# Existence, uniqueness, universality and functoriality of the perfect locality over a Frobenius $\mathcal{P}$-category

**Lluis Puig**

CNRS, Institut de Mathématiques de Jussieu, puig@math.jussieu.fr

6 Av Bizet, 94340 Joinville-le-Pont, France

*To Pierre Cartier on his 80th birthday*

**Abstract:** Let $p$ be a prime, $P$ a finite $p$-group and $\mathcal{F}$ a *Frobenius P-category*. The question on the existence of a suitable category $\mathcal{L}^{\mathrm{sc}}$ extending the *full* subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-*selfcentralizing* subgroups of $P$ goes back to Dave Benson in 1994. In 2002 Carles Broto, Ran Levi and Bob Oliver formulate the existence and the uniqueness of the cateogry $\mathcal{L}^{\mathrm{sc}}$ in terms of the nullity of an *obstruction* 3-*cohomology element* and of the vanishing of a 2-*cohomology group*, and they state a sufficient condition for the vanishing of these *n-cohomology groups*. Recently, Amy Chermak has proved the existence and the uniqueness of $\mathcal{L}^{\mathrm{sc}}$ *via* his *objective partial groups*, and Bob Oliver, following some of Chermak's methods, has also proved the vanishing of those n-cohomology groups for n > 1, both applying the *Classification of the finite simple groups*. Here we give *direct* proofs of the existence and the uniqueness of $\mathcal{L}^{\mathrm{sc}}$, and of Oliver's result; moreover, we complete $\mathcal{L}^{\mathrm{sc}}$ in a suitable category $\mathcal{L}$ extending $\mathcal{F}$ in such a way that the correpondence sending $\mathcal{F}$ to $\mathcal{L}$ is *functorial*.

## 1. Introduction

1.1. Let $p$ be a prime, $P$ a finite $p$-group and $\mathcal{F}$ a *Frobenius P-category* [8]. The question on the existence of a suitable category $\mathcal{L}^{^{\mathrm{sc}}}$ extending the *full* subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-*selfcentralizing* subgroups of $P$ [8, §3] goes back to Dave Benson in 1994 [1]. Indeed, considering our suggestion of constructing a *topological space* from the family of *classifying spaces* of the $\mathcal{F}$-*localizers* — a family of finite groups indexed by the $\mathcal{F}$-*selfcentralizing* subgroups of $P$ we had just introduced at that time [6] — Benson, in his tentative construction, had foreseen the interest of this extension, actually as a generalization for Frobenius $P$-categories of our old *O-locality* for finite groups [5]

1.2. In [2] Carles Broto, Ran Levi and Bob Oliver formulate the existence and the uniqueness of the cateogry $\mathcal{L}^{^{\mathrm{sc}}}$ in terms of the nullity of an *obstruction* 3-*cohomology element* and of the vanishing of a 2-*cohomology group*, respectively. They actually state a sufficient condition for the vanishing of the corresponding *n-cohomology groups* and moreover, assuming the existence of $\mathcal{L}^{^{\mathrm{sc}}}$, they succeed in the construction of a *classifying space*.

1.3. As a matter of fact, if $G$ is a finite group and $P$ a Sylow $p$-subgroup of $G$, the corresponding *Frobenius P-category* $\mathcal{F}_G$ [5] admits an extension $\mathcal{L}_G$ defined over *all* the subgroups of $P$ where, for any pair of subgroups $Q$ and $R$ of $P$, the set of *morphisms* from $R$ to $Q$ is the following quotient set of the $G$-*transporter*

$$\mathcal{L}_G(Q,R) = T_G(R,Q)/\mathbb{O}^p\big(C_G(R)\big) \qquad 1.3.1.$$

Thus, in the general setting, if we are interested in the *functoriality* of our constructions, we need not only the existence of $\mathcal{L}^{^{\rm sc}}$ but the existence of a suitable category $\mathcal{L}$ extending $\mathcal{F}$ and containing $\mathcal{L}^{^{\rm sc}}$ as a *full* subcategory. Soon after [2], we showed that the functor from $\mathcal{F}_G$ mapping $Q$ on $C_G(Q)/\mathbb{O}^p\big(C_G(Q)\big)$ can be generalized to a functor $\mathfrak{c}^{\mathfrak{h}}_{\mathcal{F}}$ from any *Frobenius $P$-category* $\mathcal{F}$ (see 2.4 below), and that the existence of $\mathcal{L}^{^{\rm sc}}$ forces the exis-tence of a unique extension $\mathcal{L}$ of $\mathcal{F}$ by $\mathfrak{c}^{\mathfrak{h}}_{\mathcal{F}}$, namely the so-called *perfect $\mathcal{F}$-locality*, already introduced in [7].

1.4. Recently, Amy Chermak [3] has proved the existence and the uniqueness of $\mathcal{L}^{^{\rm sc}}$ *via* his *objective partial groups*, and Bob Oliver [4], following some of Chermak's methods, has also proved, for $n \geq 2$, the vanishing of the *$n$-cohomology groups* mentioned above. In reading their preprints, we were disappointed not only because their proofs depend on the so-called *Classification of the finite simple groups* (CFSG), but because in their arguments they need strongly properties of finite groups. Indeed, since [5] we are convinced that a previous classification of the so-called "local structures" will be the way to clarify CFSG in future versions; thus, our effort in creating the *Frobenius $P$-categories* was directed to provide a precise formal support to the vague notion of "local structures", independent of "environmental" finite groups and of most of finite group properties.

1.5. Here we will show that, till now, our intuition was correct, namely that there is a direct proof of the existence and the uniqueness of $\mathcal{L}^{^{\rm sc}}$; that is to say, a proof that can be qualified of *inner* or *tautological* in the sense that only pushes far enough the initial axioms. But, as we mention above, the existence and the uniqueness of $\mathcal{L}^{^{\rm sc}}$ will guarantee the existence and the uniqueness of the *perfect $\mathcal{F}$-locality* $\mathcal{L}$ defined over all the subgroups of $P$ and then it makes sense to discuss the *functoriality* of the correspondence mapping $\mathcal{F}$ on $\mathcal{L}$. Moreover, as a kind of converse of the mentioned result in [2], the existence of $\mathcal{L}^{^{\rm sc}}$ allows us to get a direct proof of Oliver's result in [4].

1.6. Let us explain how our method works. In [9, Chap. 18] we introduce the *$\mathcal{F}$-localizers* mentioned above and, as a matter of fact, we already introduce the $\mathcal{F}$-localizer $L_{\mathcal{F}}(Q)$ for *any* subgroup $Q$ of $P$ (see Theorem 2.10 below), which is indeed an extension of the group $\mathcal{F}(Q)$ of $\mathcal{F}$-automorphisms of $Q$, by the $p$-group $\mathfrak{c}^{\mathfrak{h}}_{\mathcal{F}}(Q)$ (cf. 1.3). More precisely, it makes sense to consider the $\mathcal{F}$-localizer $L_{\mathcal{F}}(\mathfrak{q})$ for any *$\mathcal{F}$-chain* $\mathfrak{q}$ (cf. 3.3 below) and then, with the quotients

$$\bar L_{\mathcal{F}}(\mathfrak{q}) = L_{\mathcal{F}}(\mathfrak{q})/[\mathfrak{c}^{\mathfrak{h}}_{\mathcal{F}}(Q), \mathfrak{c}^{\mathfrak{h}}_{\mathcal{F}}(Q)] \qquad 1.6.1,$$

we succeed in building the *$\mathcal{F}$-localizing functor* $\mathfrak{loc}_{\mathcal{F}}$ (see Proposition 3.7 below), which will play a critical role in our proof of the existence of $\mathcal{L}^{^{\rm sc}}$.

1.7. More generally, in [9, Chap. 17] we introduce the *$\mathcal{F}$-localities* as a wider framework where to look for the *perfect $\mathcal{F}$-locality*. Considering the category $\mathcal{T}_P$ where the objects are all the subgroups of $P$, where the set of morphisms from $R$ to $Q$ is the *$P$-transporter* $T_P(R,Q)$ and where the composition is induced by the product in $P$, we call *$\mathcal{F}$-locality* any extension $\pi : \mathcal{L} \to \mathcal{F}$ of the category $\mathcal{F}$ endowed with a functor $\tau : \mathcal{T}_P \to \mathcal{L}$ such that the composition $\pi \circ \tau : \mathcal{T}_P \to \mathcal{F}$ is the canonical functor defined by the conjugation in $P$; of course, we add some suitable conditions as *divisibility* and *p-coherence* (see 2.8 below). As a matter of fact, a *perfect $\mathcal{F}$-locality* is just a *divisible $\mathcal{F}$-locality* $\mathcal{L}$ where the group $\mathcal{L}(Q)$ of $\mathcal{L}$-automorphisms of any subgroup $Q$ of $P$ coincides with the *$\mathcal{F}$-localizer* of $Q$ (see 2.11 below).

1.8. It turns out that there are indeed other $\mathcal{F}$-localities, easier to construct, which deserve consideration; they depend on the existence of the *$\mathcal{F}$-basic $P \times P$-sets* $\Omega$ introduced in [9, Chap. 21] which allows the realization of $\mathcal{F}$ inside the *symmetric group* of $\Omega$, and then, the consideration of $\mathcal{F}$-localities as defined in [5]. In [9, Chap. 22] we introduce the so-called *basic $\mathcal{F}$-locality* which, although too "big", is canonically associated with $\mathcal{F}$ and will be a "support" for the construction of the *perfect $\mathcal{F}$-locality*. More precisely, in [9, Chap. 24] we show that the very structure of a *perfect $\mathcal{F}^{^{\rm sc}}$-locality $\mathcal{L}^{^{\rm sc}}$* supplies a particular *$\mathcal{F}$-basic $P \times P$-set* that, from the corresponding *basic $\mathcal{F}$-locality*, allows the construction of the *reduced $\mathcal{F}^{^{\rm sc}}$-locality* which contains $\mathcal{L}^{^{\rm sc}}$ [9, Corollary 24.18].

1.9. At this point, on the one hand this particular *$\mathcal{F}$-basic $P \times P$-set* can be described *directly*, without assuming the existence of $\mathcal{L}^{^{\rm sc}}$. On the other hand, any *p-coherent $\mathcal{F}$-locality* $\hat{\mathcal{L}}$ determines a functor mapping any *$\hat{\mathcal{L}}$-chain* $\hat{\mathfrak{q}}$ on the group $\hat{\mathcal{L}}(\hat{\mathfrak{q}})$ of $\hat{\mathcal{L}}$-automorphisms of $\hat{\mathfrak{q}}$ (see 3.2 below); assuming that the kernels of the structural group homomorphisms $\hat{\mathcal{L}}(\hat{\mathfrak{q}}) \to \mathcal{F}(\mathfrak{q})$ are Abelian, this functor enable us to construct a new functor $\mathfrak{loc}_{\hat{\mathcal{L}}}$ analogous to the *$\mathcal{F}$-localizing functor* $\mathfrak{loc}_{\mathcal{F}}$ mentioned above and, as a matter of fact, in this context $\mathfrak{loc}_{\mathcal{F}}$ becomes "universal" in the sense that there is a unique suitable *natural map* $\lambda_{\hat{\mathcal{L}}} : \mathfrak{loc}_{\mathcal{F}} \to \mathfrak{loc}_{\hat{\mathcal{L}}}$ (see Proposition 3.9 below).

1.10. What about the "image" of $\lambda_{\hat{\mathcal{L}}}$? More precisely, is there a *coherent $\mathcal{F}$-sublocality* $\mathcal{L}$ of $\hat{\mathcal{L}}$ such that the corresponding *natural map* $\lambda_{\mathcal{L}}$ is *surjective*? In this case, restricting our attention to the *full* subcategories $\mathcal{F}^{^{\rm sc}}$, $\hat{\mathcal{L}}^{^{\rm sc}}$ and $\mathcal{L}^{^{\rm sc}}$ over the set of $\mathcal{F}$-selfcentralizing subgroups $Q$ of $P$ and assuming that $\hat{\mathcal{L}}$ is "big enough" — for instance, that it is the *basic $\mathcal{F}$-locality* — it turns out that the group $\mathcal{L}(Q)$ of $\mathcal{L}$-automorphisms of $Q$ coincides with the *$\mathcal{F}$-localizer* of $Q$ (cf. 1.6.1) and therefore that $\mathcal{L}^{^{\rm sc}}$ would be a *perfect $\mathcal{F}^{^{\rm sc}}$-locality* (cf. 1.7). Note that a positive answer to this question would extend the "universal" character of the *$\mathcal{F}$-localizing functor* to some *universality* of the *perfect $\mathcal{F}^{^{\rm sc}}$-locality*.

