Existence and optimality of accessible and approximatable global minimizers

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Abstract The concepts of robustness of sets and functions are proposed in view of the theory of integral global minimization. These concepts are generalized, and global minimization of quasi and pseudo upper robust function is investigated in this paper. With the deviation integral optimality condition of global minimum, the existence of accessible minimizer of quasi upper functions and approximatable minimizer of pseudo upper robust function is examined.

Keywords global optimization, robust minimizer, quasi upper robust, deviation integral

Chinese Library Classification O224

2010 Mathematics Subject Classification 49K35,49J35

可达到和可逼近总极小点的存在性和最优性

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摘要 针对积分总极值，讨论并拓展了丰满集和丰满函数的概念，研究了拟上丰满和伪上丰满函数的总极值问题. 在总极值的变差积分最优性条件下，证明了拟上丰满函数的可达到极小点和伪上丰满函数的可逼近极小点的存在性.

关键词 总极值问题，丰满极小点，拟上半丰满，变差积分

中图分类号 O224

数学分类号 49K35,49J35

0 Introduction

Let X be a topological space, S a subset of X, and $f : X \to R^1$ a real valued function. Consider the following minimization problem: find the minimum value of f over S

$$c^* = \inf_{x \in S} f(x), \tag{0.1}$$

and the set of global minimizers

$$H^* = \{x \in S : f(x) = c^*\}. \tag{0.2}$$

收稿日期： 2011 年 2 月 28 日

* Supported by the grant of National Natural Science Foundation of China (No. 10771133), Key Disciplines of Shanghai Municipality (Operations Research and Control Theory S 30104).

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In [1], several existence theorems of global minimizer were introduced, which almost all conditions ensure only the emptyness of the set of global minimizers. However, the nonemptyness of the set of global minimizers cannot ensure accessibility and approximatability of minimizers[2] and finding them numerical.

We still know another kind of existence theorem[3]. They are quite useful because the requirement of compactness is put on that of the objective function itself.

$$\text{If } X \text{ is a Banach space,} \quad f \in C^1 \text{ and satisfies Palais} - \text{Smale condition,} \tag{0.3}$$

then there exists a point x^* such that

$$f(x^*) = c^* \quad \text{and} \quad df(x^*) = \theta, \tag{0.4}$$

where $\mathrm{d}f(x^*)$ is the differential of f at x^* and θ is the null vector. Palais-Smale condition means that for each sequence $\{x_n\} \subset X$,

$$\{f(x_n)\} \text{ is bounded and } \mathrm{d}f(x_n) \to \theta \Rightarrow \text{ the sequence } \{x_n\} \text{ has a convergent subsequence.} \tag{0.5}$$

In this paper, we investigate minimization problems of quasi and pseudo upper robust functions, and examine the optimality conditions and existence of robust, accessible, and approximatable minimizers with deviation integral. We recall some basic definitions and properties of robust sets, functions and three kind of minimizers in Section 0. We establish optimality conditions for global minimum of a quasi (or pseudo) upper robust function with deviation integral (see [4]) in Section 1. In Section 2, we prove the existence theorems of accessible and approximatable minimizers based on these optimality conditions. Several examples are given to show that these theorems are useful to prove the existence of global minimizer. We conclude our paper in Section 3.

1 Robust sets, robust and upper robust functions

1.1 Robust sets and functions

We begin with recalling concepts of *robust sets, points,* and *semineighborhood* (see [5-7]). Let X be a topological space, and D a subset of X. A set $D \subset X$ is said to be robust iff

$$\text{cl } D = \text{cl int } D, \tag{1.1}$$

where int D denotes the interior of D and cl D the closure of D.

A point $x \in$ cl D is said to be robust to D, if for each neighborhood $N(x)$ of x, $N(x) \cap$ int $D \neq \emptyset$. A point $x \in D$ is a robust point of D if and only if there exists a net $\{x_\lambda\} \subset$ int D such that

$$x_\lambda \to x.$$

If x is a robust point of set D, then D is called a semineighborhood of x.

Function $f : X \to R^1$ is said to be *upper robust* iff the set

$$F_c = \{x : f(x) < c\}$$

is robust for each real number c; it is said to be robust if for each open set $G \subset R^1, f^{-1}(G)$ is a robust set in X.

