Sub-b-convex Functions

and Sub-b-convex Programming *

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Abstract: The paper studied a new generalized convex function which called sub-b-convex function, and introduced a new concept of sub-b-convex set. The basic properties of sub-b-convex functions were discussed in general case and differentiable case, respectively. And, obtained the sufficient conditions that the sub-b convex function become quasi-convex function or pseudo-convex function. Furthermore, the sufficient conditions of optimality for unconstrained and inequality constrained programming were obtained under the sub-b-convexity.

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Key Words: convex function, sub-b-convex set, sub-b-convex function, pseudo-sub-b-convex function, optimality conditions.

次 b 凸函数和次 b 凸规划

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摘要: 本文研究了一种称为次 b 凸函数的广义凸函数, 并介绍了次 b 凸集的概念. 分别在一般情形及可微情形下讨论了次 b 凸函数的相关性质, 得到了次 b 凸函数成为拟凸函数及伪凸函数的充分条件. 最后, 在次 b 凸函数的条件下给出了无约束及带不等式约束规划的最优性条件.

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关键词:凸函数, 次 b 凸集合, 次 b 凸函数, 伪次 b 凸函数, 最优性条件.

1. Introduction

Convexity plays a vital role in many aspects of mathematical programming, for example, sufficient optimality conditions and duality theorems. Over the years, many generalized convexity were presented (see [1]-[11]). In this paper, we introduce a new class of functions, which are called sub-b-convex functions and present some results about them.

In the following, we review several concepts of generalized convexity which have some relationships with this work. Though out the paper, we assume that the set $S \subseteq R^n$ is a nonempty convex set.

Definition 1.1 (see [2]) *A real function $f : S \subseteq R^n \to R$ is said to be a quasi-convex function, if*

$$f(\lambda x + (1-\lambda)y) \leq max\{f(x), f(y)\}, \ \forall x, y \in S, \forall \lambda \in (0,1).$$

Definition 1.2 (see [2]) *Let $f : S \subseteq R^n \to R$ be a differentiable function on S. The function f is said to be pseudo-convex function, if for each $x, y \in S$ with $\nabla f(x)^T(y-x) \geq 0$, we have $f(x) \geq f(y)$; or equivalently, if $f(x) < f(y)$, then $\nabla f(x)^T(y-x) < 0$.*

Theorem 1.1 (see [2]) *Let $f : S \subseteq R^n \to R$ be differentiable on S. Then f is a quasi-convex function if and only if the following equivalent statements holds:*

(1) If $x, y \in S$ and $f(x) \leq f(y)$, then $\nabla f(y)^T(x-y) \leq 0$;

(2) If $x, y \in S$ and $\nabla f(y)^T(x-y) > 0$, then $f(x) > f(y)$.

2. Sub-b-convex Functions and Their Properties

In this section, we present the definition of sub-b-convex function and discuss its some basic properties.

Definition 2.1 *Let S be a nonempty convex set in R^n. The function $f : S \to R$ is said to be a sub-b-convex function on S with respect to map $b : S \times S \times [0,1] \to R$, if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + b(x, y, \lambda), \forall x, y \in S, \lambda \in [0,1].$$

Remark 2.1 *If one defines map*

$$b(x,y,\lambda)=f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y),$$

then, each function is sub-b-convex with this map. However, our interesting is to study some $b(x,y,\lambda)$ *with special properties.And, obviously, each convex function* f *is sub-b-convex function with respect to the map* $b(x,y,\lambda)=0$.

Proposition 2.1 *If* $f_i : S \to R,\ (i=1,2,3,\dots,m)$ *are sub-b-convex functions with respect to maps* $b_i : S\times S\times[0,1]\to R,\ (i=1,2,3,\dots,m)$ *respectively. Then function*

$$f=\sum_{i=1}^{m} a_i f_i,\ a_i\geq 0,\ i=1,2,3,\dots,m$$

is sub-b-convex with respect to $b=\sum_{i=1}^{m} a_i b_i$.

Proof. For all $x,y\in S$ and $\lambda\in[0,1]$,

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &= \textstyle\sum_{i=1}^{m} a_i f_i(\lambda x+(1-\lambda)y) \\ &\leq \textstyle\sum_{i=1}^{m} a_i[\lambda f_i(x)+(1-\lambda)f_i(y)+b_i(x,y,\lambda)] \\ &= \lambda \textstyle\sum_{i=1}^{m} a_i f_i(x)+(1-\lambda)\sum_{i=1}^{m} a_i f_i(y)+\sum_{i=1}^{m} a_i b_i(x,y,\lambda) \\ &= \lambda f(x)+(1-\lambda)f(y)+\textstyle\sum_{i=1}^{m} a_i b_i(x,y,\lambda). \end{aligned}$$

So, from the definition of sub-b-convexity, knows f is sub-b-convex with respect to $b=\sum_{i=1}^{m} a_i b_i$.□

Proposition 2.2 *If functions* $f_i : S\to R,\ (i=1,2,3,\dots,m)$ *are sub-b-convex functions with respect to* $b_i,\ (i=1,2,3,\dots,m)$, *respectively. Then* $f=\max\{f_i,\ i=1,2,\cdots,m\}$ *is sub-b-convex with respect to* $b=\max\{b_i,\ i=1,2,\cdots,m\}$.