1.11. As in [2], we have been able to formulate this question in *cohomological terms* [9 Proposition 18.28]; but, our formulation only needs to consider the so-called *stable cohomology groups* [9, A3.17], which *a priori* can be expected to be *smaller* than the ordinary ones. Denoting by $\tilde{\mathcal{F}}^{^{\rm sc}}$ the *exterior* quotient of the category $\mathcal{F}^{^{\rm sc}}$ [9, 1.3], by $k$ a perfect field of characteristic $p$, by $k$-$\mathfrak{mod}$ the category of finite dimensional $k$-vector spaces and by $\mathfrak{m}:\tilde{\mathcal{F}}^{^{\rm sc}} \to k\text{-}\mathfrak{mod}$ a *contravariant* functor, our key result is that, if $n \geq 1$, the *stable cohomology groups* $\mathbb{H}^n_*(\tilde{\mathcal{F}}^{^{\rm sc}}, \mathfrak{m})$ vanish. The fact that this result only covers characteristic $p$ forbids us to apply [9, Propositions 18.28 and 18.29] as it stands, but an inductive argument will solve the problem.

1.12. Although this result and the choice of $\hat{\mathcal{L}}$ as the *basic* $\mathcal{F}$*-locality* would suffice to prove the existence of a *perfect* $\mathcal{F}^{^{\rm sc}}$*-locality*, the choice of a $\mathcal{F}^{^{\rm sc}}$*-locality* which *has* to contain $\mathcal{L}^{^{\rm sc}}$, as described above, and our key result allow us to prove the uniqueness of the *perfect* $\mathcal{F}^{^{\rm sc}}$*-locality* [9, Proposition 18.29]. Then, as mentioned above, we already get the existence and the uniqueness of the *perfect* $\mathcal{F}$*-locality* $\mathcal{L}$ [9, Chap. 20].

1.13. Once we have the existence and the uniqueness of $\mathcal{L}^{^{\rm sc}}$ and $\mathcal{L}$, the *universality* of the *perfect* $\mathcal{F}$*-locality* in the category of $p$*-coherent* $\mathcal{F}$*-localities* will follow from the "universal" character of the $\mathcal{F}$*-localizing functor* mentioned above, from our key result, and from a suitable inductive argument. Finally, from this *universality* we will obtain the *functoriality* of the *perfect* $\mathcal{F}$*-localities*, from the category formed by the pairs $(P, \mathcal{F})$ where $P$ is a finite $p$-group and $\mathcal{F}$ a *Frobenius* $P$*-category*, and by the morphisms

$$(\alpha, \mathfrak{f}_\alpha) : (P, \mathcal{F}) \longrightarrow (P', \mathcal{F}') \qquad 1.13.1$$

where $\alpha : P \to P'$ is a $(\mathcal{F}, \mathcal{F}')$*-functorial* group homomorphism and $\mathfrak{f}_\alpha$ the corresponding *Frobenius functor* [9, 12.1].

1.14. Moreover, denoting by $\mathcal{O}$ a complete discrete valuation ring of characteristic zero lifting $k$, by $\mathcal{O}$-$\mathfrak{mod}$ the category of finitely generated $\mathcal{O}$-modules and by $\mathfrak{m}:\tilde{\mathcal{F}}^{^{\rm sc}} \to \mathcal{O}\text{-}\mathfrak{mod}$ a *contravariant* functor sending the $\tilde{\mathcal{F}}^{^{\rm sc}}$-morphisms to *injective* $\mathcal{O}$-module homomorphisms, we consider a new *contravariant* functor $\widehat{\mathfrak{m}}:\tilde{\mathcal{F}}^{^{\rm sc}} \to \mathcal{O}\text{-}\mathfrak{mod}$ containing $\mathfrak{m}$, in such a way that the *existence* of $\mathcal{L}^{^{\rm sc}}$ enable us to prove that, for any $n \geq 1$, we have

$$\mathbb{H}^n(\tilde{\mathcal{F}}^{^{\rm sc}}, \widehat{\mathfrak{m}}) = \{0\} \quad \text{and} \quad \mathbb{H}^n(\tilde{\mathcal{F}}^{^{\rm sc}}, \widehat{\mathfrak{m}}/\mathfrak{m}) = \{0\} \qquad 1.14.1,$$

which, for any $n \geq 2$, forces

$$\mathbb{H}^1(\tilde{\mathcal{F}}^{^{\rm sc}}, \mathfrak{m}) \cong \varprojlim(\widehat{\mathfrak{m}}/\mathfrak{m})/(\varprojlim \widehat{\mathfrak{m}}/\varprojlim \mathfrak{m}) \quad \text{and} \quad \mathbb{H}^n(\tilde{\mathcal{F}}^{^{\rm sc}}, \mathfrak{m}) = \{0\} \qquad 1.14.2.$$

1.15. After recalling our terminology and some quoted results, we follow the pattern of the Introduction since 1.6, except that, for inductive purposes, we replace the whole set of $\mathcal{F}$-selfcentralizing subgroups of $P$ by a nonempty set $\mathfrak{X}$ of $\mathcal{F}$-selfcentralizing subgroups containing any subgroup of $P$ admitting an $\mathcal{F}$-morphism from some subgroup in $\mathfrak{X}$.

**2. Frobenius $P$-categories and coherent $\mathcal{F}$-localities**

2.1. Denote by $\mathfrak{iGr}$ the category formed by the finite groups and by the injective group homomorphisms. Recall that, for any category $\mathfrak{C}$, $\mathfrak{C}^\circ$ denotes the *opposite* category and, for any $\mathfrak{C}$-object $C$, $\mathfrak{C}_C$ (or $(\mathfrak{C})_C$ to avoid confusion) denotes the category of "$\mathfrak{C}$-morphisms to $C$" [9, 1.7] ; if any $\mathfrak{C}$-object admits *inner* automorphisms we denote by $\tilde{\mathfrak{C}}$ the corresponding quotient and call it the *exterior* quotient of $\mathfrak{C}$ [9, 1.3]. Let $p$ be a prime; for any finite $p$-group $P$ we denote by $\mathcal{F}_P$ the subcategory of $\mathfrak{iGr}$ where the objects are all the subgroups of $P$ and the morphisms are the group homomorphisms induced by conjugation by elements of $P$.

2.2. A *Frobenius $P$-category* $\mathcal{F}$ is a subcategory of $\mathfrak{iGr}$ containing $\mathcal{F}_P$ where the objects are all the subgroups of $P$ and the morphisms fulfill the following three conditions [9, 2.8 and Proposition 2.11]

2.2.1 *For any subgroup $Q$ of $P$ the inclusion functor $(\mathcal{F})_Q \to (\mathfrak{iGr})_Q$ is full.*

2.2.2 *$\mathcal{F}_P(P)$ is a Sylow $p$-subgroup of $\mathcal{F}(P)$.*

2.2.3 *Assume that $Q$ is a subgroup of $P$ fulfilling $\xi\big(C_P(Q)\big) = C_P\big(\xi(Q)\big)$ for any $\mathcal{F}$-morphism $\xi : Q{\cdot}C_P(Q) \to P$, that $\varphi : Q \to P$ is an $\mathcal{F}$-morphism and that $R$ is a subgroup of $N_P\big(\varphi(Q)\big)$ containing $\varphi(Q)$ such that $\mathcal{F}_P(Q)$ contains the action of $\mathcal{F}_R\big(\varphi(Q)\big)$ over $Q$ via $\varphi$. Then there is an $\mathcal{F}$-morphism $\zeta : R \to P$ fulfilling $\zeta\big(\varphi(u)\big) = u$ for any $u \in Q$.*

As in [9, 1.2], for any pair of subgroups $Q$ and $R$ of $P$, we denote by $\mathcal{F}(Q,R)$ the set of $\mathcal{F}$-morphisms from $Q$ to $R$ and set $\mathcal{F}(Q) = \mathcal{F}(Q,Q)$. If $G$ is a finite subgroup admitting $P$ as a Sylow $p$-subgroup, we denote by $\mathcal{F}_G$ the Frobenius $P$-category where the morphisms are the group homomorphisms induced by conjugation by elements of $G$.

2.3. Fix a Frobenius $P$-category $\mathcal{F}$; for any subgroup $Q$ of $P$ and any subgroup $K$ of the group $\mathrm{Aut}(Q)$ of automorphisms of $Q$, we say that $Q$ is *fully $K$-normalized* in $\mathcal{F}$ if we have [9, 2.6]

$$\xi\big(N_P^K(Q)\big) = N_P^{{}^\xi K}\big(\xi(Q)\big) \qquad 2.3.1$$

for any $\mathcal{F}$-morphism $\xi : Q{\cdot}N_P^K(Q) \to P$, where $N_P^K(Q)$ is the converse image of $K$ in $N_P(Q)$ *via* the canonical group homomorphism $N_P(Q) \to \mathrm{Aut}(Q)$ and ${}^\xi K$ is the image of $K$ in $\mathrm{Aut}\big(\xi(Q)\big)$ *via* $\xi$. Recall that if $Q$ is fully $K$-normalized in $\mathcal{F}$ then we have a new Frobenius $N_P^K(Q)$-category $N_\mathcal{F}^K(Q)$ where, for any pair of subgroups $R$ and $T$ of $N_P^K(Q)$, $\big(N_\mathcal{F}^K(Q)\big)(R,T)$ is the set of group homomorphisms *from $T$ to $R$* induced by the $\mathcal{F}$-morphisms $\psi : Q{\cdot}T \to Q{\cdot}R$ which stabilize $Q$ and induce on it an element of $K$ [9, 2.14 and Proposition 2.16].

2.4. We denote by $H_{\mathcal{F}}$ the $\mathcal{F}$*-hyperfocal* subgroup of $P$ which is the subgroup generated by the sets $\{u^{-1}\sigma(u)\}_{u\in Q}$ where $Q$ runs over the set of subgroups of $P$ and $\sigma$ over the set of $p'$-elements of $\mathcal{F}(Q)$ [9, 13.2]. As above, for any subgroup $Q$ of $P$ *fully centralized* in $\mathcal{F}$ — namely, with $K=\{1\}$ — we have the Frobenius $C_P(Q)$-category $C_{\mathcal{F}}(Q)$ and therefore we can consider the $C_{\mathcal{F}}(Q)$-hyperfocal subgroup $H_{C_{\mathcal{F}}(Q)}$ of $C_P(Q)$ ; then, in [9, Proposition 13.14] we exhibit a unique *contravariant* functor

$$\mathfrak{c}^{\mathfrak{h}}_{\mathcal{F}} : \mathcal{F} \longrightarrow \widetilde{\mathfrak{Gr}} \tag{2.4.1}$$

where $\widetilde{\mathfrak{Gr}}$ denotes the *exterior* quotient of the category $\mathfrak{Gr}$ of finite groups, mapping any subgroup $Q$ of $P$ fully centralized in $\mathcal{F}$ on $C_P(Q)/H_{C_{\mathcal{F}}(Q)}$ and any $\mathcal{F}$-morphism $\varphi : R \to Q$ from a subgroup $R$ of $P$ fully centralized in $\mathcal{F}$ on a $\widetilde{\mathfrak{Gr}}$-morphism induced by an $\mathcal{F}$-morphism

$$\varphi(R){\cdot}C_P(Q) \longrightarrow R{\cdot}C_P(R) \tag{2.4.2}$$

sending $\varphi(v)$ to $v$ for any $v\in R$.

2.5. We say that a subgroup $U$ of $P$ is $\mathcal{F}$*-stable* if we have $\varphi(Q\cap U)\subset U$ for any subgroup $Q$ of $P$ and any $\mathcal{F}$-morphism $\varphi : Q \to P$ ; then, setting $\bar P = P/U$ , there is a Frobenius $\bar P$-category $\bar{\mathcal{F}} = \mathcal{F}/U$ such that the canonical homomorphism $\varpi : P \to \bar P$ is $(\mathcal{F},\bar{\mathcal{F}})$*-functorial* and the corresponding *Frobenius functor* $\mathfrak{f}_\varpi : \mathcal{F} \to \bar{\mathcal{F}}$ is *surjective* over the subgroups of $P$ containing $U$ [9 Proposition 12.3]. In particular, if $Q$ is a subgroup of $P$ fully normalized in $\mathcal{F}$, it follows from [9, Proposition 13.9] that $H_{C_{\mathcal{F}}(Q)}$ is a $N_{\mathcal{F}}(Q)$-stable subgroup of $N_P(Q)$ and therefore we can consider the quotients

$$\overline{N_P(Q)} = N_P(Q)/H_{C_{\mathcal{F}}(Q)} \quad \text{and} \quad \overline{N_{\mathcal{F}}(Q)} = N_{\mathcal{F}}(Q)/H_{C_{\mathcal{F}}(Q)} \qquad 2.5.1.$$

2.6. We say that a subgroup $Q$ of $P$ is $\mathcal{F}$*-selfcentralizing* if we have

$$C_P\big(\varphi(Q)\big) \subset \varphi(Q) \tag{2.6.1}$$

for any $\varphi\in\mathcal{F}(P,Q)$ ; we denote by $\mathcal{F}^{^{\mathrm{sc}}}$ the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$. More generally, as mentioned above we consider a nonempty set $\mathfrak{X}$ of subgroups of $P$ containing any subgroup of $P$ admitting an $\mathcal{F}$-morphism from some subgroup in $\mathfrak{X}$ and then we denote by $\mathcal{F}^{^{\mathfrak{X}}}$ the *full* subcategory of $\mathcal{F}$ over the set $\mathfrak{X}$ of objects; in most situations, the subgroups in $\mathfrak{X}$ will be $\mathcal{F}$-selfcentralizing.

