A Newton Method for a Nonsmooth Nonlinear Complementarity Problem*

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Abstract This paper is devoted to a nonlinear complementarity problem with nonsmooth data. The nonlinear complementarity problem is reformulated as a system of nonsmooth equations. Then, a Newton method for solving the nonsmooth equations is proposed. In each iteration of the Newton method, an element of the B-differential of related functions, not nonlinear complementarity function, is required. The superlinear convergence is shown.

Keywords nonlinear complementarity problem, nonsmooth analysis, optimization, Newton methods

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摘要

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0 Introduction

The nonlinear complementarity problem

$$F(x) \geqslant 0, x \geqslant 0, x^{\mathrm{T}}F(x) = 0, \tag{0.1}$$

where $F : \Re^n \to \Re^n$ and $x \in \Re^n$ is to find a solution $x \in \Re^n$, which satisfies (0.1). The complementarity problem plays an important role in economics equilibrium, system engineering, optimization and others. It has been studied extensively when F is smooth, see

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for instance [3-4, 6] and references therein. Based on nolinear complementarity function, a nonlinear complementarity problem is equivalently transformed into a system of nonsmooth equations. A generalized Newton method is used to solve the system of nonsmooth equations.

For the case where F is nonsmooth function, to our knowledge, only [1-2, 5] dealt with the problem (0.1). They transform (0.1) into a unconstrained optimization, then solve it by nonsmooth optimization method. In the present paper, we try to study nonlinear complementarity with nonsmooth data. We first reformulate the nonlinear complementarity problem as a system of nonsmooth equation, then propose a Newton to solve the nonsmooth equations.

Let us consider the following nonlinear complementarity problem

$$F(x) \geqslant 0, Z(x) \geqslant 0, Z(x)^{\mathrm{T}} F(x) = 0, \tag{0.2}$$

where $F : \Re^n \to \Re^n$ is locally Lipschitzian, $Z : \Re^n \to \Re^n$ is continuously differentiable. When $Z(x) = x$, the problem (1.2) happens to be the problem (0.1). Throughtout of the paper, we denote $F(x) = (f_1(x), \dots, f_n(x))^{\mathrm{T}}$, $Z(x) = (z_1(x), \dots, z_n(x))$, $B(x, \delta)$ the open ball with x and δ as its center and radium, respectively.

1 Preliminaries

We start with a brief review of nonsmooth analysis and Newton method for solving nonsmooth equations.

Let both X and Y be subsets of finite dimensional spaces. As in [3], the set-valued mapping $x \to S(x)$ from X to 2^Y is said to be upper-semicontinuous at $x_0 \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|x - x_0\| \leqslant \delta$ implies that

$$S(x) \subset S(x_0) + \varepsilon B(0, 1).$$

We say the set-valued mapping $x \to S(x)$ to be upper-semicontinuous on X if it is upper-semicontinuous at all $x \in X$.

Let $H : \Re^n \to \Re^m$ be locally Lipschitzian. By the definition in [3-4],

$$\partial_B H(x) = \{\lim_{y \to x} JH(y) \mid y \in D_H\},$$

where D_H is the set of differentiable points of H, is called the B-differential of H at x; $\partial_{Cl} H(x) = \mathrm{co}\partial_B H(x)$ is called the Clarke generalized Jacobian of H at x; when H is from $\Re^n$ to $\Re$, $\partial_{Cl} H(x)$ is said to be the Clarke generalized gradient at x.

Both set-valued mappings $x \to \partial_{Cl} H(x)$ and $x \to \partial_B H(x)$ are upper-semicontinuous, see [3].

As in [3-4], the locally Lipschitzian function $H : \Re^n \to \Re^m$ is said to be semismooth at x if

$$\lim_{\substack{V \in \partial_{Cl} H(x+th') \\ h' \to h,\ t \to 0^+}} Vh'$$

exists for any $h \in \Re^n$.

