On the smoothing of the lower order exact penalty function for inequality constrained optimization*

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Abstract In this paper, we propose a method to smooth the general lower order exact penalty function for inequality constrained optimization. Error estimations are obtained among the optimal objective function values of the smoothed penalty problem, of the nonsmooth penalty problem and of the original optimization problem. It is shown that under mild assumption, an approximate global solution of the original problem can be obtained by searching a global solution of the smoothed penalty problem. We develop an algorithm for solving the original optimization problem based on the smoothed penalty function and prove the convergence of the algorithm. Some numerical examples are given to illustrate the applicability of the present smoothing method.

Keywords constrained nonlinear programming, exact penalty function, lower order penalty function, smooth exact penalty function, second order sufficient condition

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不等式约束优化问题的低阶精确罚函数的光滑化算法

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摘要 对不等式约束优化问题提出了一个低阶精确罚函数的光滑化算法. 首先给出了光滑罚问题、非光滑罚问题及原问题的目标函数值之间的误差估计，进而在弱的假设之下证明了光滑罚问题的全局最优解是原问题的近似全局最优解. 最后给出了一个基于光滑罚函数的求解原问题的算法，证明了算法的收敛性，并给出数值算例说明算法的可行性.

关键词 约束非线性规划，精确罚函数，低阶罚函数，光滑精确罚函数，二阶充分条件

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0 Introduction

Consider the constrained optimization problem

$$[P] \qquad \begin{aligned} &\min f(x),\\ &\text{s.t. } g_i(x) \leqslant 0, \quad i=1,2,\cdots,m,\\ &\qquad x\in R^n, \end{aligned} \tag{0.1}$$

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where $f : R^n \to R$ and $g_i(x) : R^n \to R,\ i \in I = \{1, 2, \cdots, m\}$ are twice continuously differentiable functions. Let

$$G_0 = \{x \in R^n |\ g_i(x) \leqslant 0, \quad i = 1, 2, \cdots, m\}.$$

In Zangwill[1], the l_1 exact penalty function for problem [P] is proposed as follows

$$f_\beta(x) = f(x) + \beta \sum_{i=1}^{m} g_i^+(x), \tag{0.2}$$

where $\beta > 0$ is a penalty parameter and

$$g_i^+(x) = \max\{0, g_i(x)\},\ i \in I.$$

Obviously, it is a nondifferentiable function. Nondifferentiable penalty functions have been the first ones for which some exactness properties have been established by [1]. The obvious difficulty with the exact penalty function is that it is non-differentiable, which prevents the use of efficient minimization algorithms, see, e.g., [2-5]. From an algorithmic viewpoint, this nondifferentiablility can induce the so-called Maratos effect which prevents rapid local convergence. In order to avoid the drawback related to the nondifferentiability, the smoothing of the l_1 exact penalty function attracts much attention, see, e.g., [6-12].

Recent research on lower-order penalty functions shows that lower-order (order lower than 1) penalty functions require weaker conditions to guarantee exactness than the l_1 penalty function(see, e.g., [13-19]. In [15], Luo gave a global exact penalty result for a lower order penalty function of the form

$$f(x) + \alpha r(x)^{1/\gamma},$$

where $\alpha > 0,\ \gamma \geqslant 1$ are the penalty parameters, $r(x) = \sum\limits_{i=1}^{m} g_i^+(x)$.

Nonlinear penalty function of the following form has been investigated (see, e.g. [14,16-17])

$$L^k(x, d) = \left[f(x)^k + \sum_{i=1}^{m} d_i (g_i^+(x))^k \right]^{1/k},$$

where $f(x)$ is assumed to be positive, $k > 0$ is a given number, and $d = (d_1, d_2, \cdots, d_m) \in R_+^m$ is the penalty parameter. Obviously, $L^k(x, d)$ is the l_1 penalty function when $k = 1$. In [16], Rubinov gave a sufficient and necessary condition for $L^k(x, d)$ to be an exact penalty function by using a generalized calmness when $m = 1$ and $k \in (0, 1]$. Then it was shown in [16], that the least exact penalty parameter corresponding to $k \in (0, 1]$ is substantially smaller than that of the classical l_1 exact penalty function.

The lower order penalty function

$$\varphi_{q,v}(x) = f(x) + q \sum_{i=1}^{m} (g_i^+(x))^v, \qquad v \in (0, 1) \tag{0.3}$$

has been introduced and investigated in [20] and [21]. It is shown in [21] that any strict local minimum satisfying the second order sufficiency condition for the original problem is a strict local minimum of the lower order penalty function with any positive penalty parameters. The smoothing of the $\frac{1}{2}$-order penalty function, i.e. $f(x) + q\sum_{i=1}^{m}(g_i^+(x))^{\frac{1}{2}}$ has been investigated in [21] and [22].

In this paper, we aim to smooth the lower order penalty function of the general form (0.3). The main contribution of the paper is the extensions of the results of [21] and [22] from the case $v = \frac{1}{2}$ to the general case $v \in (0,1)$.

The paper is organized as follows. In section 1, a smoothing function to the lower order penalty function is introduced, and some fundamental properties of the smoothing function are discussed. In section 2, we present an algorithm to compute an approximate solution to [P] based on the smooth penalty function and show the convergence of the algorithm. Some numerical examples are given in section 3.