Proof. For forall $x,y\in S$ and $\lambda\in[0,1]$, from the sub-b-convexity of f_i, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &= \max\{f_i(\lambda x+(1-\lambda)y), i=1,2,\dots,m\} \\ &= f_t(\lambda x+(1-\lambda)y) \\ &\leq \lambda f_t(x)+(1-\lambda)f_t(y)+b_t(x,y,\lambda) \\ &\leq \lambda f(x)+(1-\lambda)f(y)+b(x,y,\lambda). \end{aligned}$$

So, $f(x)$ is is sub-b-convex with respect to $b=\max\{b_i,\ i=1,2,\cdots,m\}$. □

Here, we introduced a new concept of sub-b-convex set.

Definition 2.2 *Let* $X\subseteq R^{n+1}$ *be a nonempty set.* X *is said to be sub-b-convex with respect to* $b : R^n\times R^n\times[0,1]\to R$, *if*

$$(\lambda x+(1-\lambda)y, \lambda\alpha+(1-\lambda)\beta+b(x,y,\lambda))\in X, \forall\ (x,\alpha),(y,\beta)\in X,\quad x,y\in R^n, \lambda\in[0,1].$$

We now give a characterization of sub-b-convex function $f : S \to R$ in the term of their epigraph $E(f)$ given by

$$E(f) = \{(x, \alpha)|\ x \in S, \alpha \in R, f(x) \leq \alpha\}.$$

Theorem 2.1 *A function $f : S \to R$ is sub-b-convex with respect to $b : R^n \times R^n \times [0, 1] \to R$ if and only if $E(f)$ is a sub-b-convex set with respect to b.*

Proof. Suppose that f is sub-b-convex with respect to b. Let $(x_1, \alpha_1), (x_2, \alpha_2) \in E(f)$. Then, $f(x_1) \leq \alpha_1, f(x_2) \leq \alpha_2$. Since f is sub-b-convex with respect to b, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) + b(x_1, x_2, \lambda) \leq \lambda\alpha_1 + (1 - \lambda)\alpha_2 + b(x_1, x_2, \lambda).$$

Hence, from Definition 2.2 one has

$$(\lambda x_1 + (1 - \lambda)x_2, \lambda\alpha_1 + (1 - \lambda)\alpha_2 + b(x_1, x_2, \lambda)) \in E(f).$$

Thus $E(f)$ is a sub-b-convex set with respect to b.

Conversely, assume that $E(f)$ is a sub-b-convex set with respect to b. Let $x_1, x_2 \in S$, then $(x_1, f(x_1)), (x_2, f(x_2)) \in E(f)$. Thus, for $\lambda \in [0, 1]$,

$$(\lambda x_1 + (1 - \lambda)x_2, \lambda f(x_1) + (1 - \lambda)f(x_2) + b(x_1, x_2, \lambda)) \in E(f).$$

This further follows that

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) + b(x_1, x_2, \lambda).$$

That is, f is sub-b-convex with respect to b. □

Proposition 2.3 *If X_i $(i \in I)$ is a family of sub-b-convex set with respect to a same map $b(x, y, \lambda)$, then $\cap_{i\in I}X_i$ is a sub-b-convex set with respect to $b(x, y, \lambda)$.*

Proof. Let $(x, \alpha), (y, \beta) \in \cap_{i\in I}X_i$, $\lambda \in [0, 1]$. Then, for each $i \in I, (x, \alpha), (y, \beta) \in X_i$. Since X_i is a sub-b-convex set with respect to b, it follows that

$$(\lambda x + (1 - \lambda)y, \lambda\alpha + (1 - \lambda)\beta + b(x, y, \lambda)) \in X_i,\ \forall i \in I.$$

Thus,

$$(\lambda x + (1 - \lambda)y, \lambda\alpha + (1 - \lambda)\beta + b(x, y, \lambda)) \in \cap_{i\in I}X_i.$$

Hence, $\cap_{i\in I}X_i$ is a sub-b-convex set with respect to $b(x, y, \lambda)$. □

Theorem 2.2 *If f_i $(i \in I)$ is a family of numerical functions, and each f_i sub-b-convex with respect to the same map $b(x, y, \lambda)$, then the numerical function $f = \sup_{i \in I} f_i(x)$ is a sub-b-convex function with respect to $b(x, y, \lambda)$.*