Let X be a metric space and $f : X \to R^1$ a real valued function. Suppose C is the set of points of continuity of f. Then f is said to be upper approximatable iff

1. C is dense in X;
2. for each point $\overline{x} \in X$, there is a sequence $\{x_n\} \subset C$ such that $\lim\limits_{n\to\infty} x_n = \overline{x}$ and $\limsup\limits_{n\to\infty} f(x_n) = f(\overline{x})$;

1.2 Robust, accessible, and approximatable minimizers, quasi-upper robust function

In this subsection, we first recall the definitions of three kinds of minimizer and their relationships. When $c^* = f(x^*)$ is the global minimum value, we have

$$f^{-1}((c^* - \varepsilon, c^* + \varepsilon)) = F_{c^*+\varepsilon} = \{x : f(x) < c^* + \varepsilon\}.$$

Thus, the definition of a robust minimizer can be modified as follows.

Definition 1.1 Let X be a topological space, S is a subset of X, and $f : X \to R^1$ is a real valued function. A point $x^* \in S$ is said to be an accessible minimizer if

1. $f(x^*) \leqslant f(x), \forall x \in S$;
2. for each $c > c^*$,there is a sequence of points $\{x_\alpha\} \subset \text{int}(S \cap F_c)$ such that

$$x_\alpha \to x^* \quad \textit{and} \quad f(x_\alpha) \to f(x^*).$$

Definition 1.2 Let X be a topological space, S is a subset of X, and $f : X \to R^1$ is a real valued function. Point $x^* \in X$ is said to be an approximatable minimizer if

1. $f(x^*) \leqslant f(x), \forall x \in S$;
2. there is a sequence of points $\{x_\alpha\} \subset C \cap S$, such that

$$x_\alpha \to x^* \quad \textit{and} \quad f(x_\alpha) \to f(x^*),$$

where C is the set of points of continuity of f.

Proposition 1.3(see [2]) Let X be a topological space, $f : X \to R^1$ a real valued function. Suppose that x^* is a global minimizer of f over S. Then the function f is upper robust at $x^* \in S$ if and only if for each number $c > c^* = f(x^*)$, there is a sequence of points $\{x_\alpha\} \subset \text{int}(S \cap F_c)$, such that $x_\alpha \to x^*$ and $f(x_\alpha) \to f(x^*)$.

Hence a global minimizer is robust if and only if it is accessible.

Proposition 1.4 (see [2]) Let X be a topological space, $f : X \to R^1$ a real valued function. Suppose that $x^* \in S$ is a global minimizer of f. If the minimizer x^* is approximatable then it is accessible.

However, an accessible minimizer may be not approximatable.

Now, we recall the concept of quasi upper robustness.

Definition 1.5 (see [2]) Function $f : X \to R^1$ is said to be quasi upper robust on S iff for each $c > c^*$, the level set $S \cap F_c = \{x \in S : f(x) < c\}$ contains a nonempty robust subset.

2 Optimality Conditions

Let X be a topological space, S a subset of X, $f : X \to R^1$ a real valued function, and

$$c^* = \inf_{x \in S} f(x). \tag{2.1}$$

We now examine the optimality conditions for global minimum using an integral approach under weaker assumptions. To do so, some of the following assumptions are required.

Assumption (M): (X, Ω, μ) is a Q-measure space.

Assumption (A): S is a measurable set and f is a measurable and bounded below function.

Assumption (R): f is a quasi upper robust on S.

Note that measure space (X, Ω, μ) is a Q-measure space if

1. Each open set in X is measurable;

2. The measure $\mu(G)$ of a nonempty open set G in X is positive: $\mu(G) > 0$;

3. The measure $\mu(K)$ of a compact set K in X is finite.

2.1 A Sufficient Condition for Global Minimum

The following lemma leads to a sufficient optimality condition for global minimum.

Lemma 2.1 Suppose that Conditions (M), (A), and (R) hold. If $c > c^* = \inf_{x \in S} f(x)$, then $\mu(H_c \cap S) > 0$, where $H_c \cap S = \{x \in S : f(x) \leqslant c\}$.