Lemma 1.1(see [3-4]). Suppose that $H : \Re^n \to \Re^m$ is locally Lipschitzian on $\Re^n$ and semismooth at x. Then, one has that

$$\xi h - H'(x; h) = o(\|h\|),\ \xi \in \partial_{Cl} H(x+h), \tag{1.1}$$

$$H(x+h) - H(x) - H'(x; h) = o(\|h\|). \tag{1.2}$$

Let us consider the nonsmooth equations:

$$H(z) = 0,$$

where $H : \Re^n \to \Re^n$ is locally Lipschitzian. Newton method for solving the nonsmooth equations is given by

$$x_{k+1} = x_k - \xi_k^{-1} H(x_k), \tag{1.3}$$

where ξ_k is an element of $\partial_B H(x_k)$, $\partial_{Cl} H(x_k)$ or $\partial_B h_1(x_k) \times \cdots \times \partial_B h_n(x_k)$, and $H(x) = (h_1(x), \ldots, h_n(x))^{\mathrm{T}}$. The locally superlinear convergence of Newton methods are shown when F is semismooth and all elements of corresponding subdifferentials, mentioned above, of H at the solution are nonsingular. The Newton of (1.3) can be performed if an element of related subdifferential can be computed.

2 A Newton and its convergence analysis

Evidently, the nonlinear complentarity problem (0.2) can be reformulated as the following

$$\min\{z_i(x), f_i(x)\} = 0, i = 1, \ldots, n. \tag{2.1}$$

Denote $G(x) = (g_1(x) \ldots g_n(x))^{\mathrm{T}}$, where

$$g_i(x) = \min\{z_i(x), f_i(x)\}, i = 1, \ldots, n \tag{2.2}$$

Of course, as mentioned in [1], the Newton method (1.3) can be used to solve the equations (2.2) directly. As is known, that method need to compute an element of some kind subdifferential of G at each iteration.

Define the set-valued mapping $x \to V(x)$ from $\Re^n$ to subsets of $\Re^{n\times n}$ as the following

$$V(x) = V_1(x) \times \cdots \times V_n(x),$$

where

$$V_i(x) = \begin{cases} \{\nabla z_i(x)\}, & \text{if } z_i(x) \leqslant f_i(x), \\ \partial_B f_i(x), & \text{if } z_i(x) > f_i(x). \end{cases} \tag{2.3}$$

We give a Newton method for solving the nonlinear complentarity problem (0.2), equivalently solving the equations $G(x) = 0$ as the following:

$$x_{k+1} = x_k - \xi_k^{-1} G(x_k), \quad \xi_k \in V(x_k). \tag{2.4}$$

It should be mentioned that $V(x)$ is not a subdifferential of $G(x)$ and is even not upper-semicontinuous as a set-valued mapping. We next define another set-valued mapping $x \to \bar{V}(x)$ from $\Re^n$ to subsets of $\Re^{n\times n}$ as follows

$$\bar{V}(x) = \bar{V}_1(x) \times \cdots \times \bar{V}_n(x),$$

where

$$\bar{V}_i(x) = \begin{cases} \{\nabla z_i(x)\}, & \text{if } z_i(x) < f_i(x), \\ \{\nabla z_i(x)\} \bigcup \partial_B f_i(x), & \text{if } z_i(x) = f_i(x), \\ \partial_B f_i(x), & \text{if } z_i(x) > f_i(x). \end{cases} \tag{2.5}$$

It is easy to see that $\bar{V}(x) \subset \bar{V}(x)$ for any $x \in \Re^n$. Next two lemmas characterize the upper-semicontinuity of the set-valued mapping $x \to \bar{V}(x)$.

Lemma 2.1 The set-valued mapping $x \to \bar{V}(x)$ is upper-semicontinuous.

