An Approximation Algorithm for the Stochastic Fault-Tolerant Facility Placement Problem*

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Abstract In the deterministic fault-tolerant facility placement problem (FTFP), we are given a set of locations and a set of clients. We can open any number of different facilities with the same opening cost at each location. Each client j has an integer requirement r_j. Connecting client j to different facilities at the same location is permitted. The objective is to open some facilities and connect each client j to r_j different facilities such that the total cost is minimized. In this paper, we consider the *two-stage stochastic fault-tolerant facility placement problem* (SFTFP) with recourse in which the set of clients are unknown in advance. But there are finite scenarios materialized by a probability distribution. Each scenario specifies a subset of clients to be assigned. Moreover, each facility has two kinds of opening cost. In the first stage, we open a subset of facilities according to the stochastic information of the clients. In the second stage, we can open additional number of facilities when the actual information of the clients is revealed. We give a linear integral program and an LP-rounding based 5-approximation algorithm for the SFTFP.

Keywords facility placement problem, approximation algorithm, LP-rounding
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随机容错设施布局问题的近似算法

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摘要 在确定性的容错设施布局问题中，给定顾客的集合和地址的集合. 在每个地址上可以开设任意数目的不同设施. 每个顾客 j 有连接需求 r_j. 允许将顾客 j 连到同一地址的不同设施上. 目标是开设一些设施并将每个顾客 j 连到 r_j 个不同的设施上，使得总开设费用和连接费用最小. 研究两阶段随机容错设施布局问题 (SFTFP), 顾客的集合事先不知道，但是具有有限多个场景并知道其概率分布. 每个场景指定需要服务的顾客的子集. 并且每个设施有两种类型的开设费用. 在第一阶段根据顾客的随机信息确定性地开设一些设施，在第二阶段根据顾客的真实信息再增加开设一些设施. 给出随机容错布局问题的线性整数规划和基于线性规划舍入的 5- 近似算法.

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0 Introduction

In the classical *facility location problem* (FLP), we are given a set of locations F and a set of clients D. We need to open some facilities at some locations and connect each client to an open facility. We can only open one facility at each location $i \in F$ with an opening cost f_i. Connecting client j to facility i (or location i) incurs a connection cost c_{ij}. All the connection costs constitute a metric. The objective is to minimize the total cost including the opening cost and connection cost. Since the problem is NP-hard, many researchers are interested in designing approximation algorithms[1−3]. The currently best approximation ratio is achieved by the 1.488-approximation algorithm of Li[4]. Guha and Khuller[5] prove that the approximation lower bound is 1.463. The LP-rounding technique is one of the main techniques for designing the approximation algorithms[6−8]. For other variants of the FLP, we refer to [9-14] and the references therein.

One of the variants of the FLP is the *fault-tolerant facility location problem* (FTFL), which is introduced by Jain and Vazirani[15]. In the FTFL, each client j has an integer requirement r_j. The objective is to open some facilities and connect each client j to r_j different open facilities so that the total opening cost and connection cost is minimized. The FTFL is reduced to the FLP if $r_j = 1$ for all j. There are some approximation algorithms for the FTFL[16−18]. The currently best ratio for the FTFL is 1.7245[16].

Another variant of the FLP is the *fault-tolerant facility placement problem* (FTFP) which is different from the FTFL in that we can open any number of different facilities with the same opening cost at each location ([19]). Similarly to the FTFL, each client j has an integer requirement r_j. But connecting client j to different facilities at the same location is permitted. The objective is to open some facilities and connect each client j to r_j different facilities such that the total cost is minimized.

Apart from the deterministic FLP, there are many works in the stochastic version. Among the most popular models in stochastic facility location problem is the *two-stage stochastic facility location problem* (SFLP) with recourse in which the set of clients are unknown in advance ([20]). But there are finite scenarios materialized by a probability distribution. Each scenario specifies a subset of clients to be assigned. Moreover, each facility has two kinds of opening cost. One arises in Stage I and the other is scenario-dependent arising in Stage II called recourse cost. Typically, the recourse cost varies under different scenarios and is greater than that in Stage I. The objective is to open some facilities in Stage I and II and assign each client in each scenario to an open facility in Stage I or the corresponding scenario in Stage II so as to minimize the total expected cost over both stages. Ravi and Sinha[20] give an LP-rounding 8-approximation algorithm for the SFLP.

