i = 1, \ldots, n.
\end{aligned}
$$

Theorem 2.1 states that the necessary global optimality condition $[NC1]$ is a sufficient and necessary condition for a local minimizer of problem $(POP)_I$. In the following, we will give a local optimization method for problem $(POP)_I$ by using the condition $[NC1]$.

Algorithm 2.2 (Local Optimization Method for Problem $(POP)_I$)

Step 1 Take an initial point $x_0 \in U_I$. Let $\bar{x} := x_0,\ k := 1$.

Step 2 Check whether the following condition holds:

$$[NC1] \qquad -\mathrm{diag}(\alpha_{\bar{x}}) \preceq \tfrac{1}{2}\mathrm{diag}\ (A), \quad \text{i.e., } \frac{1}{2}a_{ii} + \alpha_{\bar{x}_i} \geqslant 0,\ \forall i = 1, \cdots, n.$$

If [NC1] does not hold, let $I_{\bar{x}} := \{i \mid \frac{1}{2}a_{ii} + \alpha_{\bar{x}_i} < 0, i = 1, \ldots, n\}$, go to Step 3; otherwise go to Step 4.

Step 3 Let $x^* = (x_1^*, \cdots, x_n^*)^{\mathrm{T}} := argmin\{f(x)|x \in \bigcup_{i\in I_{\bar{x}}} N_i(\bar{x})\}$. Let k:=k+1 and let $\bar{x} := x^*$, go to Step 2.

Step 4 Stop. $\bar{x}$ is a local minimizer.

Remark 2.1 From the proof of Theorem 2.1, we know that for any $i = 1, \ldots, n$, $\frac{1}{2}a_{ii} + \alpha_{\bar{x}_i} \geqslant 0$ if and only if $\bar{x} = argmin\{f(x)|x \in N_i(\bar{x})\}$. Hence $x^* = (x_1^*, \cdots, x_n^*)^{\mathrm{T}} := argmin\{f(x)|x \in \bigcup_{i\in I_{\bar{x}}} N_i(\bar{x})\}$ means $x^* = argmin\{f(x)|x \in \bigcup_{i=1}^{n} N_i(\bar{x})\}$ since for any $x \in \bigcup_{i\in\{1,\ldots,n\}\setminus I_{\bar{x}}} N_i(\bar{x})$, $f(x) \geqslant f(\bar{x})$ and for any $x \in \bigcup_{i\in I_{\bar{x}}} N_i(\bar{x})$, we have that $f(x^*) \leqslant f(x)$ which implies that $f(x^*) \leqslant f(\bar{x})$. Therefore, for any $x \in \bigcup_{i=1}^{n} N_i(\bar{x})$, we have that $f(x^*) \leqslant f(x)$, i.e., $x^* = argmin\{f(x)|x \in \bigcup_{i=1}^{n} N_i(\bar{x})\}$.

3 Global optimization method for problem $(POP)_I$

To introduce the global optimization method, firstly we need to introduce the following auxiliary function. The auxiliary function will be used to escape the current local minimizer and find a better solution of problem $(POP)_I$.

3.1 An auxiliary function for problem $(POP)_I$ and its properties

For any $r > 0$, set

$$g_r(t) = \begin{cases} 0, & \text{if } t \leqslant -r \\ \frac{1}{r}t + 1, & \text{if } -r < t \leqslant 0 \\ 1, & \text{if } t > 0 \end{cases} \qquad f_r(t) = \begin{cases} t + r, & \text{if } t \leqslant -r \\ \frac{1}{r}t + 1 & \text{if } -r < t \leqslant 0 \\ 1, & \text{if } t > 0 \end{cases}$$

and set

$$F_{r,\bar{x}}(x) = \frac{1}{1 + \|x - \bar{x}\|} g_r(f(x) - f(\bar{x})) + f_r(f(x) - f(\bar{x})),$$

where $r > 0$ is a parameter, $\bar{x}$ is the current local minimizer of problem $(POP)_I$ and $\|x\| = \sum_{i=1}^{n} |x_i|$.

