Angles and a Classification of Normed Spaces

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July 3, 2012

2010 AMS-classification: 46B20, 52A10
Keywords: generalized angle, normed space

Abstract

We suggest a concept of generalized 'angles' in arbitrary real normed vector spaces. We give for each real number a definition of an 'angle' by means of the shape of the unit ball. They all yield the well known Euclidean angle in the special case of real inner product spaces. With these different angles we achieve a classification of normed spaces, and we obtain a characterization of inner product spaces. Finally we consider this construction also for a generalization of normed spaces, i.e. for spaces which may have a non-convex unit ball.

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1 Introduction

In a real inner product space $(X, <.|.>)$ it is well-known that the inner product can be expressed by the norm, namely for $\vec{x}, \vec{y} \in X$, $\vec{x} \neq \vec{0} \neq \vec{y}$, we can write

$$< \vec{x} \mid \vec{y} > = \frac{1}{4} \cdot \left(\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 \right) = \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] .$$

Furthermore we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the usual Euclidean angle

$$\angle_{Euclid}(\vec{x}, \vec{y}) := \arccos \frac{< \vec{x} \mid \vec{y} >}{\|\vec{x}\| \cdot \|\vec{y}\|} = \arccos \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right) ,$$

which is defined in terms of the norm, too.

In this paper we deal with generalized real normed vector spaces. We consider vector spaces X provided with a *'weight'* or *'functional'* $\|\cdot\|$, that means we have a continuous map $\|\cdot\| : X \longrightarrow \mathbb{R}^+ \cup \{0\}$. We assume that the weights are *'absolute homogeneous'* or *'balanced'*, i.e. $\|r \cdot \vec{x}\| = |r| \cdot \|\vec{x}\|$ for $\vec{x} \in X, r \in \mathbb{R}$. We call such pairs $(X, \|\cdot\|)$ *'balancedly weighted vector spaces'*, or for short 'BW spaces' .

To avoid problems with a denumerator 0 we restrict our considerations to BW spaces which are positive definite, i.e. $\|\vec{x}\| = 0$ only for $\vec{x} = \vec{0}$.

Following the lines of an inner product we define for each real number ϱ a continuous product $< . \mid . >_\varrho$ on X. Let $\vec{x}, \vec{y}$ be two arbitrary elements of X. In the case of $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$ we set $< \vec{x} \mid \vec{y} >_\varrho := 0$, and if $\vec{x}, \vec{y} \neq \vec{0}$ (i.e. $\|\vec{x}\| \cdot \|\vec{y}\| > 0$) we define the real number

$$< \vec{x} \mid \vec{y} >_\varrho :=$$

$$\|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \cdot \left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right)^\varrho .$$

It is easy to show that such product fulfils the symmetry ($< \vec{x}|\vec{y} >_\varrho \; = \; < \vec{y}|\vec{x} >_\varrho$), the positive semidefiniteness ($< \vec{x}|\vec{x} >_\varrho \geqslant 0$), and the homogeneity ($< r \cdot \vec{x}|\vec{y} >_\varrho = r \cdot < \vec{x}|\vec{y} >_\varrho$), for $\vec{x}, \vec{y} \in X$, $r \in \mathbb{R}$.

Let us fix a number $\varrho \in \mathbb{R}$ and a positive definite BW space $(X, \|\cdot\|)$. We are able to define for two vectors $\vec{x}, \vec{y} \neq \vec{0}$ with the additional property $| < \vec{x} \mid \vec{y} >_\varrho | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|$ an 'angle' $\angle_\varrho(\vec{x}, \vec{y})$, according to the Euclidean angle in inner product spaces. Let

$$\angle_\varrho(\vec{x}, \vec{y}) := \arccos \frac{< \vec{x} \mid \vec{y} >_\varrho}{\|\vec{x}\| \cdot \|\vec{y}\|} .$$

We consider mainly those pairs $(X, \|\cdot\|)$ where the triple $(X, \|\cdot\|, < . \mid . >_\varrho)$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality, that means for all $\vec{x}, \vec{y} \in X$ we have the inequality

$$| < \vec{x} \mid \vec{y} >_\varrho | \leqslant \|\vec{x}\| \cdot \|\vec{y}\| ,$$

for a fixed real number ϱ. In this case we get that the 'ϱ*-angle*' $\angle_\varrho(\vec{x}, \vec{y})$ is defined for all $\vec{x}, \vec{y} \neq \vec{0}$, and we shall express this by

' The space $(X, \|\cdot\|)$ has the angle $\angle_\varrho$',

This new 'angle' has seven comfortable properties which are known from the Euclidean angle in inner product spaces, and it corresponds for all $\varrho \in \mathbb{R}$ to the Euclidean angle in the case that $(X, \|\cdot\|)$ already is an inner product space.

Let $(X, \|\cdot\|)$ be a real positive definite BW space with $\dim(X) > 1$. Assume that the triple $(X, \|\cdot\|, <.|.>_\varrho)$ satisfies the CSB inequality for a fixed number ϱ. Hence we are able to define the ϱ-angle, and for elements $\vec{x}, \vec{y} \neq \vec{0}$ we have the properties that

- $\angle_\varrho$ is a continuous surjective map from $(X \backslash \{\vec{0}\})^2$ onto the closed interval $[0, \pi]$,
- $\angle_\varrho(\vec{x}, \vec{x}) = 0$,
- $\angle_\varrho(-\vec{x}, \vec{x}) = \pi$,
- $\angle_\varrho(\vec{x}, \vec{y}) = \angle_\varrho(\vec{y}, \vec{x})$,
- For all $r, s > 0$ we have $\angle_\varrho(r \cdot \vec{x}, s \cdot \vec{y}) = \angle_\varrho(\vec{x}, \vec{y})$,
- $\angle_\varrho(-\vec{x}, -\vec{y}) = \angle_\varrho(\vec{x}, \vec{y})$,
- $\angle_\varrho(\vec{x}, \vec{y}) + \angle_\varrho(-\vec{x}, \vec{y}) = \pi$,

which are easy to prove.

Then we define some classes of real vector spaces. Let NORM be the class of all real normed vector spaces. For all fixed real numbers ϱ let

$$\mathsf{NORM}_\varrho \;:=\; \{(X, \|\cdot\|) \in \mathsf{NORM} \mid \text{ The normed space } (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}\,.$$

We prove the statements

$$\mathsf{NORM} \;=\; \mathsf{NORM}_\varrho$$

for all real numbers ϱ from the closed interval $[-1, 1]$, and also

$$\mathsf{IPspace} \;=\; \bigcap_{\varrho \in \mathbb{R}} \mathsf{NORM}_\varrho\;,$$

where $\mathsf{IPspace}$ denotes the class of all real inner product spaces. Furher, if we assume four positive real numbers $\alpha, \beta, \gamma, \delta$ such that

$$-\delta < -\gamma < -1 < 1 < \alpha < \beta,$$

we obtain the chain of inclusions

$$\mathsf{NORM}_{-\delta} \subset \mathsf{NORM}_{-\gamma} \subset \mathsf{NORM} \supset \mathsf{NORM}_\alpha \supset \mathsf{NORM}_\beta\,.$$

We prove the inequalities

$$\mathsf{NORM}_{-\gamma} \neq \mathsf{NORM} \neq \mathsf{NORM}_\alpha$$

and we strongly believe, but we have no proof that the inclusions $\mathsf{NORM}_{-\delta} \subset \mathsf{NORM}_{-\gamma}$ and $\mathsf{NORM}_\alpha \supset \mathsf{NORM}_\beta$ are proper.

After that we return to the more general situation. We abandon the restriction of the triangle inequality, again we consider positive definite BW spaces $(X, \|\cdot\|)$, i.e. its weights $\|\cdot\|$ have to be positive definite and absolute homogeneous only. We say *'positive definite balancedly weighted spaces'* or pdBW for the class of all such pairs, and we define for all fixed real number ϱ the class

$$\mathsf{pdBW}_\varrho \;:=\; \{(X, \|\cdot\|) \in \mathsf{pdBW} \mid \text{ The space } (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}\,.$$

We show $\mathsf{pdBW}_{\mathbf{-1}} = \mathsf{pdBW}$. Roughly speaking this means that for the angle $\angle_\varrho$ the 'best' choice is $\varrho = \mathbf{-1}$, since the angle $\angle_{\mathbf{-1}}$ is defined in every element of pdBW.

For real numbers $\alpha, \beta, \gamma, \delta$ with $-\delta < -\gamma < -1 < \alpha < \beta$ we get the inclusions

$$\mathsf{pdBW}_{-\delta} \subset \mathsf{pdBW}_{-\gamma} \subset \mathsf{pdBW} \supset \mathsf{pdBW}_{\alpha} \supset \mathsf{pdBW}_{\beta} .$$

Since $\mathsf{NORM}_{-\gamma} \neq \mathsf{NORM}$ we already know the fact $\mathsf{pdBW}_{-\gamma} \neq \mathsf{pdBW}$. Further we prove $\mathsf{pdBW} \neq \mathsf{pdBW}_{\alpha}$, and we conjecture that the inclusions $\mathsf{pdBW}_{-\delta} \subset \mathsf{pdBW}_{-\gamma}$ and also $\mathsf{pdBW}_{\alpha} \supset \mathsf{pdBW}_{\beta}$ are proper.

To prove the above statements we define and use '*convex corners*' which can occur even in normed vector spaces, and '*concave corners*' which can be vectors in BW spaces which are not normed spaces. Both expressions are mathematical descriptions of a geometric shape exactly what the names associate. For instance, the well-known normed space $(\mathbb{R}^2, \|\cdot\|_1)$ with the norm $\|(x,y)\|_1 = |x| + |y|$ has four convex corners at its unit sphere, they are just the corners of the generated square.

Further we introduce a function Υ,

$$\Upsilon : \mathsf{pdBW} \longrightarrow [-\infty, -1] \times [-1, +\infty] \ , \quad \Upsilon(X, \|\cdot\|) := (\nu, \mu) \ ,$$

which maps every real positive definite BW space to a pair of extended real numbers (ν, μ), where

$$\begin{aligned} \nu &:= \inf\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_{\varrho}\}, \quad \text{and} \\ \mu &:= \sup\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_{\varrho}\} \ . \end{aligned}$$

For an inner product space $(X, \|\cdot\|)$ we get immediately $\Upsilon(X, \|\cdot\|) = (-\infty, \infty)$.

If $(X, \|\cdot\|)$ is an arbitrary space from the class pdBW with $-\infty < \nu, \mu < \infty$, we show that the infimum and the supremum are attained, i.e.

$$\begin{aligned} \nu &= \min\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_{\varrho}\}, \quad \text{and} \\ \mu &= \max\{\varrho \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_{\varrho}\} \ . \end{aligned}$$

Let $(X, \|\cdot\|) \in \mathsf{NORM}$. We assume that $(X, \|\cdot\|)$ has a convex corner. Then we get

$$\Upsilon(X, \|\cdot\|) = (-1, 1) \ .$$

For instance, for the normed space $\big(\mathbb{R}^2, \|\cdot\|_1\big)$ we have $\Upsilon(\mathbb{R}^2, \|\cdot\|_1) = (-1, 1)$.

At the end we consider products. For two spaces $(A, \|\cdot\|_A), (B, \|\cdot\|_B) \in \mathsf{pdBW}$ we take its Cartesian product $A \times B$, and we get a set of balanced weights $\|\cdot\|_p$ on $A \times B$, for $p > 0$. For a positive number p for each element $\left(\vec{a}, \vec{b}\right) \in A \times B$ we define the non-negative number

$$\left\| \left(\vec{a}, \vec{b}\right) \right\|_p \ := \ \sqrt[p]{\|\vec{a}\|_A^p + \|\vec{b}\|_B^p} \ .$$

This makes the pair $\Big(A \times B, \|\cdot\|_p\Big)$ to an element of the class pdBW, and with this construction we finally ask two more interesting and unanswered questions.

2 General Definitions

Let $X = (X, \tau)$ be an arbitrary real topological vector space, that means that the real vector space X is provided with a topology τ such that the addition of two vectors and the multiplication with real numbers are continuous. Further let $\|\cdot\|$ denote a *positive functional* or a *weight* on X, that means that $\|\cdot\|$: $X \longrightarrow \mathbb{R}^+ \cup \{0\}$ is continuous, the non-negative real numbers $\mathbb{R}^+ \cup \{0\}$ carry the usual Euclidean topology.

We consider some conditions.

$\widehat{(1)}$: For all $r \in \mathbb{R}$ and all $\vec{x} \in X$ we have: $\|r \cdot \vec{x}\| = |r| \cdot \|\vec{x}\|$ ('absolute homogeneity'),
$\widehat{(2)}$: $\|\vec{x}\| = 0$ if and only if $\vec{x} = \vec{0}$ ('positive definiteness'),
$\widehat{(3)}$: For $\vec{x}, \vec{y} \in X$ it holds $\|\vec{x} + \vec{y}\| \leqslant \|\vec{x}\| + \|\vec{y}\|$ ('triangle inequality'),
$\widehat{(4)}$: For $\vec{x}, \vec{y} \in X$ it holds $\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2 \cdot [\|\vec{x}\|^2 + \|\vec{y}\|^2]$ ('parallelogram identity').