In this paper, we study the *stochastic fault-tolerant facility placement problem* (SFTFP).

Following the approaches of [20 − 21], we give an LP-rounding 5-approximation algorithm for the SFTFP.

The organization of this paper is as follows. In Section 2, we give some definitions and a formulation for the SFTFP. In Section 3, we introduce an LP-rounding based algorithm for the SFTFP. In Section 4, we analyze the algorithm and prove our main result. Some discussions are given in the last section.

1 Formulation

In the SFTFP, we denote F a set of given locations, D a set of potential clients. We also give a set of scenarios S each of which specifies a subset of clients $D_s \subseteq D$ and materializes with probability p_s where $s \in S$. We use f_i^0 and f_i^s to denote the cost of opening one facility at location i in Stage I and scenario s of Stage II, respectively. As usual, c_{ij} is the connection cost between location i and client j, which is nonnegative, symmetry and satisfies the so-called triangle inequality. For simplification, we redefine the set of locations to be $\mathcal{F} = \{(i,t) : i \in F, t \in \{0\} \cup S\}$, and the set of clients to be $\mathcal{D} = \{(j,s) : s \in S, j \in D_s\}$ (each of which has an integer requirement r_j^s). We use c_{ij}^{ts} to denote the connection cost between location (i,t) and client (j,s). Define $c_{ij}^{ts} = c_{ij}$ if $t = 0$ or s for any $(i,t) \in \mathcal{F}$, $(j,s) \in \mathcal{D}$ and $+\infty$ otherwise. Let $p_0 = 1$. We give the linear integer program for the SFTFP as follows:

$$
\begin{aligned}
\min \quad & \sum_{(i,t)\in\mathcal{F}} p_t f_i^t y_i^t + \sum_{(i,t)\in\mathcal{F},(j,s)\in\mathcal{D}} p_s c_{ij}^{ts} x_{ij}^{ts} \\
\text{s.t.} \quad & \sum_{(i,t)\in\mathcal{F}} x_{ij}^{ts} \geqslant r_j^s, \qquad \forall (j,s) \in \mathcal{D}, \\
& x_{ij}^{ts} \leqslant y_i^t, \qquad \forall (i,t) \in \mathcal{F}, (j,s) \in \mathcal{D}, \\
& x_{ij}^{ts}, y_i^t \text{ nonnegative integer}, \qquad \forall (i,t) \in \mathcal{F}, (j,s) \in \mathcal{D},
\end{aligned}
$$

where y_i^t is the number of facilities opened at location (i,t), x_{ij}^{ts} is the number of connections between location (i,t) and client (j,s). The first constrain means that the total number of connections from client (j,s) to all locations cannot be smaller than its requirement. The second constrain means that the number of connections from client (j,s) to location (i,t) cannot exceed the number of facilities opened at this location. By relaxing the integrality constraints, we give the linear programming relaxation for the SFTFP as follows:

$$
\begin{aligned}
\min \quad & \sum_{(i,t)\in\mathcal{F}} p_t f_i^t y_i^t + \sum_{(i,t)\in\mathcal{F},(j,s)\in\mathcal{D}} p_s c_{ij}^{ts} x_{ij}^{ts} \\
\text{s.t.} \quad & \sum_{(i,t)\in\mathcal{F}} x_{ij}^{ts} \geqslant r_j^s, \qquad \forall (j,s) \in \mathcal{D}, \\
& x_{ij}^{ts} \leqslant y_i^t, \qquad \forall (i,t) \in \mathcal{F}, (j,s) \in \mathcal{D}, \\
& x_{ij}^{ts}, y_i^t \geqslant 0, \qquad \forall (i,t) \in \mathcal{F}, (j,s) \in \mathcal{D}.
\end{aligned} \tag{1.1}
$$

Let us denote (x, y) the optimal solution of (1.1). It is easy to prove that

$$\sum_{(i,t)\in\mathcal{F}} x_{ij}^{ts} = r_j^s, \qquad \forall (j,s) \in \mathcal{D}. \tag{1.2}$$