Consider the following problem:

$$\begin{aligned} (AFP) \quad & min \quad F_{r,\bar{x}}(x) \\ & s.t. \quad x \in U_I. \end{aligned}$$

In the following, we will discuss some important properties of the auxiliary function.

Theorem 3.1 Let $\bar{x}$ is a local minimizer of problem $(POP)_I$, then for any $r > 0$, $\bar{x}$ is a strict local maximizer of problem (AFP).

Proof Since $\bar{x}$ is a local minimizer of problem $(POP)_I$, $f(x) \geqslant f(\bar{x})$ $\forall x \in \bigcup_{i=1}^{n} N_i(\bar{x})$. $t = f(x) - f(\bar{x}) \geqslant 0$, hence

$$F_{r,\bar{x}}(x) = \frac{1}{1 + \|x - \bar{x}\|} + 1 < 1 + 1 = F_{r,\bar{x}}(\bar{x}), \ \forall x \in \bigcup_{i=1}^{n} N_i(\bar{x}) \setminus \{\bar{x}\}.$$

Therefore, $\bar{x}$ is a strict local maximizer of problem (AFP).

Let x^* be the global minimizer of problem $(POP)_I$ and let $\beta = f(\bar{x}) - f(x^*)$.

Theorem 3.2 If $\bar{x}$ is not a global minimizer of problem $(POP)_I$, then x^* is a local minimizer of problem (AFP) when $r \leqslant \beta$ and satisfies $F_{r,\bar{x}}(x^*) < F_{r,\bar{x}}(\bar{x})$.

Proof When $r \leqslant \beta$, we have that $f(x^*) - f(\bar{x}) \leqslant -r$, hence,

$$F_{r,\bar{x}}(x^*) = f(x^*) - f(\bar{x}) + r \leqslant 0.$$

Then, we can prove $F_{r,\bar{x}}(x) \geqslant F_{r,\bar{x}}(x^*)$, $\forall x \in \bigcup_{i=1}^{n} N_i(x^*)$ by considering the following two cases:

1. If $f(x) - f(\bar{x}) \geqslant -r$, then
$$F_{r,\bar{x}}(x) \geqslant 0 \geqslant F_{r,\bar{x}}(x^*).$$

2. If $f(x) - f(\bar{x}) < -r$, then
$$F_{r,\bar{x}}(x) = f(x) - f(\bar{x}) + r \geqslant f(x^*) - f(\bar{x}) + r = F_{r,\bar{x}}(x^*).$$

Thus x^* is a local minimizer of problem (AFP) and satisfies

$$F_{r,\bar{x}}(x^*) \leqslant 0 < F_{r,\bar{x}}(\bar{x}) = 2.$$

Theorem 3.3 *For any* $x_1,\ x_2 \in \bigcup_{i=1}^{n} N_i(\bar{x})$ *satisfying* $f(x_1) \geqslant f(\bar{x}),\ f(x_2) \geqslant f(\bar{x}),\ \|x_2 - \bar{x}\| > (\geqslant)\|x_1 - \bar{x}\|$ *if and only if* $F_{r,\bar{x}}(x_2) < (\leqslant) F_{r,\bar{x}}(x_1)$ *for any* $r > 0$.

Proof For any $x_1,\ x_2 \in \bigcup_{i=1}^{n} N_i(\bar{x})$ satisfying $f(x_1) \geqslant f(\bar{x}),\ f(x_2) \geqslant f(\bar{x})$, we have that

$$F_{r,\bar{x}}(x_2) = \frac{1}{1 + \|x_2 - \bar{x}\|} + 1,$$
$$F_{r,\bar{x}}(x_1) = \frac{1}{1 + \|x_1 - \bar{x}\|} + 1.$$

Thus for any $r > 0$, $\|x_2 - \bar{x}\| > (\geqslant)\|x_1 - \bar{x}\|$if and only if $F_{r,\bar{x}}(x_2) < (\leqslant) F_{r,\bar{x}}(x_1)$.

