Vertex Vulnerability Parameters of Kronecker Products of Complete Multipartite Graphs and Complete Graphs*

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Abstract Let G_1 and G_2 be two graphs. The Kronecker product $G_1 \times G_2$ is defined as $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper we compute several vertex vulnerability parameters of Kronecker product of a complete p-partite graph $K_{m_1,m_2,\ldots,m_p}$ and a complete graph K_n on n vertices, where $m_1 \leqslant m_2 \leqslant \ldots \leqslant m_p$, $2 \leqslant p \leqslant n$, and $n \geqslant 3$. This result generalizes the previous result by Mamut and Vumar.

Keywords Kronecker product, vertex vulnerability parameter, cut set, complete p-partite graph, complete graph

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完全图与完全多部图的 Kronecker 积的点脆弱性参数

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摘要 两个图 G_1 和 G_2 的 Kronecker 积 $G_1 \times G_2$ 定义为：点集 $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, 且边集 $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. 计算了完全 p- 部图 $K_{m_1,m_2,\ldots,m_p}$ 与 n 阶完全图 K_n 的 Kronecker 积的几类点脆弱性参数的值，其中 $m_1 \leqslant m_2 \leqslant \ldots \leqslant m_p$, $2 \leqslant p \leqslant n$, 且 $n \geqslant 3$. 这些结论推广了 Mamut 和 Vumar 得到的相关结果.

关键词：Kronecker 积，点脆弱性参数，割集，完全 p- 部图，完全图

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0 Introduction

All graphs considered in this paper are finite and simple. For notation and terminology not defined here we refer to West[1].

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Given two graphs and the cartesian product of their vertex sets we can define several graph products. Four standard products defined this way are: the *cartesian product*, the *Kronecker product*, the *strong product*, and the *lexicographic product*. Here we consider the Kronecker product of two graphs G_1 and G_2 which defined as: $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$ (see [?]). The Kronecker product of graphs has been extensively investigated concerning graph colorings, graph recognition and decomposition, graph embed-dings, matching theory and stability in graphs (see, for example, [3-4]). This product has many interesting applications, for instance, Leskovec et al. used Kronecker product to give an approach to modeling networks in [?].

Many graph theoretical parameters such as connectivity, toughness, scattering number, integrity, tenacity and their edge-analogues (see, for example, [6-8]) have been defined to measuring the stability of networks. For most of these parameters, the corresponding computing problems have been proved to be NP-hard. So it is interesting to determine vulnerability parameters for some particular graphs. This paper gives the values on several vulnerability parameters for the Kronecker product of a complete multipartite graph and a complete graph.

We first recall the definitions of some graph parameters. Let $G = (V, E)$ is a graph. A set $S \subseteq V$ is a *cut set* of G, if either $G - S$ is disconnected or has only one vertex. For $S \subseteq V$, let $\omega(G - S)$ and $\tau(G - S)$ denote the number of components and the order of a largest component of $G - S$, respectively. The *independent number* $\alpha(G)$ of a graph G is defined to be the maximum number of mutually nonadjacent vertices in G. The *covering number* $\beta(G)$ of the graph G is defined to be the minimum number of subset of $V(G)$, such that every edge of $V(G)$ has at least one endpoint in this subset. The vulnerability parameters are defined as follows:

(a) Vertex connectivity $\kappa(G)$

$$\kappa(G) = \min\{|S| : S \subseteq V \ \text{ is a cut set of } G\};$$

(b) Vertex toughness $t(G)$ (Chvátal, 1973, [?])

$$t(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V \ \text{ is a cut set of } G\right\};$$

(c) Scattering number $s(G)$ (Jung, 1978, [?])

$$s(G) = \max\{\omega(G-S) - |S| : S \subseteq V \ \text{ is a cut set of } G\};$$

(d) Vertex integrity $I(G)$ (Barefoot, et al., 1987, [?])

$$I(G) = \min\{|S| + \tau(G-S) : S \subseteq V \ \text{ is a cut set of } G\};$$

(e) Vertex Tenacity $T(G)$ (Cozzens, et al., 1995, [?])

$$T(G) = \min\left\{\frac{|S| + \tau(G-S)}{\omega(G-S)} : S \subseteq V \text{ is a cut set of } G\right\}.$$

Edge analogues of these parameters can be defined similarly (see, [9-12]).