1 Smoothing exact lower order penalty function

Consider the following lower order penalty problem

$$[LOP]_v \qquad \min_{x\in R^n} \varphi_{q,v}(x).$$

In this paper, we say that the pair (x^*, λ^*) satisfies the second order sufficient condition[23] if

$$\begin{aligned} \nabla_x L(x^*,\lambda^*) &= 0,\\ g_i(x^*) &\leqslant 0,\ i\in I,\\ \lambda_i^* &\geqslant 0, i\in I,\\ \lambda_i^* g_i(x^*) &= 0,\ i\in I,\\ y^{\mathrm{T}}\nabla^2 L(x^*,\lambda^*)y &> 0, \quad for\ \ any\ \ y\in V(x^*), \end{aligned} \tag{1.1}$$

where $L(x,\lambda) = f(x) + \sum_{i=1}^{m}\lambda_i g_i(x)$, and

$$\begin{aligned} &V(x^*) = \{y\in R^n|\ \nabla^{\mathrm{T}} g_i(x^*)y = 0, i\in A(x^*),\quad \nabla^{\mathrm{T}} g_i(x^*)y \leqslant 0, i\in B(x^*)\},\\ &A(x^*) = \{i\in I|\ g_i(x^*) = 0,\ \lambda_i^* > 0\},\\ &B(x^*) = \{i\in I|\ g_i(x^*) = 0,\ \lambda_i^* = 0\}. \end{aligned}$$

In order to establish the global exact penalization, we need the following assumptions.

Assumption 1 $f(x)$ *satisfies the following coercive condition*

$$\lim_{\|x\|\to+\infty} f(x) = +\infty.$$

Under Assumption 1, there exists a box X such that $G([P]) \subset int(X)$, where $G([P])$ is the set of global minima of problem [P], $int(X)$ denotes the interior of the set X. Consider the following problem

$$[P'] \qquad \begin{aligned} &\min f(x)\\ &s.t.\ g_i(x)\leqslant 0, \quad i=1,\cdots,m,\\ &\quad\ x\in X, \end{aligned}$$

Let $G([P'])$ denote the set of global minima of problem $[P']$. Then $G([P']) = G([P])$.

Assumption 2 *The set $G([P])$ is a finite set.*

Then for any $v \in (0,1)$, we consider the penalty problem of the form

$$[LOP']_v \quad \min_{x\in X} \varphi_{q,v}(x).$$

We know that the lower order penalty function $\varphi_{q,k}(x)$ $(k \in (0,1))$ is an exact penalty function in [21] under Assumption 1, Assumption 2 and the second order sufficient condition. But the lower order exact penalty function $\varphi_{q,v}(x)$ $(v \in (0,1))$ is a nondifferentiable function. Now we consider its smoothing. We have the following definition.

Let $p_v(u) = (max\{0,u\})^v$, that is,

$$p_v(u) = \begin{cases} u^v & if\ u > 0, \\ 0 & otherwise. \end{cases} \tag{1.2}$$

Then

$$\varphi_{q,v}(x) = f(x) + q\sum_{i=1}^{m} p_v(g_i(x)). \tag{1.3}$$

For any $\varepsilon > 0$, let

$$p_{\varepsilon,v}(u) = \begin{cases} 0, & if\ \ u < 0; \\ \dfrac{2v-1}{(v+2)\varepsilon^2}u^{v+2}, & if\ \ 0 \leqslant u < \varepsilon; \\ u^v + \varepsilon u^{v-1} - \dfrac{5}{v+2}\varepsilon^v, & if\ \ u \geqslant \varepsilon. \end{cases} \tag{1.4}$$

It is easy to see that the function $p_{\varepsilon,v}(u)$ is continuously differentiable on R and

$$\lim_{\varepsilon\to 0^+} p_{\varepsilon,v}(u) = p_v(u).$$

If $v = \frac{1}{2}$, we have

$$p_{\varepsilon,\frac{1}{2}}(u) = \begin{cases} 0, & if\ \ u < \varepsilon; \\ u^{\frac{1}{2}} + \varepsilon u^{-\frac{1}{2}} - 2\varepsilon^{\frac{1}{2}}, & if\ \ u \geqslant \varepsilon. \end{cases}$$

It is easy to see that $p_{\varepsilon,\frac{1}{2}}(u)$ is twice continuously differentiable.

Let

$$\varphi_{q,\varepsilon,v}(x) = f(x) + q\sum_{i=1}^{m} p_{\varepsilon,v}(g_i(x)). \tag{1.5}$$

Then $\varphi_{q,\varepsilon,v}(x)$ is continuously differentiable on R^n. Consider the following smoothed optimization problem,

$$[SP] \quad \min_{x\in R^n} \varphi_{q,\varepsilon,v}(x).$$

Theorem 1.1 *For any $x \in R^n,\ v \in (0,1)$ and $\varepsilon > 0$,*

$$0 \leqslant \varphi_{q,v}(x) - \varphi_{q,\varepsilon,v}(x) \leqslant \frac{5}{v+2}mq\varepsilon^v.$$

Proof Note that

$$p_v(g_i(x)) - p_{\varepsilon,v}(g_i(x)) = \begin{cases} 0, & if \quad g_i(x) < 0; \\ (g_i(x))^v - \dfrac{2v-1}{(v+2)\varepsilon^2}(g_i(x))^{v+2}, & if \quad 0 \leqslant g_i(x) < \varepsilon; \\ \dfrac{5}{(v+2)}\varepsilon^v - \varepsilon(g_i(x))^{v-1}, & if \quad g_i(x) \geqslant \varepsilon. \end{cases}$$