Theorem 3.4 If $\hat{x}$ is a local minimizer of problem (AFP), then $\hat{x}$ satisfies one of the following conditions:

1. $f(\hat{x}) < f(\bar{x})$;

2. $\hat{x} := (\hat{x}_1, \cdots, \hat{x}_n)^{\mathrm{T}}$, where $\hat{x}_i \in \{0, J\} \setminus \{\bar{x}_i\}$.

Proof We can prove that if $f(\hat{x}) \geqslant f(\bar{x})$, then $\hat{x}_i \in \{0, J\} \setminus \{\bar{x}_i\}$. In fact if there exists $i_0 \in \{1, \cdots, n\}$ such that $\hat{x}_{i_0} \notin \{0, J\} \setminus \{\bar{x}_{i_0}\}$.

Let

$$k_{i_0} := \begin{cases} 1, & \text{if } \bar{x}_{i_0} = 0 \\ -1 & \text{if } \bar{x}_{i_0} = J \\ 1, & \text{if } \bar{x}_{i_0} \in (0, J) \textit{ and } \hat{x}_{i_0} - \bar{x}_{i_0} \geqslant 0 \\ -1, & \text{if } \bar{x}_{i_0} \in (0, J) \textit{ and } \hat{x}_{i_0} - \bar{x}_{i_0} < 0 \end{cases}$$

Let $\tilde{x} := \hat{x} + k_{i_0} e_{i_0}$, then $\tilde{x} \in \bigcup_{i=1}^{n} N_i(\hat{x})$.

Since $\hat{x}$ is a local minimizer of problem (AFP), then we should have $F_{r,\bar{x}}(\tilde{x}) \geqslant F_{r,\bar{x}}(\hat{x})$, which contradicts

$$\begin{aligned}
F_{r,\bar{x}}(\tilde{x}) &\leqslant \frac{1}{1+\|\tilde{x}-\bar{x}\|}+1 \\
&= \frac{1}{1+\sum\limits_{i=1}^{n}|\tilde{x}_i-\bar{x}_i|}+1 \\
&= \frac{1}{1+\sum\limits_{i\neq i_0}|\tilde{x}_i-\bar{x}_i|+|\tilde{x}_{i_0}-\bar{x}_{i_0}|}+1 \\
&= \frac{1}{1+\sum\limits_{i\neq i_0}|\tilde{x}_i-\bar{x}_i|+|\hat{x}_{i_0}-\bar{x}_{i_0}+k_{i_0}|}+1 \\
&= \frac{1}{1+\sum\limits_{i=1}^{n}|\hat{x}_i-\bar{x}_i|+|k_{i_0}|}+1 \\
&< \frac{1}{1+\sum\limits_{i=1}^{n}|\hat{x}_i-\bar{x}_i|}+1 \\
&= F_{r,\bar{x}}(\hat{x}).
\end{aligned}$$

3.2 Global optimization method for problem $(POP)_I$

In this subsection, we will introduce a global optimization method to find a global minimizer of the problem $(POP)_I$. This method combines the sufficient global optimality condition $[SC1]$, the local optimization methods (Algorithm 2.1, Algorithm 2.2) and the auxiliary function $F_{r,\bar{x}}(x)$.

Algorithm 3.1 (Global Optimization Method for Problem $(POP)_I$)

Step 0 Take an initial point $x_1 \in U_I$, a sufficiently small positive number μ, and an initial $r_1 > 0$. Set $r = r_1$ and $k := 1$.

Step 1 Use the local minimization method: Algorithm 2.2 to solve problem $(POP)_I$ starting from x_k. Let x_k^* be the obtained local minimizer.

Step 2 Verify whether x_k^* satisfies the following global optimality sufficient condition:

$$[SC1] \qquad -\mathrm{diag}(\alpha_{\bar{x}}) \preceq \tfrac{1}{2}\mathrm{A}.$$

If $[SC1]$ holds, then go to step 6; otherwise, let $r := r_1$, go to step 3.