The dual program of (1.1) is

$$\begin{aligned} \max \quad & \sum_{(j,s)\in\mathcal{D}} r_j^s \alpha_j^s \\ \text{s.t.} \quad & \sum_{(j,s)\in\mathcal{D}} \beta_{ij}^{ts} \leqslant p_t f_i^t, \qquad \forall (i,t) \in \mathcal{F}, \\ & \alpha_j^s - \beta_{ij}^{ts} \leqslant p_s c_{ij}^{ts}, \qquad \forall (i,t) \in \mathcal{F}, (j,s) \in \mathcal{D}, \\ & \alpha_j^s, \beta_{ij}^{ts} \geqslant 0, \qquad \forall (i,t) \in \mathcal{F}, (j,s) \in \mathcal{D}. \end{aligned} \tag{1.3}$$

2 Algorithm

In this section, we give our LP-rounding algorithm for the SFTFP. First, we solve (1.1) and (1.3) to obtain the optimal solution (x, y) and (α, β). Second, we choose the client (j, s) with minimum α_j^s/p_s as *center* among all not-fully-connected clients and open facilities and create new connections iteratively. Here *a not-fully-connected client* is a client whose requirement is not all satisfied. Finally, all requirements of clients in $\mathcal{D}$ are satisfied and an integral feasible solution of (1.1) is obtained.

Algorithm 2.1

Step 0. Set $\mathcal{F}_o := \emptyset$, $\mathcal{D}_a := \mathcal{D}$, $(\bar{x}, \bar{y}) := (0, 0)$, $d_j^s := r_j^s$, $\forall (j,s) \in \mathcal{D}$.

Step 1. Solve (1.1) and (1.3) to obtain the optimal solution (x, y) and (α, β), respectively. For each $(j, s) \in \mathcal{D}_a$, denote

$$N_0(j,s) = \left\{(i,0) \in \mathcal{F} : x_{ij}^{0s} > 0\right\}, \ N_s(j,s) = \left\{(i,s) \in \mathcal{F} : x_{ij}^{ss} > 0\right\}.$$

Set

$$N_{alg}(j,s) = \begin{cases} N_0(j,s), & \text{if} \sum\limits_{(i,0)\in N_0(j,s)} x_{ij}^{0s} \geqslant \dfrac{1}{2} r_j^s; \\ N_s(j,s), & \text{if} \sum\limits_{(i,s)\in N_s(j,s)} x_{ij}^{ss} \geqslant \dfrac{1}{2} r_j^s. \end{cases}$$

Step 2. Choose $(\bar{j}, \bar{s}) := \arg\min\{\alpha_j^s/p_s : (j,s) \in \mathcal{D}_a\}$. Choose the location

$$(\bar{i}, \bar{t}) := \arg\min\{p_t f_i^t : (i,t) \in N_{alg}(\bar{j}, \bar{s})\}.$$

Open $d_{\bar{j}}^{\bar{s}}$ facilities at location $(\bar{i}, \bar{t})$ and connect $d_{\bar{j}}^{\bar{s}}$ requirement of $(\bar{j}, \bar{s})$ to different facilities at this location **directly**. Set

$$\bar{x}_{\bar{i}\bar{j}}^{\bar{t}\bar{s}} := \bar{x}_{\bar{i}\bar{j}}^{\bar{t}\bar{s}} + d_{\bar{j}}^{\bar{s}}, \bar{y}_{\bar{i}}^{\bar{t}} := \bar{y}_{\bar{i}}^{\bar{t}} + d_{\bar{j}}^{\bar{s}}, \mathcal{F}_o := \mathcal{F}_o \cup \{(\bar{i}, \bar{t})\}, d_{\bar{j}}^{\bar{s}} := 0, \mathcal{D}_a := \mathcal{D}_a \setminus \{(\bar{j}, \bar{s})\}.$$

If there exists $(j,s) \in \mathcal{D}_a$ such that $N_{alg}(j,s) \cap N_{alg}(\bar{j},\bar{s}) \neq \emptyset$, we connect $\min\{d_{\bar{j}}^{\bar{s}}, d_j^s\}$ requirement of (j,s) to different facilities at $(\bar{i},\bar{t})$ **indirectly**. Set

$$\bar{x}_{\bar{i}j}^{\bar{t}s} := \bar{x}_{\bar{i}j}^{\bar{t}s} + \min\{d_{\bar{j}}^{\bar{s}}, d_j^s\}$$

and

$$d_j^s := \max\{d_j^s - d_{\bar{j}}^{\bar{s}}, 0\}.$$

If $d_j^s = 0$, then $\mathcal{D}_a := \mathcal{D}_a \setminus \{(j,s)\}$.