If $\|\cdot\|$ fulfils $\widehat{(1)}$, then we call $\|\cdot\|$ a *balanced weight* on X,
if $\|\cdot\|$ fulfils $\widehat{(1)}, \widehat{(3)}$ then $\|\cdot\|$ is called a *seminorm* on X, and
if $\|\cdot\|$ fulfils $\widehat{(1)}, \widehat{(2)}, \widehat{(3)}$ then $\|\cdot\|$ is called a *norm* on X, and
if $\|\cdot\|$ fulfils $\widehat{(1)}, \widehat{(2)}, \widehat{(3)}, \widehat{(4)}$ the pair $(X, \|\cdot\|)$ is called an *inner product space*.

According to this four cases we call the pair $(X, \|\cdot\|)$ a *balancedly weighted vector space* (or BW space), a *seminormed vector space*, a *normed vector space*, or an *inner product space* (or IP space), respectively. See also the interesting paper **[10]** where it has been shown that $\widehat{(1)}, \widehat{(2)}, \widehat{(4)}$ is sufficient to get $\widehat{(3)}$, and therefore to get an inner product space.

We shall restrict our considerations mostly to BW spaces which are positive definite, i.e. $\|\vec{x}\| = 0$ only for $\vec{x} = \vec{0}$, i.e. they satisfy $\widehat{(2)}$. That means in a pair $(X, \|\cdot\|)$ the weight $\|\cdot\|$ is always positive definite, except we say explicitly the contrary.

Remark 1. In a positive definite BW space $(X, \|\cdot\|)$ we can generate a 'distance' d by $d(\vec{x}, \vec{y}) := \|\vec{x} - \vec{y}\|$. Note that generally the pair (X, d) is not a metric space.

Now let $< . \,|\, . > : X^2 \longrightarrow \mathbb{R}$ be continuous as a map from the product space $X \times X$ into the Euclidean space $\mathbb{R}$.

We consider some conditions.

$\overline{(1)}$: For all $r \in \mathbb{R}$ and $\vec{x}, \vec{y} \in X$ it holds $< r \cdot \vec{x} \mid \vec{y} > = r \cdot < \vec{x} \mid \vec{y} >$ ('homogeneity'),
$\overline{(2)}$: For all $\vec{x}, \vec{y} \in X$ it holds $< \vec{x} \mid \vec{y} > = < \vec{y} \mid \vec{x} >$ ('symmetry'),
$\overline{(3)}$: For all $\vec{x} \in X$ we have $< \vec{x} \mid \vec{x} > \geqslant 0$ ('positive semidefiniteness'),
$\overline{(4)}$: $< \vec{x} \mid \vec{x} > = 0$ if and only if $\vec{x} = \vec{0}$ ('definiteness'),
$\overline{(5)}$: For all $\vec{x}, \vec{y}, \vec{z} \in X$ it holds $< \vec{x} \mid \vec{y} + \vec{z} > = < \vec{x} \mid \vec{y} > + < \vec{x} \mid \vec{z} >$ ('linearity').

If $< . \,|\, . >$ fulfils $\overline{(1)}, \overline{(2)}, \overline{(3)}$, we call $< . \,|\, . >$ a *homogeneous product* on X,
if $< . \,|\, . >$ fulfils $\overline{(1)}, \overline{(2)}, \overline{(3)}, \overline{(4)}, \overline{(5)}$, then $< . \,|\, . >$ is an *inner product* on X.

According to these cases we call the pair $(X, < . \,|\, . >)$ a *homogeneous product vector space*, or an *inner product space* (or IP space), respectively.

Remark 2. We use the term 'IP space' twice, but both definitions coincide: It is well-known that a norm is based on an inner product if and only if the parallelogram identity holds.

Let $\|\cdot\|$ be denote a positive functional on X. Then we define two closed subsets of X:
$\mathbf{S} := \mathbf{S}_{(X, \|\cdot\|)} := \{ \vec{x} \in X \mid \|\vec{x}\| = 1 \}$, the *unit sphere* of X,
$\mathbf{B} := \mathbf{B}_{(X, \|\cdot\|)} := \{ \vec{x} \in X \mid \|\vec{x}\| \leqslant 1 \}$, the *unit ball* of X.

Now assume that the real vector space X is provided with a positive functional $\|\cdot\|$ and a product $< . \,|\, . >$. Then the triple $(X, \|\cdot\|, < . \,|\, . >)$ satisfies the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality if and only if for all $\vec{x}, \vec{y} \in X$ we have the inequality

$$| < \vec{x} \,|\, \vec{y} > | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|.$$

Let A be an arbitrary subset of a real vector space X. Let A has the property that for arbitrary $\vec{x}, \vec{y} \in A$ and for every $0 \leqslant t \leqslant 1$ we have $t \cdot \vec{x} + (1-t) \cdot \vec{y} \in A$. Such a set A is called *convex.* The unit ball $\mathbf{B}$ in a seminormed space is convex because of the triangle inequality.

A convex set A is called *strictly convex* if and only if for each number $0 < t < 1$ it holds that the linear combination $t \cdot \vec{x} + (1-t) \cdot \vec{y}$ lies in the interior of A, for all vectors $\vec{x}, \vec{y} \in A$.

For two real numbers $a < b$ the term $[a, b]$ means the closed interval of a and b, while (a, b) means the pair of two numbers or the open interval between a and b.

3 Some Examples of Balancedly Weighted Vector Spaces

We describe some easy examples of balanced weights on the usual vector space $\mathbb{R}^2$. At first, for each $p \in \{\infty, -\infty\} \cup \mathbb{R}$ we construct a balanced weight $\|\cdot\|_p$ on $\mathbb{R}^2$. For $\vec{x} = (x, y) \in \mathbb{R}^2$ for a real number $p > 0$ we set $\|\vec{x}\|_p := \sqrt[p]{|x|^p + |y|^p}$, and for the negative real number $-p$ we define correspondingly

$$\|\vec{x}\|_{-p} := \begin{cases} \sqrt[-p]{|x|^{-p} + |y|^{-p}} & \text{if} \quad x \cdot y \neq 0 \\ 0 & \text{if} \quad x \cdot y = 0 \end{cases}$$

and let $\|\vec{x}\|_\infty := \max\{|x|\,, |y|\}$, and $\|\vec{x}\|_{-\infty} := \min\{|x|\,, |y|\}$. For convenience, we define for $p = 0$ the trivial seminorm $\|\vec{x}\|_0 = 0$ for all $\vec{x} \in \mathbb{R}^2$.

These weights $\|\cdot\|_p$ are called the *Hölder weights* on $\mathbb{R}^2$. Since $\|\cdot\|_p$ fulfils $\widehat{(1)}$, the pair $(\mathbb{R}^2, \|\cdot\|_p)$ is a BW space for each $p \in \{\infty, -\infty\} \cup \mathbb{R}$.

Remark 3. The above definition for negative real numbers $-p$ may be clearer if one notes

$$\|\vec{x}\|_{-p} \;=\; \sqrt[-p]{|x|^{-p} + |y|^{-p}} \;=\; \left[\sqrt[p]{\frac{1}{|x|^p} + \frac{1}{|y|^p}}\,\right]^{-1} \;=\; \frac{|x| \cdot |y|}{\sqrt[p]{|x|^p + |y|^p}} \qquad \text{if} \quad x \cdot y \neq 0\,.$$

For $-p < 0$ we have $\|(x, y)\|_{-p} = 0$ if and only if $x = 0$ or $y = 0$. That means (x, y) lies on one of the two axes. Furthermore, for arbitrary numbers $p \in \{\infty, -\infty\} \cup \mathbb{R}$, we have that the pair $(\mathbb{R}^2, \|\cdot\|_p)$ is a normed space if and only if $p \geqslant 1$, (the *Hölder norms*), and the weight $\|\cdot\|_p$ is positive definite if and only if $p > 0$. For $p = 2$ we get the usual and well-known Euclidean norm $\|\cdot\|_{Euclid}$.

Other interesting non-trivial examples are the following, the first is not positive definite. Let $(\mathbb{R}^2, \|\cdot\|_A)$ be a BW space with the unit sphere $\mathbf{S}_A$, the set of unit vectors, and define

$$\mathbf{S}_A \;:=\; \big\{(x, y) \in \mathbb{R}^2 \mid |x| \cdot |y| = 1\big\},$$

and extend the weight $\|\cdot\|_A$ by homogeneity. Every point on the axes has the weight zero.

The next two examples are positive definite, but weird. At first we define a weight $\|\cdot\|_B$ on $\mathbb{R}^2$ by defining the unit sphere $\mathbf{S}_B$,

$$\mathbf{S}_B \;:=\; \Big\{(x, y) \in \mathbb{R}^2 \mid \sqrt{|x|^2 + |y|^2} = 1 \;\wedge (x, y) \notin \{(1, 0), (-1.0)\}\Big\} \;\bigcup\; \{(2, 0), (-2, 0)\},$$

and we extend the weight $\|\cdot\|_B$ by homogeneity.

In a similar way the next weight $\|\cdot\|_C$ is constructed by fixing the unit sphere $\mathbf{S}_C$,

$$\mathbf{S}_C \;:=\; \Big\{(x, y) \in \mathbb{R}^2 \mid \sqrt{|x|^2 + |y|^2} = 1 \;\wedge (x, y) \notin \{(1, 0), (-1.0)\}\Big\} \;\bigcup\; \big\{\big(\tfrac{1}{2}, 0\big)\,, \big(-\tfrac{1}{2}, 0\big)\big\},$$

and extending the weight $\|\cdot\|_C$ by homogeneity.

The pairs $\big(\mathbb{R}^2, \|\cdot\|_B\big)$ and $\big(\mathbb{R}^2, \|\cdot\|_C\big)$ are positive definite BW spaces.

Some definitions and discussions to separate these strange examples $\big(\mathbb{R}^2, \|\cdot\|_B\big)$ and $\big(\mathbb{R}^2, \|\cdot\|_C\big)$ from the others would be desirable.

4 On Angle Spaces

In the usual Euclidean plane $\mathbb{R}^2$ angles are considered for more than 2000 years. With the idea of 'metrics' and 'norms' others than the Euclidean one the idea came to have also orthogonality and angles in generalized metric and normed spaces, respectively. The first attempt to define a concept of generalized 'angles' on metric spaces was made by Menger **[5]**, p.749 . Since then a few ideas have been developed, see the references **[2]**, **[3]**, **[4]**, **[6]**, **[7]**, **[8]**, **[12]**, **[14]**, **[15]** . In this paper we focus our attention on real normed spaces as a generalizitation of real inner product spaces. Let $(X, <\,.\,|\,.\,>)$ be an IP space, and let $\|\cdot\|$ be the associated norm, $\|\vec{x}\| := \sqrt{<\vec{x}|\vec{x}>}$, then the triple $(X, \|\cdot\|, <\,.\,|\,.\,>)$ fulfils the CSB inequality, and we have for all $\vec{x}, \vec{y} \neq \vec{0}$ the well-known Euclidean angle $\angle_{Euclid}(\vec{x}, \vec{y}) := \arccos \frac{<\vec{x}\,|\,\vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|}$ with all its comfortable properties (`An` 1) - (`An` 11).

Definition 1. Let $(X, \|\cdot\|)$ be a real positive definite BW space. We call the triple $(X, \|\cdot\|, \angle_X)$ an *angle space* if and only if the following seven conditions (`An` 1) $-$ (`An` 7) are fulfilled. We regard vectors $\vec{x}, \vec{y} \neq \vec{0}$.

- (`An` 1) $\angle_X$ is a continuous map from $(X \backslash \{\vec{0}\})^2$ into the interval $[0, \pi]$. We say that '$(X, \|\cdot\|)$ has the angle $\angle_X$'.
- (`An` 2) For all $\vec{x}$ we have $\angle_X(\vec{x}, \vec{x}) = 0$.
- (`An` 3) For all $\vec{x}$ we have $\angle_X(-\vec{x}, \vec{x}) = \pi$.
- (`An` 4) For all $\vec{x}, \vec{y}$ we have $\angle_X(\vec{x}, \vec{y}) = \angle_X(\vec{y}, \vec{x})$.
- (`An` 5) For all $\vec{x}, \vec{y}$ and all $r, s > 0$ we have $\angle_X(r \cdot \vec{x}, s \cdot \vec{y}) = \angle_X(\vec{x}, \vec{y})$.
- (`An` 6) For all $\vec{x}, \vec{y}$ we have $\angle_X(-\vec{x}, -\vec{y}) = \angle_X(\vec{x}, \vec{y})$.
- (`An` 7) For all $\vec{x}, \vec{y}$ we have $\angle_X(\vec{x}, \vec{y}) + \angle_X(-\vec{x}, \vec{y}) = \pi$.

Furthermore we write down some more properties which seem to us 'desirable', but 'not absolutely necessary'.

