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Density matrix of the harmonic oscillator in thermostat based on coordinate-momentum intermediate representation

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Abstract: The density matrix of the harmonic oscillator in thermostat is calculated by virtue of the coherent states method and coordinate-momentum intermediate representation. The density matrix of the harmonic oscillator in thermostat is presented only through choosing different parameter values without making use of Fourier-transformation relations and direct calculation. Furthermore, the direct way to calculate the density matrix is also shown and the corresponding result is closely related to Hermite polynomials. As a useful application of the above conclusions, a fully new relation on Hermite polynomials can be revealed by comparing both of the above equivalent results.

Key words: quantum optics; density matrix; intermediate representation; Hermite polynomial

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基于坐标 - 动量中介表象的热平衡态谐振子密度矩阵

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摘 要： 利用相干态方法，借助坐标 - 动量中介表象研究了处于热平衡状态下谐振子的密度矩阵。在此基础上，不必利用傅里叶变换关系，而仅仅通过选择不同的参数值，就可以获得坐标和动量表象下处于热平衡状态下的谐振子密度矩阵的表示式。另外，给出了直接求解密度矩阵的过程，发现此结果与厄米多项式密切相关，作为以上所求表示式的一个有效应用，通过比较两种方法的结果，揭示了一个全新的厄密多项式的关系式。

关键词： 量子光学；密度矩；中间表象；厄密多项式

1 Introduction

It is well known that the state vector contains the full information about a quantum system. However, in many cases we do not know every detail of the quantum system, so we need the help of density matrix[1,2]. It is difficult to solve the density matrix for many real quantum systems, because there are a lot of mathematical difficulties. Thus finding a laconic way in solving the density matrix is a meaningful work. In a recent paper[3], Avakyan *et al* calculated the density matrix of harmonic oscillator in thermostat with the application of the coherent states method. It is really showed that the coherent states method is mathematically refined and

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physically transparent for the interpretation of quantum phenomena in classical language. In their paper they derived the density matrix of harmonic oscillator in thermostat in the coordinate representation in detail, and then told us that there are two ways to calculate the density matrix in the momentum representation: either from a coordinate representation with the use of Fourier-transformation or by direct, either from a coordinate representation with the use of Fourier-transformation or by direct calculation. In this work, we use the basis of coordinate-momentum intermediate representation to obtain density matrix of harmonic oscillator in thermostat. Then, by choosing different parameter values, the achieved density matrix will reduce to that of the coordinate-momentum representations, respectively, i.e., we do not recalculate the density matrix in the momentum representation by means of Fourier-transformation or direct calculation.

2 Calculating the density matrix

The coordinate-momentum intermediate representation has the form in Fock space[4~6]

$$|x\rangle_{\lambda,\nu} = [\pi(\lambda^2+\nu^2)]^{-\frac{1}{4}} \exp\Big[-\frac{x^2}{2(\lambda^2+\nu^2)} + \frac{\sqrt{2}x}{\lambda-\mathrm{i}\nu}a^+ - \frac{\lambda+\mathrm{i}\nu}{2(\lambda-\mathrm{i}\nu)}a^{+^2}\Big]|0\rangle \ , \tag{1}$$

where a^+ is the bosonic creation operator, λ, ν are two independent real parameters. Eq.(1) obeys the eigen-value equation

$$(\lambda Q + \nu P)|x\rangle_{\lambda,\nu} = x|x\rangle_{\lambda,\nu} \ , \tag{2}$$

where Q, P are coordinate and momentum operators, respectively. Note that $|x\rangle_{\lambda,\nu}$ reduces to the coordinate operator for $(\lambda,\nu) = (1,0)$ and to the momentum operator for $(\lambda,\nu) = (0,1)$, so it is named coordinate-momentum intermediate representation. From Eq.(1) we see

$$\int_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}p'\delta(x-\lambda x'-\nu p')\Delta(x',p') = |x\rangle_{\lambda,\nu\lambda,\nu}\langle x| \ ,$$

which shows that the pure state density operator $|x\rangle_{\lambda,\nu\lambda,\nu}\langle x|$ is just a Radon transform of the Wigner operator. Using the IWOP technique[7~12], we see that $|x\rangle_{\lambda,\nu}$ satisfies the completeness relation

$$\int_{-\infty}^{\infty} \mathrm{d}x|x\rangle_{\lambda,\nu\lambda,\nu}\langle x| = \frac{1}{\sqrt{\pi(\lambda^2+\nu^2)}}\int_{-\infty}^{\infty}\mathrm{d}x : \exp\Big[-\frac{1}{\lambda^2+\nu^2}(x-\lambda Q-\nu P)\Big] := 1 \ .$$