Kirlangic[13] gave some results on rupture degree of gear graphs and the relationships between the rupture degree and some vulnerability parameters. Choudum and Priya[14] studied the tenacity of Cartesian products of complete graphs. Recently, Mamut and Vumar[15] computed some vulnerability parameters of Kronecker product of two complete graphs as follows.

Theorem 1 [15] *Let m, n be integers with $n \geqslant m \geqslant 2$ and $n \geqslant 3$. Then*

(a) $\kappa(K_m \times K_n) = (m-1)(n-1)$.

(b) $t(K_m \times K_n) = m-1$.

(c) $s(K_m \times K_n) = \begin{cases} 2-(m-1)(n-1), & \text{if } m=n; \\ 2n-mn, & \text{otherwise.} \end{cases}$

(d) $I(K_m \times K_n) = mn - n + 1$.

(e) $T(K_m \times K_n) = m + 1/n - 1$.

In this paper we consider the vulnerability parameters of $G = K_{m_1,m_2,\ldots,m_p} \times K_n$ $(2 \leqslant p \leqslant n,\ m_1 \leqslant m_2 \leqslant \ldots \leqslant m_p,\ n \geqslant 3)$ and get the following results.

Theorem 2 *Let $m_1, m_2, \ldots, m_p, n$ be integers with $m_1 \leqslant m_2 \leqslant \ldots \leqslant m_p$, $2 \leqslant p \leqslant n$, and $n \geqslant 3$. Then*

(a) $\kappa(K_{m_1,m_2,\ldots,m_p} \times K_n) = (n-1)\sum_{j=1}^{p-1} m_j$.

(b) $t(K_{m_1,m_2,\ldots,m_p} \times K_n) = \sum_{j=1}^{p-1} m_j \Big/ m_p$.

(c) $s(K_{m_1,m_2,\ldots,m_p} \times K_n) = \begin{cases} 1 + m_p - (n-1)\sum_{j=1}^{p-1} m_j, & \text{if } p = n, m_1 = m_p; \\ 2n \cdot m_p - n\sum_{j=1}^{p} m_j, & \text{otherwise.} \end{cases}$

(d) $I(K_{m_1,m_2,\ldots,m_p} \times K_n) = n \cdot \sum_{j=1}^{p-1} m_j + 1$.

(e) $T(K_{m_1,m_2,\ldots,m_p} \times K_n) = n \cdot \sum_{j=1}^{p-1} m_j + 1 \Big/ n \cdot m_p$.

Obviously, if $m_1 = m_2 = \cdots = m_p = 1$ in Theorem **??**, then we can obtain Theorem **??**. Thus Theorem **??** is a special case of Theorem **??**.

1 Proof of the main result

To show Theorem **??**, we first introduce more notations. When consider the Kronecker product of $K_{m_1,\ldots,m_p}$ and K_n $(m_1 \leqslant m_2 \leqslant \ldots \leqslant m_p, 2 \leqslant p \leqslant n, n \geqslant 3)$, we shall always let

$$\begin{aligned} V_1 &= V(K_{m_1,m_2,\ldots,m_p}) = \{u_{11}, u_{12}, \ldots, u_{1m_1}, \ldots, u_{i1}, \ldots, u_{im_i}, \ldots, u_{p1}, \ldots, u_{pm_p}\}, \\ V_2 &= V(K_n) = \{1, 2, \ldots, n\}, \\ S_k &= V_1 \times \{k\},\ k = 1, 2, \ldots, n, \\ T_i &= \{u_{i1}, u_{i2}, \ldots, u_{im_i}\} \times V_2,\ i = 1, 2, \ldots, p. \end{aligned}$$