2.7. Denote by $\mathcal{T}_P^{^{\mathfrak{X}}}$ the *full* subcategory of $\mathcal{T}_P$ over the set $\mathfrak{X}$ and by $\kappa^{\mathfrak{x}} : \mathcal{T}_P^{^{\mathfrak{X}}} \to \mathcal{F}^{^{\mathfrak{X}}}$ the canonical functor determined by the conjugation. An $\mathcal{F}^{^{\mathfrak{X}}}$*-locality* $\mathcal{L}^{^{\mathfrak{X}}}$ is a category where $\mathfrak{X}$ is the set of objects, endowed with two functors

$$\tau^{\mathfrak{x}} : \mathcal{T}_P^{^{\mathfrak{X}}} \longrightarrow \mathcal{L}^{^{\mathfrak{X}}} \quad \text{and} \quad \pi^{\mathfrak{x}} : \mathcal{L}^{^{\mathfrak{X}}} \longrightarrow \mathcal{F}^{^{\mathfrak{X}}} \tag{2.7.1}$$

which are the identity on the set of objects and fulfill $\pi^{\mathfrak{x}} \circ \tau^{\mathfrak{x}} = \kappa^{\mathfrak{x}}$ , $\pi^{\mathfrak{x}}$ being *full*; as above, for any pair of subgroups $Q$ and $R$ in $\mathfrak{X}$ , we denote by $\mathcal{L}^{\mathfrak{x}}(Q,R)$ the set of $\mathcal{L}^{\mathfrak{x}}$-morphisms from $R$ to $Q$ and by

$$\tau^{\mathfrak{x}}_{Q,R} : \mathcal{T}^{\mathfrak{x}}_P(Q,R) \to \mathcal{L}^{\mathfrak{x}}(Q,R) \quad \text{and} \quad \pi^{\mathfrak{x}}_{Q,R} : \mathcal{L}^{\mathfrak{x}}(Q,R) \to \mathcal{F}^{\mathfrak{x}}(Q,R) \qquad 2.7.2$$

the corresponding maps; we write $Q$ only once if $Q = R$ .

2.8. We say that $\mathcal{L}^{\mathfrak{x}}$ is *divisible* if, for any pair of subgroups $Q$ and $R$ in $\mathfrak{X}$ , $\mathrm{Ker}(\pi^{\mathfrak{x}}_R)$ acts regularly on the “fibers” of $\pi^{\mathfrak{x}}_{Q,R}$ , and that $\mathcal{L}^{\mathfrak{x}}$ is *coherent* if moreover, for any $x \in \mathcal{L}^{\mathfrak{x}}(Q,R)$ and any $v \in R$ . we have [9, 17.8 and 17.9]

$$x \cdot \tau^{\mathfrak{x}}_R(v) = \tau^{\mathfrak{x}}_Q\Big(\big(\pi^{\mathfrak{x}}_{Q,R}(x)\big)(v)\Big) \cdot x \qquad 2.8.1.$$

More precisely, we say that $\mathcal{L}^{\mathfrak{x}}$ is *p-coherent* if moreover, for any subgroup $Q$ in $\mathfrak{X}$ , the kernel $\mathrm{Ker}(\pi^{\mathfrak{x}}_Q)$ is a $p$-group; in this case, it follows from [9, 17.13] that if $Q$ is fully centralized in $\mathcal{F}$ then we have

$$H_{C_{\mathcal{F}}(Q)} \subset \mathrm{Ker}(\tau^{\mathfrak{x}}_Q) \qquad 2.8.2.$$

Finally, we say that $\mathcal{L}^{\mathfrak{x}}$ is *perfect* if it is *p-coherent* and for any subgroup $Q$ in $\mathfrak{X}$ fully centralized in $\mathcal{F}$ we have [9, 17.13]

$$H_{C_{\mathcal{F}}(Q)} = \mathrm{Ker}(\tau^{\mathfrak{x}}_Q) \qquad 2.8.3.$$

2.9. With the notation in 2.5.1, we are interested in the $\overline{N_{\mathcal{F}}(Q)}$-locality $\overline{N_{\mathcal{F},Q}(Q)}$ where the morphisms are the pairs formed by an $\overline{N_{\mathcal{F}}(Q)}$-morphism and by an automorphism of $Q$ , both determined by the *same* $\mathcal{F}$-morphism [9, 18.3], and where the composition and the structural functors are the obvious ones. Similarly, if $L$ is a finite group acting on $Q$ , we are interested in the $\mathcal{F}_L$-locality $\mathcal{F}_{L,Q}$ where the morphisms are the pairs formed by an $\mathcal{F}_L$-morphism and by an automorphism of $Q$ , both determined by the *same* element of $L$ . We are ready to describe the $\mathcal{F}$*-localizer* of $Q$ [9, Theorem 18.6].

**Theorem 2.10.** *For any subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$ there is a triple formed by a finite group $L_{\mathcal{F}}(Q)$ and by two group homomorphisms*

$$\tau_Q : N_P(Q) \longrightarrow L_{\mathcal{F}}(Q) \quad \textit{and} \quad \pi_Q : L_{\mathcal{F}}(Q) \longrightarrow \mathcal{F}(Q) \qquad 2.10.1$$

*such that $\pi_Q \circ \tau_Q$ is induced by the $N_P(Q)$-conjugation, that we have the exact sequence*

$$1 \longrightarrow H_{C_{\mathcal{F}}(Q)} \longrightarrow C_P(Q) \xrightarrow{\tau_Q} L_{\mathcal{F}}(Q) \xrightarrow{\pi_Q} \mathcal{F}(Q) \longrightarrow 1 \qquad 2.10.2$$

*and that $\pi_Q$ and $\tau_Q$ induce an equivalence of categories*

$$\overline{N_{\mathcal{F},Q}(Q)} \cong \mathcal{F}_{L_{\mathcal{F}}(Q),Q} \qquad 2.10.3.$$

*Moreover, for another such a triple $L'$ , $\tau'_Q$ and $\pi'_Q$ , there is a group isomorphism $\lambda : L_{\mathcal{F}}(Q) \cong L'$ , unique up to $\mathfrak{c}^{\mathfrak{h}}_{\mathcal{F}}(Q)$-conjugation, fulfilling $\lambda \circ \tau_Q = \tau'_Q$ and $\pi'_Q \circ \lambda = \pi_Q$ .*

2.11. For any subgroup $Q$ of $P$ fully normalized in $\mathcal{F}$, we call *$\mathcal{F}$-localizer* of $Q$ any finite group $L$ endowed with two group homomorphisms as in 2.10.1 fulfilling the conditions 2.10.2 and 2.10.3. Note that, if $\mathcal{L}^{\mathfrak{X}}$ is an $\mathcal{F}^{\mathfrak{X}}$-locality then, for any $Q \in \mathfrak{X}$, the structural functors $\tau^{\mathfrak{X}}$ and $\pi^{\mathfrak{X}}$ determine two group homomorphisms (cf. 2.7.2)

$$\tau_Q^{\mathfrak{X}} : N_P(Q) \longrightarrow \mathcal{L}^{\mathfrak{X}}(Q) \quad \text{and} \quad \pi_Q^{\mathfrak{X}} : \mathcal{L}^{\mathfrak{X}}(Q) \longrightarrow \mathcal{F}(Q) \qquad 2.11.1$$

and $\pi_Q^{\mathfrak{X}}$ is surjective; in particular, if $Q$ is fully normalized in $\mathcal{F}$ then, since $\mathcal{F}_P(Q)$ is a Sylow $p$-subgroup of $\mathcal{F}(Q)$ [9, Proposition 2.11], $\tau_Q^{\mathfrak{X}}\big(N_P(Q)\big)$ is a Sylow $p$-subgroup of $\mathcal{L}^{\mathfrak{X}}(Q)$ if and only if it contains a Sylow $p$-subgroup of $\mathrm{Ker}(\pi_Q^{\mathfrak{X}})$. Consequently, if $\mathcal{L}^{\mathfrak{X}}$ is *divisible* and, for any $Q \in \mathfrak{X}$ fully normalized in $\mathcal{F}$, the group $\mathcal{L}^{\mathfrak{X}}(Q)$ endowed with $\tau_Q^{\mathfrak{X}}$ and $\pi_Q^{\mathfrak{X}}$ is an $\mathcal{F}$-localizer of $Q$, it is easily checked from [9, Proposition 17.10] that $\mathcal{L}^{\mathfrak{X}}$ is *coherent* and therefore that it is a *perfect $\mathcal{F}^{\mathfrak{X}}$-locality*. Actually, the converse statement is true and it is easily checked from [9, Proposition 18.4].

## 3. The $\mathcal{F}$-localizing functor

3.1. For any $n \in \mathbb{N}$, let us consider the *$n$-simplex* $\Delta_n$ as a category where the objects are the elements of $\Delta_n$ and the set of morphisms from $i \in \Delta_n$ to $j \in \Delta_n$ is either the set of one element $i\bullet j$ or the empty set according to $i \le j$ or $i > j$ [9, A1.7]. Then, the *proper category of chains* $\mathfrak{ch}^*(\mathcal{F})$ of $\mathcal{F}$ [9, A2.8] is the category formed by the pairs $(\mathfrak{q}, \Delta_n)$ where $n \in \mathbb{N}$ and $\mathfrak{q} : \Delta_n \to \mathcal{F}$ is a functor, with the morphisms from $(\mathfrak{q}, \Delta_n)$ to another object $(\mathfrak{r}, \Delta_m)$ given by the pairs $(\nu, \delta)$ where $\delta : \Delta_m \to \Delta_n$ is a *functor* or, equivalently, an order-preserving map, and $\nu : \mathfrak{q} \circ \delta \cong \mathfrak{r}$ is a *natural isomorphism*, the composition with another morphism $(\mu, \varepsilon) : (\mathfrak{r}, \Delta_m) \to (\mathfrak{t}, \Delta_\ell)$ being defined by [9, A2.6.3]

$$(\mu, \varepsilon) \circ (\nu, \delta) = \big(\mu \circ (\nu * \varepsilon), \delta \circ \varepsilon\big) \qquad 3.1.1.$$

Occasionally, we write $(\nu, \delta)_{\mathfrak{q}}$ instead of $(\nu, \delta)$ to avoid confusion.

3.2. Then, it is easily checked that we have a functor [9, Proposition A2.10]

$$\mathfrak{aut}_{\mathcal{F}} : \mathfrak{ch}^*(\mathcal{F}) \longrightarrow \mathfrak{Gr} \qquad 3.2.1$$

mapping any $\mathfrak{ch}^*(\mathcal{F})$-object $(\mathfrak{q}, \Delta_n)$ on its group of automorphisms in $\mathfrak{ch}^*(\mathcal{F})$, denoted by $\mathcal{F}(\mathfrak{q})$. Similarly, for any $\mathcal{F}^{\mathfrak{X}}$-locality $\mathcal{L}^{\mathfrak{X}}$ we have the *proper category of chains* $\mathfrak{ch}^*(\mathcal{L}^{\mathfrak{X}})$ of $\mathcal{L}^{\mathfrak{X}}$ and the corresponding functor

$$\mathfrak{aut}_{\mathcal{L}^{\mathfrak{X}}} : \mathfrak{ch}^*(\mathcal{L}^{\mathfrak{X}}) \longrightarrow \mathfrak{Gr} \qquad 3.2.2;$$

once again, we denote by $\mathcal{L}^{\mathfrak{X}}(\hat{\mathfrak{q}})$ the group of $\mathfrak{ch}^*(\mathcal{L}^{\mathfrak{X}})$-automorphisms of an $\mathfrak{ch}^*(\mathcal{L}^{\mathfrak{X}})$-object $(\hat{\mathfrak{q}}, \Delta_n)$.