Step 3 Construct the following auxiliary function:

$$F_{r,x_k^*}(x) = \frac{1}{1+\|x-x_k^*\|}g_r(f(x)-f(x_k^*)) + f_r(f(x)-f(x_k^*)).$$

Consider the following problem:

$$(AFP) \quad min \quad F_{r,x_k^*}(x)$$
$$s.t. \quad x \in U_1.$$

Let $\bar{x}_k := x_k^*$, go to Step 4.

Step 4 Use the local minimization method: Algorithm 2.1 to solve problem (AFP) starting from $\bar{x}_k$. Let $\bar{x}_k^*$ be the local minimizer of problem (AFP). If $f(\bar{x}_k^*) < f(x_k^*)$, let $x_{k+1} = \bar{x}_k^*$, $k := k+1$, go to Step1; otherwise go to Step 5.

Step 5 If $r \geqslant \mu$, decrease r, such as, let $r := r/10$, go to Step 3; otherwise, go to Step 6.

Step 6 Stop and x_k^* is the obtained global minimizer or an approximate global minimizer.

Remark 3.1 If the sufficient global optimality condition $[SC1]$ holds, then $\bar{x}$ is a global minimizer of problem $(POP)_I$. If the sufficient global optimality condition $[SC1]$ does not hold, then we use the auxiliary problem (AFP) to improve the current local minimizer $\bar{x}$ to obtain a better local minimizer if $\bar{x}$ is not the global minimizer and finally we can obtain an approximate global minimizer (which is the best solution that we can obtain by this algorithm) for problem $(POP)_I$.

4 Numerical examples

In this section, we apply Algorithm 3.1 to the following test examples. In all the instances, we set $\mu = 10^{-16}$ and $r_1 = 10^{-2}$.

Notation:

x_k:	the $k-th$ initial point
x_k^*:	the $k-th$ local minimizer of problem $(POP)_I$ starting from x_k
$f(x_k)$:	the function value of $f(x)$ at the $k-th$ initial point
$f(x_k^*)$:	the function value of $f(x)$ at the $k-th$ local minimizer of problem $(POP)_I$
$\bar{x}_k^*$:	the $k-th$ local minimizer of problem (AFP) starting from x_k^*
$f(\bar{x}_k^*)$:	the function value of $f(x)$ at the $k-th$ local minimizer of problem $(POP)_I$
r:	the changed value of r_1 after solving the problem of (AFP)

Example 4.1 Consider the problem

$$(EP1) \quad \min \quad f(x) := 3x_1^3 - 5x_2^3 + 2x_3^3 - x_1^4 + 4x_2^4 - 5x_3^4 - 2x_1^5 + 3x_2^5 + x_3^5 + \frac{1}{2}x^{\mathrm{T}}Ax + a^{\mathrm{T}}x$$
$$s.t. \quad x \in \{0,1,2,3,4,5,6\}^3$$

Here $A = \begin{pmatrix} 2 & 4 & 3 \\ -2 & -4 & 8 \\ 3 & -1 & -5 \end{pmatrix}$, $a = (2,-1,3)^{\mathrm{T}}$ and $J = 6$.

Table 4.1 records the numerical results of solving Example [$EP1$] by Algorithm 3.1. From Table 4.1, we see that $(6,0,4)^{\mathrm{T}}$ is the obtained global minimizer starting from the different initial points: $(1,2,3)^{\mathrm{T}},(0,5,6)^{\mathrm{T}},(5,1,0)^{\mathrm{T}}$ and $(2,1,1)^{\mathrm{T}}$. And the local minimizer for problem ($EP1$) starting from $(1,2,3)^{\mathrm{T}},(0,5,6)^{\mathrm{T}}$ is already the global minimizer, but the local minimizers for problem ($EP1$) starting from $(5,1,0)^{\mathrm{T}}$ and $(2,1,1)^{\mathrm{T}}$ are not the global minimizer. Here we have to use the auxiliary function (AFP) to improve them and find another starting points and the second local minimizer is the global minimizer. The sufficient global optimality condition [$SC1$] holds at this global minimizer $(6,0,4)^{\mathrm{T}}$.