Then S_k, T_i are independent sets in $G = K_{m_1,m_2,\dots,m_p} \times K_n$, and $V(K_{m_1,m_2,\dots,m_p} \times K_n)$ has partitions $V_1 \times V_2 = S_1 \cup S_2 \cup \cdots \cup S_n = T_1 \cup T_2 \cup \cdots \cup T_p$. Moreover, for notational convenience, we abbreviate the partition of $\{u_{i1}, u_{i2}, \dots, u_{il}, \dots, u_{im_i}\} \times \{v_k\}$ as A_{ik}, $i = 1, 2, \dots, p$, $k = 1, 2, \dots, n$, $l = 1, 2, \dots, m_i$. For two vertices u and v in a graph G, we write $u \sim v$ if $uv \in E(G)$ and $u \nsim v$ otherwise. See Fig 1.

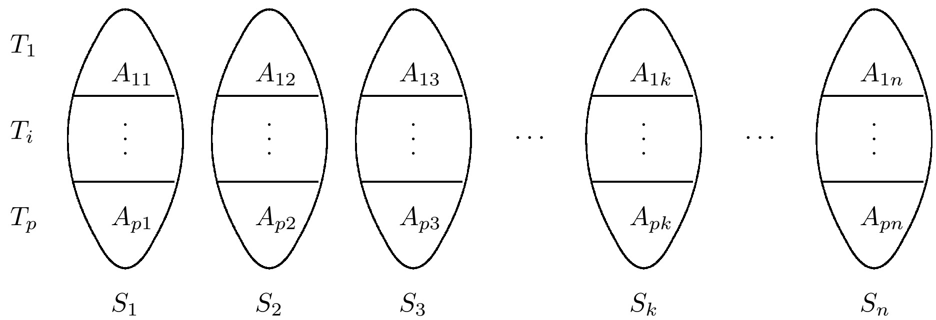

Fig. 1: $G \times K_n$

Proposition 1 *Let $m_1, m_2, \dots, m_p, n$ be integers with $m_1 \leqslant m_2 \leqslant \dots \leqslant m_p$, $2 \leqslant p \leqslant n$, and $n \geqslant 3$, Then $\alpha(K_{m_1,m_2,\dots,m_p} \times K_n) = n \cdot m_p$, $\beta(K_{m_1,m_2,\dots,m_p} \times K_n) = n \cdot \sum_{j=1}^{p-1} m_j$.*

Next we give a lemma on the components after removing a cut set from $K_{m_1,m_2,\dots,m_p} \times K_n$, which plays a key role in the proof of our main result.

Lemma 1 *Let $m_1, m_2, \dots, m_p, n$ be integers with $m_1 \leqslant m_2 \leqslant \dots \leqslant m_p$, $2 \leqslant p \leqslant n$, and $n \geqslant 3$, and let S be a cut set of $G = K_{m_1,m_2,\dots,m_p} \times K_n$.*

(a) *When $\omega(G - S) = 2$, and let G_1, G_2 be the components of $G - S$.*

(1) *If $\min\{|G_1|, |G_2|\} = 1$, then*

$$\tau(G - S) \leqslant (n-1)m_p + \sum_{j=1}^{p-1} m_j, \quad |S| \geqslant (n-1) \cdot \sum_{j=1}^{p-1} m_j + m_p - 1.$$

(2) *If $\min\{|G_1|, |G_2|\} \geqslant 2$, then*

$$\tau(G - S) \leqslant m_{p-1} + m_p, \quad |S| \geqslant n \cdot \sum_{j=1}^{p} m_j - 2(m_{p-1} + m_p).$$

(b) *When $\omega(G - S) \geqslant 3$,*

(1) *if $\tau(G - S) = 1$, then*

$$\omega(G - S) \leqslant n \cdot m_p, \quad |S| \geqslant n \cdot \sum_{j=1}^{p-1} m_j.$$

(2) *if $\tau(G - S) \geqslant 2$, then there is only one component with vertices greater than 2, and $\omega(G) \leqslant m_p + 1$,*

$$\tau(G - S) \leqslant \sum_{j=1}^{p-1} m_j + (n-1)m_p, \quad |S| \geqslant (n-1) \sum_{j=1}^{p-1} m_j.$$

Proof (a) Suppose that $\omega(G-S)=2$ and G_1,G_2 be the two components of $G-S$.