Proof Suppose, on the contrary, that

$$\mu(H_c \cap S) = 0. \tag{2.2}$$

By Assumption (R), there exists a nonempty robust set $D \subset S \cap F_c$. Now we have

$$\emptyset \neq D \subset S \cap H_c, \tag{2.3}$$

and $\text{int}D \neq \emptyset$. It follows from Assumptions (M) and (A) that

$$\mu(S \cap H_c) \geqslant \mu(S \cap F_c) \geqslant \mu(\text{int}D) > 0. \tag{2.4}$$

This contradicts (2.2).

Corollary 2.2 Under the assumption of Lemma 2.1, if $H_c \cap S \neq \emptyset$ and $\mu(H_c \cap S) = 0$, then c is the global minimum value of f over S.

2.2 Deviation integral

Definition 2.3 Under the assumptions (M), (A) and (R), let $\phi : R^1 \to R^1$ be a strictly increasing continuous function and $\phi(0) = 0$. We define deviation integral of f as following:

$$V_\phi(c) = \int_{H_c \cap S} \phi(c - f(x)) \mathrm{d}\mu, \tag{2.5}$$

where the integration is with respect to x over $H_c \cap S$.

For $c > c^*$, $\phi(c - f(x))$ is measurable and well defined. We can obtain properties of the integral $V_\phi(c)$.

Proposition 2.4 Integral $V_\phi(c)$ has the following properties:

1. $V_\phi(c) > 0, \forall c > c^*$, $V_\phi(c) = 0, \forall c < c^*$ and $V_\phi(c)$ is continuous;
2. $V_\phi(c)$ is non-decreasing on $(-\infty, +\infty)$ and strictly increasing on (c^*, ∞);
3. Suppose that, in addition, ϕ is differentiable on $(-\infty, \infty)$ and $\phi'(0) = 0$, then the integral $V_\phi(c)$ is differential on $(-\infty, \infty)$, and $V'_\phi(c) = V_{\phi'}(c)$. Moreover, $V_\phi(c)$ is convex.

They can be proved in a similar way as corresponding properties in [4].

2.3 Optimality conditions with deviation integral

We now examine optimality conditions of global minimization. Let $\{c_n\}(> c^* = \inf_{x \in S} f(x))$ be a sequence of deceasing real numbers, and $\lim_{n\to\infty} c_n = c'$.

Theorem 2.5 Under the assumptions (M), (A) and (R), c' is the global minimum value if and only if for $c_n \downarrow c'$,

$$\lim_{n\to\infty} V_\phi(c_n) = 0 \tag{2.6}$$

Proof Necessity: $c' = \inf_{x\in S} f(x)$, because of continuity of $V_\phi(c)$, and $V_\phi(c) = 0$ when $c < c'$, From continuity of $V_\phi(c)$, we obtain $\lim_{n\to\infty} V_\phi(c_n) = 0$.

Sufficiency: Suppose c' is not the global minimum value of f but $\hat{c}$ is. Then $c' - \hat{c} = 2\eta > 0$. We have,

$$\begin{aligned} V_\phi(c') &= \int_{H_{c'} \cap S} \phi(c' - f(x)) \mathrm{d}\mu \\ &= \int_{(H_{c'} \setminus H_{\hat{c}+\eta}) \cap S} \phi(c' - f(x)) \mathrm{d}\mu + \int_{H_{\hat{c}+\eta} \cap S} \phi(c' - f(x)) \mathrm{d}\mu \\ &\geqslant \int_{H_{\hat{c}+\eta} \cap S} \phi(c' - \hat{c} - \eta) \mathrm{d}\mu = \phi(\eta) \cdot \mu(H_{\hat{c}+\eta} \cap S) > 0, \end{aligned} \tag{2.7}$$

which is a contradiction.

3 Existence theorems of robust minimizers

The Palais-Smale condition ensures the existence of robust minimizers. However, the analytic requirement added on the objective function f is quite demanding. This existence theorem cannot be applied to a nondifferentiable continuous function, nor a discontinuous objective function. In this section we will modify Palais-Smale condition into one which fits the framework of robust analysis.