Let

$$F(t) = t^v - \frac{2v-1}{v+2}t^{v+2} - \frac{5}{v+2}.$$

It is easy to see that $F(t)$ is monotone increasing in [0,1). So we have $-\frac{5}{v+2} \leqslant F(t) \leqslant 0$ when $t \in [0,1)$. Let $t = g_i(x)/\varepsilon$, then

$$0 \leqslant (g_i(x))^v - \frac{2v-1}{(v+2)\varepsilon^2}(g_i(x))^{v+2} \leqslant \frac{5}{v+2}\varepsilon^v, \quad when \quad 0 \leqslant g_i(x) < \varepsilon.$$

By $v \in (0,1)$, we have

$$0 \leqslant \frac{5}{v+2}\varepsilon^v - \varepsilon(g_i(x))^{v-1} \leqslant \frac{5}{v+2}\varepsilon^v, \quad when \quad g_i(x) \geqslant \varepsilon.$$

So we have

$$0 \leqslant \varphi_{q,v}(x) - \varphi_{q,\varepsilon,v}(x) = q\sum_{i=1}^{m}(p_v(g_i(x)) - p_{\varepsilon,v}(g_i(x))) \leqslant \frac{5}{v+2}mq\varepsilon^v.$$

As a direct result of Theorem 1.1, we have the following two theorems.

Theorem 1.2 *Let* $\{\varepsilon_k\} \to 0$ *be a sequence of positive numbers and assume that* x^k *is a solution to* $\min_{x\in R^n} \varphi_{q,\varepsilon_k,v}(x)$ *for* $q > 0$ *and* $v \in (0,1)$. *Let* $\bar{x}$ *be an accumulating point of the sequence* $\{x^k\}$. *Then* $\bar{x}$ *is an optimal solution to* $\min_{x\in R^n} \varphi_{q,v}(x)$.

Theorem 1.3 *Let* $x^*_{q,v}$ *be an optimal solution of problem* $[LOP]_v$ *and* $\bar{x}_{q,\varepsilon,v}$ *be an optimal solution of problem* $[SP]$ *for* $q > 0$, $v \in (0,1)$ *and* $\varepsilon > 0$. *Then we have*

$$0 \leqslant \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \leqslant \frac{5}{v+2}mq\varepsilon^v.$$

Proof By Theorem 1.1, we have

$$\begin{aligned} 0 &\leqslant \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(x^*_{q,v}) \\ &\leqslant \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \\ &\leqslant \varphi_{q,v}(\bar{x}_{q,\varepsilon,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \\ &\leqslant \frac{5}{v+2}mq\varepsilon^v. \end{aligned}$$

We complete the proof.

Theorem 1.1 and Theorem 1.2 show that an approximate solution to $[SP]$ is also an approximate solution to $[LOP]_v$ when the error $\varepsilon > 0$ is sufficiently small.

Corollary 1.1 *Suppose that Assumption 1 and Assumption 2 hold, for any $x^* \in G([P])$, there exists a $\lambda^* \in R_+^m$ such that the pair (x^*, λ^*) satisfies the second order sufficient condition. Let $x^* \in X$ be a global solution of problem $[P]$ and $\bar{x}_{q,\varepsilon,v} \in X$ be a global solution of problem $[SP]$ for given $v \in (0,1)$ and $\varepsilon > 0$. Then there exists $q^* > 0$ such that for any $q > q^*$,*

$$0 \leqslant f(x^*) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \leqslant \frac{5}{v+2} mq\varepsilon^v,$$

where q^* is defined in Corollary 2.3 in [21].

Proof By Corollary 2.3 in [21], we have that x^* is a global solution of problem $[LOP]_v$. Then by Theorem 1.3, we have

$$0 \leqslant \varphi_{q,v}(x^*) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \leqslant \frac{5}{v+2} mq\varepsilon^v.$$

Since $\sum\limits_{i=1}^{m} p_v(g_i(x^*)) = 0$, we have

$$\varphi_{q,v}(x^*) = f(x^*) + q\sum_{i=1}^{m} p_v(g_i(x^*)) = f(x^*).$$

We complete the proof.

Definition 1.1 *For $\varepsilon > 0$, a point $x_\varepsilon \in R^n$ is an $\varepsilon - feasible\ solution$, if*

$$g_i(x_\varepsilon) \leqslant \varepsilon, \quad i = 1, \cdots, m \tag{1.6}$$

Theorem 1.4 *Let $x^*_{q,v}$ be an optimal solution of problem $[LOP]_v$ and $\bar{x}_{q,\varepsilon,v}$ be an optimal solution of problem $[SP]$ for $q > 0$, $v \in (0,1)$ and $\varepsilon > 0$. Furthermore, let $x^*_{q,v}$ be a feasible solution of problem [P] and $\bar{x}_{q,\varepsilon,v}$ be an $\varepsilon - feasible$ solution of problem [P], then we have*

$$\frac{2v-1}{v+2} mq\varepsilon^v \leqslant f(x^*_{q,v}) - f(\bar{x}_{q,\varepsilon,v}) \leqslant \frac{5}{v+2} mq\varepsilon^v, \quad v \in (0, \frac{1}{2}),$$

and

$$0 \leqslant f(x^*_{q,v}) - f(\bar{x}_{q,\varepsilon,v}) \leqslant \frac{6}{v+2} mq\varepsilon^v, \quad v \in (\frac{1}{2}, 1).$$