Step 3. If $\mathcal{D}_a = \emptyset$, then stop and output $(\bar{x}, \bar{y})$ and $\mathcal{F}_o$; otherwise, go to Step 2.

In Algorithm 2.1, $\mathcal{D}_a$ is the set of clients which are not fully connected. $\mathcal{F}_o$ is the set of locations at which some facilities are opened.

3 Analysis

In this section, we prove our main result that Algorithm 2.1 is a 5-approximation algorithm. We will estimate the connection cost and opening cost respectively.

Lemma 3.1 For any $(j,s) \in \mathcal{D}$, $\alpha_j^s \geqslant p_s c_{ij}^{ts}$ for all $(i,t) \in N_{alg}(j,s)$.

Proof Recall that (x,y) and (α,β) are the optimal solutions of (1.1) and (1.3), respectively. From complementary slackness conditions, we have $\alpha_j^s \geqslant p_s c_{ij}^{ts}$ for all $(i,t) \in \mathcal{F}$, $(j,s) \in \mathcal{D}$ such that $x_{ij}^{ts} > 0$. Combining the definition of $N_{alg}(j,s)$, we conclude the lemma.

Let C_{alg} be the connection cost of the solution output by Algorithm 2.1. We bound C_{alg} as follows.

Lemma 3.2

$$C_{alg} \leqslant 3 \sum_{(j,s)\in\mathcal{D}} r_j^s \alpha_j^s.$$

Proof Note that at Step 2 of Algorithm 2.1, all clients' requirements are satisfied directly or indirectly. Now we consider the following two possibilities.

Case 1. Client (j,s)'s requirement is satisfied directly. Suppose that (j,s) is connected to a facility at a location $(i,t) \in N_{alg}(j,s)$. In this case the connection cost of such a requirement is c_{ij}^{ts}. From Lemma 3.1, we have

$$p_s c_{ij}^{ts} \leqslant \alpha_j^s.$$

Case 2. Client (j,s)'s requirement is satisfied indirectly. Suppose that (j,s) is indirectly connected to a facility at location $(\bar{i},\bar{t}) \in N_{alg}(\bar{j},\bar{s})$ for some center $(\bar{j},\bar{s})$. By the choice of centers, we have

$$\alpha_{\bar{j}}^{\bar{s}}/p_{\bar{s}} \leqslant \alpha_j^s/p_s \quad \text{and} \quad N_{alg}(\bar{j},\bar{s}) \cap N_{alg}(j,s) \neq \emptyset.$$

Let $(i,t) \in N_{alg}(\bar{j},\bar{s}) \cap N_{alg}(j,s)$. Using the triangle inequality, we get

$$c_{\bar{i}j}^{\bar{t}s} \leqslant c_{\bar{i}\bar{j}}^{\bar{t}\bar{s}} + c_{i\bar{j}}^{t\bar{s}} + c_{ij}^{ts}.$$

From Lemma 3.1, we have

$$c_{\bar{i}\bar{j}}^{\bar{t}\bar{s}} \leqslant \alpha_{\bar{j}}^{\bar{s}}/p_{\bar{s}}, \quad c_{i\bar{j}}^{t\bar{s}} \leqslant \alpha_{\bar{j}}^{\bar{s}}/p_{\bar{s}}, \quad c_{ij}^{ts} \leqslant \alpha_{j}^{s}/p_{s}.$$

In this case, the connection cost $c_{\bar{i}j}^{\bar{t}s}$ of such a requirement satisfies

$$c_{\bar{i}j}^{\bar{t}s} \leqslant 2\alpha_{\bar{j}}^{\bar{s}}/p_{\bar{s}} + \alpha_{j}^{s}/p_{s} \leqslant 3\alpha_{j}^{s}/p_{s}.$$

Thus the total connection cost for client (j,s) is no more than $3r_j^s\alpha_j^s$. Adding up all the connections costs of clients in $\mathcal{D}$, we conclude the lemma.

In order to proceed our analysis, we need the following lemma.