- (`An` 8) For all $\vec{x}, \vec{y},\ \vec{x} + \vec{y} \in X \backslash \{\vec{0}\}$ we have $\angle_X(\vec{x}, \vec{x} + \vec{y}) + \angle_X(\vec{x} + \vec{y}, \vec{y}) = \angle_X(\vec{x}, \vec{y})$.
- (`An` 9) For all $\vec{x}, \vec{y},\ \vec{x} - \vec{y} \in X \backslash \{\vec{0}\}$ we have $\angle_X(\vec{x}, \vec{y}) + \angle_X(-\vec{x}, \vec{y} - \vec{x}) + \angle_X(-\vec{y}, \vec{x} - \vec{y}) = \pi$.
- (`An` 10) For all $\vec{x}, \vec{y},\ \vec{x} - \vec{y} \in X \backslash \{\vec{0}\}$ we have $\angle_X(\vec{y}, \vec{y} - \vec{x}) + \angle_X(\vec{x}, \vec{x} - \vec{y}) = \angle_X(-\vec{x}, \vec{y})$.
- (`An` 11) For any two linear independent vectors $\vec{x}, \vec{y} \in X$ we have a decreasing homeomorphism
$$\Theta : \mathbb{R} \longrightarrow (0, \pi), \quad t \mapsto \angle_X(\vec{x}, \vec{y} + t \cdot \vec{x}).$$

Remark 4. We add another condition. If $(Y, \|\cdot\|)$ is an IP space, and if the triple $(Y, \|\cdot\|, \angle_Y)$ is an 'angle space' , then it should hold that $\angle_Y = \angle_{Euclid}$, i.e. the new angle should be a generalization of the Euclidean angle.

5 An Infinite Set of Angles

Now assume that the real topological vector space (X,τ) is provided with a positive functional $\|\cdot\|$ and a product $<.\,|\,.>$. Take two elements $\vec{x},\vec{y}\in X$ with the two properties $\|\vec{x}\|\cdot\|\vec{y}\|\neq 0$ and $|<\vec{x}\,|\,\vec{y}>|\leqslant\|\vec{x}\|\cdot\|\vec{y}\|$. Then we could define an angle between these two elements, $\angle(\vec{x},\vec{y}) := \arccos\frac{<\vec{x}\,|\,\vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|}$. If the triple $(X,\|\cdot\|,<.\,|\,.>)$ satisfy the *Cauchy-Schwarz-Bunjakowsky Inequality* or CSB inequality, then we would be able to define for all $\vec{x},\vec{y}\in X$ with $\|\vec{x}\|\cdot\|\vec{y}\|\neq 0$ this angle $\angle(\vec{x},\vec{y}) := \arccos\frac{<\vec{x}\,|\,\vec{y}>}{\|\vec{x}\|\cdot\|\vec{y}\|}\in[0,\pi]$.

Let the pair $(X,\|\cdot\|)$ be a real BW space, i.e. the weight $\|\cdot\|$ is absolute homogeneous, or *'balanced'*. Notice again that we only deal with positive definite weights $\|\cdot\|$.

Definition 2. Let ϱ be an arbitrary real number. We define a continuous product $<.\,|\,.>_\varrho$ on X. Let $\vec{x},\vec{y}$ be two arbitrary elements of X. In the case of $\vec{x}=\vec{0}$ or $\vec{y}=\vec{0}$ we set $<\vec{x}\,|\,\vec{y}>_\varrho := 0$, and if $\vec{x},\vec{y}\neq\vec{0}$ (i.e. $\|\vec{x}\|\cdot\|\vec{y}\|\neq 0$) we define the real number $<\vec{x}\,|\,\vec{y}>_\varrho :=$

$$\|\vec{x}\|\cdot\|\vec{y}\|\cdot\frac{1}{4}\cdot\left[\left\|\frac{\vec{x}}{\|\vec{x}\|}+\frac{\vec{y}}{\|\vec{y}\|}\right\|^2-\left\|\frac{\vec{x}}{\|\vec{x}\|}-\frac{\vec{y}}{\|\vec{y}\|}\right\|^2\right]\cdot\left(\frac{1}{4}\cdot\left[\left\|\frac{\vec{x}}{\|\vec{x}\|}+\frac{\vec{y}}{\|\vec{y}\|}\right\|^2+\left\|\frac{\vec{x}}{\|\vec{x}\|}-\frac{\vec{y}}{\|\vec{y}\|}\right\|^2\right]\right)^\varrho \tag{1}$$

□

For the coming discussions it is very useful to introduce some abbreviations. We define for arbitrary vectors $\vec{x},\vec{y}\neq\vec{0}$ (hence $\|\vec{x}\|\cdot\|\vec{y}\|\neq 0$) two non-negative real numbers $\mathbf{s}$ and $\mathbf{d}$,

$$\mathbf{s} := \mathbf{s}(\vec{x},\vec{y}) := \left\|\frac{\vec{x}}{\|\vec{x}\|}+\frac{\vec{y}}{\|\vec{y}\|}\right\|, \quad\text{and}\quad \mathbf{d} := \mathbf{d}(\vec{x},\vec{y}) := \left\|\frac{\vec{x}}{\|\vec{x}\|}-\frac{\vec{y}}{\|\vec{y}\|}\right\|,$$

and also two real numbers $\mathbf{\Sigma}$ and $\mathbf{\Delta}$, the latter can be negative,

$$\mathbf{\Sigma} := \mathbf{\Sigma}(\vec{x},\vec{y}) := \mathbf{s}^2+\mathbf{d}^2 = \left\|\tfrac{\vec{x}}{\|\vec{x}\|}+\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^2+\left\|\tfrac{\vec{x}}{\|\vec{x}\|}-\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^2,$$

and

$$\mathbf{\Delta} := \mathbf{\Delta}(\vec{x},\vec{y}) := \mathbf{s}^2-\mathbf{d}^2 = \left\|\tfrac{\vec{x}}{\|\vec{x}\|}+\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^2-\left\|\tfrac{\vec{x}}{\|\vec{x}\|}-\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^2.$$

All defined four variables depend on two vectors $\vec{x},\vec{y}\neq\vec{0}$. Since $(X,\|\cdot\|)$ is positive definite, $\mathbf{\Sigma}$ must be a positive number. Note the inequality $0\leqslant|\mathbf{\Delta}|\leqslant\mathbf{\Sigma}$.

With this abbreviations the above formula of the product $<\vec{x}\,|\,\vec{y}>_\varrho$ is shortened to

$$<\vec{x}\,|\,\vec{y}>_\varrho = \begin{cases} 0 & \text{for } \vec{x}=\vec{0} \text{ or } \vec{y}=\vec{0} \\ \|\vec{x}\|\cdot\|\vec{y}\|\cdot\frac{1}{4}\cdot\mathbf{\Delta}\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^\varrho & \text{for } \vec{x},\vec{y}\neq\vec{0}, \end{cases} \qquad \text{for all } \varrho\in\mathbb{R}.$$

Lemma 1. *In the case that* $(X,\|\cdot\|)$ *is already an* IP *space with the inner product* $<.\,|\,.>_{IP}$*, then our product corresponds to the inner product, i.e. for all* $\vec{x},\vec{y}\in X$ *we have*

$$<\vec{x}\,|\,\vec{y}>_{IP} = <\vec{x}\,|\,\vec{y}>_\varrho \qquad \textit{for all } \varrho\in\mathbb{R}. \tag{2}$$

Proof. Let the normed space $(X,\|\cdot\|)$ be an inner product space or 'IP space'. For two elements $\vec{x},\vec{y}\in X$ we can express its inner product by its norms, i.e. we have

$$<\vec{x}\,|\,\vec{y}>_{IP} = \frac{1}{4}\cdot\left(\|\vec{x}+\vec{y}\|^2-\|\vec{x}-\vec{y}\|^2\right),$$

and by the homogeneity and the symmetry of the inner product we can write for $\vec{x},\vec{y}\neq\vec{0}$

$$<\vec{x}\,|\,\vec{y}>_{IP} = \|\vec{x}\|\cdot\|\vec{y}\|\cdot<\frac{\vec{x}}{\|\vec{x}\|}\,|\,\frac{\vec{y}}{\|\vec{y}\|}>_{IP} = \|\vec{x}\|\cdot\|\vec{y}\|\cdot\frac{1}{4}\cdot\left[\left\|\frac{\vec{x}}{\|\vec{x}\|}+\frac{\vec{y}}{\|\vec{y}\|}\right\|^2-\left\|\frac{\vec{x}}{\|\vec{x}\|}-\frac{\vec{y}}{\|\vec{y}\|}\right\|^2\right].$$

This shows the equation $< \vec{x}\,|\,\vec{y} >_{IP} = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta}$. Further, in inner product spaces the parallelogram identity holds, that means for unit vectors $\vec{v}$ and $\vec{w}$ we have

$$\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 \;=\; 2 \cdot \left(\|\vec{v}\|^2 + \|\vec{w}\|^2\right) \;=\; 4\;.$$

Then it follows for the unit vectors $\frac{\vec{x}}{\|\vec{x}\|}$ and $\frac{\vec{y}}{\|\vec{y}\|}$

$$\frac{1}{4} \cdot \left(\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right) \;=\; \frac{1}{4} \cdot (\mathbf{s}^2 + \mathbf{d}^2) \;=\; \frac{1}{4} \cdot \mathbf{\Sigma} \;=\; 1\;,$$

and the lemma is proven. □

Lemma 2. *For a positive definite* BW *space* $(X, \|\cdot\|\,)$ *the pair* $(\,X, < .\,|\,. >_\varrho\,)$ *is a homogeneous product vector space, with* $\|\vec{x}\| = \sqrt{< \vec{x}\,|\,\vec{x} >_\varrho}$, *for all* $\vec{x} \in X$ *and for all real numbers* ϱ.

Proof. We have $< .\,|\,. >_\varrho\colon X^2 \longrightarrow \mathbb{R}$, and the properties $\overline{(2)}$ (symmetry) and $\overline{(3)}$ (positive semidefiniteness) are rather trivial. Clearly, $\|\vec{x}\| = \sqrt{< \vec{x}\,|\,\vec{x} >_\varrho}$ for all $\vec{x} \in X$. We show $\overline{(1)}$, the homogeneity. For a real number $r > 0$ holds that $< r \cdot \vec{x} \mid \vec{y} >_\varrho = r \cdot < \vec{x} \mid \vec{y} >_\varrho$, because $(X, \|\cdot\|\,)$ satisfies $\widehat{(1)}$. Now we prove $< -\vec{x} \mid \vec{y} >_\varrho = \; - < \vec{x} \mid \vec{y} >_\varrho$. Let $\vec{x}, \vec{y} \neq \vec{0}$. Note that the factor $\mathbf{\Sigma}$ is not affected by a negative sign at $\vec{x}$ or $\vec{y}$. We have

$$\begin{aligned} - < \vec{x} \mid \vec{y} >_\varrho \;&=\; -\frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma} \right)^\varrho, \text{ and} \\ < -\vec{x} \mid \vec{y} >_\varrho \;&=\; \frac{1}{4} \cdot \|-\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{-\vec{x}}{\|-\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{-\vec{x}}{\|-\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma} \right)^\varrho \\ &=\; \frac{1}{4} \cdot \|\vec{x}\| \cdot \|\vec{y}\| \cdot \left[\left\| \frac{\vec{y}}{\|\vec{y}\|} - \frac{\vec{x}}{\|\vec{x}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma} \right)^\varrho, \end{aligned}$$

hence $< -\vec{x} \mid \vec{y} >_\varrho = \; - < \vec{x} \mid \vec{y} >_\varrho$. Then easily follows also for every real number $r < 0$ that $< r \cdot \vec{x} \mid \vec{y} >_\varrho = r \cdot < \vec{x} \mid \vec{y} >_\varrho$, and the homogeneity $\overline{(1)}$ is proven. □

Definition 3. Let ϱ be a real number. For positive definite BW spaces $(X, \|\cdot\|)$ for two elements $\vec{x}, \vec{y} \in X \backslash \{\vec{0}\}$ with the additional property $| < \vec{x} \mid \vec{y} >_\varrho | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|$, or equivalently, $\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho \leqslant 1$, we define the '$\varrho$-angle' $\angle_\varrho(\vec{x}, \vec{y})$. Let

$$\angle_\varrho(\vec{x}, \vec{y}) \;:=\; \arccos \tfrac{< \vec{x} \mid \vec{y} >_\varrho}{\|\vec{x}\| \cdot \|\vec{y}\|} \;=\; \arccos\left(\tfrac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\tfrac{1}{4} \cdot \mathbf{\Sigma}\right)^\varrho\,\right) \;=$$
$$\arccos\left(\tfrac{1}{4} \cdot \left[\left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \cdot \left\langle \tfrac{1}{4} \cdot \left[\left\| \tfrac{\vec{x}}{\|\vec{x}\|} + \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \tfrac{\vec{x}}{\|\vec{x}\|} - \tfrac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right\rangle^\varrho \right).$$

Proposition 1. *Let* $(X, < .\,|\,. >_{IP})$ *be an* IP *space with the inner product* $< .\,|\,. >_{IP}$ *and the generated norm* $\|\cdot\|$. *Then the triple* $(X, \|\cdot\|, < .\,|\,. >_\varrho)$ *fulfils the* CSB *inequality, and we have that the well-known Euclidean angle corresponds to the* ϱ*-angle, i.e. for all* $\vec{x}, \vec{y} \neq \vec{0}$ *it holds* $\angle_\varrho(\vec{x}, \vec{y}) = \angle_{Euclid}(\vec{x}, \vec{y})$, *for every real number* ϱ.