Proof. Since f_i is a sub-b-convex function on S with respect to $b(x, y, \lambda)$, its epigraph $E(f_i) = \{(x, \alpha)|\ x \in S, f_i(x) \leq \alpha\}$ is a sub-b-convex set with respect to b. Therefor, their intersection

$$\cap_{i \in I} E(f_i) = \{(x, \alpha)|\ x \in S, f_i(x) \leq \alpha,\ i \in I\} = \{(x, \alpha)|\ x \in S, f(x) \leq \alpha\} = E(f)$$

is also a sub-b-convex set with respect to b. So, from Theorem 2.1 and Proposition 2.2, one know that $f = \sup_{i \in I} f_i(x)$ is a sub-b-convex function with respect to $b(x, y, \lambda)$. □

In the follows, we consider continuously differentiable functions which are sub-b-convex function with respect to a map b. Further, we assume that the limit $\lim\limits_{\lambda \to 0+} \frac{b(x,y,\lambda)}{\lambda}$ is exists for fixed $x, y \in S$.

Theorem 2.3 *Suppose that $f : S \to R$ is differentiable and sub-b-convex with respect to map b. Then*

$$\nabla f(y)^T (x - y) \leq f(x) - f(y) + \lim_{\lambda \to 0+} \frac{b(x, y, \lambda)}{\lambda}.$$

Proof. From Taylor expansion and the sub-b-convexity of f, we have

$$f(y) + \lambda \nabla f(y)(x - y) + o(\lambda) = f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y) + b(x, y, \lambda).$$

This implies that

$$\lambda \nabla f(y)(x - y) + o(\lambda) \leq \lambda f(x) - \lambda f(y) + b(x, y, \lambda).$$

Dividing the inequality above by λ and taking $\lambda \to 0+$, we have

$$\nabla f(y)^T (x - y) \leq f(x) - f(y) + \lim_{\lambda \to 0+} \frac{b(x, y, \lambda)}{\lambda}.$$

The proof of this theorem is completed. □

Base on Theorem 3.3.3 of [2] and Theorem 2.3 of this paper, it is easy to get the following result.

Corollary 2.1 *Let $f : S \to R$ be differentiable and sub-b-convex with respect to b. If for each $x, y \in S$, $\lim\limits_{\lambda \to 0+} \frac{b(x,y,\lambda)}{\lambda} \leq 0$, then f is a convex function on S.*

Theorem 2.4 *Suppose that $f : S \to R$ is differentiable and sub-b-convex with respect to b, then*

$$(\nabla f(y) - \nabla f(x))^T (x - y) \leq \lim_{\lambda \to 0+} \frac{b(x, y, \lambda)}{\lambda} + \lim_{\lambda \to 0+} \frac{b(y, x, \lambda)}{\lambda}.$$

Proof. From Theorem 2.3, we have

$$\nabla f(y)^T(x-y) \le f(x) - f(y) + \lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda},$$

$$\nabla f(x)^T(y-x) \le f(y) - f(x) + \lim_{\lambda\to 0+} \frac{b(y,x,\lambda)}{\lambda}.$$

Adding the two inequalities above, we have

$$(\nabla f(y) - \nabla f(x))^T(x-y) \le \lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda} + \lim_{\lambda\to 0+} \frac{b(y,x,\lambda)}{\lambda}.$$

The proof of this theorem is completed. □

Theorem 2.5 *Let $f : S \to R$ be differentiable and sub-b-convex with respect to b. If $\lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda} \le |f(x) - f(y)|$, $\forall x, y \in S$, then f is quasi-convex. Furthermore, if $\lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda} < |f(x) - f(y)|$, $\forall x, y \in S$, then f is pseudo-convex.*

Proof. For any $x, y \in S$ and $\lambda \in (0, 1)$. Suppose that $f(x) \le f(y)$, then from Theorem 2.3, we have

$$\nabla f(y)^T(x-y) \le f(x) - f(y) + \lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda}.$$

If $\lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda} \le |f(x) - f(y)|$, then $f(x) - f(y) + \lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda} \le 0$. So, $\nabla f(y)^T(x-y) \le 0$. Therefor, we know from Theorem 1.1, we get that f is quasi-convex.

Similarly, if $f(x) < f(y)$, we also have $\nabla f(y)^T(x-y) < 0$. So, from the Definition 1.2, we get that f is pseudo-convex. □

3. Optimality Conditions

In this section, we apply the associated results above to the nonlinear programming problem. First, we consider the unconstraint problem.

Theorem 4.1 *Let $f : S \to R$ be differentiable and sub-b-convex with respect to b. Consider the optimal problem* $\min\{f(x)|\ x \in S\}$. *If $\bar{x} \in S$ and relation*

$$\nabla f(\bar{x})^T(x-\bar{x}) - \lim_{\lambda\to 0+} \frac{b(x,y,\lambda)}{\lambda} \ge 0 \tag{4.1}$$

holds for each $x \in S$, then $\bar{x}$ is the optimal solution of f on S.