3.3. Actually, we identify $\mathcal{F}(\mathfrak{q})$ with the stabilizer in $\mathcal{F}\big(\mathfrak{q}(n)\big)$ of all the subgroups $\mathrm{Im}\big(\mathfrak{q}(i\bullet n)\big)$ when $i$ runs over $\Delta_n$ and we say that $\mathfrak{q}$ is *fully normalized* in $\mathcal{F}$ if $\mathfrak{q}(n)$ is fully $\mathcal{F}(\mathfrak{q})$-normalized in $\mathcal{F}$; in this case, we set

$$N_P(\mathfrak{q}) = N_P^{\mathcal{F}(\mathfrak{q})}\big(\mathfrak{q}(n)\big) \quad \text{and} \quad N_{\mathcal{F}}(\mathfrak{q}) = N_{\mathcal{F}}^{\mathcal{F}(\mathfrak{q})}\big(\mathfrak{q}(n)\big) \qquad 3.3.1$$

and we know that $N_{\mathcal{F}}(\mathfrak{q})$ is a Frobenius $N_P(\mathfrak{q})$-category [9, Proposition 2.16]; moreover, since there is an $\mathcal{F}$-morphism $\zeta : \mathfrak{q}(n){\cdot}N_P(\mathfrak{q}) \to P$ such that $\zeta\big(\mathfrak{q}(n)\big)$ is fully normalized in $\mathcal{F}$ [9, Proposition 2.7], it easily follows from Theorem 2.10 that we also have an $\mathcal{F}$*-localizer* of $\mathfrak{q}$, namely a finite group $L_{\mathcal{F}}(\mathfrak{q})$ endowed with two group homomorphisms

$$\tau_{\mathfrak{q}} : N_P(\mathfrak{q}) \longrightarrow L_{\mathcal{F}}(\mathfrak{q}) \quad \textit{and} \quad \pi_{\mathfrak{q}} : L_{\mathcal{F}}(\mathfrak{q}) \longrightarrow \mathcal{F}(\mathfrak{q}) \qquad 3.3.2$$

such that $\pi_{\mathfrak{q}} \circ \tau_{\mathfrak{q}}$ is induced by the $N_P(\mathfrak{q})$-conjugation, that we have the exact sequence

$$1 \longrightarrow H_{C_{\mathcal{F}}(\mathfrak{q}(n))} \longrightarrow C_P\big(\mathfrak{q}(n)\big) \xrightarrow{\tau_{\mathfrak{q}}} L_{\mathcal{F}}(\mathfrak{q}) \xrightarrow{\pi_{\mathfrak{q}}} \mathcal{F}(\mathfrak{q}) \longrightarrow 1 \qquad 3.3.3$$

and that $\pi_{\mathfrak{q}}$ and $\tau_{\mathfrak{q}}$ induce an equivalence of categories

$$\overline{N_{\mathcal{F},\mathfrak{q}(n)}(\mathfrak{q})} \cong \mathcal{F}_{L_{\mathcal{F}}(\mathfrak{q}),\mathfrak{q}(n)} \qquad 3.3.4.$$

3.4. On the other hand, let us denote by $\mathfrak{Loc}$ the category where the objects are the pairs $(L,Q)$ formed by a finite group $L$ and a normal $p$-subgroup $Q$ of $L$ and where the morphisms from $(L,Q)$ to $(L',Q')$ are the group homomorphisms $f : L \to L'$ fulfilling $f(Q) \subset Q'$. Actually, any object $(L,Q)$ admit as *inner automorphisms* the automorphisms determined by the $Q$-conjugation; we denote by $\widetilde{\mathfrak{Loc}}$ the corresponding *exterior quotient* (cf. 2.1). That is to say, the category $\widetilde{\mathfrak{Loc}}$ has the same objects as $\mathfrak{Loc}$ and the morphisms from $(L,Q)$ to $(L',Q')$ are the $Q'$-conjugacy classes of group homomorphisms $f : L \to L'$ fulfilling $f(Q) \subset Q'$. Note that we have an evident functor

$$\mathfrak{lo} : \widetilde{\mathfrak{Loc}} \longrightarrow \mathfrak{Gr} \qquad 3.4.1$$

mapping $(L,Q)$ on $L/Q$.

3.5. Assume that any element of $\mathfrak{X}$ is $\mathcal{F}$-selfcentralizing; in particular, any $Q \in \mathfrak{X}$ is fully centralized in $\mathcal{F}$ and we have [9, 4.3 and 13.2.2]

$$C_{\mathcal{F}}(Q) = \mathcal{F}_{Z(Q)} \quad \text{and} \quad H_{C_{\mathcal{F}}(Q)} = \{1\} \qquad 3.5.1.$$

Moreover, assuming the existence of a *perfect* $\mathcal{F}^{\mathfrak{x}}$*-locality* $\mathcal{P}^{\mathfrak{x}}$, it follows from 2.11 that the functor 3.2.2 induces a new functor

$$\mathfrak{loc}_{\mathcal{P}^{\mathfrak{x}}} : \mathfrak{ch}^*(\mathcal{F}^{\mathfrak{x}}) \longrightarrow \widetilde{\mathfrak{Loc}} \qquad 3.5.2$$

mapping any $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{x}})$-object $(\mathfrak{q},\Delta_n)$ on $\big(\mathcal{P}^{\mathfrak{x}}(\hat{\mathfrak{q}}), Z(\mathfrak{q}(n))\big)$ where $\hat{\mathfrak{q}} : \Delta_n \to \mathcal{P}^{\mathfrak{x}}$ is a functor lifting $\mathfrak{q}$ and we identify $Z(\mathfrak{q}(n))$ with its structural image; as a matter of fact, from the existence of the $\mathcal{F}$*-localizers* we can obtain this functor — called the $\mathcal{F}^{\mathfrak{x}}$*-localizing functor* — *without* assuming the existence of $\mathcal{P}^{\mathfrak{x}}$.

3.6. Indeed, let $(\nu,\delta):(\mathfrak{r},\Delta_m)\to(\mathfrak{q},\Delta_n)$ be a $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})$-morphism; thus, the functor $\mathfrak{aut}_{\mathcal{F}}$ gives a group homomorphism

$$\mathfrak{aut}_{\mathcal{F}}(\nu,\delta):\mathcal{F}(\mathfrak{r})\longrightarrow\mathcal{F}(\mathfrak{q}) \qquad 3.6.1$$

and, assuming that $\mathfrak{q}$ is fully normalized in $\mathcal{F}$, there is $\rho\in\mathcal{F}(\mathfrak{q})$ such that [9, Proposition 2.11]

$$\big(\mathfrak{aut}_{\mathcal{F}}(\nu,\delta)\big)\big(\mathcal{F}_P(\mathfrak{r})\big)\subset\mathcal{F}_P(\mathfrak{q})^\rho \qquad 3.6.2.$$

Then, assuming that $\mathfrak{r}$ is fully normalized in $\mathcal{F}$ too, it follows from [9, Proposition 18.16] that there is a group homomorphism

$$\lambda_{(\nu,\delta)}:L_{\mathcal{F}}(\mathfrak{r})\longrightarrow L_{\mathcal{F}}(\mathfrak{q}) \qquad 3.6.3$$

such that

$$\pi_{\mathfrak{q}}\circ\lambda_{(\nu,\delta)}=\mathfrak{aut}_{\mathcal{F}}(\nu,\delta)\circ\pi_{\mathfrak{r}} \qquad 3.6.4$$

and that, identifying $N_P(\mathfrak{q})$ and $N_P(\mathfrak{r})$ with their respective images *via* $\tau_{\mathfrak{q}}$ and $\tau_{\mathfrak{r}}$, for some $r\in L_{\mathcal{F}}(\mathfrak{q})$ lifting $\rho$ we have

$$\lambda_{(\nu,\delta)}(v)=\zeta_{(\nu,\delta)}(v)^r \qquad 3.6.5.$$

for any $v\in N_P(\mathfrak{r})$, where the $\mathcal{F}$-morphism

$$\zeta_{(\nu,\delta)}:N_P(\mathfrak{r})\cdot\mathrm{Im}\big(\mathfrak{r}(\delta(n)\bullet m)\big)\longrightarrow N_P(\mathfrak{q})\cdot\mathfrak{q}(n) \qquad 3.6.6$$

comes from [9, Proposition 2.11] and fulfillis $\zeta_{(\nu,\delta)}\big(\mathfrak{r}(\delta(n)\bullet m)(v)\big)=\rho\big(\nu_n(v)\big)$ for any $v\in\mathfrak{r}\big(\delta(n)\big)$. Now, the following statement is easily checked from [9, Proposition 18.19].

**Proposition 3.7** *Assume that any element of $\mathfrak{X}$ is $\mathcal{F}$-selfcentralizing. There is a unique isomorphism class of functors*

$$\mathfrak{loc}_{\mathcal{F}^{\mathfrak{X}}}:\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})\longrightarrow\widetilde{\mathfrak{Loc}} \qquad 3.7.1$$

*mapping any $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})$-object $(\mathfrak{q},\Delta_n)$ such that $\mathfrak{q}$ is fully normalized in $\mathcal{F}$, on $\big(L_{\mathcal{F}}(\mathfrak{q}),Z(\mathfrak{q}(n))\big)$ and any $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})$-morphism $(\nu,\delta):(\mathfrak{r},\Delta_m)\to(\mathfrak{q},\Delta_n)$ such that $\mathfrak{r}$ is also fully normalized in $\mathcal{F}$, on the $\widetilde{\mathfrak{Loc}}$-morphism*

$$\mathfrak{loc}_{\mathcal{F}^{\mathfrak{X}}}(\nu,\delta):\big(L_{\mathcal{F}}(\mathfrak{r}),Z(\mathfrak{r}(n))\big)\longrightarrow\big(L_{\mathcal{F}}(\mathfrak{q}),Z(\mathfrak{q}(n))\big) \qquad 3.7.2$$

*determined by $\lambda_{(\nu,\delta)}$. In particular, we have $\mathfrak{lo}\circ\mathfrak{loc}_{\mathcal{F}^{\mathfrak{X}}}\cong\mathfrak{aut}_{\mathcal{F}^{\mathfrak{X}}}$.*

3.8. More generally, let a category $\mathcal{L}^{\mathfrak{X}}$ endowed with two functors

$$\tau^{\mathfrak{X}}:\mathcal{T}_P^{\mathfrak{X}}\longrightarrow\mathcal{L}^{\mathfrak{X}}\quad\text{and}\quad\pi^{\mathfrak{X}}:\mathcal{L}^{\mathfrak{X}}\longrightarrow\mathcal{F}^{\mathfrak{X}} \qquad 3.8.1$$

be a $p$-coherent $\mathcal{F}^{\mathfrak{X}}$-locality and assume that $\mathrm{Ker}(\pi_Q^{\mathfrak{X}})$ is Abelian for any $Q\in\mathfrak{X}$; then, the functor $\mathfrak{aut}_{\mathcal{L}^{\mathfrak{X}}}$ still induces a new functor

$$\mathfrak{loc}_{\mathcal{L}^{\mathfrak{X}}}:\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})\longrightarrow\widetilde{\mathfrak{Loc}} \qquad 3.8.2$$

mapping any $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})$-object $(\mathfrak{q},\Delta_n)$ on $\big(\mathcal{L}^{\mathfrak{X}}(\hat{\mathfrak{q}}),\mathrm{Ker}(\pi^{\mathfrak{X}}_{\mathfrak{q}(n)})\big)$ where as above $\hat{\mathfrak{q}}:\Delta_n\to\mathcal{L}^{\mathfrak{X}}$ is a functor lifting $\mathfrak{q}$, and we still have $\mathfrak{lo}\circ\mathfrak{loc}_{\mathcal{L}^{\mathfrak{X}}}\cong\mathfrak{aut}_{\mathcal{F}^{\mathfrak{X}}}$. A critical point for our argument is the following "universal" property of the $\mathcal{F}^{\mathfrak{X}}$-*localizing functor* which is easily checked from [9, Proposition 18.21] and from 2.11 above.