Proof. In Lemma 1 it was shown that $< .\,|\,. >_{IP} = < .\,|\,. >_\varrho$, for all real numbers ϱ. □

Lemma 3. *For positive definite* BW *spaces* $(X, \|\cdot\|)$ *for any element* $\vec{x} \in X$, $\vec{x} \neq \vec{0}$ *(i.e.* $\|\vec{x}\| > 0$*) the 'angles'* $\angle_\varrho(\vec{x}, \vec{x})$ *and* $\angle_\varrho(\vec{x}, -\vec{x})$ *always exist, with* $\angle_\varrho(\vec{x}, \vec{x}) = 0$ *and* $\angle_\varrho(\vec{x}, -\vec{x}) = \pi$. *That means* (An 2) *and* (An 3) *from Definition 1 is fulfilled, for every number* $\varrho \in \mathbb{R}$.

Proof. Trivial if we use that $\| \cdot \|$ is balanced and $\|\vec{0}\| = 0$. $\square$

Now the reader should take a short look on (An 4) - (An 7) from Definition 1 to prepare the following proposition.

Proposition 2. *Assume a positive definite* BW *space* $(X, \| \cdot \|)$ *and two fixed vectors* $\vec{x}, \vec{y} \in X$, *such that the* ϱ*-angle* $\angle_\varrho(\vec{x}, \vec{y})$ *is defined, for a fixed number* ϱ. *Then the following* ϱ*-angles are also defined, and we have*

- *(a)* $\angle_\varrho(\vec{x}, \vec{y}) = \angle_\varrho(\vec{y}, \vec{x})$.
- *(b)* $\angle_\varrho(r \cdot \vec{x}, s \cdot \vec{y}) = \angle_\varrho(\vec{x}, \vec{y})$, *for all positive real numbers* r, s.
- *(c)* $\angle_\varrho(-\vec{x}, -\vec{y}) = \angle_\varrho(\vec{x}, \vec{y})$.
- *(d)* $\angle_\varrho(\vec{x}, \vec{y}) + \angle_\varrho(-\vec{x}, \vec{y}) = \pi$.

Proof. Easy. We defined $\angle_\varrho(\vec{x}, \vec{y}) = \arccos \frac{<\vec{x}\,|\,\vec{y}>_\varrho}{\|\vec{x}\| \cdot \|\vec{y}\|}$, and in Lemma 2 we proved that the space $(X, < . \,|\, . >_\varrho)$ is a homogeneous product vector space. Then (a) is true since $< . \,|\, . >_\varrho$ is symmetrical. We have (b) and (c) because the product is homogeneous, i.e. $< r \cdot \vec{x} \,|\, s \cdot \vec{y} >_\varrho = r \cdot s \cdot < \vec{x} \,|\, \vec{y} >_\varrho$, for all $r, s \in \mathbb{R}$ as well as $\|r \cdot \vec{y}\| = |r| \cdot \|\vec{y}\|$, for real numbers r. And (d) follows because $\arccos(v) + \arccos(-v) = \pi$, for all v from the interval $[-1, 1]$. $\square$

We consider three special cases, let us take ϱ from the set $\{1, 0, -1\}$. For vectors $\vec{x}, \vec{y} \neq \vec{0}$ of a positive definite BW space $(X, \| \cdot \|)$ let us assume $| < \vec{x} \,|\, \vec{y} >_{\mathbf{1}} | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|$ or $| < \vec{x} \,|\, \vec{y} >_{\mathbf{0}} | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|$ or $| < \vec{x} \,|\, \vec{y} >_{\mathbf{-1}} | \leqslant \|\vec{x}\| \cdot \|\vec{y}\|$, respectively. In Definition 3 we defined the 'ϱ-angle' $\angle_\varrho$, including the cases $\angle_{\mathbf{1}}, \angle_{\mathbf{0}}, \angle_{\mathbf{-1}}$. We have

$$
\begin{aligned}
\angle_{\varrho=\mathbf{1}}(\vec{x}, \vec{y}) &= \arccos\left(\frac{1}{16} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^4 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^4 \right] \right) = \arccos\left(\frac{1}{16} \cdot \left(\mathbf{s}^4 - \mathbf{d}^4\right) \right), \\
\angle_{\varrho=\mathbf{0}}(\vec{x}, \vec{y}) &= \arccos\left(\frac{1}{4} \cdot \left[\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 \right] \right) = \arccos\left(\frac{1}{4} \cdot \mathbf{\Delta} \right), \\
\angle_{\varrho=\mathbf{-1}}(\vec{x}, \vec{y}) &= \arccos\left(\frac{\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 - \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2}{\left\| \frac{\vec{x}}{\|\vec{x}\|} + \frac{\vec{y}}{\|\vec{y}\|} \right\|^2 + \left\| \frac{\vec{x}}{\|\vec{x}\|} - \frac{\vec{y}}{\|\vec{y}\|} \right\|^2} \right) = \arccos\left(\frac{\mathbf{s}^2 - \mathbf{d}^2}{\mathbf{s}^2 + \mathbf{d}^2} \right) = \arccos\left(\frac{\mathbf{\Delta}}{\mathbf{\Sigma}} \right).
\end{aligned}
$$

Remark 5. The angle $\angle_{\mathbf{0}}$ reflects the fact that the cosine of an inner angle in a rhombus with the side lenght 1 can be expressed as the fourth part of the difference of the squares of the two diagonals, while $\angle_{\mathbf{-1}}$ means that the cosine of an inner angle in a rhombus with the side lenght 1 is the difference of the squares of the two diagonals divided by its sum.

Proposition 3. *For this proposition let* $(X, \| \cdot \|)$ *be a real normed vector space, and we consider vectors* $\vec{x}, \vec{y} \in X \backslash \{\vec{0}\}$. *Let us take* ϱ *from the set* $\{1, 0, -1\}$. *We have*
(a) The triple $(X, \| \cdot \|, < . | . >_\varrho)$ *fulfils the* CSB *inequality, hence the '*ϱ*- angle'* $\angle_\varrho(\vec{x}, \vec{y})$ *is defined for all* $\vec{x}, \vec{y} \neq \vec{0}$. *It means that the space* $(X, \| \cdot \|)$ *has the angle* $\angle_\varrho$, *for* $\varrho \in \{1, 0, -1\}$.
(b) The triple $(X, \| \cdot \|, \angle_\varrho)$ *fulfils all seven demands* (An *1)* - (An *7). Hence* $(X, \| \cdot \|, \angle_\varrho)$ *is an angle space as it was defined in* Definition 1.
(c) The triple $(X, \| \cdot \|, \angle_\varrho)$ *generally does not fulfil* (An *8),* (An *9),* (An *10).*
(d) In the special case of $\varrho = 0$ *the triple* $(X, \| \cdot \|, \angle_{\mathbf{0}})$ *fulfils* (An *11).*

Proof. (a) We show the CSB inequality for $\varrho = 1$. If $(X, \|\cdot\|)$ is a normed vector space, then because of the triangle inequality and $\left\|\frac{\vec{x}}{\|\vec{x}\|}\right\| = 1$ we get that

$$|< \vec{x} \mid \vec{y} >_{\mathbf{1}}| \;=\; \left|\tfrac{1}{16}\cdot\|\vec{x}\|\cdot\|\vec{y}\|\cdot\left[\;\left\|\tfrac{\vec{x}}{\|\vec{x}\|}+\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4 \;-\; \left\|\tfrac{\vec{x}}{\|\vec{x}\|}-\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4\;\right]\right|$$

$$\leqslant\; \tfrac{1}{16}\cdot\|\vec{x}\|\cdot\|\vec{y}\|\cdot\max\left\{\left\|\tfrac{\vec{x}}{\|\vec{x}\|}+\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4, \left\|\tfrac{\vec{x}}{\|\vec{x}\|}-\tfrac{\vec{y}}{\|\vec{y}\|}\right\|^4\right\} \leqslant \tfrac{1}{16}\cdot\|\vec{x}\|\cdot\|\vec{y}\|\cdot 2^4 \;=\; \|\vec{x}\|\cdot\|\vec{y}\| \;.$$

The same way works with $\varrho = 0$, and for $\varrho = -1$ the CSB inequality is obvious.
(b) The demands (An 2), (An 3) are shown in Lemma 3. The map $\angle_\varrho : [X\backslash\{\vec{0}\}]^2 \longrightarrow [0,\pi]$ is continuous, hence (An 1) is fulfilled. For (An 4), (An 5), (An 6) and (An 7) see Proposition 2.
(c) We repeat counterexamples from the online publication **[13]**. Recall the pairs $(\mathbb{R}^2, \|\cdot\|_p)$, with the *Hölder weights* $\|\cdot\|_p$, $p > 0$, we have defined $\|(x_1|x_2)\|_p := \sqrt[p]{|x_1|^p + |x_2|^p}$. The pairs $(\mathbb{R}^2, \|\cdot\|_p)$ are normed spaces if and only if $p \geqslant 1$. For $p = 2$ we get the usual Euclidean norm. Let us take, for instance, $p = 1$, because it is easy to calculate with. Let $\vec{x} := (1|0)$, $\vec{y} := (0|1)$, both vectors have a Hölder weight $\|\cdot\|_1 = 1$. We choose $\varrho := 0$. Then we have

$$\begin{aligned}
\angle_{\mathbf{0}}(\vec{x},\vec{y}) \;&=\; \arccos\left(\frac{1}{4}\cdot\left[\;\left\|\frac{\vec{x}}{\|\vec{x}\|_1}+\frac{\vec{y}}{\|\vec{y}\|_1}\right\|_1^2 \;-\; \left\|\frac{\vec{x}}{\|\vec{x}\|_1}-\frac{\vec{y}}{\|\vec{y}\|_1}\right\|_1^2\;\right]\right)\\
&=\; \arccos\left(\frac{1}{4}\cdot\left[\;\|(1|0)+(0|1)\|_1^2 \;-\; \|(1|0)-(0|1)\|_1^2\;\right]\right)\\
&=\; \arccos\left(\frac{1}{4}\cdot[\;4\;-\;4\;]\right) \;=\; \arccos(0) \;=\; \pi/2 \;=\; 90\,\mathrm{deg}\;,\\
\angle_{\mathbf{0}}(\vec{x},\vec{x}+\vec{y}) \;&=\; \arccos\left(\frac{1}{4}\cdot\left[\;\left\|(1|0)+\frac{1}{2}\cdot(1|1)\right\|_1^2 \;-\; \left\|(1|0)-\frac{1}{2}\cdot(1|1)\right\|_1^2\;\right]\right)\\
&=\; \arccos\left(\frac{1}{4}\cdot\left[\;(2)^2\;-\;(1)^2\;\right]\right) = \arccos\left(\frac{3}{4}\right) \;\approx\; 41.41\,\mathrm{deg}\,.
\end{aligned}$$

With similar calculations, we get $\angle_{\mathbf{0}}(\vec{x}+\vec{y},\vec{y}) \;=\; \arccos\left(\frac{3}{4}\right)$, hence $\angle_{\mathbf{0}}(\vec{x},\vec{x}+\vec{y}) \;+\; \angle_{\mathbf{0}}(\vec{x}+\vec{y},\vec{y}) \;\neq\; \angle_{\mathbf{0}}(\vec{x},\vec{y})$, and that contradicts (An 8).

The property (An 9) means that the sum of the inner angles of a triangle is π. We can use the same example of the normed space $(\mathbb{R}^2, \|\cdot\|_1)$ with unit vectors $\vec{x} = (1|0)$, and $\vec{y} = (0|1)$. Again we get $\angle_{\mathbf{0}}(\vec{x},\vec{y}) = \pi/2$, $\angle_{\mathbf{0}}(-\vec{x},\vec{y}-\vec{x}) = \angle_{\mathbf{0}}(-\vec{y},\vec{x}-\vec{y}) = \arccos\left(\frac{3}{4}\right)$, hence $\angle_{\mathbf{0}}(\vec{x},\vec{y}) + \angle_{\mathbf{0}}(-\vec{x},\vec{y}-\vec{x}) + \angle_{\mathbf{0}}(-\vec{y},\vec{x}-\vec{y}) < \pi$, hence (An 9) is not fulfilled.

For the condition (An 10) we use the same space and the same vectors $\vec{x} = (1|0)$, and $\vec{y} = (0|1)$. We get $\angle_{\mathbf{0}}(-\vec{x},\vec{y}) = \pi/2$, $\angle_{\mathbf{0}}(\vec{y},\vec{y}-\vec{x}) = \angle_{\mathbf{0}}(\vec{x},\vec{x}-\vec{y}) = \arccos\left(\frac{3}{4}\right)$, hence (An 10) is not fulfilled.
(d) This was shown in **[13]** on 'ArXiv'.
Now the proof of the proposition is complete. □

Remark 6. Note that one ϱ-angle was considered first by Pavle M. Miličić, see the references **[6]**, **[7]**, **[8]**, where he dealt with the case $\varrho = \mathbf{1}$. He named his angle as the 'g-angle'. In the recent paper **[9]** it is shown that the different definitions of the angle $\angle_{\mathbf{1}}$ and the 'g-angle' are equivalent at least in *quasi-inner-product spaces*. The case $\varrho = \mathbf{0}$ was introduced by the author in **[13]**. There it was called the 'Thy-angle'. In **[9]** some properties of the g-angle and the Thy-angle are compared.