Moreover, one can prove

$$_{\lambda,\nu}\langle x'|x\rangle_{\lambda,\nu} = \delta(x-x') \ , \tag{3}$$

so $|x\rangle_{\lambda,\nu}$ is eligible to make up a new quantum mechanical representation. Then for any state $|\Psi\rangle$, the relationship between its Wigner function $W(x',p') = \langle\Psi|\Delta(x',p')|\Psi\rangle$ and the module square of the wave function of $|\Psi\rangle$ in the intermediate representation $_{\lambda,\nu}\langle x|$ is established,

$$\int_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}p'\delta(x-\lambda x'-\nu p')W(x',p') = |\langle\Psi|x\rangle_{\lambda,\nu}|^2 \ . \tag{4}$$

According to the definition of Tomogram of quantum states[13,14], $|\langle\Psi|x\rangle_{\lambda,\nu}|^2$ is just the Tomogram of $|\Psi\rangle$.

Based on the above preliminaries, we now turn to calculate the matrix element of the density operator in $|x\rangle_{\lambda,\nu}$ representation. Considering that the state of a quantum system in a thermostat can be described by the following statistical operator

$$\rho = [1-\exp(-\beta)]\sum_{n}^{\infty}\exp(-n\beta)|n\rangle\langle n| \ , \tag{5}$$

in which $\beta = 1/kT$, and $|n\rangle$ is the Fock state, we have

$$\langle z|\rho|z'\rangle = [1-\exp(-\beta)]\exp\Big(-\frac{|z|^2}{2}-\frac{|z'|^2}{2}+z^*z'\mathrm{e}^{-\beta}\Big)\ , \tag{6}$$

where we have employed the expression

$$|z\rangle = \exp\Big(-\frac{1}{2}|z|^2\Big)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle\ , \tag{7}$$

and $\langle m|n\rangle = \delta_{mn}$. From Eq.(1) we can re-express $|x\rangle_{\lambda,\nu}$ as

$$|x\rangle_{\lambda,\nu} = [\pi(\lambda^2+\nu^2)]^{-1/4}\exp\Big[-\frac{x^2}{2(\lambda^2+\nu^2)}\Big]\sum_{n}^{\infty}\frac{H_n\Big(\frac{x}{\sqrt{\lambda^2+\nu^2}}\Big)}{\sqrt{n!}}\Bigg[\sqrt{\frac{\lambda+\mathrm{i}\nu}{2(\lambda-\mathrm{i}\nu)}}\Bigg]^n|n\rangle\ , \tag{8}$$

in which $H_n(x)$ is the Hermite polynomial, whose generating function

$$\sum_{n}^{\infty}\frac{H_n(x)}{n!}t^n = \exp(2xt-t^2) \tag{9}$$

has been utilized. From Eqs.(7) and (8) we derive

$${}_{\lambda,\nu}\langle x|z\rangle = [\pi(\lambda^2+\nu^2)]^{-1/4}\exp\Big[-\frac{|z|^2}{2}-\frac{x^2}{2(\lambda^2+\nu^2)}\Big]\exp\Big[z\frac{\sqrt{2}x}{\lambda+\mathrm{i}\nu}-z^2\frac{\lambda-\mathrm{i}\nu}{2(\lambda+\mathrm{i}\nu)}\Big]\ . \tag{10}$$

By using the completeness of the coherent states,

$$\int\frac{\mathrm{d}^2z}{\pi}|z\rangle\langle z| = 1\ , \tag{11}$$

the matrix element of the density operator in coordinate-momentum intermediate representation ${}_{\lambda,\nu}\langle x|\hat{\rho}|x'\rangle_{\lambda,\nu}$ can be presented in terms of the coherent states as

$${}_{\lambda,\nu}\langle x|\hat{\rho}|x'\rangle_{\lambda,\nu} = \int\frac{\mathrm{d}^2z_1\mathrm{d}^2z_2}{\pi^2}{}_{\lambda,\nu}\langle x|z_1\rangle\langle z_1|\hat{\rho}|z_2\rangle\langle z_2|x'\rangle_{\lambda,\nu}\ . \tag{12}$$