Example 4.2 Consider the problem

$$(EP2) \qquad \min \quad f(x) := 6x_1^3 - x_1^4 + x_2^4 - x_2^5 + 3x_3^3 - x_3^4 - 4x_4^3 + x_4^5 + \frac{1}{2}x^{\mathrm{T}}Ax + a^{\mathrm{T}}x$$
$$s.t. \quad x \in \{0,1,2,3,4,5,6\}^4$$

Here $A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 5 & -6 & 7 \\ 3 & -6 & 8 & -1 \\ -4 & 7 & -1 & 9 \end{pmatrix}$, $a = (2,-1,3,-4)^{\mathrm{T}}$ and $J = 6$.

Table 4.2 records the numerical results of solving Example [$EP2$] by Algorithm 3.1. From Table 4.2, we see that $(0,\ 6,\ 6,\ 0)^{\mathrm{T}}$ is the obtained global minimizer starting from the different initial points: $(3,\ 0,\ 5,\ 4)^{\mathrm{T}},(4,\ 6,\ 1,\ 2)^{\mathrm{T}},(6,\ 1,\ 0,\ 5)^{\mathrm{T}}$ and $(6,\ 2,\ 2,\ 3)^{\mathrm{T}}$. And the local minimizer for problem ($EP2$) starting from $(3,\ 0,\ 5,\ 4)^{\mathrm{T}}$ and $(4,\ 6,\ 1,\ 2)^{\mathrm{T}}$ is already the global minimizer, but the local minimizers for problem ($EP2$) starting from $(6,\ 1,\ 0,\ 5)^{\mathrm{T}}$ and $(6,\ 2,\ 2,\ 3)^{\mathrm{T}}$ are not the global minimizer. Here we have to use the auxiliary function (AFP) to improve them and find another starting points and the second local minimizer is the global minimizer. The sufficient global optimality condition [$SC1$] holds at this global minimizer $(0,\ 6,\ 6,\ 0)^{\mathrm{T}}$.

Example 4.3 Consider the problem

$$(EP3) \qquad \min \quad f(x) := -3x_1^3 + 2x_2^3 - 6x_3^3 + 4x_1^4 - 4x_2^4 + x_3^4 + 2x_4^4$$
$$+2x_1^5 + 5x_2^5 + 3x_3^5 - x_4^5 + \frac{1}{2}x^{\mathrm{T}}Ax + a^{\mathrm{T}}x$$
$$s.t. \quad x \in \{0,1,2,3,4,5,6,7,8\}^4$$

Here $A = \begin{pmatrix} -4 & 2 & -3 & 1 \\ 2 & 3 & 7 & 5 \\ -3 & 7 & -5 & 6 \\ 1 & -3 & 4 & -8 \end{pmatrix}$, $a = (1,-2,3,-1)^{\mathrm{T}}$ and $J = 8$.

Table 4.3 records the numerical results of solving Example [$EP3$] by Algorithm 3.1. From Table 4.3, we see that $(0,\ 0,\ 0,\ 8)^{\mathrm{T}}$ is the obtained global minimizer starting from the different initial points: $(5,\ 7,\ 2,\ 4)^{\mathrm{T}},(2,\ 3,\ 5,\ 6)^{\mathrm{T}},(0,\ 0,\ 1,\ 2)^{\mathrm{T}},(8,\ 5,\ 6,\ 0)^{\mathrm{T}},(1,\ 1,\ 8,\ 0)^{\mathrm{T}}$ and $(8,\ 2,\ 6,\ 8)^{\mathrm{T}}$. And the local minimizer for problem ($EP3$) starting from $(5,\ 7,\ 2,\ 4)^{\mathrm{T}}$,(2, 3, 5,

$6)^{\mathrm{T}}$ and $(0,\ 0,\ 1,\ 2)^{\mathrm{T}}$ is already the global minimizer, but the local minimizers for problem $(EP3)$ starting from $(8,\ 5,\ 6,\ 0)^{\mathrm{T}}, (1,\ 1,\ 8,\ 0)^{\mathrm{T}}$ and $(8,\ 2,\ 6,\ 8)^{\mathrm{T}}$ are not the global minimizer. Here we have to use the auxiliary function (AFP) to improve them and find another starting points and the second local minimizer is the global minimizer. The sufficient global optimality condition $[SC1]$ holds at this global minimizer $(0,\ 0,\ 0,\ 8)^{\mathrm{T}}$.