6 On Classes and Corners

Now we define some classes of real BW spaces and real normed spaces.

Definition 4. Let pdBW be the class of all real positive definite BW spaces.
Let NORM be the class of all real normed vector spaces.
Let $\mathsf{IPspace}$ be the class of all real inner product spaces (or IP spaces).
For a fixed real number ϱ let

$$\begin{aligned} \mathsf{pdBW}_\varrho &:= \{(X, \|\cdot\|) \in \mathsf{pdBW} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\} \\ \mathsf{NORM}_\varrho &:= \{(X, \|\cdot\|) \in \mathsf{NORM} \mid \text{ The normed space } (X, \|\cdot\|) \text{ has the angle } \angle_\varrho\}. \end{aligned}$$

□

We have $\mathsf{IPspace} \subset \mathsf{NORM} \subset \mathsf{pdBW} \subset \mathsf{BW}$ spaces and $\mathsf{NORM}_\varrho \subset \mathsf{pdBW}_\varrho$, of course.

Proposition 4. *For all real numbers ϱ it holds that every element $(X, \|\cdot\|)$ of pdBW_ϱ is an angle space as it was defined in Definition 1.*

Proof. By definition of the class pdBW_ϱ each element $(X, \|\cdot\|)$ has the angle $\angle_\varrho$. Further, by definition of the angle $\angle_\varrho$ all seven properties (`An` 1) - (`An` 7) of Definition 1 are fulfilled. □

Proposition 5. *It holds* $\mathsf{pdBW} = \mathsf{pdBW}_{-\mathbf{1}}$ *and* $\mathsf{NORM} = \mathsf{NORM}_{-\mathbf{1}} = \mathsf{NORM}_{\mathbf{0}} = \mathsf{NORM}_{\mathbf{1}}$.

Proof. We had defined $\angle_{-\mathbf{1}}(\vec{x}, \vec{y}) = \arccos\left(\frac{\mathbf{s}^2 - \mathbf{d}^2}{\mathbf{s}^2 + \mathbf{d}^2}\right)$. By this definition this angle always exists for all $\vec{x}, \vec{y} \neq \vec{0}$. For the second claim see Proposition 3. □

Theorem 1. *Let us take four real numbers $\alpha, \beta, \gamma, \delta$ with*

$$-\delta < -\gamma < -1 < \alpha < \beta.$$

We get the inclusions $\mathsf{pdBW}_{-\delta} \subset \mathsf{pdBW}_{-\gamma} \subset \mathsf{pdBW} \supset \mathsf{pdBW}_\alpha \supset \mathsf{pdBW}_\beta$.

Proof. First we consider $-1 < \alpha < \beta$.

Let $(X, \|\cdot\|) \in \mathsf{pdBW}_\beta$. By Definition 4, for each pair of two vectors $\vec{x}, \vec{y} \neq \vec{0}$ the angle $\angle_\beta(\vec{x}, \vec{y})$ is defined. By Definition 3, this means that the triple $(X, \|\cdot\|, <.\,|\,.>_\beta)$ fulfils the CSB inequality, i.e. for any pair $\vec{x}, \vec{y} \neq \vec{0}$ of vectors we have the inequality

$$|< \vec{x} \mid \vec{y} >_\beta| \leqslant \|\vec{x}\| \cdot \|\vec{y}\| \ , \quad \text{that means} \quad \left| \|\vec{x}\| \cdot \|\vec{y}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\beta \right| \leqslant \|\vec{x}\| \cdot \|\vec{y}\| \ , \tag{3}$$

$$\text{or equivalently} \quad \left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\beta \leqslant 1 \ . \tag{4}$$

To prove that the angle $\angle_\alpha(\vec{x}, \vec{y})$ exists we have to show the corresponding inequality

$$\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\alpha \leqslant 1 \, .$$

We distinguish two cases. In the first case of $\frac{1}{4} \cdot \mathbf{\Sigma} \geqslant 1$ we have for all real numbers $\kappa \leqslant \beta$

$$\left(\frac{1}{4} \cdot \mathbf{\Sigma}\right\rangle^\kappa \leqslant \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^\beta .$$

Since $\alpha < \beta$ it follows that

$$0 \leqslant \left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\alpha} \leqslant \left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\beta} \leqslant 1,$$

and the angle $\angle_\alpha(\vec{x},\vec{y})$ exists.

For the second case we assume $\frac{1}{4}\cdot\mathbf{\Sigma} < 1$. That means for any positive exponent κ $\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\kappa} < 1$. Now note the inequality $0 \leqslant \left|\frac{1}{4}\cdot\mathbf{\Delta}\right| \leqslant \frac{1}{4}\cdot\mathbf{\Sigma} < 1$. In the subcase of a positive α it follows the inequality $\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\alpha} < 1$. and the angle $\angle_\alpha(\vec{x},\vec{y})$ exists.

If α is from the interval $[-1,0]$, i.e. $-\alpha \in [0,1]$, we can write the inequality

$$1 \geqslant \frac{|\mathbf{\Delta}|}{\mathbf{\Sigma}} = \frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\frac{1}{4}\cdot\mathbf{\Sigma}} \geqslant \frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\alpha}} \geqslant \left|\frac{1}{4}\cdot\mathbf{\Delta}\right| \ .$$

Again we get the desired inequality $\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\alpha} \leqslant 1$, and the angle $\angle_\alpha(\vec{x},\vec{y})$ exists. We get that $(X,\|\cdot\|)$ is an element of pdBW_α, too.

We look at $-\delta < -\gamma < -1$. We have $1 < \gamma < \delta$.
Let $(X,\|\cdot\|) \in \mathsf{pdBW}_{-\delta}$, and take two vectors $\vec{x},\vec{y} \in X$, $\vec{x},\vec{y} \neq \vec{0}$. The angle $\angle_{-\delta}(\vec{x},\vec{y})$ exists. Hence we have the inequality $\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\delta} \leqslant 1$.

As above we distinguish two cases. The first case is $\frac{1}{4}\cdot\mathbf{\Sigma} \geqslant 1$. We have

$$1 \geqslant \frac{|\mathbf{\Delta}|}{\mathbf{\Sigma}} = \frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\frac{1}{4}\cdot\mathbf{\Sigma}} \geqslant \frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\gamma}} \geqslant \frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\delta}} = \left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\delta} \ .$$

We get the inequality $\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|\cdot\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{-\gamma} \leqslant 1$. It follows that the angle $\angle_{-\gamma}(\vec{x},\vec{y})$ exists.
The second case is $\frac{1}{4}\cdot\mathbf{\Sigma} < 1$. We get

$$0 \leqslant \left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\delta} \leqslant \left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\gamma} \leqslant \frac{1}{4}\cdot\mathbf{\Sigma}\ , \quad \text{hence} \quad \frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\gamma}} \leqslant \frac{\left|\frac{1}{4}\cdot\mathbf{\Delta}\right|}{\left(\frac{1}{4}\cdot\mathbf{\Sigma}\right)^{\delta}} \leqslant 1\ ,$$

and the angle $\angle_{-\gamma}(\vec{x},\vec{y})$ exists. This proves $(X,\|\cdot\|) \in \mathsf{pdBW}_{-\gamma}$, and Theorem 1 is shown. $\square$

Corollary 1.

We have $\mathsf{NORM} = \mathsf{NORM}_\varrho$ *for all real numbers* ϱ *from the closed interval* $[-1,1]$.

Proof. See both the above Proposition 5 and Theorem 1. $\square$

Corollary 2. *Let us take four positive numbers* $\alpha,\beta,\gamma,\delta$ *with*
$$-\delta < -\gamma < -1 < 1 < \alpha < \beta.$$

We have $\mathsf{NORM}_{-\delta} \subset \mathsf{NORM}_{-\gamma} \subset \mathsf{NORM} \supset \mathsf{NORM}_\alpha \supset \mathsf{NORM}_\beta$.

Proof. This follows directly from Theorem 1. $\square$

Theorem 2. *We have the equality*

$$\mathsf{IPspace} \ = \bigcap_{\varrho\in\mathbb{R}} \mathsf{NORM}_\varrho \ .$$

Proof. " $\subset$ ": This is trivial with Proposition 1.

" $\supset$ ": This is not trivial, but easy. We show that a real normed space $(X, \|\cdot\|)$ which in not an inner product space is not an element of NORM_ϱ for at least one real number ϱ.

Let $(X, \|\cdot\|)$ be a real normed space which in not an inner product space. Then it must exist a two dimensional subspace U of X such that its unit sphere $\mathbf{S} \cap \mathsf{U}$ is not an ellipse. Hence there are two unit vectors $\vec{v}, \vec{w} \in \mathbf{S} \cap \mathsf{U}$ such that the parallelegram identity is not fulfilled; i.e. it holds

$$\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 \neq 4 = 2 \cdot \left[\|\vec{v}\|^2 + \|\vec{w}\|^2\right] \ .$$

(Case A): First we assume $\|\vec{v}+\vec{w}\| \neq \|\vec{v}-\vec{w}\|$, hence $\mathbf{\Delta} := \mathbf{\Delta}(\vec{v}, \vec{w}) \neq 0$. In the case of $\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 > 4$, i.e. $\frac{1}{4} \cdot \mathbf{\Sigma} > 1$, we can choose a very big number β such that

$$\left| \ \|\vec{v}\| \cdot \|\vec{w}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left\langle \frac{1}{4} \cdot \mathbf{\Sigma} \right\rangle^{\beta} \ \right| > \|\vec{v}\| \cdot \|\vec{w}\| = 1 \ ,$$

and if $|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 < 4$, i.e. $\frac{1}{4} \cdot \mathbf{\Sigma} < 1$, we can find a big γ such that

$$\left| \ \|\vec{v}\| \cdot \|\vec{w}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left\langle \frac{1}{4} \cdot \mathbf{\Sigma} \right\rangle^{-\gamma} \ \right| > \|\vec{v}\| \cdot \|\vec{w}\| = 1 \ .$$

We get that the angle $\angle_\beta(\vec{v}, \vec{w})$ or $\angle_{-\gamma}(\vec{v}, \vec{w})$, respectively, does not exist.

(Case B): If we have $\|\vec{v}+\vec{w}\| = \|\vec{v}-\vec{w}\|$, hence $\mathbf{\Delta} = 0$, we have to replace $\vec{w}$ by another unit vector $\widetilde{w}$. Note that $\{\vec{v}, \vec{w}\}$ is linear independent since $\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2 \neq 4$. We regard the continuous map $\underline{E} : \mathbb{R} \longrightarrow (-1, +1)$, we define

$$\underline{E}(t) \ := \ \frac{1}{4} \cdot \left[\ \left\| \vec{v} + \frac{\vec{w} + t \cdot \vec{v}}{\|\vec{w} + t \cdot \vec{v}\|} \right\|^2 \ - \ \left\| \vec{v} - \frac{\vec{w} + t \cdot \vec{v}}{\|\vec{w} + t \cdot \vec{v}\|} \right\|^2 \ \right] \ .$$

For $t = 0$ we get $\underline{E}(0) = \frac{1}{4} \cdot \left[\ \|\vec{v}+\vec{w}\|^2 \ - \ \|\vec{v}-\vec{w}\|^2 \ \right] = \frac{1}{4} \cdot \mathbf{\Delta} = 0$. In **[13]** on the internet platform 'arXiv' it is proven that the map $\underline{E}$ yields a homeomorphism from $\mathbb{R}$ onto the open interval $(-1, 1)$. Hence we can replace the factor $t = 0$ by any $\widetilde{t} \neq 0$ such that

$$\underline{E}(\widetilde{t}) \ = \ \frac{1}{4} \cdot \left[\ \left\| \vec{v} + \frac{\vec{w} + \widetilde{t} \cdot \vec{v}}{\|\vec{w} + \widetilde{t} \cdot \vec{v}\|} \right\|^2 \ - \ \left\| \vec{v} - \frac{\vec{w} + \widetilde{t} \cdot \vec{v}}{\|\vec{w} + \widetilde{t} \cdot \vec{v}\|} \right\|^2 \ \right] \ \neq \ 0 \ .$$

For each $\widetilde{t}$ we abbreviate the unit vector

$$\widetilde{w} := \frac{\vec{w} + \widetilde{t} \cdot \vec{v}}{\|\vec{w} + \widetilde{t} \cdot \vec{v}\|} \ ,$$

and since $\underline{E}$ is a homeomorphism we can choose a very small $\widetilde{t} \neq 0$ such that still holds $\|\vec{v}+\widetilde{w}\|^2 + \|\vec{v}-\widetilde{w}\|^2 \neq 4$, but $\mathbf{\Delta} := \mathbf{\Delta}(\vec{v}, \widetilde{w}) \neq 0$. At this point we can continue as in (Case A).