It then follows by substituting Eqs.(6) and (10) into Eq.(12) and after carrying out Gaussian integration that

$$\begin{aligned}{}_{\lambda,\nu}\langle x|\hat{\rho}|x'\rangle_{\lambda,\nu} &= [\pi(\lambda^2+\nu^2)]^{-1/2}\sqrt{\frac{1-\mathrm{e}^{-\beta}}{1+\mathrm{e}^{-\beta}}}\exp\Big[-\frac{x^2+x'^2+x^2\mathrm{e}^{-2\beta}+x'^2\mathrm{e}^{-2\beta}-4xx'\mathrm{e}^{-\beta}}{2(\lambda^2+\nu^2)(1-\mathrm{e}^{-2\beta})}\Big] = \\ &\sqrt{\frac{1}{[\pi(\lambda^2+\nu^2)]}\tanh\Big(\frac{\beta}{2}\Big)}\exp\Big[-\frac{x^2+x'^2}{2(\lambda^2+\nu^2)}\frac{1}{\tanh\beta}+\frac{xx'}{\lambda^2+\nu^2}\frac{1}{\sinh\beta}\Big]\ .\end{aligned} \tag{13}$$

In particular, Eq.(13) reduces to the matrix elements of the coordinate and momentum representations for $(\lambda,\nu)=(1,0)$ and $(\lambda,\nu)=(0,1)$, respectively, which are just corresponding to the results of Eqs. (25) and (26) in Ref.[1]. Up to this point, we have achieved the main goal in this paper, i.e., we unify the results of Eqs.(25) and(26) in Ref.[1].

On the other hand, we can also directly calculate ${}_{\lambda,\nu}\langle x|\hat{\rho}|x'\rangle_{\lambda,\nu}$ from Eqs.(5) and (8), i.e.,

$$\begin{aligned}{}_{\lambda,\nu}\langle x|\hat{\rho}|x'\rangle_{\lambda,\nu} &= [1-\exp(-\beta)]\sum_{n}^{\infty}\exp(-n\beta)_{\lambda,\nu}\langle x|n\rangle\langle n|x'\rangle_{\lambda,\nu} = [1-\exp(-\beta)][\pi(\lambda^2+\nu^2)]^{-1/2}\times \\ &\exp\Big[-\frac{x^2+x'^2}{2(\lambda^2+\nu^2)}\Big]\sum_{n}^{\infty}\Big(\frac{\mathrm{e}^{\beta}}{2}\Big)^n\frac{1}{n!}H_n\Big(\frac{x}{\sqrt{\lambda^2+\nu^2}}\Big)H_n\Big(\frac{x'}{\sqrt{\lambda^2+\nu^2}}\Big)\ .\end{aligned} \tag{14}$$

Comparing Eqs.(13) and (14), we have

$$\sum_{n}^{\infty}\left(\frac{\mathrm{e}^{-\beta}}{2}\right)^{n}\frac{1}{n!}H_n\left(\frac{x}{\sqrt{\lambda^2+\nu^2}}\right)H_n\left(\frac{x'}{\sqrt{\lambda^2+\nu^2}}\right)=\sqrt{\frac{1}{1-\mathrm{e}^{-2\beta}}}\exp\left[-\frac{x^2\mathrm{e}^{-2\beta}-2xx'\mathrm{e}^{-\beta}+x'^2\mathrm{e}^{-2\beta}}{(\lambda^2+\nu^2)(1-\mathrm{e}^{-2\beta})}\right],\tag{15}$$

thus a new relation on Hermite polynomials is obtained.

3 Conclusion

In summary, by virtue of the coherent states method, we have derived the density matrix of the harmonic oscillator in thermostat in coordinate-momentum intermediate representation ${}_{\lambda,\nu}\langle x|\hat{\rho}|x'\rangle_{\lambda,\nu}$. We did not recalculate the density matrix in the momentum representation by means of Fourier-transformation or direct calculation. Namely, merely through choosing different parameter values, ${}_{1,0}\langle x|\hat{\rho}|x'\rangle_{1,0}$ and ${}_{0,1}\langle x|\hat{\rho}|x'\rangle_{0,1}$, the achieved density matrix will reduce to that of the coordinate and momentum representations, respectively. Furthermore, we also adopted a direct method to calculate ${}_{\lambda,\nu}\langle x|\hat{\rho}|x'\rangle_{\lambda,\nu}$ and then obtained a fully new expression with respect to Hermite polynomials, it is significantly different from the result in Eq.(13). By comparing both of them we obtain a new relation on Hermite polynomials, which is not been seen in the literature before.

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