**Proposition 3.9** *With the notation and hypothesis above, there is a unique natural map*

$$\lambda_{\mathcal{L}^{\mathfrak{X}}}:\mathfrak{loc}_{\mathcal{F}^{\mathfrak{X}}}\longrightarrow\mathfrak{loc}_{\mathcal{L}^{\mathfrak{X}}}\qquad 3.9.1$$

*such that* $\mathfrak{lo}*\lambda_{\mathcal{L}^{\mathfrak{X}}}=\mathrm{id}_{\mathfrak{aut}_{\mathcal{F}^{\mathfrak{X}}}}$ *and that, for any* $\mathcal{F}^{\mathfrak{X}}$*-chain* $\mathfrak{q}:\Delta_n\to\mathcal{F}^{\mathfrak{X}}$ *fully normalized in* $\mathcal{F}$, *we have* $(\lambda_{\mathcal{L}^{\mathfrak{X}}})_{(\mathfrak{q},\Delta_n)}\circ\tau^{\mathfrak{X}}_{\mathfrak{q}}=\tau^{\mathfrak{X}}_{\hat{\mathfrak{q}}}$ *where* $\hat{\mathfrak{q}}:\Delta_n\to\mathcal{L}^{\mathfrak{X}}$ *is a functor lifting* $\mathfrak{q}$. *In particular,* $\lambda_{\mathcal{L}^{\mathfrak{X}}}$ *is an isomorphism if and only if* $\mathcal{L}^{\mathfrak{X}}$ *is a perfect* $\mathcal{F}^{\mathfrak{X}}$*-locality.*

3.10. Till the end of this section, we will seek for the possible existence of a $\mathcal{F}^{\mathfrak{X}}$-sublocality $\mathcal{P}^{\mathfrak{X}}$ of $\mathcal{L}^{\mathfrak{X}}$ realizing the "image" of $\lambda_{\mathcal{L}^{\mathfrak{X}}}$ [9, 18.23-18.29]. Note that, denoting by $\bar{\mathcal{L}}^{\mathfrak{X}}$ the *quotient* $\mathcal{F}^{\mathfrak{X}}$-locality of $\mathcal{L}^{\mathfrak{X}}$ defined by

$$\bar{\mathcal{L}}^{\mathfrak{X}}(Q,R)=\mathcal{L}^{\mathfrak{X}}(Q,R)/\tau^{\mathfrak{X}}_{R}\big(Z(R)\big)\qquad 3.10.1$$

for any $Q,R\in\mathfrak{X}$, which is easily checked to be *coherent*, it follows from Proposition 3.9 that the existence of $\mathcal{P}^{\mathfrak{X}}$ is equivalent to the existence of a *section* $\mathcal{F}^{\mathfrak{X}}\to\bar{\mathcal{L}}^{\mathfrak{X}}$ of the structural functor $\bar{\pi}^{\mathfrak{X}}:\bar{\mathcal{L}}^{\mathfrak{X}}\to\mathcal{F}^{\mathfrak{X}}$, and we will do a careful choice of liftings for any $\bar{\mathcal{L}}^{\mathfrak{X}}$-morphism $\bar{x}:R\to Q$.

3.11. Thus, till the end of this section, we assume that $\tau^{\mathfrak{X}}_{Q}\big(Z(Q)\big)=\{1\}$ for any $Q\in\mathfrak{X}$. Denoting by $(Q,\Delta_0)$ the obvious $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})$-object and choosing a representative $\lambda_Q$ of the $\widetilde{\mathfrak{Loc}}$-morphism

$$(\lambda_{\mathcal{L}^{\mathfrak{X}}})_{(Q,\Delta_0)}:\big(L_{\mathcal{F}}(Q),Z(Q)\big)\longrightarrow\big(\mathcal{L}^{\mathfrak{X}}(Q),\mathrm{Ker}(\pi^{\mathfrak{X}}_{Q})\big)\qquad 3.11.1,$$

it is clear that the image of the $\mathcal{F}$-localizer in $\mathcal{L}^{\mathfrak{X}}(Q)$ is isomorphic to $\mathcal{F}(Q)$ and therefore we have

$$\mathcal{L}^{\mathfrak{X}}(Q)\cong\mathrm{Ker}(\pi^{\mathfrak{X}}_{Q})\rtimes\mathcal{F}(Q)\qquad 3.11.2.$$

Similarly, for any $\mathcal{F}^{\mathfrak{X}}$-morphism $\varphi:R\to Q$, let us denote by $(\varphi,\Delta_1)$ the $\mathfrak{ch}^*(\mathcal{F}^{\mathfrak{X}})$-object determined by the $\mathcal{F}^{\mathfrak{X}}$-*chain* mapping 0 on $R$, 1 on $Q$ and the $\Delta_1$-morphism $0\bullet 1$ on $\varphi$; then, the *divisibility* of $\mathcal{F}$ [9, 2.4] forces a canonical isomorphism

$$\mathfrak{aut}_{\mathcal{F}^{\mathfrak{X}}}(\varphi,\Delta_1)\cong\mathcal{F}(Q)_{R'}\qquad 3.11.3$$

where we set $R' = \varphi(R)$ and, as usual, $\mathcal{F}(Q)_{R'}$ denotes the stabilizer of $R'$ in $\mathcal{F}(Q)$, and moreover the functor $\mathfrak{aut}_{\mathcal{F}^{\mathfrak{x}}}$ [9, Proposition A2.10] maps the obvious $\mathfrak{ch}^*(\mathcal{F}^{^\mathfrak{x}})$-morphism

$$(\mathrm{id}_R, \delta_1^0)_\varphi : (\varphi, \Delta_1) \longrightarrow (R, \Delta_0) \qquad 3.11.4,$$

in the group homomorphism $\mathcal{F}(Q)_{R'} \to \mathcal{F}(R)$ mapping $\rho \in \mathcal{F}(Q)_{R'}$ on the unique $\sigma \in \mathcal{F}(R)$ fulfilling $\varphi \circ \sigma = \rho \circ \varphi$.

3.12. *Mutatis mutandis*, for any $\mathcal{L}^{^\mathfrak{x}}$-morphism $x : R \to Q$, let us denote by $(x, \Delta_1)$ the $\mathfrak{ch}^*(\mathcal{L}^{^\mathfrak{x}})$-object determined by the $\mathcal{L}^{^\mathfrak{x}}$-*chain* mapping 0 on $R$, 1 on $Q$ and the $\Delta_1$-morphism $0 \bullet 1$ on $x$; then, the *divisibility* of $\mathcal{L}^{^\mathfrak{x}}$ [9, 17.7] forces a canonical isomorphism

$$\mathfrak{aut}_{\mathcal{L}^{\mathfrak{x}}}(x, \Delta_1) \cong \mathcal{L}^{^\mathfrak{x}}(Q)_{R'} \qquad 3.12.1$$

where we set $R' = \big(\pi^{\mathfrak{x}}_{Q,R}(x)\big)(R)$ and, as above, $\mathcal{L}^{^\mathfrak{x}}(Q)_{R'}$ denotes the stabilizer of $R'$ in $\mathcal{L}^{^\mathfrak{x}}(Q)$, and moreover the functor $\mathfrak{aut}_{\mathcal{L}^{\mathfrak{x}}}$ [9, Proposition A2.10] maps the obvious $\mathfrak{ch}^*(\mathcal{L}^{^\mathfrak{x}})$-morphism

$$(\tau_R(1), \delta_1^0)_x : (x, \Delta_1) \longrightarrow (R, \Delta_0) \qquad 3.12.2,$$

in the group homomorphism $c_x : \mathcal{L}^{^\mathfrak{x}}(Q)_{R'} \to \mathcal{L}^{^\mathfrak{x}}(R)$ mapping $r \in \mathcal{L}^{^\mathfrak{x}}(Q)_{R'}$ on the unique $s \in \mathcal{L}^{^\mathfrak{x}}(R)$ fulfilling $x{\cdot}s = r{\cdot}x$. Then, the functor $\mathfrak{loc}_{\mathcal{L}^{\mathfrak{x}}}$ in 3.8 above maps $(x, \Delta_1)$ on $\big(\mathcal{L}^{^\mathfrak{x}}(Q)_{R'}, \mathrm{Ker}(\pi^{\mathfrak{x}}_Q)\big)$ and, setting $\varphi = \pi^{\mathfrak{x}}_{Q,R}(x)$, it maps $(\mathrm{id}_R, \delta_1^0)_\varphi$ in the $\widetilde{\mathfrak{Loc}}$-morphism

$$\big(\mathcal{L}^{^\mathfrak{x}}(Q)_{R'}, \mathrm{Ker}(\pi^{\mathfrak{x}}_Q)\big) \longrightarrow \big(\mathcal{L}^{^\mathfrak{x}}(R), \mathrm{Ker}(\pi^{\mathfrak{x}}_R)\big) \qquad 3.12.3$$

determined by $c_x$.

3.13. Moreover, the structural functor $\pi^{\mathfrak{x}} : \mathcal{L}^{^\mathfrak{x}} \to \mathcal{F}^{^\mathfrak{x}}$ determines a *natural map* [9. Proposition A2.10]

$$\mathfrak{aut}_{\pi^{\mathfrak{x}}} : \mathfrak{aut}_{\mathcal{L}^{\mathfrak{x}}} \longrightarrow \mathfrak{aut}_{\mathcal{F}^{\mathfrak{x}}} \circ \mathfrak{ch}^*(\pi^{\mathfrak{x}}) \qquad 3.13.1$$

which, identifying $\mathfrak{Gr}$ with the *full subcategory* of $\widetilde{\mathfrak{Loc}}$ over the objects $(G, \{1\})$ where $G$ runs over the finite groups, it factorizes through a *natural map*

$$\mathfrak{loc}_{\pi^{\mathfrak{x}}} : \mathfrak{loc}_{\mathcal{L}^{\mathfrak{x}}} \longrightarrow \mathfrak{aut}_{\mathcal{F}^{\mathfrak{x}}} \qquad 3.13.2;$$

then, this natural map and the natural map $\lambda_{\mathcal{L}^{\mathfrak{x}}} : \mathfrak{loc}_{\mathcal{F}^{\mathfrak{x}}} \to \mathfrak{loc}_{\mathcal{L}^{\mathfrak{x}}}$ in Proposition 3.9, both applied to the $\mathfrak{ch}^*(\mathcal{F}^{^\mathfrak{x}})$-morphism 3.11.4, yield the following commutative $\widetilde{\mathfrak{Loc}}$-diagram

$$\begin{array}{ccccc}\big(L_{\mathcal{F}}(R),Z(R)\big) & \xrightarrow{(\lambda_{\mathcal{L}^{\mathfrak{X}}})_{(R,\Delta_0)}} & \big(\mathcal{L}^{\mathfrak{X}}(R),\mathrm{Ker}(\pi^{\mathfrak{X}}_R)\big) & \longrightarrow & \mathcal{F}(R)\\ \big\uparrow\mathfrak{loc}_{\mathcal{F}^{\mathfrak{X}}}((\mathrm{id}_R,\delta^0_1)_\varphi) & & \mathfrak{loc}_{\mathcal{L}^{\mathfrak{X}}}((\mathrm{id}_R,\delta^0_1)_\varphi)\big\uparrow & & \mathfrak{aut}_{\mathcal{F}^{\mathfrak{X}}}((\mathrm{id}_R,\delta^0_1)_\varphi)\big\uparrow\\ \big(L_{\mathcal{F}}(Q)_{R'},Z(Q)\big) & \xrightarrow{(\lambda_{\mathcal{L}^{\mathfrak{X}}})_{(\varphi,\Delta_1)}} & \big(\mathcal{L}^{\mathfrak{X}}(Q)_{R'},\mathrm{Ker}(\pi^{\mathfrak{X}}_Q)\big) & \longrightarrow & \mathcal{F}(Q)_{R'}\end{array} \qquad 3.13.3;$$

hence, for a suitable $x_\varphi\in\mathcal{L}^{\mathfrak{X}}(Q,R)$ lifting $\varphi\in\mathcal{F}(Q,R)$, we still have the commutative diagram of *group homomorphisms*

$$\begin{array}{ccccc}L_{\mathcal{F}}(R) & \xrightarrow{\lambda_R} & \mathcal{L}^{\mathfrak{X}}(R) & \longrightarrow & \mathcal{F}(R)\\ \uparrow & & c_{x_\varphi}\uparrow & & \uparrow\\ L_{\mathcal{F}}(Q)_{R'} & \longrightarrow & \mathcal{L}^{\mathfrak{X}}(Q)_{R'} & \longrightarrow & \mathcal{F}(Q)_{R'}\end{array} \qquad 3.13.4.$$

3.14. Now, it is clear that the images of the horizontal left-hand homomorphisms supplie sections for the horizontal right-hand homomorphisms; in particular, identifying $\mathcal{F}(R)$ and $\mathcal{F}(Q)_{R'}$ with their respective images in $\mathcal{L}^{\mathfrak{X}}(R)$ and $\mathcal{L}^{\mathfrak{X}}(Q)_{R'}$, for any $\rho\in\mathcal{F}(Q)_{R'}$ and any $\sigma\in\mathcal{F}(R)$ fulfilling $\varphi\circ\sigma=\rho\circ\varphi$, in $\mathcal{L}^{\mathfrak{X}}(Q,R)$ we still have $x_\varphi\cdot\sigma=\rho\cdot x_\varphi$. Consequently, considering a set of representatives in $\mathfrak{X}$ for the set of its $\mathcal{F}$-isomorphism classes, and the action of $\mathcal{F}(Q)\times\mathcal{F}(R)$ on $\mathcal{L}^{\mathfrak{X}}(Q,R)$ defined by the composition on the left and on the right *via* the inclusions $\mathcal{F}(Q)\subset\mathcal{L}^{\mathfrak{X}}(Q)$ and $\mathcal{F}(R)\subset\mathcal{L}^{\mathfrak{X}}(R)$ chosen in 3.10, for any $\varphi\in\mathcal{F}(Q,R)$ we can choose a lifting $x_\varphi\in\mathcal{L}^{\mathfrak{X}}(Q,R)$ in such a way that [9, 18.27.3]

3.14.1 *We have $x_{\rho\circ\varphi\circ\sigma}=\rho\cdot x_\varphi\cdot\sigma$ for any $\rho\in\mathcal{F}(Q)$ and any $\sigma\in\mathcal{F}(R)$.*

## 4. The key result

4.1. Let $k$ be a field of characteristic $p$ and denote by $k$-$\mathfrak{mod}$ the category of finite dimensional $k$-vector spaces. Assume that any element of $\mathfrak{X}$ is $\mathcal{F}$-selfcentralizing, denote by $\tilde{\mathcal{F}}^{\mathfrak{X}}$ the usual *exterior* quotient of $\mathcal{F}^{\mathfrak{X}}$ (cf. 2.1) and consider a *contravariant* functor $\mathfrak{m}^{\mathfrak{X}}:\tilde{\mathcal{F}}^{\mathfrak{X}}\to k\text{-}\mathfrak{mod}$. In this section, we will prove the *nullity* of the *stable cohomology groups* $\mathbb{H}^n_*(\tilde{\mathcal{F}}^{\mathfrak{X}},\mathfrak{m}^{\mathfrak{X}})$ for $n\geq 1$.