Proof It is easy to see that $\sum\limits_{i=1}^{m} p_v(g_i(x^*_{q,v})) = 0$ and by Theorem 1.3, we have

$$\begin{aligned}
0 &\leqslant \varphi_{q,v}(x^*_{q,v}) - \varphi_{q,\varepsilon,v}(\bar{x}_{q,\varepsilon,v}) \\
&= f(x^*_{q,v}) + q\sum_{i=1}^{m} p_v(g_i(x^*_{q,v})) - \Big(f(\bar{x}_{q,\varepsilon,v}) + q\sum_{i=1}^{m} p_{\varepsilon,v}(g_i(\bar{x}_{q,\varepsilon,v}))\Big) \\
&\leqslant \frac{5}{v+2} mq\varepsilon^v.
\end{aligned}$$

It follows that

$$q\sum_{i=1}^{m} p_{\varepsilon,v}(g_i(\bar{x}_{q,\varepsilon,v})) \leqslant f(x^*_{q,v}) - f(\bar{x}_{q,\varepsilon,v}) \leqslant q\sum_{i=1}^{m} p_{\varepsilon,v}(g_i(\bar{x}_{q,\varepsilon,v})) + \frac{5}{v+2} mq\varepsilon^v.$$

From (1.4), we have

$$\sum_{i=1}^{m}\frac{2v-1}{v+2}\varepsilon^{v}\leqslant\sum_{i=1}^{m}p_{\varepsilon,v}(g_i(\bar{x}_{q,\varepsilon,v}))\leqslant 0,\quad v\in(0,\frac{1}{2})$$

and

$$0\leqslant\sum_{i=1}^{m}p_{\varepsilon,v}(g_i(\bar{x}_{q,\varepsilon,v}))\leqslant\sum_{i=1}^{m}\frac{2v-1}{v+2}\varepsilon^{v}=\frac{2v-1}{v+2}m\varepsilon^{v}\leqslant\frac{1}{v+2}m\varepsilon^{v},\quad v\in(\frac{1}{2},1).$$

Then it follows that

$$\frac{2v-1}{v+2}mq\varepsilon^{v}\leqslant f(x^{*}_{q,v})-f(\bar{x}_{q,\varepsilon,v})\leqslant\frac{5}{v+2}mq\varepsilon^{v},\quad v\in(0,\frac{1}{2}).$$

and

$$0\leqslant f(x^{*}_{q,v})-f(\bar{x}_{q,\varepsilon,v})\leqslant\frac{1}{v+2}mq\varepsilon^{v}+\frac{5}{v+2}mq\varepsilon^{v}\leqslant\frac{6}{v+2}mq\varepsilon^{v},\quad v\in(\frac{1}{2},1).$$

By Theorem 1.4, an approximately optimal solution to $[SP]$ becomes an approximately optimal solution to $[P]$ if the solution to $[SP]$ is $\varepsilon-feasible$.

2 A smoothing method

For $x\in R^n$, we define

$$\begin{aligned}I^0(x)&=\{i|\ g_i(x)=0,\ i\in I\},\\ I^-(x)&=\{i|\ g_i(x)<0,\ i\in I\},\\ I^+_\varepsilon(x)&=\{i|\ g_i(x)\geqslant\varepsilon,\ i\in I\},\\ I^-_\varepsilon(x)&=\{i|\ g_i(x)<\varepsilon,\ i\in I\}.\end{aligned}$$

We propose the following algorithm to solve [P].

Algorithm 2.1

Step 1 Choose an initial point x^0, and a stopping tolerance $\varepsilon>0$. Given $\varepsilon_0>0,\ q_0>0,\ 0<\eta<1$, and $N>1$, let $k=0$ and go to Step 2.

Step 2 Use x^k as the starting point to solve $\min_{x\in R^n}\varphi_{q_k,\varepsilon_k,v}(x)$. Let x^*_k be the optimal solution obtained (x^*_k is obtained by a quasi-Newton method and a finite difference gradient). Go to Step 3.

Step 3 If x^*_k is $\varepsilon-feasible$ to [P], then stop and we have obtained an approximate solution x^*_k of the original problem [P]. Otherwise, let $q_{k+1}=Nq_k,\ \varepsilon_{k+1}=\eta\varepsilon_k,\ x^{k+1}=x^*_k$, and $k=k+1$, then go to Step 2.

Remark Since $0<\eta<1$ and $N>1$, hence, as $k\to+\infty$, the sequence $\{\varepsilon_k\}$ is decreasing to 0 and the sequence $\{q_k\}$ is increasing to $+\infty$.

Theorem 2.1 *Suppose that Assumption 1 holds. Let $\{x^*_k\}$ be the sequence generated by Algorithm 2.1. Suppose that the sequence $\{\varphi_{q_k,\varepsilon_k,v}(x^*_k)\}$ is bounded where $v\in(\frac{1}{2},1)$. Then $\{x^*_k\}$ is bounded and any limit point x^* of x^*_k is feasible to [P] and satisfies*

$$\lambda\nabla f(x^*)+\sum_{i\in I^0(x^*)}u_i\nabla g_i(x^*)=0,\tag{2.1}$$

where $\lambda \geqslant 0$ *and* $u_i \geqslant 0,\ i \in I$.