(1) If $|G_1|=|G_2|=1$, we are done. Without loss of generality, we suppose $|G_1|=1$, $|G_2|\geqslant 2$ and $(u_{i_1l_1},k_1)\in V(G_1)$, $i_1\in\{1,2,\ldots,p\}$, $l_1\in\{1,2,\ldots,i_1\}$, $k_1\in\{1,2,\ldots,n\}$. By the definition of Kronecker product, $(u_{i_1l_1},k_1)$ is adjacent to all the vertices of G except $T_{i_1}\cup S_{k_1}$. Since G_2 is a component, G_2 contains at most all the vertices in $(T_{i_1}\cup S_{k_1})\setminus A_{i_1k_1}$, hence

$$|G_1|+|G_2|\leqslant 1+(n-1)m_{i_1}+\sum_{j\neq i_1}m_j\leqslant 1+(n-1)m_p+\sum_{j=1}^{p-1}m_j,$$

Thus,

$$|S|\geqslant n\cdot\sum_{j=1}^{p}m_j-(n-1)m_p-\sum_{j=1}^{p-1}m_j-1=(n-1)\cdot\sum_{j=1}^{p-1}m_j+m_p-1,$$

Therefore,

$$\tau(G-S)\leqslant(n-1)m_p+\sum_{j=1}^{p-1}m_j.$$

(2) If $\min\{|G_1|,|G_2|\}\geqslant 2$, we suppose $(u_{i_1l_1},k_1),(u_{i_2l_2},k_2)\in V(G_1)$, $i_1,i_2\in\{1,2,\ldots,p\}$, $l_1\in\{1,2,\ldots,i_1\}$, $l_2\in\{1,2,\ldots,i_2\}$, $k_1,k_2\in\{1,2,\ldots,n\}$. By the definition of Kronecker product, $(u_{i_1l_1},k_1),(u_{i_2l_2},k_2)$ are adjacent to all the vertices of G except for $A_{i_1k_2}\cup A_{i_2k_1}$. Since $|G_2|\geqslant 2$, there must exist two vertices $(u_{i_1l_3},k_2),(u_{i_2l_4},k_1)\in V(G_2)$, $l_3\in\{1,2,\ldots,i_1\}$, $l_4\in\{1,2,\ldots,i_2\}$. Obviously, G_1,G_2 have at most all the vertices in $A_{i_1k_1},A_{i_1k_2}$, $A_{i_2k_1},A_{i_2k_2}$, so we have

$$\tau(G-S)\leqslant m_{p-1}+m_p$$

and

$$|G_1|+|G_2|\leqslant 2(m_{j_1}+m_{j_2})\leqslant 2(m_{p-1}+m_p).$$

This implies that

$$|S|\geqslant n\cdot\sum_{j=1}^{p}m_j-2(m_{p-1}+m_p).$$

(b) When $\omega(G-S)\geqslant 3$, we consider $\tau(G-S)$.