In both cases (Case A) and (Case B) it follows that $(X, \|\cdot\|)$ is not an element of the classes NORM_β or $\mathsf{NORM}_{-\gamma}$, respectively. Now the proof of Theorem 2 is finished. $\square$

We define a function Υ which maps every real positive definite BW space to a pair of extended numbers (ν, μ),

$$\Upsilon : \mathsf{pdBW} \longrightarrow [-\infty, -1] \times [-1, +\infty].$$

Definition 5. Let $(X, \|\cdot\|)$ be a positive definite balancedly weighted vector space. We define

$$\begin{aligned} \nu &:= \inf\{\kappa \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\kappa\}, \\ \mu &:= \sup\{\kappa \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\kappa\}, \\ \Upsilon(X, \|\cdot\|) &:= (\nu, \mu). \end{aligned}$$

With Theorem 1 we get that ν is from the interval $[-\infty, -1]$ and μ is from the interval $[-1, +\infty]$. If $(X, \|\cdot\|)$ is even a normed vector space we have $\mu \in [+1, +\infty]$. If $(X, \|\cdot\|)$ is even an inner product space it follows from Theorem 2 the identity $\Upsilon(X, \|\cdot\|) = (-\infty, +\infty)$.

Proposition 6. *Let $(X, \|\cdot\|) \in \mathsf{pdBW}$, i.e. $(X, \|\cdot\|)$ is a positive definite balancedly weighted vector space. We defined $\Upsilon(X, \|\cdot\|) = (\nu, \mu)$. Let us assume $\nu \neq -\infty$ and $\mu \neq \infty$. We claim that the infimum and the supremum will be attained, i.e. we claim*

$$\nu = \min\{\kappa \in \mathbb{R} \mid (X, \|\cdot\|) \textit{ has the angle } \angle_\kappa\}, \quad \mu = \max\{\kappa \in \mathbb{R} \mid (X, \|\cdot\|) \textit{ has the angle } \angle_\kappa\}.$$

Proof. We show the first claim

$$\nu = \min\{\kappa \in \mathbb{R} \mid (X, \|\cdot\|) \text{ has the angle } \angle_\kappa\}.$$

Let us assume the opposite, i.e. we assume that the angle $\angle_\nu$ does not exist in $(X, \|\cdot\|)$. This means $\nu < -1$, since $\mathsf{pdBW} = \mathsf{pdBW}_{-\mathbf{1}}$. Hence there are two unit vectors $\vec{v}, \vec{w} \in X$ such that

$$| < \vec{v} \,|\, \vec{w} >_\nu | \;=\; \frac{1}{4} \cdot |\mathbf{\Delta}(\vec{v}, \vec{w})| \cdot \left\langle \frac{1}{4} \cdot \mathbf{\Sigma}(\vec{v}, \vec{w}) \right\rangle^{\nu} \;=\; 1 + \varepsilon \quad \text{for a positive } \varepsilon\ . \tag{5}$$

We make the exponent ν ‘less negative’. Since $\nu < -1$ and $0 \leqslant |\mathbf{\Delta}| \leqslant \mathbf{\Sigma}$ it has to be $\mathbf{\Sigma} < 4$. Since $\mathsf{pdBW} = \mathsf{pdBW}_{-\mathbf{1}}$ we have $| < \vec{v} \,|\, \vec{w} >_{-1} | \leqslant 1$. By the continuity of the left hand side of the above Equation (5) we can find two positive numbers $\overline{\eta}, \overline{\lambda}$ with $\nu < \nu + \overline{\eta} < -1$ and $0 < \overline{\lambda} < \varepsilon$ such that

$$1 \;<\; | < \vec{v} \,|\, \vec{w} >_{\nu+\overline{\eta}} | \;=\; \frac{1}{4} \cdot |\mathbf{\Delta}(\vec{v}, \vec{w})| \cdot \left\langle \frac{1}{4} \cdot \mathbf{\Sigma}(\vec{v}, \vec{w}) \right\rangle^{\nu+\overline{\eta}} \;=\; 1 + \overline{\lambda} < \; 1 + \varepsilon\ . \tag{6}$$

By the continuity of the product $< . \,|\, . >$ we can choose the positive number $\overline{\eta}$ such that

$$1 \;<\; 1 + \overline{\lambda} \;\leqslant\; 1 + \lambda \;=\; | < \vec{v} \,|\, \vec{w} >_{\nu+\eta} | \;\leqslant\; 1 + \varepsilon$$

holds for all η, for $0 \leqslant \eta \leqslant \overline{\eta}$ for positive λ with $\overline{\lambda} \leqslant \lambda \leqslant \varepsilon$. We conclude that for all $0 \leqslant \eta \leqslant \overline{\eta}$ the angle $\angle_{\nu+\eta}(\vec{v}, \vec{w})$ does not exist in $(X, \|\cdot\|)$. This contradicts the definition of ν as an infimum. This proves the first claim of the proposition. ☐

For the next proposition we need the term of a *‘strictly convex normed space’*.

Definition 6. A BW space $(X, \|\cdot\|)$ is called *‘strictly convex’* if and only if the interior of the line $\{t \cdot \vec{u} + (1-t) \cdot \vec{v} \mid 0 \leqslant t \leqslant 1\}$ lies in the interior of the unit ball of $(X, \|\cdot\|)$ for each pair of distinct unit vectors $(\vec{u}, \vec{v})$, $\vec{u} \neq \vec{v}$. That means it holds

$$\|t \cdot \vec{u} + (1-t) \cdot \vec{v}\| < 1 \quad \text{for} \quad 0 < t < 1.$$

Definition 7. We call a BW space $(X, \|\cdot\|)$ *‘strictly curved’* if and only if for each pair of distinct unit vectors $(\vec{u}, \vec{v})$, $\vec{u} \neq \vec{v}$ the line $\{t \cdot \vec{u} + (1-t) \cdot \vec{v} \mid 0 \leqslant t \leqslant 1\}$ contains at least one element $\widehat{t}$ with $0 < \widehat{t} < 1$ which is not a unit vector, i.e. for $\widehat{t}$ holds

$$\left\|\widehat{t} \cdot \vec{u} + (1-\widehat{t}) \cdot \vec{v}\right\| \neq 1.$$

Note that in normed spaces both definitions are equivalent. Further, a positive definite BW space which is strictly convex is a normed space, and it is strictly curved.

Further, a BW space $(X, \|\cdot\|)$ which is not strictly curved must contain a piece of a straight line which is completely in the unit sphere of $(X, \|\cdot\|)$. As examples we can take the two Hölder norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$. The unit spheres of both spaces have the shape of a square. The Hölder weights $\|\cdot\|_p$ on $\mathbb{R}^2$ with $0 < p < 1$ yield examples of BW spaces which are strictly curved, but not strictly convex.

Proposition 7. *Let* $(X, \|\cdot\|)$ *be a real positive definite* BW *space. Let* $\Upsilon(X, \|\cdot\|) = (\nu, \mu)$. *If* $(X, \|\cdot\|)$ *is not strictly curved we have* $\mu = 1$.

Proof. Let us consider a BW space $(X, \|\cdot\|)$ which is not strictly curved. As we said above it contains a piece of a straight line which is completely in the unit sphere. This fact described in formulas means that we have two unit vectors $\vec{z}, \vec{w}$ and a positive number $0 < \mathbf{r} < 1$ such that

$$\|\vec{z} + t \cdot \vec{w}\| = 1 \quad \text{holds for all} \quad t \in [-\mathbf{r}, \mathbf{r}] \ .$$

Now we show that for each exponent $\varrho > 1$ we can find two unit vectors $\vec{x}, \vec{y}$ with the property $\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left\langle\frac{1}{4} \cdot \mathbf{\Sigma}\right\rangle^{\varrho} > 1$. That means that the ϱ-angle $\angle_{\varrho}(\vec{x}, \vec{y})$ does not exist.

Let us take the unit vectors $\vec{x} := \vec{z} + t \cdot \vec{w}$ and $\vec{y} := \vec{z} - t \cdot \vec{w}$ for $0 < t < \mathbf{r}$. With $\mathbf{\Delta} = \mathbf{\Delta}(\vec{x}, \vec{y})$ and $\mathbf{\Sigma} = \mathbf{\Sigma}(\vec{x}, \vec{y})$ we consider the desired inequality $\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left\langle\frac{1}{4} \cdot \mathbf{\Sigma}\right\rangle^{\varrho} > 1$, i.e.

$$\left|\frac{1}{4} \cdot \mathbf{\Delta}\right| \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^{\varrho} = \left|\frac{1}{4} \cdot \left(\mathbf{s}^2 - \mathbf{d}^2\right)\right| \cdot \left(\frac{1}{4} \cdot \left(\mathbf{s}^2 + \mathbf{d}^2\right)\right)^{\varrho} = \left(1 - t^2\right) \cdot \left(1 + t^2\right)^{\varrho} > 1 \ . \tag{7}$$

This is equivalent to

$$\varrho > - \frac{\log\left(1 - t^2\right)}{\log\left(1 + t^2\right)} \ . \tag{8}$$

The right hand side is greater than 1 for all $0 < t < \mathbf{r}$. By the rules of L'Hospital we get the limit

$$\lim_{t \searrow 0} \left(- \frac{\log\left(1 - t^2\right)}{\log\left(1 + t^2\right)} \right) = 1 \ .$$

This means that we can find for all $\varrho > 1$ a suitable t such that Inequality (7) is fulfilled. Hence, for each $\varrho > 1$, we are able to find a pair of unit vectors $\vec{x} = \vec{z} + t \cdot \vec{w}$ and $\vec{y} = \vec{z} - t \cdot \vec{w}$ such that the ϱ-angle $\angle_{\varrho}(\vec{x}, \vec{y})$ does not exist. Proposition 7 is proven. □

Now we introduce the concept of a *'convex corner'*. The word 'convex' seems to be superfluous in normed spaces. But later we define also something that we shall call *'concave corner'*. These can occur in BW spaces which have a non-convex unit ball. This justifies the adjective 'convex'.

Definition 8. Let the pair $(X, \|\cdot\|)$ be a BW space, let $\widehat{y} \in X$. The vector $\widehat{y}$ is called a *convex corner* if and only if there is another vector $\overline{x} \in X$ and there are two real numbers $m_- < m_+$ such that we have a pair of unit vectors for each $\delta \in [0, 1]$, we have

$$\|\delta \cdot \overline{x} + (1 + \delta \cdot m_-) \cdot \widehat{y}\| = 1 = \| - \delta \cdot \overline{x} + (1 - \delta \cdot m_+) \cdot \widehat{y}\| \ . \tag{9}$$

□

Remark 7. A convex corner is only the mathematical description of something what everybody already has in his mind. We can imagine it as an intersection of two straight lines of unit vectors which meet with an Euclidean angle of less than 180 degrees.

Note that from the definition follows $\|\widehat{y}\| = 1$ and that $\{\widehat{y}, \overline{x}\}$ is linear independent. Further note that a space with a convex corner is not strictly curved.

As examples we can take the *Hölder weights* $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$. Both spaces have four convex corners, e.g. $\left(\mathbb{R}^2, \|\cdot\|_1\right)$ has one at $(0|1)$, and $\left(\mathbb{R}^2, \|\cdot\|_\infty\right)$ has one at $(1|1)$. They are just the corners of the corresponding unit spheres, i.e. the corners of the squares.

Proposition 8. *Let $(X, \|\cdot\|)$ be a positive definite balancedly weighted vector space which has a convex corner. Let $\Upsilon(X, \|\cdot\|) = (\nu, \mu)$. We claim $\nu = -1$.*

Proof. We assume in the proposition a convex corner $\widehat{y} \in X$ and another element $\overline{x} \in X$ and two real numbers $m_- < m_+$ with the properties of Definition 8. We get with Proposition 5 the inequality $\nu \leqslant -1$. Let us fix a number $\varrho > 1$, hence $-\varrho < -1$. We want to find two vectors $\widetilde{v}, \widetilde{w} \in X$ with $| < \widetilde{v} | \widetilde{w} >_{-\varrho} | > 1$. This would mean that the angle $\angle_{-\varrho}(\widetilde{v}, \widetilde{w})$ does not exist.