Proof By the assumptions, there is some number L such that

$$L > \varphi_{q_k,\varepsilon_k,v}(x_k^*),\ k = 0,1,2,\cdots \tag{2.2}$$

Suppose to the contrary that $\{x_k^*\}$ is unbounded. Without loss of generality, we assume that $\|x_k^*\| \to \infty$ as $k \to \infty$. Then, from (1.4), (1.5), (2.2) and $v \in (\frac{1}{2}, 1)$, we have

$$L > f(x_k^*),\ k = 0,1,2,\cdots,$$

which results in a contradiction since $\lim\limits_{\|x\|\to+\infty} f(x) = +\infty$.

Now we show that any limit point of $\{x_k^*\}$ belong to G_0. Without loss of generality, we assume that $\lim_{k\to+\infty} x_k^* = x^*$. Suppose to the contrary that $x^* \notin G_0$. Then there exists some i such that $p_v(g_i(x^*)) > 0$. As g_i $(i \in I)$ are continuous, so are $\varphi_{q_k,\varepsilon_k,v}(\cdot),\ k = 0,1,2,\cdots$. Note that

$$\begin{aligned}\varphi_{q_k,\varepsilon_k,v}(x_k^*) =& f(x_k^*) + q_k \sum_{i\in I^+_{\varepsilon_k}(x_k^*)} \left(g_i^+(x_k^*)^v + \varepsilon_k g_i^+(x_k^*)^{v-1} - \frac{5}{v+2}\varepsilon_k^v\right)\\ &+ q_k \sum_{i\in I^-_{\varepsilon_k}(x_k^*)\backslash I^-(x_k^*)} \frac{2v-1}{(v+2)\varepsilon_k^2} g_i^+(x_k^*)^{v+2}.\end{aligned}$$

Then, as $k \to \infty,\ \varphi_{q_k,\varepsilon_k,v}(x_k^*) \to \infty$, which contradicts the assumption.

Finally, we show that (2.1) holds. By Step 2, $\nabla\varphi_{q_k,\varepsilon_k,v}(x_k^*) = 0$, that is

$$\begin{aligned}\nabla f(x_k^*) + q_k &\sum_{i\in I^+_{\varepsilon_k}(x_k^*)} \left(v g_i^+(x_k^*)^{v-1} + \varepsilon_k(v-1) g_i^+(x_k^*)^{v-2}\right)\nabla g_i(x_k^*)\\ + q_k &\sum_{i\in I^-_{\varepsilon_k}(x_k^*)\backslash I^-(x_k^*)} (2v-1)\varepsilon_k^2 g_i^+(x_k^*)^{v+1}\nabla g_i(x_k^*) = 0.\end{aligned} \tag{2.3}$$

For $k = 0,1,2,\cdots$, let

$$\begin{aligned}\gamma_k = 1 + q_k &\sum_{i\in I^+_{\varepsilon_k}(x_k^*)} \left(v g_i^+(x_k^*)^{v-1} + \varepsilon_k(v-1) g_i^+(x_k^*)^{v-2}\right)\\ + q_k &\sum_{i\in I^-_{\varepsilon_k}(x_k^*)\backslash I^-(x_k^*)} (2v-1)\varepsilon_k^2 g_i^+(x_k^*)^{v+1}.\end{aligned}$$

Then $\gamma_k > 0$ for enough large k. From (2.3), we have

$$\begin{aligned}\frac{1}{\gamma_k}\nabla f(x_k^*) + &\sum_{i\in I^+_{\varepsilon_k}(x_k^*)} \frac{q_k\left(v g_i^+(x_k^*)^{v-1} + \varepsilon_k(v-1) g_i^+(x_k^*)^{v-2}\right)}{\gamma_k}\nabla g_i(x_k^*)\\ + &\sum_{i\in I^-_{\varepsilon_k}(x_k^*)\backslash I^-(x_k^*)} \frac{q_k(2v-1)\varepsilon_k^2 g_i^+(x_k^*)^{v+1}}{\gamma_k}\nabla g_i(x_k^*) = 0.\end{aligned} \tag{2.4}$$

Let $\lambda_k = \frac{1}{\gamma_k}$,

$$u_i^k = \frac{q_k\left(v g_i^+(x_k^*)^{v-1} + \varepsilon_k(v-1) g_i^+(x_k^*)^{v-2}\right)}{\gamma_k},\quad i \in I^+_{\varepsilon_k}(x_k^*),$$

$$u_i^k = \frac{q_k(2v-1)\varepsilon_k^2 g_i^+(x_k^*)^{v+1}}{\gamma_k}, \quad i \in I_{\varepsilon_k}^-(x_k^*) \setminus I^-(x_k^*),$$
$$u_i^k = 0, \quad i \in I^-(x_k^*)$$

Then

$$\lambda_k + \sum_{i\in I} u_i^k = 1, \quad \forall\ k. \tag{2.5}$$

By $v \in (\frac{1}{2}, 1)$ and $\varepsilon_k \to 0$ as $k \to \infty$, we have

$$u_i^k \geqslant 0,\ i \in I,$$

for enough large k.

Clearly, as $k \to +\infty,\ \lambda_k \to \lambda \geqslant 0,\ u_i^k \to u_i \geqslant 0,\ \forall\ i \in I$. By (2.4) and (2.5), as $k \to \infty$, we have

$$\lambda \nabla f(x^*) + \sum_{i\in I} u_i \nabla g_i(x^*) = 0, \qquad \lambda + \sum_{i \in I} u_i = 1.$$

For $i \in I^-(x^*)$, as $k \to \infty$, we have $u_i^k \to 0$. Therefore, $u_i = 0,\ \forall\ i \in I^-(x^*)$. Then we complete the proof.