Example 4.4 Consider the problem

$$(EP4) \qquad \min \quad f(x) := 2x_1^3 - 3x_2^3 + x_3^3 + x_1^4 + 2x_2^4 - 3x_3^4 + \frac{1}{2}x^{\mathrm{T}}Ax + a^{\mathrm{T}}x$$
$$s.t. \quad x \in \{0,1,2\}^3$$

Here $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & -1 \end{pmatrix}$, $a = (1,-4,1)^{\mathrm{T}}$ and $J = 2$.

Table 4.4 records the numerical results of solving Example $[EP4]$ by Algorithm 3.1. From Table 4.4, we see that $[EP4]$ has two global minima $(0,\ 1,\ 2)^{\mathrm{T}}$ which is obtained from the following initial points $(1\ 1\ 2)^{\mathrm{T}}$, $(2\ 2\ 1)^{\mathrm{T}}$ and $(2\ 1\ 0)^{\mathrm{T}}$ and $(0\ 0\ 2)^{\mathrm{T}}$ which is obtained from the following initial points $(1\ 0\ 2)^{\mathrm{T}}$, $(2\ 0\ 0)^{\mathrm{T}}$ and $(0\ 0\ 0)^{\mathrm{T}}$. Both of the global minima $(0,\ 1,\ 2)^{\mathrm{T}}$ and $(0,\ 0,\ 2)^{\mathrm{T}}$ are the first local minima from different initial points. However the sufficient global optimality condition $[SC1]$ does not hold at these two global minimal points.

Table 4.1 Numerical results for Example (EP)

k	x_k	$f(x_k)$	$k-th$ local minimizer x_k^* of $f(x)$	$f(x_k^*)$	r	$k-th$ local minimizer $\bar{x}_k^*$ of F_{r,x_k^*}	$f(\bar{x}_k^*)$
1	$(1\ 2\ 3)^{\mathrm{T}}$	23.5	$(6\ 0\ 4)^{\mathrm{T}}$	−16236			
1	$(0\ 5\ 6)^{\mathrm{T}}$	12956	$(6\ 0\ 4)^{\mathrm{T}}$	−16236			
1	$(5\ 1\ 0)^{\mathrm{T}}$	−6461	$(6\ 1\ 4)^{\mathrm{T}}$	−16217	10^{-2}	$(6\ 0\ 4)^{\mathrm{T}}$	−16236
2	$(6\ 0\ 4)^{\mathrm{T}}$	−16236	$(6\ 0\ 4)^{\mathrm{T}}$	−16236			
1	$(2\ 1\ 1)^{\mathrm{T}}$	−39	$(6\ 1\ 4)^{\mathrm{T}}$	−16217	10^{-2}	$(6\ 0\ 4)^{\mathrm{T}}$	−16236
2	$(6\ 0\ 4)^{\mathrm{T}}$	−16236	$(6\ 0\ 4)^{\mathrm{T}}$	−16236			

Table 4.2 Numerical results for Example (EP)

k	x_k	$f(x_k)$	$k-th$ local minimizer x_k^* of $f(x)$	$f(x_k^*)$	r	$k-th$ local minimizer $\bar{x}_k^*$ of F_{r,x_k^*}	$f(\bar{x}_k^*)$
1	$(3\ 0\ 5\ 4)^{\mathrm{T}}$	757.5	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098			
1	$(4\ 6\ 1\ 2)^{\mathrm{T}}$	−6255	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098			
1	$(6\ 1\ 0\ 5)^{\mathrm{T}}$	2652	$(6\ 6\ 6\ 0)^{\mathrm{T}}$	−7032	10^{-2}	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098
2	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098			
1	$(6\ 2\ 2\ 3)^{\mathrm{T}}$	167.5	$(6\ 6\ 6\ 0)^{\mathrm{T}}$	−7032	10^{-2}	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098
2	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098	$(0\ 6\ 6\ 0)^{\mathrm{T}}$	−7098			