3.1 Existence of accessible minimizers

Theorem 3.1 Let X be a metric space, S a closed subset of X, $f : S \subset X \to R^1$ a bounded below, lower semicontinuous and quasi upper robust function and

$$G = \text{int}(S \cap F_{c^*+\varepsilon}) = \text{int}\{x \in S : f(x) < c^* + \varepsilon\}, \quad \varepsilon > 0.$$

If for each sequence $\{x_n\} \subset G$, from $V_\phi(f(x_n)) \to 0$, it follows that there is a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, then there exists a minimizer x^* such that

$$x_{n_k} \to x^*, \ \textit{and} \quad f(x_{n_k}) \to f(x^*) = \inf_{x \in S} f(x). \tag{3.1}$$

Moreover, the minimizer x^* is accessible.

Proof Since c^* is the infimum of f over S, for each integer n, there is a point $y_n \in S$ such that

$$f(y_n) < c^* + \frac{1}{2n}.$$

With the quasi robustness of the objective function f, we also have $x_n \in G(G \subset \text{int}(S \cap F_{c^*+\varepsilon})$ and $1/n < \varepsilon)$ such that

$$f(x_n) < f(y_n) + \frac{1}{2n} < c^* + \frac{1}{n}. \tag{3.2}$$

Indeed, we can take $G = \text{int } D$, where D is a nonempty robust set contained in $S \cap F_{c^*+\varepsilon}$. Thus, $\text{int } D \subset \text{int}(S \cap F_{c^*+\varepsilon})$. We then obtain a sequence of point $\{x_n\} \subset \text{int } F_{c^*+\varepsilon}$ satisfied (3.2). Furthermore, we can assume that $\{f(x_n)\}$ is a monotone sequence without loss of generality. Therefore, we obtain a sequence of point $\{x_n\} \subset G$ such that

$$f(x_n) \downarrow c^* = \inf_{x \in S} f(x). \tag{3.3}$$

From the deviation integral optimality condition of global minimum, we have

$$V_\phi(f(x_n)) \to 0. \tag{3.4}$$

Thus, from condition (3.1), there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. It implies that there is a point $x^* \in X$ such that $x_{n_k} \to x^*$. The point x^* is also in S since $\{x_{n_k}\} \subset S$ and S is closed. We now prove that x^* is a global minimizer of f satisfying (3.2). Since c^* is the global minimum value of f, we have

$$f(x^*) \geqslant c^*. \tag{3.5}$$

Furthermore, by lower semicontinuity of f, for each $\eta > 0$, there is a neighborhood $U(x^*)$ of x^* such that

$$f(x) > f(x^*) - \eta,\ \forall x \in U(x^*)$$

Because $x_{n_k} \to x^*$, there exists a positive integer N such that for $n_k > N, x_{n_k} \in U(x^*)$ and then

$$f(x_{n_k}) > f(x^*) - \eta,\ \forall n_k > N.$$

Let $n \to \infty$ in the above inequality, we obtain from (3.5) that $c^* \geqslant f(x^*) - \eta$. Subsequently, by the arbitrariness of η, we obtain $f(x^*) \leqslant c^*$ It implies $f(x^*) = c^* = \min_{x \in S} f(x)$. Furthermore, by the above construction, x^* is a accessible minimizer of f. Therefore, x^* is a robust minimizer.

3.2 Existence of approximatable minimizers

We now consider the existence of approximatable minimizers. An accessible minimizer may be not approximatable. To ensure the existence of approximatable minimizers, we need more conditions such as pseudo upper robustness.

Definition 3.2 Let X be a topological space, S a subset of X. Function $f : X \to R^1$ is said to be pseudo upper robust on S iff for each $\varepsilon > 0$, the level set

$$S \cap F_{c^*+\varepsilon} = \{x \in S : f(x) < c^* + \varepsilon\}$$

contains a nonempty robust subset D on which f is upper robust.

Theorem 3.3 Let X be a metric space, S a closed subset of X, $f : X \to R^1$ a bounded below, lower semicontinuous and pseudo upper robust function, and C is the set of points of continuity of f. If for each sequence $\{x_n\} \subset C$, from $V_\phi(f(x_n)) \to 0$, it follows that there is a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, then there exists a minimizer x^* such that

$$x_{n_k} \to x^*, and \quad f(x_{n_k}) \to f(x^*) = \inf_{x \in S} f(x). \tag{3.6}$$

Moreover, the minimizer x^* is approximatable.