3 Numerical examples

In this section, we give several numerical examples to show the applicability of the presented algorithm with $v = \frac{1}{3}$, $v = \frac{1}{2}$ and $v = \frac{2}{3}$.

Example 3.1 (*See Eg. 4.2.9 in page 146 of [23]*)

$$\begin{aligned} &\min\ f(x) = (x_1-3)^2 + (x_2-2)^2, \\ &\text{s.t. } g_1(x) = x_1^2 + x_2^2 - 5 \leqslant 0, \\ &\qquad g_2(x) = x_1 + 2x_2 - 4 \leqslant 0, \\ &\qquad x_1, x_2 \geqslant 0. \end{aligned}$$

Starting point $x^0 = (1,1)$, $q_0 = 0.01$, $\varepsilon_0 = 0.01$, $\eta = 0.01$, $N = 20$, $\varepsilon = 1 \times 10^{-15}$. Numerical results are given in Table 1, 2 and 3.

Table 1 Numerical results of Example 3.1 with $v = \frac{1}{3}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(2.996704, 1.996741)	0.01	0.01	7.967210	2.990186	0.0000215
1	(2.931383,1.932444)	0.2	0.0001	7.327350	2.796272	0.009272
2	(2.000158, 0.9992276)	4.0	0.000001	-0.000914	-0.001387	2.001230

Table 2 Numerical results of Example 3.1 with $v = \frac{1}{2}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(2.993250, 1.993579)	0.01	0.01	7.933899	2.980407	0.000087
1	(2.858075,1.865501)	0.2	0.0001	6.648686	2.589077	0.038233
2	(2.000120,0.9992266)	4.0	0.000001	-0.001065	-0.001426	2.001307

Table 3 Numerical results of Example 3.1 with $v=\frac{2}{3}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(2.987660,1.988704)	0.01	0.01	7.881060	2.965069	0.000279
1	(2.743870,1.766386)	0.2	0.0001	5.648941	2.276642	0.1201781
2	(2.000263, 0.9992877)	4.0	0.000001	-0.000370	-0.001161	2.000898

From [23], we know that the global solution is (2,1) with global optimal value 2. It is clear from these three tables that the obtained approximate global solution is $x^*=(2.000263, 0.9992877)$ with corresponding objective function value 2.000898.

Example 3.2 (*The Rosen – Suzki problem in [22] and [24]*)

$$\begin{aligned}\min\ & f(x)=x_1^2+x_2^2+2x_3^2+x_4^2-5x_1-5x_2-21x_3+7x_4,\\ \text{s.t. } & g_1(x)=2x_1^2+x_2^2+x_3^2+2x_1+x_2+x_4-5\leqslant 0,\\ & g_2(x)=x_1^2+x_2^2+x_3^2+x_4^2+x_1-x_2+x_3-x_4-8\leqslant 0,\\ & g_3(x)=x_1^2+2x_2^2+x_3^2+2x_4^2-x_1-x_4-10\leqslant 0.\end{aligned}$$

Starting point $x^0=(0,0,0,0)$, $q_0=5.0$, $\varepsilon_0=0.01$, $\eta=0.1$, $N=2$, $\varepsilon=1\times10^{-15}$. Numerical results are given in Table 4 and 5.

Table 4 Numerical results of Example 3.2 with $v=\frac{1}{3}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$g_3(x_k^*)$	$f(x_k^*)$
0	$(1.065070, 1.203614, 3.707415, -1.911620)$	5	0.01	13.88449	17.46277	15.93181	-68.85327
1	$(0.2190443, 0.9523497, 1.913380, -1.053221)$	10	0.001	0.001171	-0.041455	-1.424329	-44.02423
2	$(0.2092652, 0.9399132, 1.916596, -1.069591)$	20	0.0001	-0.066786	0.000136	-1.367615	-44.06362
3	$(0.1701657, 0.8347672, 1.999031, -0.9810361)$	40	0.00001	-0.055064	-0.000185	-1.845512	-44.19109

Table 5 Numerical results of Example 3.2 with $v=\frac{1}{2}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$g_3(x_k^*)$	$f(x_k^*)$
0	$(0.2315499, 0.8870426, 1.947699, -1.027912)$	5	0.01	0.009838	0.010714	-1.669594	-44.20590
1	$(0.1590134, 0.8141334, 1.942331, -1.082092)$	10	0.001	-0.4638979	0.000974	-1.611514	-43.82501
2	$(0.1645929, 0.8394377, 2.008706, -0.9659035)$	20	0.0001	-0.003545	-0.000622	-1.861450	-44.22978

Starting point $x^0=(2.0, 2.0, 2.0, 2.0)$, $q_0=5.0$, $\varepsilon_0=0.1$, $\eta=0.1$, $N=2$, $\varepsilon=1\times10^{-15}$. Numerical results are given in Table 6.

It is clear from these three tables that the obtained approximate global solution is $x^*=(0.1645929, 0.8394377, 2.008706, -0.9659035)$ with corresponding objective function value -44.22978. From [22], the obtained approximate global solution is $x^*=(0.169234, 0.835656, 2.008690, -0.964901)$ with corresponding objective function value -44.233582. The solution we obtained is similar with that obtained in [22].