Proof It is enough to prove that each $x \to \bar{V}_i(x)$ is upper-semicontinuous. Given a fixed point $x_0 \in \Re^n$ and a fixed index i, in what follows, we prove that $x \to \bar{V}_i(x)$ is upper-semicontiuous at x_0. If $z_i(x_0) < f_i(x_0)$, then there is a neighbourhood $B(x_0, \delta)$ of x_0 such that $z_i(x) < f_i(x)$, $\forall x \in B(x_0, \delta)$. Hence, $\bar{V}_i(x) = \{\nabla z_i(x)\}$, $\forall x \in B(x_0, \delta)$. By the continuity of the function $\nabla z_i(x)$, the set-valued mapping $\bar{V}_i(x)$ is upper-semicontinuously at x_0. If $z_i(x_0) > f_i(x_0)$, then there is a neighbourhood $B(x_0, \delta_1)$ of x_0 such that $z_i(x) > f_i(x), \forall x \in B(x_0, \delta_1)$. Therefore, $\bar{V}_i(x) = \partial_B f_i(x)$, $\forall x \in B(x_0, \delta_1)$. By the upper-semicontinuity of the set-valued mapping $x \to \partial_B f_i(x)$, the set-valued mapping $x \to \bar{V}_i(x)$ is upper-semicontinuous at x_0. Now we consider the case where $z_i(x_0) = f_i(x_0)$. Suppose that $x_k \to x_0$ and $\xi_k \to \xi$ with $\xi_k \in \bar{V}_i(x_k)$. Then, $\xi_k \in \{\nabla z_i(x_k)\} \bigcup \partial_B f_i(x_k)$. Evidently, the set-valued mapping $x \to \{\nabla z_i(x)\} \bigcup \partial_B f_i(x)$ is upper-semicontinuous. Therefore, $\xi \in \{\nabla z(x_0)\} \bigcup \partial_B f_i(x_0)\} = \bar{V}_i(x_0)$, i.e. $\bar{V}(x)$ is upper-semicontinuous at x_0. This completes the proof of the lemma.

Lemma 2.2 Let $x_0 \in \Re^n$. If all $\xi \in \bar{V}(x_0)$ are nonsingular, then there exists $\beta > 0$ such that

$$\|\xi^{-1}\| \leqslant \beta, \quad \forall \xi \in V(x_0). \tag{2.6}$$

Proof Since the set-valued mapping $x \to \bar{V}(x)$ is upper-semicontinuous, the set $\bar{V}(x_0)$ is compact and all $\xi \in \bar{V}(x_0)$ are nonsingular, there exists $\beta > 0$ such that $\|\xi^{-1}\| \leqslant \beta, \forall \xi \in \bar{V}(x_0)$. Noticing $V(x_0) \subset \bar{V}_0(x)$, (2.6) holds. This completes the proof of the lemma.

The next theorem is on the convergence analysis for the Newton method (2.4).

Theorem 2.1 Suppose that x^* is a solution of the complementarity problem (0.3), F is semismooth at x^* and all $\xi \in \bar{V}(x^*)$ are nonsingular. Then, the iteration (2.4) is well-defined and generates the sequence $\{x_k\}$ converging to x^* superlinearly in a neighborhood of x^*.

Proof We first prove

$$V(x) \subset \partial_{Cl} g_1(x) \times \cdots \partial_{Cl} g_n(x). \tag{2.7}$$

It is enough to prove that $V_i(x) \subset \partial_{Cl} g_i(x), i = 1, \ldots, n$. Let $x_0 \in \Re^n$ and i be fixed point and index, respectively. If $z_i(x_0) < f_i(x_0)$, then there is a neighbourhood $B(x_0, \delta)$ of x_0 such that $z_i(x) < f_i(x)$, $\forall x \in B(x_0, \delta)$. Hence, $g_i(x) = z_i(x)$ and $g_i(x)$ is continuously differentiable with $\nabla g_i(x) = \nabla z_i(x)$ for all $x \in B(x_0, \delta)$. Therefore,

$$V_i(x_0) = \{\nabla z_i(x_0)\} = \{\nabla g_i(x_0)\} = \partial_B g_i(x_0),$$

that is $V_i(x_0) \subset \partial_{Cl} g_i(x_0)$.

If $z_i(x_0) > f_i(x_0)$, then there is a neighbourhood of $B(x_0, \delta_1)$ of x_0 such that $z_i(x) > f_i(x), \forall x \in B(x_0, \delta_1)$. Therefore,

$$g_i(x) = f_i(x) \quad \text{and} \quad \partial_B g_i(x) = \partial_B f_i(x)$$

for all $x \in B(x_0, \delta_1)$. This leads to

$$V_i(x_0) = \partial_B f_i(x_0) = \partial_B g_i(x_0) \subset \partial_{Cl} g_i(x_0),$$

that is $V_i(x_0) \subset \partial_{Cl} g_i(x_0)$.

Now we suppose that $z_i(x_0) = f_i(x_0)$. There are two cases.