We define for each $\delta \in [0,1]$ the pair of unit vectors $\vec{v}, \vec{w}$,

$$\vec{v} := \delta \cdot \overline{x} + (1 + \delta \cdot m_-) \cdot \widehat{y} \quad \text{and} \quad \vec{w} := -\delta \cdot \overline{x} + (1 - \delta \cdot m_+) \cdot \widehat{y} \ .$$

We use the abbreviations $\mathbf{\Delta} = \mathbf{\Delta}(\vec{v}, \vec{w}) = \mathbf{s}^2 - \mathbf{d}^2$ and $\mathbf{\Sigma} = \mathbf{\Sigma}(\vec{v}, \vec{w}) = \mathbf{s}^2 + \mathbf{d}^2$ from the beginning of the section 'An Infinite Set of Angles', and we compute

$$\begin{aligned}
< \vec{v} \,|\, \vec{w} >_{-\varrho} &= \ < \delta \cdot \overline{x} + (1 + \delta \cdot m_-) \cdot \widehat{y} \ | - \delta \cdot \overline{x} + (1 - \delta \cdot m_+) \cdot \widehat{y} >_{-\varrho} &(10)\\
&= \|\vec{v}\| \cdot \|\vec{w}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma} \right)^{-\varrho} &(11)\\
&= 1 \cdot 1 \cdot \frac{\frac{1}{4} \cdot \mathbf{\Delta}}{\left(\frac{1}{4} \cdot \mathbf{\Sigma}\right)^{\varrho}} = \frac{\frac{1}{4} \cdot (\mathbf{s}^2 - \mathbf{d}^2)}{\left[\frac{1}{4} \cdot (\mathbf{s}^2 + \mathbf{d}^2)\right]^{\varrho}} &(12)\\
&= \frac{\frac{1}{4} \cdot \left(\| (2 + \delta \cdot (m_- - m_+)) \cdot \widehat{y} \|^2 - \| 2 \cdot \delta \cdot \overline{x} + \delta \cdot (m_- + m_+) \cdot \widehat{y} \|^2 \right)}{\left[\frac{1}{4} \cdot (\| (2 + \delta \cdot (m_- - m_+)) \cdot \widehat{y} \|^2 + \| 2 \cdot \delta \cdot \overline{x} + \delta \cdot (m_- + m_+) \cdot \widehat{y} \|^2) \right]^{\varrho}} &(13)\\
&= \frac{\frac{1}{4} \cdot \left((2 + \delta \cdot (m_- - m_+))^2 \cdot \|\widehat{y}\|^2 \ - \ \delta^2 \cdot \| 2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y} \|^2 \right)}{\left[\frac{1}{4} \cdot \left((2 + \delta \cdot (m_- - m_+))^2 \cdot \|\widehat{y}\|^2 \ + \ \delta^2 \cdot \| 2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y} \|^2 \right) \right]^{\varrho}} \ . &(14)
\end{aligned}$$

$$\text{Hence} \quad < \vec{v} \,|\, \vec{w} >_{-\varrho} = \frac{\frac{1}{4} \cdot \mathbf{\Delta}}{\left[\frac{1}{4} \cdot \mathbf{\Sigma}\right]^{\varrho}} = \frac{1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_-}{\left[1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_+\right]^{\varrho}} \ , \tag{15}$$

if we define two real constants K_- , K_+ by setting

$$\mathsf{K}_- := (m_- - m_+)^2 - \|2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y}\|^2 \ , \quad \mathsf{K}_+ := (m_- - m_+)^2 + \|2 \cdot \overline{x} + (m_- + m_+) \cdot \widehat{y}\|^2 \ .$$

The above chain of identities holds for all $\delta \in [0,1]$. For a shorter display we abbreviate the parts of the fraction by

$$\begin{aligned}
\mathsf{T} &:= \frac{1}{4} \cdot \mathbf{\Delta} = 1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_- \ ,\\
\mathsf{B} &:= \frac{1}{4} \cdot \mathbf{\Sigma} = 1 + \delta \cdot (m_- - m_+) + \frac{1}{4} \cdot \delta^2 \cdot \mathsf{K}_+ \ .
\end{aligned}$$

Since $\mathsf{K}_- < \mathsf{K}_+$ and $m_- - m_+ < 0$ we can find a positive number s with $0 < \mathsf{s} < 1$ such that we have for all positive δ with $0 < \delta \leqslant \mathsf{s}$ the inequality

$$0 < \mathsf{T} < \mathsf{B} < 1, \quad \text{i.e.} \quad 1 < \frac{\log(\mathsf{T})}{\log(\mathsf{B})} \ . \tag{16}$$

Our aim is to find vectors $\vec{v}, \vec{w}$ such that the product $< \vec{v} | \vec{w} >_{-\varrho}$ is greater than 1. With Equation (15) this is equivalent to

$$\begin{aligned}
< \vec{v} \,|\, \vec{w} >_{-\varrho} = \frac{\mathsf{T}}{[\mathsf{B}]^{\varrho}} > 1 \quad &\Longleftrightarrow \quad \log(\mathsf{T}) > \varrho \cdot \log(\mathsf{B})\\
&\Longleftrightarrow \quad \frac{\log(\mathsf{T})}{\log(\mathsf{B})} < \varrho \ , \quad \text{note that } \log(\mathsf{B}) \text{ is negative, see (16) .}
\end{aligned}$$

By the rules of L'Hospital we get the limit

$$\lim_{\delta\searrow 0}\left(\frac{\log(\mathsf{T})}{\log(\mathsf{B})}\right) \;=\; 1$$

With Inequation (16) it follows that we can find a very small $\widetilde{\delta}$ with $0<\widetilde{\delta}<\mathsf{s}$ such that

$$1 \;<\; \frac{\log(\mathsf{T})}{\log(\mathsf{B})} \;<\; \varrho$$

is fulfilled. That means with the definition of

$$\widetilde{v} := \widetilde{\delta}\cdot\overline{x} + (1+\widetilde{\delta}\cdot m_-)\cdot\widehat{y} \quad \text{and} \quad \widetilde{w} := -\widetilde{\delta}\cdot\overline{x} + (1-\widetilde{\delta}\cdot m_+)\cdot\widehat{y}$$

we get the desired inequality $<\widetilde{v}|\widetilde{w}>_{-\varrho}\; > 1$. Hence the $-\varrho$-angle $\angle_{-\varrho}(\widetilde{v},\widetilde{w})$ does not exist. Since the variable $-\varrho$ is an arbitrary number less than -1, Proposition 8 is proven. □

Corollary 3. *Let $(X,\|\cdot\|)\in\mathsf{NORM}$. Further we assume that $(X,\|\cdot\|)$ has a convex corner. It follows $\Upsilon(X,\|\cdot\|)=(-1,1)$.*

Proof. This is a direct consequence of Proposition 7 and Proposition 8. □

Corollary 4. *It holds for the Hölder weights $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^2$*

$$\Upsilon(\mathbb{R}^2,\|\cdot\|_1) \;=\; \Upsilon(\mathbb{R}^2,\|\cdot\|_\infty) \;=\; (-1,1).$$

Now we introduce a corresponding definition of 'concave corners'. Note that they can not occur in normed spaces. In a normed space the triangle inequality $\widehat{(3)}$ holds, as a consequence its unit ball is 'everywhere' convex.

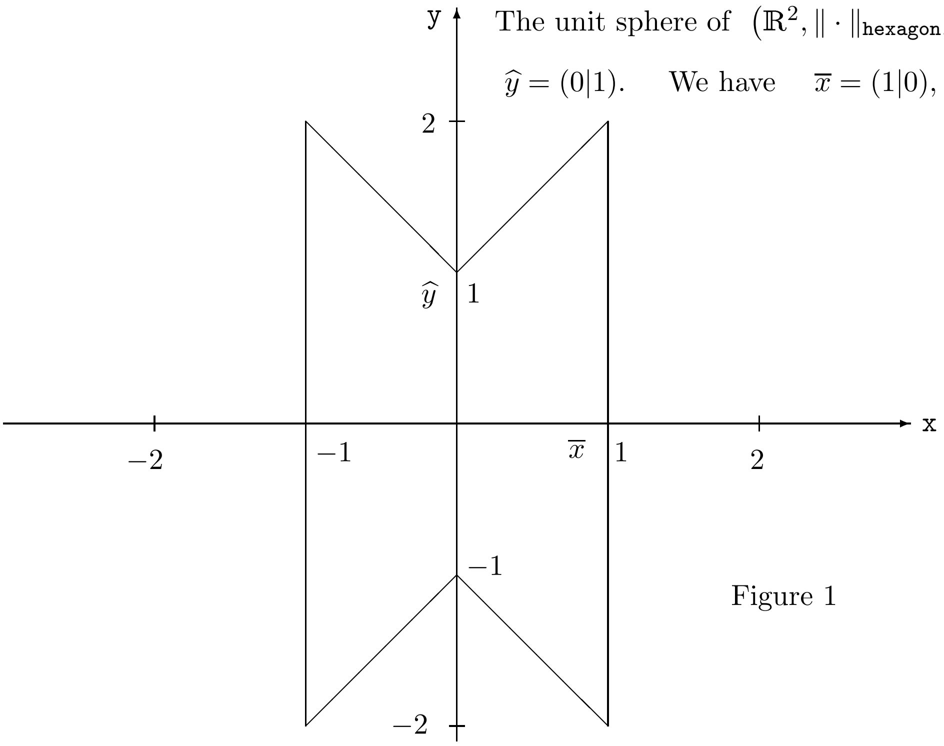
Definition 9. Let the pair $(X,\|\cdot\|)$ be a BW space, let $\widehat{y}\in X$. $\widehat{y}$ is called a *concave corner* if and only if there is an $\overline{x}\in X$, and there are two real numbers $m_- < m_+$ such that we have a pair of unit vectors for each $\delta\in[0,1]$, we have

$$\|\delta\cdot\overline{x}+(1+\delta\cdot m_+)\cdot\widehat{y}\| \;=\; 1 \;=\; \|-\delta\cdot\overline{x}+(1-\delta\cdot m_-)\cdot\widehat{y}\|\;. \tag{17}$$

□

Remark 8. Note that from the definition follows $\|\widehat{y}\|=1$ and that $\{\widehat{y},\overline{x}\}$ is linear independent. Further note that a space $(X,\|\cdot\|)$ with a concave corner contains a piece of a straight line which is completely in its unit sphere, i.e. $(X,\|\cdot\|)$ is not strictly curved.

We get a set of balanced weights on $\mathbb{R}^2$ if we define for every $r\geqslant 0$ a weight $\|\cdot\|_{\mathtt{hexagon},r}:\mathbb{R}^2\longrightarrow\mathbb{R}^+\cup\{0\}$, if we fix the unit sphere $\mathbf{S}$ of $(\mathbb{R}^2,\|\cdot\|_{\mathtt{hexagon},r})$ with the polygon through the six points $\{(0|1),(1|r),(1|-r),(0|-1),(-1|-r),(-1|r)\}$ and returning to $(0|1)$, and then extending $\|\cdot\|_{\mathtt{hexagon},r}$ by homogeneity. (See Figure 1).



The unit sphere of $\big(\mathbb{R}^2, \|\cdot\|_{\texttt{hexagon},2}\big)$ with the concave corner $\widehat{y} = (0|1)$. We have $\overline{x} = (1|0)$, $m_- = -1 < m_+ = +1$.

Figure 1

Note that the balanced weights $\|\cdot\|_{\texttt{hexagon},0}$ and $\|\cdot\|_{\texttt{hexagon},1}$ on $\mathbb{R}^2$ coincide with the Hölder weights $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively, which have been defined in the third section. Further, the pairs $\big(\mathbb{R}^2, \|\cdot\|_{\texttt{hexagon},r}\big)$ are normed spaces if and only if $0 \leqslant r \leqslant 1$.

Lemma 4. *For all* $r > 1$ *the space* $\big(\mathbb{R}^2, \|\cdot\|_{\texttt{hexagon},r}\big)$ *has a concave corner at* $\widehat{y} := (0|1)$, *with* $\overline{x} := (1|0)$, $m_- := 1 - r < 0 < m_+ := r - 1$.

Proof. Follow Definition 9 of a concave corner. □

Proposition 9. *Here we consider the special angle* $\angle_{\mathbf{0}}$.
Let the pair $(X, \|\cdot\|\ )$ *be a* BW *space, let* $\widehat{y} \in X$ *be a concave corner. Then the triple* $(X, \|\cdot\|, <.\,|\,.>_{\mathbf{0}})$ *does not fulfil the* CSB *inequality, i.e.* $(X, \|\cdot\|) \notin \mathsf{pdBW}_{\mathbf{0}}$.

Proof. We use the vectors $\widehat{y}$, $\overline{x}$ from the above definition of a concave corner, and then for all $\delta \in [0,1]$ we take the unit vectors $\vec{v} := \delta\cdot\overline{x} + (1+\delta\cdot m_+)\cdot\widehat{y}$ and $\vec{w} := -\delta\cdot\overline{x} + (1-\delta\cdot m_-)\cdot\widehat{y}$, and we compute

$$< \vec{v}\,|\,\vec{w} >_{\mathbf{0}} \;=\; < \delta\cdot\overline{x} + (1+\delta\cdot m_+)\cdot\widehat{y} \;\;|\; -\delta\cdot\overline{x} + (1-\delta\cdot m_-)\cdot\widehat{y} >_{\mathbf{0}} \tag{18}$$

$$= \frac{1}{4}\cdot 1\cdot 1\cdot\Big[\;\|\,[2+\delta\cdot(m_+ - m_-)]\cdot\widehat{y}\,\|^2 \;-\; \|\,2\cdot\delta\cdot\overline{x} + \delta\cdot(m_+ + m_-)\cdot\widehat{y}\,\|^2\;\Big] \tag{19}$$

$$= \frac{1}{4}\cdot\Big[\;[2+\delta\cdot(m_+ - m_-)]^2\cdot\|\,\widehat{y}\,\|^2 \;-\; \delta^2\cdot\|\,2\cdot\overline{x} + (m_+ + m_-)\cdot\widehat{y}\,\|^2\;\Big] \tag{20}$$

$$= 1 + \delta\cdot(m_+ - m_-) + \frac{1}{4}\cdot\delta^2\cdot\Big[\;(m_+ - m_-)^2 \;-\; \|\,2\cdot\overline{x} + (m_+ + m_-)\cdot\widehat{y}\,\|^2\;\Big] \tag{21}$$

$$= 1 + \delta\cdot(m_+ - m_-) + \frac{1}{4}\cdot\delta^2\cdot\mathsf{K}\,, \qquad \text{if we define the real constant } \mathsf{K} \text{ by} \tag{22}$$

$$\mathsf{K} \;:=\; (m_+ - m_-)^2 \;-\; \| \, 2 \cdot \overline{x} + (m_+ + m_-) \cdot \widehat{y} \, \|^2 .$$