It can be proved in a similar way as Theorem 3.1.

3.3 Examples

We now examine some examples.

Example 3.1 Let $X = R^1$, $A = \bigcup\limits_{n=1}^{\infty} \Big(\dfrac{1}{n+1}, \dfrac{1}{n}\Big) \bigcup \Big(-\dfrac{1}{n}, -\dfrac{1}{n+1}\Big) \bigcup \{0\}$,

$$B = \bigcup_{n=1}^{\infty} \Big\{\frac{1}{n}\Big\} \bigcup \Big\{-\frac{1}{n}\Big\} \bigcup \Big(-\infty, -1\Big) \bigcup (1, \infty) \text{ and}$$

$$f(x) = \begin{cases} |x|, & x \in A, \\ 1, & x \in B. \end{cases} \tag{3.7}$$

The function f is lower semicontinuous and quasi upper robust. The function has a unique global minimizer $x^* = 0$. The Palais-Smale condition requires that sequences $\mathrm{d}f(x_n) \to \theta$

but it does not. However, the condition of Theorem 3.1 only requires that sequences $\{x_n\}$ with $V_\phi(f(x_n)) \to 0$, it does not require that the function is continuous.

Example 3.2 Consider the following function (see [9])

$$f(x) = \begin{cases} 1.0 + \dfrac{\sum\limits_{i=1}^{n} |x^i|}{n} + \operatorname{sgn}\left(\sin\left(\dfrac{n}{\sum\limits_{i=1}^{n} |x^i|}\right) - 0.5\right), & x \neq \theta, \\ a, & x = \theta, \end{cases} \tag{3.8}$$

where n is the dimension of the function, and $x = (x^1, \ldots, x^n)$.

The function has an infinite number of discontinuous hypersurfaces. It is lower semi-continuous and quasi upper robust. Let $a = 0$. Its unique global minimizer is at the origin θ, where the function has a discontinuity of "the second kind". The Palais-Smale condition requires that sequences $\mathrm{d}f(x_k) \to \theta$ but it does not. However, the condition of Theorem 3.1 only requires that subsequences $\{x_k\}$ with $V_\phi(f(x_k)) \to 0$, here, we can take $x_k = (\frac{1}{k\pi}, \frac{1}{k\pi}, \cdots, \frac{1}{k\pi}) \to \theta$ where take mean deviation integral

$$m(f(x_k)) = V_\phi(f(x_k)) = \int_{H_{f(x)}} (f(x_k) - f(x))\mathrm{d}\mu \to 0.$$

Note that if $a > 0$ then the function has no global minimizer.

We can apply Theorem 3.1 and 3.3 to characterize the existence of critical points.

Example 3.3 Consider the following function:

$$g(x) = \begin{cases} x^2 \cos \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The function is differentiable on $X = R^1$, and

$$g'(x) = \begin{cases} 2x \cdot \cos \dfrac{1}{x} + \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The origin $x = 0$ is a critical point of g, at which the derivative is discontinuous. Since

$$g'\left(\frac{1}{\frac{\pi}{2} + k\pi}\right) = \begin{cases} 1, & k \text{ is even}, \\ -1, & k \text{ isodd}, \end{cases}$$

there is a point x_k in $\left(\frac{1}{\frac{\pi}{2}+(k+1)\pi}, \frac{1}{\frac{\pi}{2}+k\pi}\right)$, such that $g'(x_k) = 0$; and g' is continuous at x_k. We now consider $f(x) = |g'(x)|$. The infimum of function f is 0, and $x = 0$ is a global minimizer at which f is discontinuous. The global minimizer is approximatable. Theorem 3.3 can be applied to this situation, but not Palais-Smale condition.

4 Conclusion

The concept of deviation integral of quasi and pseudo upper robust functions is introduced in this work. We obtain the optimality condition of a quasi upper robust function. The existence of robust, accessible and approximatable minimizers are studied with deviation integral.

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