(1) If $\tau(G-S)=1$, then $\alpha=n\cdot m_p\geqslant\omega(G-S)=n\cdot\sum\limits_{j=1}^{p}m_j-|S|$, we get

$$|S|\geqslant n\cdot\sum_{j=1}^{p}m_j-n\cdot m_p=n\cdot\sum_{j=1}^{p-1}m_j,\quad \omega(G-S)\leqslant n\cdot m_p.$$

(2) If $\tau(G-S)\geqslant 2$. Suppose $|G_1|\geqslant 2$, $(u_{i_1l_1},k_1),(u_{i_2l_2},k_2)\in V(G_1)$, $i_1,i_2\in\{1,2,\ldots,p\}$, $l_1\in\{1,2,\ldots,i_1\}$, $l_2\in\{1,2,\ldots,i_2\}$, $k_1,k_2\in\{1,2,\ldots,n\}$. By the definition of Kronecker product, the vertices $(u_{i_1l_1},k_1),(u_{i_2l_2},k_2)$ are adjacent to all the vertices except some in $A_{i_1k_2},A_{i_2k_1}$. The left components must contain in one partition. So there

is only one component with vertices greater than 2. If we choose the vertices in $A_{i_1k_2}$, then G_1 at most contain the vertices of $(T_{i_1} \cup S_{k_2}) \setminus A_{i_1k_2}$. Hence

$$\begin{aligned}|S| &\geqslant n \cdot \sum_{j=1}^{p} m_j - \sum_{j \neq i_1} m_j - n \cdot m_{i_1}\\ &\geqslant n \cdot \sum_{j=1}^{p} m_j - \sum_{j=1}^{p-1} m_j - n \cdot m_p\\ &= (n-1)\sum_{j=1}^{p-1} m_j,\\ \tau(G-S) &\leqslant \sum_{j=1}^{p-1} m_j + (n-1)m_p\end{aligned}$$

and $\omega(G) \leqslant m_p + 1$. If choose the vertices in $A_{i_2k_1}$, we can obtain the desired formulae by similar reason.

Now we are ready to give the proof of Theorem **??**.

Proof of Theorem ?? For convenience, we use the abbreviation $G = K_{m_1,m_2,\ldots,m_p} \times K_n$ and the abbreviation mentioned above are still in use.

(a) Vertex connectivity $\kappa(G)$. Obviously $W_1 = \bigcup_{i=1}^{p-1} \bigcup_{k=1}^{n-1} A_{ik}$ is a cut set of G , and $|W_1| = (n-1) \cdot \sum_{j=1}^{p-1} m_j$. Let S' be a cut set of G with vertices fewer than $(n-1) \cdot \sum_{j=1}^{p-1} m_j$. We shall prove that S' does not exist. Pick two arbitrary nonadjacent vertices in $G - S'$, there exist three conditions.

(1) $(u_{i_1l_1}, k_1), (u_{i_1l_2}, k_1)$,

(2) $(u_{i_1l_1}, k_1), (u_{i_1l_2}, k_2)$,

(3) $(u_{i_1l_1}, k_1), (u_{i_2l_3}, k_1)$. $i_1, i_2 \in \{1, 2, \ldots, p\}$, $l_1, l_2 \in \{1, 2, \ldots, i_1\}$, $l_3 \in \{1, 2, \ldots, i_2\}$, $k_1, k_2 \in \{1, 2, \ldots, n\}$.

In the first condition, $(u_{i_1l_1}, k_1)$ and $(u_{i_1l_2}, k_1)$ are adjacent to all the vertices except $T_{i_1} \cup S_{k_1}$, then we need to get rid of at least

$$n \cdot \sum_{j=1}^{p} m_j - n \cdot m_p - \sum_{j=1}^{p-1} m_j = |T_1| = (n-1) \cdot \sum_{j=1}^{p-1} m_j,$$

so we can not find a cut set S' with $|S'| < (n-1) \cdot \sum_{j=1}^{p-1} m_j$ to disconnected this two vertices. It is also verified under the condition (2), (3). Hence (a) holds.

In the following, the cut set $W_2 = \bigcup_{i=1}^{p-1} \bigcup_{k=1}^{n} A_{ik}$ of G will be used in (c) to get a lower bound, and in $(b), (d), (e)$ to get an upper bound. Note that $|W_2| = n \cdot \sum_{j=1}^{p-1} m_j$, $\omega(G - W_2) = n \cdot m_p = \alpha(G)$, $\tau(G - W_2) = 1$.