Table 6 Numerical results of Example 3.2 with $v=\frac{2}{3}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$g_3(x_k^*)$	$f(x_k^*)$
0	$(0.1146511, 0.8429196, 2.031888, -0.9505037)$	5	0.1	-0.012908	0.009810	-1.794490	-44.22678
1	$(0.1453490, 0.8286198, 2.018960, -0.9596925)$	10	0.01	-0.035314	0.003254	-1.873092	-44.20470
2	$(0.1476035, 0.8314404, 2.022063, -0.9505301)$	20	0.001	-0.000278	-0.005919	-1.896948	-44.21819

Example 3.3 (*Example 3.1 in* [*25*])

$$\begin{aligned} \min\ & f(x)=x_1^2+x_2^2-\cos(17x_1)-\cos(17x_2)+3,\\ \text{s.t. } & g_1(x)=(x_1-2)^2+x_2^2-1.6^2\leqslant 0,\\ & g_2(x)=x_1^2+(x_2-3)^2-2.7^2\leqslant 0,\\ & 0\leqslant x_1\leqslant 2,\\ & 0\leqslant x_2\leqslant 2. \end{aligned}$$

Starting point $x^0=(0,0),\ \eta=0.01,\ N=3,\ \varepsilon=1\times10^{-15}$.
Let $q_0=1,\ \varepsilon_0=0.1$ when $v=\frac{1}{3}$.
Let $q_0=0.1,\ \varepsilon_0=0.01$ when $v=\frac{1}{2}$.
Let $q_0=1,\ \varepsilon_0=0.01$ when $v=\frac{2}{3}$. Numerical results are given in Table 7, 8 and 9.

Table 7 Numerical results of Example 3.3 with $v=\frac{1}{3}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(0.003100, 0.004269)	1	1×10^{-1}	1.427628	1.684413	1.004048
1	(0.3833721, 0.7169942)	3	1×10^{-3}	0.5675662	-1.930911	1.758740
2	(0.7358550, 0.4022011)	9	1×10^{-5}	-0.8001720	0.000004	1.854559
3	(0.7239410, 0.3988712)	27	1×10^{-7}	-0.7725754	-0.000039	1.837919

Table 8 Numerical results of Example 3.3 with $v=\frac{1}{2}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(0.000562, 0.000772)	0.1	1×10^{-2}	1.437753	1.705369	1.000133
1	(0.001728, 0.002382)	0.3	1×10^{-4}	1.433097	1.695716	1.001260
2	(0.005178, 0.007165)	0.9	1×10^{-6}	1.419366	1.667088	1.011359
3	(0.7253758, 0.3992658)	2.7	1×10^{-8}	-0.7759201	-0.000012	1.837568

Table 9 Numerical results of Example 3.3 with $v=\frac{2}{3}$

k	x_k^*	q_k	ε_k	$g_1(x_k^*)$	$g_2(x_k^*)$	$f(x_k^*)$
0	(0.008104,0.011597)	1	1×10^{-2}	1.407784	1.640618	1.029046
1	(0.7848462,1.040875)	3	1×10^{-4}	0.000019	-2.835846	3.581485
2	(0.7276356,0.3998984)	9	1×10^{-6}	-0.7811703	-0.000019	1.838380

It is clear from these three tables that the obtained approximate global solution is $x^*=(0.7253758, 0.3992658)$ with corresponding objective function value 1.837568. By Sun and Li [25], we know that $x^*=(0.7255, 0.3993)$ is a global minimum with global optimal value $f^*=1.8376$.

For the $k'th$ iteration of the algorithm, we define a constraint error e_k by

$$e_k = \sum_{i=1}^{m} \max\{g(x^k), 0\}.$$

Example 3.4 (*Example 3.4 in [22] and Example 3.1 in [9]*)

$$\begin{aligned}
\min\ & f(x) = 10x_2 + 2x_3 + x_4 + 3x_5 + 4x_6, \\
\text{s.t. } & g_1(x) = x_1 + x_2 - 10 = 0, \\
& g_2(x) = -x_1 + x_3 + x_4 + x_5 = 0, \\
& g_3(x) = -x_2 - x_3 + x_5 + x_6 = 0, \\
& g_4(x) = 10x_1 - 2x_3 + 3x_4 - 2x_5 - 16 \leqslant 0, \\
& g_5(x) = x_1 + 4x_3 + x_5 - 10 \leqslant 0, \\
& 0 \leqslant x_1 \leqslant 12, \\
& 0 \leqslant x_2 \leqslant 18, \\
& 0 \leqslant x_3 \leqslant 5, \\
& 0 \leqslant x_4 \leqslant 12, \\
& 0 \leqslant x_5 \leqslant 1, \\
& 0 \leqslant x_6 \leqslant 16.
\end{aligned}$$

Starting point $x^0 = (0, 0, 0, 0, 0, 0)$, $\varepsilon_0 = 0.1$, $\varepsilon = 1 \times 10^{-6}$.

Let $q_0 = 700.0$, $\eta = 0.1$, $N = 9$ when $v = \frac{1}{3}$ and $v = \frac{2}{3}$.

Let $q_0 = 1000.0$, $\eta = 0.01$, $N = 4$ when $v = \frac{1}{2}$. Numerical results are given in Table 10.