4.2. Recall that the usual *cohomology* groups are defined by

$$\mathbb{H}^n(\tilde{\mathcal{F}}^{\mathfrak{X}},\mathfrak{m}^{\mathfrak{X}})=\mathrm{Ker}(d_n)/\mathrm{Im}(d_{n-1}) \qquad 4.2.1$$

where, denoting by $\mathfrak{Fct}(\Delta_n,\tilde{\mathcal{F}}^{\mathfrak{X}})$ the set of functors from $\Delta_n$ to $\tilde{\mathcal{F}}^{\mathfrak{X}}$, and setting

$$\mathrm{Im}(d_{-1})=\{0\}\quad\text{and}\quad\mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{X}},\mathfrak{m}^{\mathfrak{X}})=\prod_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\tilde{\mathcal{F}}^{\mathfrak{X}})}\mathfrak{m}^{\mathfrak{X}}\big(\mathfrak{q}(0)\big) \qquad 4.2.2,$$

for any $n \in \mathbb{N}$ we denote by

$$d_n : \mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}) \longrightarrow \mathbb{C}^{n+1}(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}) \qquad 4.2.3$$

the usual *differential* map [9, A3.11]. Then, for any $n \in \mathbb{N}$ consider the $k$-submodule $\mathbb{C}^n_*(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}})$ of *stable elements* of $\mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}})$, namely the elements $m = (m_{\mathfrak{q}})_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n, \tilde{\mathcal{F}}^{\mathfrak{X}})}$ fulfilling

$$m_{\mathfrak{q}} = \big(\mathfrak{m}(\tilde{\nu}_0)\big)(m_{\mathfrak{q}'}) \qquad 4.2.4$$

for any natural isomorphism $\tilde{\nu} : \mathfrak{q} \cong \mathfrak{q}'$ between two $\tilde{\mathcal{F}}^{\mathfrak{X}}$*-chains* $\mathfrak{q} : \Delta_n \to \tilde{\mathcal{F}}^{\mathfrak{X}}$ and $\mathfrak{q}' : \Delta_n \to \tilde{\mathcal{F}}^{\mathfrak{X}}$; it is easily checked that, for any $n \in \mathbb{N}$, the differential map $d_n$ preserve the *stable elements* and, denoting by $d_n^*$ the corresponding restriction, we define

$$\mathbb{H}^n_*(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}) = \mathrm{Ker}(d_n^*)/\mathrm{Im}(d_{n-1}^*) \qquad 4.2.5.$$

4.3. In order to prove that these groups vanish for $n \geq 1$, let us recall some features of the category $\tilde{\mathcal{F}}^{\mathfrak{X}}$. It follows from [9, Corollary 4.9] that, for any triple of subgroups $Q$, $R$ and $T$ in $\mathfrak{X}$, any $\tilde{\mathcal{F}}$-morphism $\tilde{\alpha} : Q \to R$ induces an injective map from $\tilde{\mathcal{F}}(T, R)$ to $\tilde{\mathcal{F}}(T, Q)$ and then we set

$$\tilde{\mathcal{F}}(T, Q)_{\tilde{\alpha}} = \tilde{\mathcal{F}}(T, Q) - \bigcup_{\tilde{\theta}'} \tilde{\mathcal{F}}(T, Q') \circ \tilde{\theta}' \qquad 4.3.1,$$

where $\tilde{\theta}'$ runs over the set of $\tilde{\mathcal{F}}$*-nonisomorphisms* $\tilde{\theta}' : Q \to Q'$ from $Q$ or, equivalently, the set of *nonfinal* $(\tilde{\mathcal{F}}^{\circ})_Q$-objects (cf. 2.1) fulfilling $\tilde{\alpha}' \circ \tilde{\theta}' = \tilde{\alpha}$ for some $\tilde{\alpha}' \in \tilde{\mathcal{F}}(R, Q')$; in this case, according to [9, Corollary 4.9], $\tilde{\alpha}'$ is uniquely determined, and we simply say that $\tilde{\theta}'$ *divides* $\tilde{\alpha}$ setting $\tilde{\alpha}' = \tilde{\alpha}/\tilde{\theta}'$. Note that the existence of $\tilde{\alpha}'$ is equivalent to the existence of a subgroup of $R$ which is $\mathcal{F}$-isomorphic to $Q'$ and contains $\alpha(Q)$ for a representative $\alpha \in \tilde{\alpha}$; in particular, we have $\tilde{\mathcal{F}}(T, Q)_{\tilde{\alpha}} = \tilde{\mathcal{F}}(T, Q)$ if and only if $\tilde{\alpha}$ is an isomorphism.

4.4. Actually, an element $\tilde{\beta} \in \tilde{\mathcal{F}}(T, Q)$ which can be extended to $Q'$ *via* $\tilde{\theta}'$, *a fortiori* it can be extended to $N_{Q'}\big(\theta'(Q)\big)$ for a representative $\theta' \in \tilde{\theta}'$; hence, it follows from condition 2.2.3 above that $\tilde{\beta}$ belongs to $\tilde{\mathcal{F}}(T, Q)_{\tilde{\alpha}}$ if and only if, for some representative $\beta \in \tilde{\beta}$, we have

$$^{\alpha^*}\mathcal{F}_R\big(\alpha(Q)\big) \cap {}^{\beta^*}\mathcal{F}_T\big(\beta(Q)\big) = \mathcal{F}_Q(Q) \qquad 4.4.1$$

where $\alpha^* : \alpha(Q) \cong Q$ and $\beta^* : \beta(Q) \cong Q$ denote the inverse of the isomorphisms respectively induced by $\alpha$ and $\beta$, and, in particular, we get

4.4.2 $\tilde{\beta} \in \tilde{\mathcal{F}}(T, Q)_{\tilde{\alpha}}$ *is equivalent to* $\tilde{\alpha} \in \tilde{\mathcal{F}}(R, Q)_{\tilde{\beta}}$.

Moreover, the quotient

$$N_R\big(\alpha(Q)\big)/\alpha(Q) = \bar N_R\big(\alpha(Q)\big) \cong {}^{\alpha^*}\tilde{\mathcal{F}}_R\big(\alpha(Q)\big) \qquad 4.4.3$$

clearly acts on $\tilde{\mathcal{F}}(T,Q)_{\tilde\alpha}$ by composition on the right-hand, and from equality 4.4.1 it is easily checked that [9, 6.6.4]

4.4.4 $\bar N_R\big(\alpha(Q)\big)$ *acts freely on* $\tilde{\mathcal{F}}(T,Q)_{\tilde\alpha}$ *. In particular, if* $\tilde\alpha$ *is not an* $\tilde{\mathcal{F}}$*-isomorphism then* $p$ *divides* $|\tilde{\mathcal{F}}(T,Q)_{\tilde\alpha}|$ *.*

The next result follows from [9, Proposition 6.7].

**Proposition 4.5** *For any triple of subgroups* $Q$ *,* $R$ *and* $T$ *in* $\mathfrak{X}$ *and any* $\tilde\alpha \in \tilde{\mathcal{F}}(R,Q)$ *, we have*

$$\tilde{\mathcal{F}}(T,Q) = \bigsqcup_{\tilde\theta'} \tilde{\mathcal{F}}(T,Q')_{\tilde\alpha/\tilde\theta'} \circ \tilde\theta' \qquad 4.5.1$$

*where* $\tilde\theta' : Q \to Q'$ *runs over a set of representatives for the isomorphism classes of* $(\tilde{\mathcal{F}}^\circ)_Q$*-objects dividing* $\tilde\alpha$ *. Moreover,* $p$ *does not divide* $|\tilde{\mathcal{F}}(P,Q)|$ *.*

4.6. If $Q$ is a subgroup of $P$ and $R$ a subgroup of $Q$ , we denote by $\iota_R^Q$ the group homomorphism determined by the inclusion. Now, for any $\tilde{\mathcal{F}}^{\mathfrak{X}}$*-chain* $\mathfrak{q} : \Delta_n \to \tilde{\mathcal{F}}^{\mathfrak{X}}$ (cf. 3.1), let us denote by $\mathcal{W}_{\mathfrak{q}}$ the set of pairs $(\tilde\mu, \mathfrak{q}')$ formed by an $\tilde{\mathcal{F}}^{\mathfrak{X}}$*-chain* $\mathfrak{q}' : \Delta_n \to \tilde{\mathcal{F}}^{\mathfrak{X}}$ fulfilling

$$\mathfrak{q}'(i-1) \subset \mathfrak{q}'(i) \quad \text{and} \quad \mathfrak{q}'(i{-}1 \bullet i) = \tilde\iota_{\mathfrak{q}'(i-1)}^{\mathfrak{q}'(i)} \qquad 4.6.1$$

for any $1 \le i \le n$ , and by a natural map $\tilde\mu : \mathfrak{q}' \to \mathfrak{q}$ such that $\tilde\iota_{\mathfrak{q}'(i)}^P$ belongs to $\tilde{\mathcal{F}}\big(P, \mathfrak{q}'(i)\big)_{\tilde\mu_i}$ for any $i \in \Delta_n$ (cf. 4.3.1). Note that, for any $1 \le i \le n$ , applying Proposition 4.5 to the $\mathcal{F}$-morphism $\tilde\iota_{\mathfrak{q}'(i-1)}^P$ , the composition

$$\mathfrak{q}(i-1\bullet i) \circ \tilde\mu_{i-1} : \mathfrak{q}'(i-1) \longrightarrow \mathfrak{q}(i) \qquad 4.6.2$$

determines a subgroup $\mathfrak{q}'(i)$ of $P$ containing $\mathfrak{q}'(i-1)$ and an $\tilde{\mathcal{F}}$-morphism $\tilde\mu_i \in \tilde{\mathcal{F}}\big(\mathfrak{q}(i), \mathfrak{q}'(i)\big)$ fulfilling

$$\mathfrak{q}(i-1\bullet i) \circ \tilde\mu_{i-1} = \tilde\mu_i \circ \tilde\iota_{\mathfrak{q}'(i-1}^{\mathfrak{q}(i)} \qquad 4.6.3;$$

that is to say, the pair $(\tilde\mu, \mathfrak{q}')$ is actually determined by the subgroup $\mathfrak{q}'(0)$ and the $\tilde{\mathcal{F}}$-morphism $\tilde\mu_0 : \mathfrak{q}'(0) \to \mathfrak{q}(0)$ .