(b) Vertex toughness $t(G)$. For $\omega(G-S)=2$, by Theorem **??** (a), we have

$$\frac{|S|}{\omega(G-S)} \geqslant \frac{(n-1)\cdot\sum\limits_{j=1}^{p-1} m_j}{2} \geqslant \frac{\sum\limits_{j=1}^{p-1} m_j}{m_p}.$$

For $\omega(G-S)\geqslant 3$, by Lemma **??**(b), we have

$$\begin{aligned}
t(G) &= \frac{|S|}{\omega(G-S)}\\
&= \frac{n\cdot\sum\limits_{j=1}^{p} m_j - \tau(G-S) - \omega(G-S)+1}{\omega(G-S)}\\
&= \frac{n\cdot\sum\limits_{j=1}^{p} m_j - \tau(G-S)+1}{\omega(G-S)} - 1\\
&\geqslant \begin{cases} \dfrac{n\cdot\sum\limits_{j=1}^{p} m_j - \left[\sum\limits_{j=1}^{p-1} m_j + (n-1)\cdot m_p\right]+1}{m_p+1} - 1, & \tau(G-S)\geqslant 2;\\ \dfrac{\sum\limits_{j=1}^{p-1} m_j}{m_p}, & \tau(G-S)=1. \end{cases}\\
&\geqslant \frac{\sum\limits_{j=1}^{p-1} m_j}{m_p}.
\end{aligned}$$

On the other hand, we have

$$t(G)\leqslant \frac{|W_2|}{\omega(G-W_2)} = n\cdot\sum_{j=1}^{p-1} m_j\Big/ n\cdot m_p = \sum_{j=1}^{p-1} m_j\Big/ m_p.$$

Hence $t(G)=\sum\limits_{j=1}^{p-1} m_j/m_p$.

(c) Scatting number $s(G)$. If $\omega(G-S)=2$, then we have

$$\omega(G-S)-|S| = 2-|S| \leqslant \begin{cases} 3-(n-1)\sum\limits_{j=1}^{p-1} m_j - m_p, & \min\{|G_1|,|G_2|\}=1;\\ 2(m_p+m_{p-1}+1) - n\cdot\sum\limits_{j=1}^{p} m_j, & \min\{|G_1|,|G_2|\}\geqslant 2. \end{cases}$$

If $\omega(G-S)\geqslant 3$, then

$$\begin{aligned}\omega(G-S)-|S| &= \omega(G-S)-\left[n\cdot\sum_{j=1}^{p}m_j-\tau(G-S)-\omega(G-S)+1\right]\\ &= 2\omega(G-S)+\tau(G-S)-n\cdot\sum_{j=1}^{p}m_j-1\\ &\leqslant \begin{cases}1+m_p-(n-1)\sum\limits_{j=1}^{p-1}m_j, & \tau(G-S)\geqslant 2;\\ 2n\cdot m_p-n\cdot\sum\limits_{j=1}^{p}m_j, & \tau(G-S)=1.\end{cases}\end{aligned}$$

First, note that

$$1+m_p-(n-1)\sum_{j=1}^{p-1}m_j\geqslant 3-(n-1)\sum_{j=1}^{p-1}m_j-m_p$$

and

$$1+m_p-(n-1)\sum_{j=1}^{p-1}m_j\geqslant 3-(n-1)\sum_{j=1}^{p-1}m_j-m_p.$$

This implied that we do not need to consider the condition $\omega(G-S)=2$.