Table 10 Numerical results of Example 3.4 with different v

v	*No. iter.*	q_k	*Cons. error* e_k	*Objective value*	*Solution* $(x_1, x_2, x_3, x_4, x_5, x_6)$
$\frac{1}{3}$	10	700×9^9	0.0	117.0368	(1.623695, 8.376305, 0.027677, 0.5992718, 0.9967462, 7.407235)
$\frac{1}{2}$	7	1000×4^6	1.870848×10^{-7}	117.0166	(1.615956, 8.384044, 0.004182, 0.6133483, 0.9984257, 7.389800)
$\frac{2}{3}$	7	700×9^6	4.470348×10^{-8}	117.0573	(1.635022, 8.364978, 0.060010, 0.5812775, 0.9937346, 7.431253)

It is clear from Table 10 that the obtained approximate global solution is $x^* = (1.615956, 8.384044, 0.004182, 0.6133483, 0.9984257, 7.389800)$ with corresponding objective function value 117.0166. From [22], the obtained approximate global solution is $x^* = (1.847052, 8.152948, 0.607878, 0.244707, 0.994467, 7.766359)$ with corresponding objective function value 117.038781. The solution we obtained is similar with that obtained in [22] and better than that obtained in [9] (the objection function value $f(x^*) = 124$).

References

[1] Zangwill W I. Non-linear programming via penalty functions [J]. *Manage Sci*, 1967, **13**: 344-358.

[2] Burke J. Calmness and exact penalization [J]. *SIAM J Control and Optimization*, 1991, **29**: 493-497.

[3] Evans J P, Gould F J, Tolle J W. Exact Penalty Function in Nonlinear Programming [J]. *Math Prog*, 1973, **4:** 72-97.

[4] Fletcher R. Penalty functions in mathematical programming [M]. The state of the art, A. Bachen et al(eds), Berlin: Springer-Verlag, 1983: 87-114.

[5] Pillo G D. Exact penalty methods, Algorithms for continuous Optimization [M]. E. Spedicato(ed.), Dordrecht: Kluwer Academic Publishers, 1994: 209-253

[6] Auslender A, Cominetti R, Haddou M. Asymptotic analysis for penalty and barrier methods in convex and linear programming [J]. *Mathematics of Operations Research*, 1997, **22**: 43-62.

[7] Ben-Tal A, Teboulle M. A smoothing technique for non-differentible optimization problems [C]. Lecture Notes in Mathematics, Berlin: Springer Verlag, 1989, 1405: 1-11.

[8] Gonzaga C C, Gastillo R A. A nonlinear programming algorithm based on non-coercive penalty functions [J]. *Mathematical Programming*, Ser. A, 2003, **96**: 87-101.

[9] Pinar M, Zenios S. On smoothing exact penalty functions for convex constrained optimization [J]. *SIAM Journal on Optimization*, 1994, **4:** 468-511.

[10] Wu Z Y, Lee H W J, Bai F S, et al. Quaradic smoothing approximation to l_1 exact penalty function in global optimization [J]. *Journal of Industrail and Management Optimization*, 2005, **53**: 533-547.

[11] Liu B Z. On smoothing exact penalty functions for nonlinear constrained optimization problems [J]. *Journal of Applied Mathematics and Computing*, 2009, **30**: 259-270.

[12] Yu C J, Teo K L, Zhang L S, et al. A new exact penalty function method for continuous inequality constrained optimization problems [J]. *Journal of Industrial and Management Optimization*, 2010, **6**(4): 895-910.

[13] Huang X X, Yang X Q. A Unified Augmented Lagrangian Approach to Duality and Exact Penalization [J]. *Mathematic of Operations Research*, 2003, **28**: 533-552.

[14] Huang X X, Yang X Q. Convergence analysis of a class of nonlinear penalization methods for constrained optimization via first-order necessary optimality conditions [J]. *Journal of Optimization Theory and Applications*, 2003, **116**(2): 311-332.

[15] Luo Z Q, Pang J S, Ralph D. Mathematical Programs with Equilibrium Constraints [M]. Cambridge: Cambridge University Press, 1996.

[16] Rubinov A M, Yang X Q, Bagirov A M. Penalty functions with a small penalty parameter [J]. *Optimization Methods and software*, 2002, **17**: 931-964.

[17] Rubinov A M, Yang X Q. Lagrange-type functions in nonconvex constrained optimization [M]. Dordrecht: Kluwer Academic Publishers, 2003.

[18] He Z H, Bai F S. A smoothing approximation to the lower order exact penalty function [J]. *Operations research Transactions*, 2010, **14**(2): 11-22.

[19] Shang Y L, Liu M H, Li P. A new penalty function based on non-coercive penalty functions [J]. *Operations research Transactions*, 2012, **16**(1): 56-66.

[20] Bai F S, Wu Z Y, Zhu D L. Lower order calmness and exact penalty function [J]. *Optimization Methods and Software*, 2006, **21**: 515-525.

[21] Wu Z Y, Bai F S, Yang X Q, et al. An exact lower order penalty function and its smoothing in nonlinear programming [J]. *Optimization*, 2004, **53**: 51-68.

[22] Meng Z Q, Dang C Y, Yang X Q. On the smoothing of the squareroot exact penalty function for inequality constrained optimization [J]. *Computational Optimization and Applications*, 2006, **35**: 375-398.

[23] Bazaraa M S, Sherali H D, Shetty C M. Nonlinear Programming: Theory and Algorithms [M]. second Edition, John Wiley Sons, Inc., New York, 1993.

[24] Lasserre J B. A globally convergent algorithm for exact penalty functions [J]. *European Journal of Operational Research*, 1981, **7**: 389-395.

[25] Sun X L, Li D. Value-estimation function method for for constrained global optimization [J]. *Journal of Optimization Theory and Applications*, 1999, **24**: 385-409.