4.7. For any $(\tilde\mu, \mathfrak{q}') \in \mathcal{W}_{\mathfrak{q}}$ and any $\ell \in \Delta_n$ , recall that we denote by $\mathfrak{h}_\ell^n(\tilde\mu) : \Delta_{n+1} \to \tilde{\mathcal{F}}^{\mathfrak{X}}$ the functor which coincides with $\mathfrak{q}'$ over $\Delta_\ell$ and maps $i \in \Delta_{n+1} - \Delta_\ell$ on $\mathfrak{q}(i-1)$ and $\ell \bullet \ell{+}1$ on $\tilde\mu_\ell : \mathfrak{q}'(\ell) \to \mathfrak{q}(\ell)$ [9, Lemma A4.2]; moreover, denote by

$$\mathfrak{h}_{n+1}^n(\hat\mu) : \Delta_{n+1} \longrightarrow \tilde{\mathcal{F}}^{\hat{\mathfrak{X}}} \qquad 4.7.1$$

the $\tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$-chain extending $q'$ and mapping $n+1$ on $P$ and $n \bullet n+1$ on $\tilde{\iota}^{P}_{\mathfrak{q}'(n)}$. Then, any $\mathfrak{ch}^*(\tilde{\mathcal{F}}^{\mathfrak{X}})$-isomorphism $\tilde{\nu} : \mathfrak{q} \cong \mathfrak{r}$ between two $\tilde{\mathcal{F}}^{\mathfrak{X}}$-chains clearly induces a bijection between $\mathcal{W}_{\mathfrak{q}}$ and $\mathcal{W}_{\mathfrak{r}}$ mapping $(\tilde{\mu}, \mathfrak{q}') \in \mathcal{W}_{\mathfrak{q}}$ on $(\tilde{\nu} \circ \tilde{\mu}, \mathfrak{q}')$ and, for any $\ell \in \Delta_{n+1}$, we have the natural isomorphism

$$\mathfrak{h}^n_\ell(\tilde{\nu}) : h^n_\ell(\tilde{\mu}) \cong h^n_\ell(\tilde{\nu} \circ \tilde{\mu}) \qquad 4.7.2$$

mapping $i \in \Delta_\ell$ on $\widetilde{\mathrm{id}}_{\mathfrak{q}'(i)}$ or on $\widetilde{\mathrm{id}}_P$ if $i = \ell = n+1$, and $i \in \Delta_{n+1} - \Delta_\ell$ on $\tilde{\nu}_{i-1}$; note that the bijection $\mathcal{W}_{\mathfrak{q}} \cong \mathcal{W}_{\mathfrak{r}}$ is already induced by $\tilde{\nu}_0$. Moreover, for any $u \in P$, it makes sense to consider $\mathfrak{q}'^u$ and the conjugation by $u$ defines a *natural isomorphism* $\tilde{\kappa}^{\mathfrak{q}'}_u : \mathfrak{q}'^u \cong \mathfrak{q}'$; then, the pair $(\tilde{\mu} \circ (\tilde{\kappa}^{\mathfrak{q}'}_u)^{-1}, \mathfrak{q}'^u)$ still belongs to $\mathcal{W}_{\mathfrak{q}}$ and we have an analogous natural isomorphism

$$\bar{\mathfrak{h}}^n_\ell(\tilde{\kappa}^{\mathfrak{q}'}_u) : \mathfrak{h}^n_\ell(\tilde{\mu} \circ (\tilde{\kappa}^{\mathfrak{q}'}_u)^{-1}) \cong \mathfrak{h}^n_\ell(\tilde{\mu}) \qquad 4.7.3.$$

**Teorem 4.8.** *For any contravariant functor* $\mathfrak{m}^{\mathfrak{X}} : \tilde{\mathcal{F}}^{\mathfrak{X}} \longrightarrow k\text{-}\mathfrak{mod}$ *and any* $n \geq 1$, *we have*

$$\mathbb{H}^n_*(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}) = \{0\} \qquad 4.8.1.$$

**Proof:** Since $\tilde{\mathcal{F}}(P)$ is a $p'$-group (cf. 2.2.2), if $\mathfrak{X} = \{P\}$ then we clearly have [9, Proposition A4.13]

$$\mathbb{H}^n_*\big(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}\big) = \mathbb{H}^n\big(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}\big) = \mathbb{H}^n\big(\tilde{\mathcal{F}}(P), \mathfrak{m}(P)\big) = \{0\} \qquad 4.8.2.$$

Assuming that $\mathfrak{X} \neq \{P\}$, we argue by induction on $|\mathfrak{X}|$ and, setting

$$\mathfrak{X} = \mathfrak{Y} \sqcup \{\theta(U) \mid \theta \in \mathcal{F}(P, U)\} \qquad 4.8.3,$$

for a minimal element $U \in \mathfrak{X}$, we may assume that for any $n \leq 1$ we have

$$\mathbb{H}^n_*(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \mathfrak{m}^{\mathfrak{Y}}) = \{0\} \qquad 4.8.4$$

where $\mathfrak{m}^{\mathfrak{Y}}$ denotes the restriction of $\mathfrak{m}^{\mathfrak{X}}$ to $\tilde{\mathcal{F}}^{\mathfrak{Y}}$.

With the notation in 4.2 above, consider the following commutative diagram

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{C}^0_*(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \mathfrak{m}^{\mathfrak{Y}}) & \xrightarrow{d_0^{\mathfrak{Y}}} & \mathbb{C}^1_*(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \mathfrak{m}^{\mathfrak{Y}}) & \xrightarrow{d_1^{\mathfrak{Y}}} & \mathbb{C}^2_*(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \mathfrak{m}^{\mathfrak{Y}}) & \xrightarrow{d_2^{\mathfrak{Y}}} & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{C}^0_*(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}) & \xrightarrow{d_0^{\mathfrak{X}}} & \mathbb{C}^1_*(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}) & \xrightarrow{d_1^{\mathfrak{X}}} & \mathbb{C}^2_*(\tilde{\mathcal{F}}^{\mathfrak{X}}, \mathfrak{m}^{\mathfrak{X}}) & \xrightarrow{d_2^{\mathfrak{X}}} & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{K}^0_* & \longrightarrow & \mathbb{K}^1_* & \longrightarrow & \mathbb{K}^2_* & \longrightarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array} \qquad 4.8.5$$

where the top and the vertical sequences are exact; now, in order to prove that the middle sequence is exact, it is easily checked that it suffices to show that the bottom sequence is so.

For $n \geq 1$, let $\mathrm{m} = (m_\mathfrak{q})_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\tilde{\mathcal{F}}^x)}$ be a *stable* $\mathfrak{m}^x$*-valued* $n$*-cocycle* in $\mathbb{K}_*^n$; that is to say, $m_\mathfrak{q}$ belongs to $\mathfrak{m}^x\big(\mathfrak{q}(0)\big)$, it is equal to zero whenever $\mathfrak{q}(0) \in \mathfrak{Y}$ and we have

$$d_n^x(\mathrm{m}) = 0 \quad \text{and} \quad \big(\mathfrak{m}^x(\tilde{\alpha}_0)\big)(m_{\mathfrak{q}'}) = m_\mathfrak{q} \qquad 4.8.6$$

for any natural isomorphism $\tilde{\alpha}:\mathfrak{q} \cong \mathfrak{q}'$. Consider an $\tilde{\mathcal{F}}^x$-chain $\mathfrak{r}:\Delta_{n-1} \to \tilde{\mathcal{F}}^x$ such that $\mathfrak{r}(0) \notin \mathfrak{Y}$; according to 4.6, any pair in the set $\mathcal{W}_\mathfrak{r}$ is determined by an element of $\mathcal{F}\big(P,\mathfrak{r}(0)\big)$; let us denote by $(\mathfrak{r}_\eta,\tilde{\mu}_\eta)$ the pair determined by $\eta \in \mathcal{F}\big(P,\mathfrak{r}(0)\big)$, so that we have $\mathfrak{r}_\eta(0) = \eta\big(\mathfrak{r}(0)\big)$ and the isomorphism $\eta_*:\mathfrak{r}(0) \cong \mathfrak{r}_\eta(0)$ induced by $\eta$ is a representative of $(\tilde{\mu}_\eta)_0^{-1}$.

At this point, since $p$ does not divide $\big|\tilde{\mathcal{F}}\big(P,\mathfrak{r}(0)\big)\big|$ (cf. Proposition 4.5), we define an element $n_\mathfrak{r}$ of $\mathfrak{m}^x\big(\mathfrak{r}(0)\big)$ by setting

$$n_\mathfrak{r} = \big|\tilde{\mathcal{F}}\big(P,\mathfrak{r}(0)\big)\big|^{-1}\cdot \sum_{\tilde{\eta}\in\tilde{\mathcal{F}}(P,\mathfrak{r}(0))} \mathfrak{m}^x(\widetilde{\eta_*})\Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\tilde{\mu}_\eta)}\Big) \qquad 4.8.7$$

where, for any $\tilde{\eta} \in \tilde{\mathcal{F}}(P,\mathfrak{r}(0))$, $\eta$ denotes a representative of $\tilde{\eta}$; note that, for another representative $\eta' = \eta^u$ of $\tilde{\eta}$ where $u \in P$, we have a *natural isomorphism* (cf. 4.7.3)

$$\bar{\mathfrak{h}}_\ell^n(\tilde{\kappa}_u^{\mathfrak{r}_\eta}) : \mathfrak{h}_\ell^n(\tilde{\mu}_{\eta'}) \cong \mathfrak{h}_\ell^n(\tilde{\mu}_\eta) \qquad 4.8.8$$

and therefore, since $\bar{\mathfrak{h}}_\ell^n(\tilde{\kappa}_u^{\mathfrak{r}_\eta})_0 = (\widetilde{\eta'_*})^{-1}\circ\widetilde{\eta_*}$, the *stability* of $a$ guarantees that, for any $\ell \in \Delta_n$, we have (cf. 4.2.4)

$$\big(\mathfrak{m}^x(\widetilde{\eta'_*})\big)(m_{\mathfrak{h}_\ell^{n-1}(\tilde{\mu}_{\eta'})}) = \big(\mathfrak{m}^x(\widetilde{\eta_*})\big)(m_{\mathfrak{h}_\ell^{n-1}(\tilde{\mu}_\eta)}) \qquad 4.8.9.$$

Moreover, if $\tilde{\nu}:\mathfrak{r} \cong \mathfrak{r}'$ is a natural isomorphism between $\tilde{\mathcal{F}}^x$-chains then a representative $\nu_0$ of $\tilde{\nu}_0$ induces a bijection between $\mathcal{F}\big(P,\mathfrak{r}(0)\big)$ and $\mathcal{F}\big(P,\mathfrak{r}'(0)\big)$ and, setting $\eta' = \eta\circ\nu_0^{-1}$ for any $\eta \in \mathcal{F}\big(P,\mathfrak{r}(0)\big)$, it is quite clear that we have

$$\mathfrak{r}_{\eta'} = \mathfrak{r}_\eta \quad \text{and} \quad \tilde{\mu}_{\eta'} = \tilde{\nu}\circ\tilde{\mu}_\eta \qquad 4.8.10$$

and, since $\mathfrak{h}_\ell^{n-1}(\tilde{\nu})_0 = \widetilde{\mathrm{id}}_{\mathfrak{r}_\eta(0)}$, we get $m_{\mathfrak{h}_\ell^{n-1}(\tilde{\mu}_{\eta'})} = m_{\mathfrak{h}_\ell^{n-1}(\tilde{\mu}_\eta)}$ for any $\ell \in \Delta_n$; consequently, we obtain

$$\begin{aligned} n_{\mathfrak{r}'} &= \big|\tilde{\mathcal{F}}\big(P,\mathfrak{r}'(0)\big)\big|^{-1}\cdot \sum_{\tilde{\eta}'\in\tilde{\mathcal{F}}(P,\mathfrak{r}'(0))} \mathfrak{m}^x(\widetilde{\eta'_*})\Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\tilde{\mu}_{\eta'})}\Big) \\ &= \big|\tilde{\mathcal{F}}\big(P,\mathfrak{r}(0)\big)\big|^{-1}\cdot \sum_{\tilde{\eta}\in\tilde{\mathcal{F}}(P,\mathfrak{r}(0))} \mathfrak{m}^x(\tilde{\nu}_0)^{-1}\circ\mathfrak{m}^x(\widetilde{\eta_*})\Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\tilde{\mu}_\eta)}\Big) \qquad 4.8.11. \\ &= \mathfrak{m}^x(\tilde{\nu}_0)^{-1}(n_\mathfrak{r}) \end{aligned}$$

Hence, the family $\mathrm{n} = (n_{\mathfrak{r}})_{\mathfrak{r}\in\mathfrak{Fct}(\Delta_{n-1},\tilde{\mathcal{F}}^{\hat{\mathfrak{x}}})}$, where we set $n_{\mathfrak{r}} = 0$ whenever $\mathfrak{r}(0)$ belongs to $\mathfrak{Y}$, is *stable* and we claim that $d^{\mathfrak{x}}_{n-1}(\mathrm{n}) = \mathrm{m}$.