Case one: there is a sequence $\{x_k\}$ with $x_k \to x_0$ such that $z_i(x_k) < f_i(x_k)$;

Case two: there exists a neighbourhood $B(x_0, \delta')$ of x_0 such that $z_i(x) \geqslant f_i(x)$ for all $x \in B(x_0, \delta')$.

In the case one, there exists an neighbourhood $B(x_k, \delta_k)$ of x_k such that $z_i(x) < f_i(x), x \in B(x_k, \delta_k)$. Thus, $g_i(x) = z_i(x)$ and $g_i(x)$ is continuously differentiabl with $\nabla g_i(x_k) = \nabla z_i(x_k)$ for all $x \in B(x_k, \delta_k)$. Then,

$$V_i(x_k) = \{\nabla z_i(x_k)\} = \{\nabla g_i(x_k)\} = \partial_B g_i(x_k).$$

According to the continuity of ∇z and the upper-semicontinuity of $\partial_B g_i(x)$, we have

$$V_i(x_0) = \{\nabla z_i(x_0)\} \subset \partial_B g_i(x_0) \subset \partial_{Cl} g_i(x_0).$$

This leads to $V_i(x_0) \subset \partial_{Cl} g_i(x_0)$.

For the case two, since $f_i(x) - z_i(x) \leqslant 0$ for all $x \in B(x_0, \delta')$ and $f_i(x_0) - z_i(x_0) = 0$, x_0 is a maximizer of the function $f_i(x) - z_i(x)$. According to the optimality condition of locally Lipschitizian function, one has that

$$0 \in \partial_{Cl}(f_i(x) - z_i(x)) \,|_{x=x_0} = \partial_{Cl} f_i(x_0) - \nabla z_i(x_0).$$

Thus, $\nabla z_i(x_0) \subset \partial_{Cl} f_i(x_0)$. On the other hand, the fact that $z_i(x) \geqslant f_i(x), \forall x \in B(x_0, \delta')$ means $g_i(x) = f_i(x)$ and $\partial_{Cl} g_i(x) = \partial_{Cl} f_i(x)$ for all $x \in B(x_0, \delta')$. Then, we have

$$V_i(x_0) = \{\nabla z_i(x_0)\} \subset \partial_{Cl} g_i(x_0),$$

i. e., $V_i(x_0) \subset \partial_{Cl} g_i(x_0)$.

By virtue of Lemma 2.2, (2.4) is well-defined in a neighbourhood of x^* for the first step. Let $\xi_k = (\xi_{1k}, \ldots, \xi_{nk})$. Since $g_i, i = 1, \ldots, n$ are semismooth at x^* and $\xi_{ik} \in V_i(x_k) \subset \partial_{Cl} g_i(x_k)$, it follows from (1.1) that

$$\xi_{ik}(x_k - x^*) - g_i'(x^*; x_k - x^*) = o(\|x_k - x^*\|), i = 1, \ldots, n. \tag{2.8}$$

Hence, one has that

$$\xi_k(x_k - x^*) - G'(x^*; x_k - x^*) = o(\|x_k - x^*\|). \tag{2.9}$$

Introducing $x = x^*, h = x_k - x^*$ and $H = G$ to (1.2), we have

$$G(x_k) - G(x^*) - G'(x^*; x_k - x^*) = o(\|x_k - x^*\|). \tag{2.10}$$

From Lemma 1.1, the formula (2.9) and the formula (2.10), it follows that

$$\begin{aligned}
\|x_{k+1} - x^*\| &= \|x_k - x^* - \xi_k^{-1} G(x_k)\| \\
&\leqslant \|\xi_k^{-1}[G(x_k) - G(x^*) - G'(x^*; x_k - x^*)]\| \\
&\quad + \|\xi_k^{-1}[\xi_k(x_k - x^*) - G'(x^*; x_k - x^*)]\| \\
&= o(\|x_k - x^*\|).
\end{aligned}$$

This shows the superlinear convergence of $\{x_k\}$ to x^* in a neighborhood of x^*. We thus have completed the proof of the theorem.

3 Conclusions

In this paper, a nonsmooth Newton method for a nonlinear complementarity problem with nonsmooth data is developed. In each iteration of this Newton method, an element of B-differential of nonsmooth function f_i is needed, but an element of a subdifferential of G or $\max\{z_i(x), f_i(x)\}$ not needed. The present method can be performed easily for some practice.

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