Proof. For any $x \in S$, from Theorem 2.3, one has

$$\nabla f(\bar{x})^T(x-\bar{x}) - \lim_{\lambda\to 0+} \frac{b(x,\bar{x},\lambda)}{\lambda} \le f(x) - f(\bar{x}).$$

From (4.1), we have $f(x) - f(\bar{x}) \geq 0$. So, $\bar{x}$ is an optimal solution of f on S. □

Next, we apply the associated results to the nonlinear programming problem with inequality constraints as follows:

$$(\mathrm{P}_g) \qquad \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad i \in I = \{1, 2, ..., m\}, \\ & x \in R^n. \end{array}$$

Denote the feasible set of (P_g) by $S_g = \{\, x \in R^n \mid g_i(x) \leq 0,\ i \in I\}$. For convenience of discussion, we assume that f and g_i are all differentiable and S_g is a nonempty set in R^n.

Now, we further extend the concept of sub-b-convex function, then discuss the optimality conditions of the corresponding programming.

Definition 4.4 *Let S be a nonempty convex set in R^n. The function $f : S \to R$ is said to be pseudo-sub-b-convex function on S with respect to $b : S \times S \times [0,1] \to R$, if for each $x, y \in S$ and $\lambda \in (0,1)$, from $\nabla f(y)^T(x - y) + \lim\limits_{\lambda \to 0+} \frac{b(x,y,\lambda)}{\lambda} \geq 0$ one can get $f(x) \geq f(y)$.*

Theorem 4.2 *(Karush-Kuhn-Tucker Sufficient Conditions) The function $f(x)$ is differentiable and pseudo-sub-b-convex with respect to $b : S \times S \times [0,1] \to R$, $g_i(x)$ $(i \in I)$ are differentiable and sub-b-convex with respect to $b_i : S \times S \times [0,1] \to R$ $(i \in I)$. Assume that $x^* \in S_g$ is a KKT point of (P_g) i.e., there exist multipliers $u_i \geq 0$ $(i \in I)$ such that*

$$\nabla f(x^*) + \sum_{i \in I} u_i \nabla g_i(x^*) = 0, u_i g_i(x^*) = 0. \tag{4.2}$$

If

$$\lim_{\lambda \to 0+} \frac{b(x, x^*, \lambda)}{\lambda} \geq \sum_{i \in I} u_i \lim_{\lambda \to 0+} \frac{b_i(x, x^*, \lambda)}{\lambda}, \ \forall x \in S_g. \tag{4.3}$$

Then x^* is an optimal solution of the problem (P_g).

Proof. For any $x \in S_g$, we have $g_i(x) \leq 0 = g_i(x^*)$, $i \in I(x^*) = \{i \in I \mid g_i(x^*) = 0\}$. Therefore, from the sub-b-convexity of $g_i(x)$ and Theorem 2.3, one obtain $\nabla g_i(x^*)^T(x - x^*) - \lim\limits_{\lambda \to 0+} \frac{b(x,y,\lambda)}{\lambda} \leq 0$ for $i \in I(x^*)$ From (4.2), one has $\nabla f(x^*)^T(x - x^*) = -\sum\limits_{i \in I(x^*)} u_i \nabla g_i(x^*)^T(x - x^*)$. In view of (4.3), we have $\nabla f(x^*)^T(x - x^*) + \lim\limits_{\lambda \to 0+} \frac{b(x,x^*,\lambda)}{\lambda} \geq -\sum\limits_{i \in I(x^*)} u_i \nabla g_i(x^*)^T(x - x^*) + \sum\limits_{i \in I(x^*)} u_i \lim\limits_{\lambda \to 0+} \frac{b_i(x,x^*,\lambda)}{\lambda} = -\sum\limits_{i \in I(x^*)} u_i(\nabla g_i(x^*)^T(x - x^*) - \lim\limits_{\lambda \to 0+} \frac{b_i(x,x^*,\lambda)}{\lambda})$.

From Theorem 2.3, one has $\nabla g_i(x^*)^T(x - x^*) - \lim\limits_{\lambda \to 0+} \frac{b_i(x,x^*,\lambda)}{\lambda} \leq g_i(x) - g_i(x^*) = g_i(x) \leq 0$, $i \in I(x^*)$. So,

$$\nabla f(x^*)^T(x - x^*) + \lim_{\lambda \to 0+} \frac{b(x, x^*, \lambda)}{\lambda} \geq 0.$$

This together with the pseudo-sub-b-convexity of $f(x)$, shows that $f(x) \geq f(x^*)$, therefor x^* is an optimal solution of the problem (P_g). $\square$

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