Lemma 3.3 ([21]) Given two numbers $y_i \geqslant 0$, $f_i \geqslant 0$ for each $i = 1, 2, \cdots, n$, given a non-empty set $N_t \subseteq \{1, 2, \cdots, n\}$ and a number $q_t \geqslant 0$ for each $t = 1, 2, \cdots, T$, suppose that

$$\sum_{h=1}^{t} \Lambda_{ht} q_h \leqslant \sum_{i \in N_t} y_i, \qquad \forall t = 1, 2, \cdots, T,$$

where $\Lambda_{ht} = 1$ if $N_h \cap N_t \neq \emptyset$ and 0 otherwise. Denote $\bar{f}_t = \min\{f_i : i \in N_t\}$ for $t = 1, 2, \cdots, T$. Then we have

$$\sum_{t=1}^{T} \bar{f}_t q_t \leqslant \sum_{i=1}^{n} f_i y_i.$$

Let F_{alg} and F^* be the opening cost of the solution output by Algorithm 2.1 and the optimal (fractional) opening cost of (1.1), respectively. We bound the facility cost as follows.

Lemma 3.4

$$F_{alg} \leqslant 2F^*.$$

Proof Denote by $\{(j_1, s_1), (j_2, s_2), \cdots, (j_K, s_K)\}$ the set of all centers in the chosen order of Algorithm 2.1. Let $\bar{\mathcal{F}} = \bigcup_{k=1}^{K} N_{alg}(j_k, s_k)$. For each $(i,t) \in \bar{\mathcal{F}}$, we define $\tilde{y}_i^t = 2y_i^t \geqslant 0$. For each $k \in \{1, 2, \cdots, K\}$, we have a non-empty set $N_{alg}(j_k, s_k) \subseteq \bar{\mathcal{F}}$ and a number $q_k := d_{j_k}^{s_k}$ which is the number of new facilities opened when center client (j_k, s_k) is considered. Denote $\Lambda_{hk} = 1$ if $N_{alg}(j_h, s_h) \cap N_{alg}(j_k, s_k) \neq \emptyset$ and 0 otherwise. It follows from Algorithm 2.1, (1.1) and (1.2) that

$$\sum_{h=1}^{k} \Lambda_{hk} q_h = r_{j_k}^{s_k} \leqslant \sum_{(i,t) \in \mathcal{F}} x_{ij_k}^{ts_k} \leqslant \sum_{(i,t) \in N_{alg}(j_k,s_k)} 2x_{ij_k}^{ts_k} \leqslant \sum_{(i,t) \in N_{alg}(j_k,s_k)} \tilde{y}_i^t, \qquad \forall k = 1, 2, \cdots, K.$$

Let $p_{t_k} f_{i_k}^{t_k} = \min\{p_t f_i^t : (i,t) \in N_{alg}(j_k, s_k)\}$ for $k \in \{1, 2, \cdots, K\}$. Applying Lemma 3.3 we obtain that

$$F_{alg} = \sum_{k=1}^{K} p_{t_k} f_{i_k}^{t_k} q_k \leqslant \sum_{(i,t) \in \bar{\mathcal{F}}} p_t f_i^t \tilde{y}_i^t \leqslant 2 \sum_{(i,t) \in \mathcal{F}} p_t f_i^t y_i^t = 2F^*.$$

Now we are ready to state our main result.

Theorem 3.1 Algorithm 2 is a 5-approximation algorithm for the SFTFP.

Proof From Lemmas 3.2 and 3.4, we have

$$\begin{aligned} C_{alg}+F_{alg} &\leqslant 3\sum_{(j,s)\in\mathcal{D}} r_j^s\alpha_j^s+2F^* \\ &\leqslant 3\sum_{(j,s)\in\mathcal{D}} r_j^s\alpha_j^s+2\sum_{(j,s)\in\mathcal{D}} r_j^s\alpha_j^s \\ &= 5\sum_{(j,s)\in\mathcal{D}} r_j^s\alpha_j^s. \end{aligned}$$

Recall that $\sum_{(j,s)\in\mathcal{D}} r_j^s\alpha_j^s$ is the cost of the optimal solution of (1.1), we complete the proof.

4 Discussions

In this paper, we present an LP-rounding 5-approximation algorithm for the SFTFP. It is interesting to further improve the approximation ratio for the SFTFP. Since the k-level FLP is an important variant of the FLP, it is also worth to study the k-level version for the FTFP.

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