This calculation holds for all $\delta \in [0,1]$. Hence, because of $m_+ - m_- > 0$, there is a positive but very small $\widetilde{\delta}$ such that for the two unit vectors

$$\widetilde{v} := \widetilde{\delta} \cdot \overline{x} + (1 + \widetilde{\delta} \cdot m_+) \cdot \widehat{y} \quad \text{and} \quad \widetilde{w} := -\widetilde{\delta} \cdot \overline{x} + (1 - \widetilde{\delta} \cdot m_-) \cdot \widehat{y}$$

we get $< \widetilde{v} | \widetilde{w} >_{\mathbf{0}} > 1$, i.e. the CSB inequality is not satisfied and the angle $\angle_{\mathbf{0}}(\widetilde{v}, \widetilde{w})$ does not exist. It follows $(X, \| \cdot \|) \notin \mathsf{pdBW}_{\mathbf{0}}$. □

Corollary 5. *For all $r > 1$ the space $\big(\mathbb{R}^2, \| \cdot \|_{\mathtt{hexagon},r}, < . \,|\, . >_{\mathbf{0}}\big)$ does not fulfil the CSB inequality. Hence, there are vectors $\vec{v} \neq \vec{0} \neq \vec{w}$ such that the angle $\angle_{\mathbf{0}}(\vec{v}, \vec{w})$ is not defined. Hence, for $r > 1$ it means $(\mathbb{R}^2, \| \cdot \|_{\mathtt{hexagon},r}) \notin \mathsf{pdBW}_{\mathbf{0}}$.*

Proposition 10. *Let α, β be two real numbers with $\alpha < -1 < \beta$. We have the proper inclusions*

$$\mathsf{pdBW}_\alpha \subset \mathsf{pdBW} \supset \mathsf{pdBW}_\beta \ .$$

Proof. From Proposition 8 we know $\mathsf{pdBW}_\alpha \neq \mathsf{pdBW}$. For instance, the Hölder norm $\| \cdot \|_\infty$ on $\mathbb{R}^2$ has convex corners, hence it follows $(\mathbb{R}^2, \| \cdot \|_\infty) \notin \mathsf{pdBW}_\alpha$, but $(\mathbb{R}^2, \| \cdot \|_\infty) \in \mathsf{pdBW}_{-1}$.

Now we consider $-1 < \beta$. Let us take the spaces $(\mathbb{R}^2, \| \cdot \|_{\mathtt{hexagon},r})$ which are defined above. The balanced weight $\| \cdot \|_{\mathtt{hexagon},r}$ is not a norm if $r > 1$, since it has a concave corner at $(0|1)$. We take the unit vectors $\vec{v} := (1|r)$ and $\vec{w} := (-1|r)$. We compute $< \vec{v} \,|\, \vec{w} >_{-\varrho}$ for an arbitrary positive number ϱ, i.e. $-\varrho < 0$, and we get

$$< \vec{v} \,|\, \vec{w} >_{-\varrho} \;=\; \|\vec{v}\| \cdot \|\vec{w}\| \cdot \frac{1}{4} \cdot \mathbf{\Delta} \cdot \left(\frac{1}{4} \cdot \mathbf{\Sigma} \right)^{-\varrho} \tag{23}$$

$$= \; 1 \cdot 1 \cdot \frac{1}{4} \cdot \left(\mathbf{s}^2 - \mathbf{d}^2\right) \cdot \left(\frac{1}{4} \cdot \left(\mathbf{s}^2 + \mathbf{d}^2\right) \right)^{-\varrho} \tag{24}$$

$$= \; \frac{1}{4} \cdot \left[(2 \cdot r)^2 - 2^2\right] \cdot \left(\frac{1}{4} \cdot \left((2 \cdot r)^2 + 2^2\right) \right)^{-\varrho} \tag{25}$$

$$= \; (r^2 - 1) \cdot (r^2 + 1)^{-\varrho} \;=\; \frac{r^2 - 1}{(r^2 + 1)^\varrho} \ . \tag{26}$$

We assume the inequality $< \vec{v} \,|\, \vec{w} >_{-\varrho} \; > 1$. This is equivalent to $(r^2 - 1) > (r^2 + 1)^\varrho$, and also to

$$\frac{\log(r^2 - 1)}{\log(r^2 + 1)} > \varrho \ . \tag{27}$$

By the rules of L'Hospital we can calculate the limit of the last term, and we get

$$\lim_{r \to \infty} \; \frac{\log(r^2 - 1)}{\log(r^2 + 1)} \;=\; 1 \ .$$

Hence for all $0 < \varrho < 1$, i.e. $-1 < -\varrho < 0$, we find a big number R such that Inequality (27) is fulfilled with $r := R$. This means $< \vec{v} \,|\, \vec{w} >_{-\varrho} \; > 1$, i.e. that the angle $\angle_{-\varrho}(\vec{v}, \vec{w})$ does not exist in $(\mathbb{R}^2, \| \cdot \|_{\mathtt{hexagon},R})$. We get that the space $(\mathbb{R}^2, \| \cdot \|_{\mathtt{hexagon},R})$ is not an element of $\mathsf{pdBW}_{-\varrho}$. Since $(\mathbb{R}^2, \| \cdot \|_{\mathtt{hexagon},R}) \in \mathsf{pdBW}_{-\mathbf{1}}$ we get that Proposition 10 is proven. □

Proposition 11. *Let α, β be two positive real numbers with $-\alpha < -1 < 1 < \beta$. We have the proper inclusions*

$$\mathsf{NORM}_{-\alpha} \subset \mathsf{NORM} \supset \mathsf{NORM}_\beta \ .$$

Proof. This follows directly from Proposition 7 and Proposition 8. For instance, the Hölder norm $\|\cdot\|_\infty$ on $\mathbb{R}^2$ has convex corners, hence it is not strictly convex. That means that the space $\left(\mathbb{R}^2, \|\cdot\|_\infty\right)$ neither is an element of $\mathsf{NORM}_{-\alpha}$, nor an element of NORM_β. $\square$

7 Some Conjectures

We formulate two open questions.

Conjecture 1. *Let us take four positive real numbers* $\alpha, \beta, \gamma, \delta$ *with*
$$-\delta < -\gamma < -1 < 1 < \alpha < \beta.$$
From Corollary 2 we know

$$\mathsf{NORM}_{-\delta} \subset \mathsf{NORM}_{-\gamma} \subset \mathsf{NORM} \supset \mathsf{NORM}_\alpha \supset \mathsf{NORM}_\beta \ ,$$

and from Proposition 11 we have $\mathsf{NORM}_{-\gamma} \neq \mathsf{NORM} \neq \mathsf{NORM}_\alpha$. *We are convinced that in fact all four inclusions are proper, and we believe that all five classes are different.*

Conjecture 2. *Let us assume four real numbers* $\alpha, \beta, \gamma, \delta$, *with*
$$-\delta < -\gamma < -1 < \alpha < \beta.$$
We already know from Proposition 10 the inequalities $\mathsf{pdBW}_{-\gamma} \neq \mathsf{pdBW} \neq \mathsf{pdBW}_\alpha$. *We believe that in fact we have four proper inclusions*

$$\mathsf{pdBW}_{-\delta} \subset \mathsf{pdBW}_{-\gamma} \subset \mathsf{pdBW} \supset \mathsf{pdBW}_\alpha \supset \mathsf{pdBW}_\beta \ ,$$

and that all five classes are different, too.

In Section 3 'Some Examples of Balancedly Weighted Vector Spaces' we defined the set of balanced weights $\|\cdot\|_p$ on $\mathbb{R}^2$, the *Hölder weights* . We use here only positive p, for $\vec{x} = (x, y) \in \mathbb{R}^2$ we set $\|\vec{x}\|_p := \sqrt[p]{|x|^p + |y|^p}$. For such $p > 0$ the weight $\|\cdot\|_p$ is positive definite. The pairs $(\mathbb{R}^2, \|\cdot\|_p)$ may be a supply of suitable examples to prove or disprove the above conjectures.

Now we we say something about finite products of BW spaces, and we ask interesting questions. We just have mentioned the Hölder weights, defined in third section. The method we used there can be generalized to construct products. Note that we restrict our description to products with have only two factors. But this can be extended to a finite number of factors very easily.

Assume two real vector spaces A, B provided with a balanced weight, i.e. we have two BW spaces $(A, \|\cdot\|_A), (B, \|\cdot\|_B)$, both spaces are not necessarily positive definite. Let p be any element from the extended real numbers, i.e. $p \in \mathbb{R} \cup \{-\infty, +\infty\}$. If $A \times B$ denotes the usual cartesian product of the vector spaces A and B, we define a balanced weight $\|\cdot\|_p$ for $A \times B$. If p is a positive real number we define (corresponding to the definition in the third section) for an element $\left(\vec{a}, \vec{b}\right) \in A \times B$ the real numbers

$$\left\| \left(\vec{a}, \vec{b}\right) \right\|_p := \sqrt[p]{\|\vec{a}\|_A^p + \|\vec{b}\|_B^p} \qquad \text{for the positive number } p, \text{ and} \tag{28}$$

$$\left\| \left(\vec{a}, \vec{b}\right) \right\|_{-p} := \begin{cases} \sqrt[-p]{\|\vec{a}\|_A^{-p} + \|\vec{b}\|_B^{-p}} & \text{if} \quad \|\vec{a}\|_A \cdot \|\vec{b}\|_B \neq 0 \\ 0 & \text{if} \quad \|\vec{a}\|_A \cdot \|\vec{b}\|_B = 0 \ . \end{cases} \tag{29}$$

To make the definition complete we set $\left\| \left(\vec{a}, \vec{b}\right) \right\|_0 := 0$, and

$$\left\| \left(\vec{a}, \vec{b}\right) \right\|_\infty := \max\left\{ \|\vec{a}\|_A , \|\vec{b}\|_B \right\} \ , \quad \left\| \left(\vec{a}, \vec{b}\right) \right\|_{-\infty} := \min\left\{ \|\vec{a}\|_A , \|\vec{b}\|_B \right\} \ .$$

It is easy to verify some properties of $\|\cdot\|_p$. For instance, the weight $\|\cdot\|_p$ is positive definite if and only if $p > 0$ and both $\|\cdot\|_A$ and $\|\cdot\|_B$ are positive definite. Further, the pair $((A \times B), \|\cdot\|_p)$ is a normed space if and only if $p \geqslant 1$ and both $\|\cdot\|_A$ and $\|\cdot\|_B$ are norms. Further, the pair $((A \times B), \|\cdot\|_p)$ is an inner product space if and only if $p = 2$ and both $\|\cdot\|_A$ and $\|\cdot\|_B$ are inner products.

The next conjecture deals with a more intricate problem.

Conjecture 3. *We take four real vector spaces provided with a positive definite balanced weight, i.e. we have* $(A, \|\cdot\|_A), (B, \|\cdot\|_B), (C, \|\cdot\|_C), (D, \|\cdot\|_D) \in \mathsf{pdBW}$. *Let us assume the identities*

$$\Upsilon\,(A, \|\cdot\|_A) = \Upsilon(C, \|\cdot\|_C) \quad \textit{and} \quad \Upsilon(B, \|\cdot\|_B) = \Upsilon(D, \|\cdot\|_D).$$

Then we conjecture that

$$\Upsilon((A \times B), \|\cdot\|_p) \;=\; \Upsilon((C \times D), \|\cdot\|_p) \quad \textit{holds for an arbitrary } \; p > 0.$$

At the end we try to find an 'algebraic structure' on the class pdBW_α, for a fixed number α. For two elements $(A, \|\cdot\|_A), (B, \|\cdot\|_B)$ of pdBW_α we look for a weight $\|\cdot\|_{A\times B}$ on $A \times B$ such that the pair $(A \times B, \|\cdot\|_{A\times B})$ is an element of pdBW_α, too. Before we make the conjecture we consider an example.

Take two copies of the real numbers $\mathbb{R}$ provided with the usual Euclidean metric $|\cdot|$. The pair $(\mathbb{R}, |\cdot|)$ is an inner product space and hence an element of pdBW_ϱ for all real numbers ϱ, see Theorem 2. We take the Cartesian product $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ and we provide it with a Hölder weight $\|\cdot\|_p$. But the pair $(\mathbb{R}^2, \|\cdot\|_p)$ is an inner product space only for $p = 2$. Hence it is an element of the classes pdBW_ϱ for each ϱ only for $p = 2$. This example leads to a natural question.

Conjecture 4. *Let α be a fixed real number. Let* $(A, \|\cdot\|_A), (B, \|\cdot\|_B)$ *be two elements of* pdBW_α, *i.e. they have the angle $\angle_\alpha$. We consider the product vector space $A \times B$.*

We ask whether the positive definite BW *space* $(A \times B, \|\cdot\|_2)$ *has the angle $\angle_\alpha$, too.*

Acknowledgements: We like to thank Prof. Dr. Eberhard Oeljeklaus who supported us by interested discussions and important advices, and who suggested many improvements. Further we thank Berkan Gürgec for a lot of technical aid.

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