If $p=n, m_p=m_1$, then

$$1+m_p-(n-1)\sum_{j=1}^{p-1}m_j>2n\cdot m_p-n\cdot\sum_{j=1}^{p}m_j.$$

On the other hand, let $W_3=\bigcup\limits_{i=1}^{p-1}\bigcup\limits_{k=1}^{n-1}A_{ik}$, we have

$$\omega(G-W_3)=m_p+1,\quad |W_3|=(n-1)\sum_{j=1}^{p-1}m_j.$$

Hence,

$$s(G)\geqslant\omega(G-W_3)-|W_3|=1+m_p-(n-1)\sum_{j=1}^{p-1}m_j,$$

Therefore,

$$s(G)=1+m_p-(n-1)\sum_{j=1}^{p-1}m_j.$$

Otherwise, we have

$$1+m_p-(n-1)\sum_{j=1}^{p-1}m_j\leqslant 2n\cdot m_p-n\cdot\sum_{j=1}^{p}m_j$$

and

$$s(G) \geqslant \omega(G - W_2) - |W_2| = 2n \cdot m_p - n \cdot \sum_{j=1}^{p} m_j.$$

Hence,

$$s(G) = 2n \cdot m_p - n \cdot \sum_{j=1}^{p} m_j.$$

(d) Vertex integrity $I(G)$. If $\omega(G-S) = 2$, by Lemma **??**(a), without loss of generality, we may assume that $\tau(G-S) = |G_2|$. Then

$$|S| + \tau(G - S) = n \cdot \sum_{j=1}^{p} m_j - |G_1| \geqslant n \cdot \sum_{j=1}^{p} m_j - (m_p + m_{p-1}).$$

If $\omega(G - S) \geqslant 3$, then

$$|S| + \tau(G - S) = n \cdot \sum_{j=1}^{p} -\omega(G - S) + 1 \geqslant n \sum_{j=1}^{p} m_j - n \cdot m_p + 1 = n \cdot \sum_{j=1}^{p-1} m_j + 1.$$

Since

$$n \cdot \sum_{j=1}^{p-1} m_j + 1 \leqslant n \cdot \sum_{j=1}^{p} m_j - (m_p + m_{p-1}), \quad I(G) \geqslant n \cdot \sum_{j=1}^{p-1} m_j + 1.$$

On the other hand, we have

$$I(G) \leqslant |W_2| + \tau(G - W_2) = 1 + n \cdot \sum_{j=1}^{p-1} m_j.$$

Thus $I(G) = n \cdot \sum_{j=1}^{p-1} m_j + 1$.

(e) Vertex tenacity $T(G)$. If $\omega(G - S) = 2$, then

$$\frac{|S| + \tau(G - S)}{\omega(G - S)} = \frac{|S| + \tau(G - S)}{2} \geqslant \frac{n \cdot \sum_{j=1}^{p} m_j - (m_p + m_{p-1})}{2}.$$

If $\omega(G - S) \geqslant 3$, then

$$\frac{|S| + \tau(G - S)}{\omega(G - S)} = \frac{n \cdot \sum_{j=1}^{p} m_j - \omega(G - S) + 1}{\omega(G - S)} = \frac{n \cdot \sum_{j=1}^{p} m_j + 1}{\omega(G - S)} - 1 \geqslant \frac{n \cdot \sum_{j=1}^{p-1} m_j + 1}{n \cdot m_p}.$$

From $\dfrac{n \cdot \sum_{j=1}^{p-1} m_j + 1}{n \cdot m_p} \leqslant \dfrac{n \cdot \sum_{j=1}^{p} m_j - (m_p + m_{p-1})}{2}$, it follows that

$$T(G) \geqslant \left(n \cdot \sum_{j=1}^{p-1} m_j + 1 \right) \Big/ n \cdot m_p.$$

On the other hand,

$$T(G) \leqslant (|T_2| + \tau(G - T_2))/\omega(G - T_2) = \left(n \cdot \sum_{j=1}^{p-1} m_j + 1\right)\Big/ n \cdot m_p.$$

Thus

$$T(G) = \left(n \cdot \sum_{j=1}^{p-1} m_j + 1\right)\Big/ n \cdot m_p.$$

2 Conclusion

In this paper we compute several vertex vulnerability parameters of Kronecker products of complete multipartite graph and complete graph which are very important to measure the stability of networks. This result generalizes the main result in [**?**].

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