Table 4.3 Numerical results for Example (EP)

k	x_k	$f(x_k)$	$k-th$ local minimizer x_k^* of $f(x)$	$f(x_k^*)$	r	$k-th$ local minimizer $\bar{x}_k^*$ of F_{r,x_k^*}	$f(\bar{x}_k^*)$
1	$(5\ 7\ 2\ 4)^{\mathrm{T}}$	83213	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840			
1	$(2\ 3\ 5\ 6)^{\mathrm{T}}$	5186	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840			
1	$(0\ 0\ 1\ 2)^{\mathrm{T}}$	−9.5	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840			
1	$(8\ 5\ 6\ 0)^{\mathrm{T}}$	117070	$(1\ 0\ 0\ 8)^{\mathrm{T}}$	−24830	10^{-2}	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840
2	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840			
1	$(1\ 1\ 8\ 0)^{\mathrm{T}}$	99231	$(1\ 0\ 0\ 8)^{\mathrm{T}}$	−24830	10^{-2}	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840
2	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840			
1	$(8\ 2\ 6\ 8)^{\mathrm{T}}$	79086	$(1\ 0\ 0\ 8)^{\mathrm{T}}$	−24830	10^{-2}	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840
2	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840	$(0\ 0\ 0\ 8)^{\mathrm{T}}$	−24840			

Table 4.4 Numerical results for Example (EP)

k	x_k	$f(x_k)$	$k-th$ local minimizer x_k^* of $f(x)$	$f(x_k^*)$	r	$k-th$ local minimizer $\bar{x}_k^*$ of F_{r,x_k^*}	$f(\bar{x}_k^*)$	global minimizer x_k^* of $f(x)$
1	$(1\ 0\ 2)^{\mathrm{T}}$	−36.5	$(0\ 0\ 2)^{\mathrm{T}}$	−40	10^{-16}	$(2\ 2\ 0)^{\mathrm{T}}$	52	$(0\ 0\ 2)^{\mathrm{T}}$
1	$(1\ 1\ 2)^{\mathrm{T}}$	−34.5	$(0\ 1\ 2)^{\mathrm{T}}$	−40	10^{-16}	$(2\ 2\ 0)^{\mathrm{T}}$	52	$(0\ 1\ 2)^{\mathrm{T}}$
1	$(2\ 0\ 0)^{\mathrm{T}}$	40	$(0\ 0\ 2)^{\mathrm{T}}$	−40	10^{-16}	$(2\ 2\ 0)^{\mathrm{T}}$	52	$(0\ 0\ 2)^{\mathrm{T}}$
1	$(2\ 2\ 1)^{\mathrm{T}}$	52.5	$(0\ 1\ 2)^{\mathrm{T}}$	−40	10^{-16}	$(2\ 2\ 0)^{\mathrm{T}}$	52	$(0\ 1\ 2)^{\mathrm{T}}$
1	$(2\ 1\ 0)^{\mathrm{T}}$	40	$(0\ 1\ 2)^{\mathrm{T}}$	−40	10^{-16}	$(2\ 2\ 0)^{\mathrm{T}}$	52	$(0\ 1\ 2)^{\mathrm{T}}$
1	$(0\ 0\ 0)^{\mathrm{T}}$	0	$(0\ 0\ 2)^{\mathrm{T}}$	−40	10^{-16}	$(2\ 2\ 0)^{\mathrm{T}}$	52	$(0\ 0\ 2)^{\mathrm{T}}$

5 Conclusions

In this paper, we have designed a local optimization method according to the necessary global optimality condition given in reference [7] for integer polynomial programming problem $(POP)_I$. And then we introduce an auxiliary function for this problem and design a global optimization method by combining the sufficient global optimality condition given in [7] for problem $(POP)_I$, the local optimization methods and the auxiliary function given in this paper. From the numerical results, we can see that the proposed methods are efficient and reliable. The further work will focus on the mixed polynomial integer programming problems, which will extend the field of application.

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