Indeed, for any $\tilde{\mathcal{F}}^{\mathfrak{x}}$*-chain* $\mathfrak{q}:\Delta_n \to \tilde{\mathcal{F}}^{\mathfrak{x}}$ such that $\mathfrak{q}(0) \notin \mathfrak{Y}$ and any $\theta \in \mathcal{F}\big(P, \mathfrak{q}(0)\big)$, considering the *components* of the *differential* of m at the $\tilde{\mathcal{F}}^{\mathfrak{x}}$*-chains* $\mathfrak{h}^n_\ell(\tilde{\mu}_\theta): \Delta_{n+1} \to \tilde{\mathcal{F}}^{\mathfrak{x}}$ for any $\ell \in \Delta_{n+1}$, we get [9, A3.11.2]

$$\begin{aligned}
0 &= \big(\mathfrak{m}^{\mathfrak{x}}((\tilde{\mu}_\theta)_0)\big)(m_{\mathfrak{q}}) + \sum_{i=1}^{n+1}(-1)^i m_{\mathfrak{h}^n_0(\tilde{\mu}_\theta)\circ\delta^n_i} \\
0 &= \big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}_\theta(0\bullet 1))\big)(m_{\mathfrak{h}^n_{\ell+1}(\tilde{\mu}_\theta)\circ\delta^n_0}) + \sum_{i=1}^{n+1}(-1)^i m_{\mathfrak{h}^n_{\ell+1}(\tilde{\mu}_\theta)\circ\delta^n_i}
\end{aligned} \qquad 4.8.12.$$

for any $\ell \in \Delta_n$. Let us consider the sum of all the second members of these equalities indexed by $\ell \in \Delta_{n+1}$, respectively multiplied by $(-1)^\ell$; it follows from [9, Lemma A4.2] that we have the cancellation of the $(\ell+1)$-th terms $(-1)^{\ell+1} m_{\mathfrak{h}^n_\ell(\tilde{\mu}_\theta)\circ\delta^n_{\ell+1}}$ and $(-1)^{\ell+1} m_{\mathfrak{h}^n_{\ell+1}(\tilde{\mu}_\theta)\circ\delta^n_{\ell+1}}$ in the $\ell$-th and $(\ell+1)$-th equalities for any $\ell \in \Delta_n$. Moreover, since $(\mathfrak{q}_\theta, \tilde{\mu}_\theta)$ belongs to $X_{\mathfrak{q}}$, it is clear that for any $i \in \Delta_n$ the pair $(\mathfrak{q}_\theta \circ \delta^{n-1}_i, \tilde{\mu}_\theta * \delta^{n-1}_i)$ belongs to $X_{\mathfrak{q}\circ\delta^{n-1}_i}$ and that we have

$$\mathfrak{h}^n_{n+1}(\tilde{\mu}_\theta) \circ \delta^n_i = \mathfrak{h}^{n-1}_n(\tilde{\mu}_\theta * \delta^{n-1}_i) \qquad 4.8.13.$$

It follows from [9, Lemma A4.2] and from equality 4.8.13 above that the alternating sum of all the first terms in equalities 4.8.12 above yields

$$\big(\mathfrak{m}^{\mathfrak{x}}((\tilde{\mu}_\theta)_0)\big)(m_{\mathfrak{q}}) - \big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}_\theta(0\bullet 1))\big)\Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}^{n-1}_\ell(\tilde{\mu}_\theta*\delta^{n-1}_0)}\Big) \qquad 4.8.14.$$

First of all, since $(\tilde{\mu}_\theta)_0 = (\widetilde{\theta_*})^{-1}$ and $p$ does not divide $\big|\tilde{\mathcal{F}}\big(P, \mathfrak{q}(0)\big)\big|$, we have

$$\big|\tilde{\mathcal{F}}\big(P, \mathfrak{q}(0)\big)\big|^{-1} \sum_{\tilde{\theta}\in\tilde{\mathcal{F}}(P,\mathfrak{q}(0))} \mathfrak{m}^{\mathfrak{x}}(\widetilde{\theta_*})\Big(\big(\mathfrak{m}^{\mathfrak{x}}((\tilde{\mu}_\theta)_0)\big)(m_{\mathfrak{q}})\Big) = m_{\mathfrak{q}} \qquad 4.8.15;$$

moreover, for any $\theta \in \mathcal{F}\big(P, \mathfrak{q}(0)\big)$, it is clear that

$$\mathfrak{m}^{\mathfrak{x}}(\widetilde{\theta_*}) \circ \mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}_\theta(0\bullet 1)) \circ \mathfrak{m}^{\mathfrak{x}}((\tilde{\mu}_\theta)_1) = \mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}(0\bullet 1)) \qquad 4.8.16;$$

if $\mathfrak{q}(0\bullet 1)$ is an isomorphism then we have $\mathfrak{q}_\theta(0) = \mathfrak{q}_\theta(1)$ and $\mathfrak{q}_\theta(0\bullet 1) = \widetilde{\mathrm{id}}_{\mathfrak{q}_\theta(0)}$ for any $\theta \in \mathcal{F}\big(P, \mathfrak{q}(0)\big)$, and therefore we get

$$\begin{aligned}
&\sum_{\tilde{\theta}\in\tilde{\mathcal{F}}(P,\mathfrak{q}(0))} \mathfrak{m}^{\mathfrak{x}}(\widetilde{\theta_*})\Big(\big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}_\theta(0\bullet 1))\big)\Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}^{n-1}_\ell(\tilde{\mu}_\theta*\delta^{n-1}_0)}\Big)\Big) \\
&= \mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}(0\bullet 1))\Big(\sum_{\tilde{\theta}\in\tilde{\mathcal{F}}(P,\mathfrak{q}(0))} \mathfrak{m}^{\mathfrak{x}}((\tilde{\mu}_\theta)_1)^{-1}\Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}^{n-1}_\ell(\tilde{\mu}_\theta*\delta^{n-1}_0)}\Big)\Big) \qquad 4.8.17. \\
&= \big|\tilde{\mathcal{F}}\big(P, \mathfrak{q}(0)\big)\big| \cdot \big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}(0\bullet 1))\big)(n_{\mathfrak{q}\circ\delta^{n-1}_0})
\end{aligned}$$

Otherwise, either $\mathfrak{q}_\theta(0) \neq \mathfrak{q}_\theta(1)$ and, since

$$\big(\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_0^{n-1})\big)(0) = \mathfrak{q}_\theta(1) \in \mathfrak{Y} \qquad 4.8.18$$

for any $\ell \in \Delta_n$ , we have

$$\sum_{\ell=0}^{n} (-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_0^{n-1})} = 0 \qquad 4.8.19.$$

Or $\hat{\mathfrak{q}}_\theta(1) = \hat{\mathfrak{q}}_\theta(0)$ , $\hat{\mathfrak{q}}_\theta(0\bullet 1) = \widetilde{\mathrm{id}}_{\hat{\mathfrak{q}}_\theta(0)}$ and $\tilde\theta$ belongs to $\tilde{\mathcal{F}}\big(P, \mathfrak{q}(0)\big)_{\mathfrak{q}(0\bullet 1)}$ ; but, we know that $\tilde{\mathcal{F}}_{\mathfrak{q}(1)}\big(\mathfrak{q}(0)\big)$ acts freely on $\tilde{\mathcal{F}}\big(P, \mathfrak{q}(0)\big)_{\mathfrak{q}(0\bullet 1)}$ (cf. 4.4.4) and, setting $\sigma_u = \theta_* \circ \kappa^{\mathfrak{x}}_{\mathfrak{q}(0)}(u) \circ (\theta_*)^{-1}$ for any $u \in N_{\mathfrak{q}(1)}\big(\mathfrak{q}(0)\big)$ , from the commutative $\tilde{\mathcal{F}}^{\mathfrak{x}}$-diagram

$$\begin{array}{ccccc} & & \mathfrak{q}(1) & & \\ & {\scriptstyle (\tilde\mu_\theta)_1}\nearrow & & \nwarrow{\scriptstyle (\tilde\mu_\theta)_1} & \\ \mathfrak{q}_\theta(1) & & \uparrow & & \mathfrak{q}_\theta(1) \\ & & \mathfrak{q}(0) & & \\ \| & \nearrow & & \nwarrow & \| \\ \mathfrak{q}_\theta(0) & & \overset{\tilde\sigma_u}{\cong} & & \mathfrak{q}_\theta(0) \end{array} \qquad 4.8.20$$

and from Proposition 4.5 we obtain a natural automorphism $\tilde\kappa_u^{\mathfrak{q}_\theta} : \mathfrak{q}_\theta \cong \mathfrak{q}_\theta$ (cf. 4.7) such that $(\tilde\kappa_u^{\mathfrak{q}_\theta})_0 = \tilde\sigma_u$ and then, for any $\ell \in \Delta_{n+1}$ we have the natural isomorphism (cf. 4.7.3)

$$\bar{\mathfrak{h}}_\ell^n(\tilde\kappa_u^{\mathfrak{q}_\theta}) : \mathfrak{h}_\ell^n\big(\tilde\mu_\theta \circ (\tilde\kappa_u^{\mathfrak{q}_\theta})^{-1}\big) \cong \mathfrak{h}_\ell^n(\tilde\mu_\theta) \qquad 4.8.21.$$

Moreover, in this situation it is quite clear that we have

$$\big(\tilde\mu_\theta \circ (\tilde\kappa_u^{\mathfrak{q}_\theta})^{-1}\big) * \delta_0^{n-1} = \tilde\mu_\theta \ * \delta_0^{n-1} \quad \text{and} \quad \mathfrak{q}_{\theta\circ\kappa_u} = \mathfrak{q}_\theta \qquad 78.22;$$

hence, for any $\ell \in \Delta_{n+1}$ we have

$$\begin{aligned} \big(\mathfrak{m}^{\mathfrak{x}}(\tilde\sigma_u)\big)(m_{\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_0^{n-1})}) &= m_{\mathfrak{h}_\ell^{n-1}((\tilde\mu_\theta\circ(\tilde\kappa_u^{\mathfrak{q}_\theta})^{-1}) * \delta_0^{n-1})} \\ &= m_{\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_0^{n-1})} \end{aligned} \qquad 4.8.23.$$

Finally, since $\tilde{\mathcal{F}}_{\mathfrak{q}(1)}\big(\mathfrak{q}(0)\big)$ is a nontrivial $p$-group, since $(\mathfrak{q} \circ \delta_0^{n-1})(0) \in \mathfrak{Y}$ forces $n_{\mathfrak{q}\circ\delta_0^{n-1}} = 0$ and since we have $\hat{\mathfrak{q}}_\theta(0 \bullet 1) = \widetilde{\mathrm{id}}_{\hat{\mathfrak{q}}_\theta(0)}$, we still have

$$\begin{aligned} &\sum_{\tilde\theta \in \tilde{\mathcal{F}}(P,\mathfrak{q}(0))} \mathfrak{m}^{\mathfrak{x}}(\widetilde{\theta_*})\Big(\big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}_\theta(0\bullet 1))\big)\big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_0^{n-1})}\big)\Big) \\ &= \sum_{\tilde\theta} \mathfrak{m}^{\mathfrak{x}}(\widetilde{\theta_*})\Big(\sum_u \mathfrak{m}^{\mathfrak{x}}(\tilde\sigma_u)\big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_0^{n-1})}\big)\Big) = 0 \\ &= \big|\tilde{\mathcal{F}}\big(P, \mathfrak{q}(0)\big)\big| \cdot \big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}(0\bullet 1))\big)(n_{\mathfrak{q}\circ\delta_0^{n-1}}) \end{aligned} \qquad 4.8.24$$

where $\tilde\theta$ runs over a set of representatives for $\tilde{\mathcal{F}}\big(P, \mathfrak{q}(0)\big)_{\mathfrak{q}(0\bullet 1)} / \tilde{\mathcal{F}}_{\mathfrak{q}(1)}\big(\mathfrak{q}(0)\big)$ and $u$ over a set of representatives for $\bar N_{\mathfrak{q}(1)}\big(\mathfrak{q}(0)\big)$ .

Now, according to the cancellations mentioned above, for any $\tilde{\mathcal{F}}^{\mathfrak{X}}$-*chain* $\mathfrak{q}:\Delta_n \to \tilde{\mathcal{F}}^{\mathfrak{X}}$ such that $\mathfrak{q}(0) \notin \mathfrak{Y}$ and any $\theta$ in $\mathcal{F}\big(P,\mathfrak{q}(0)\big)$ the sum of all the remaining terms in equalities 4.8.12 multiplied by $(-1)^\ell$ yields

$$\begin{aligned}\sum_{i=1}^{n+1}\ \sum_{\ell\in\Delta_{n+1}-\{i-1\}} &(-1)^{\ell+i+1} m_{\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_i^{n-1})} \\ &= \sum_{i=1}^{n+1}(-1)^{i+1}\Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\tilde\mu_\theta * \delta_i^{n-1})}\Big)\end{aligned} \qquad 4.8.25.$$

In conclusion, the alternating sum of the sum over all the $\tilde\theta \in \tilde{\mathcal{F}}\big(P,\mathfrak{q}(0)\big)$ in equalities 4.8.10 proves that, for any $\tilde{\mathcal{F}}^{\mathfrak{X}}$-*chain* $\mathfrak{q}:\Delta_n \to \tilde{\mathcal{F}}^{\mathfrak{X}}$ such that $\mathfrak{q}(0)$ does not belong to $\mathfrak{X}$, we have (cf. 4.8.14, 4.8.15, 4.8.17, 4.8.24 and 4.8.25)

$$\begin{aligned} m_{\mathfrak{q}} &= \big(\mathfrak{m}^{\mathfrak{X}}(\mathfrak{q}(0\bullet 1))\big)(n_{\mathfrak{q}\circ\delta_0^{n-1}}) + \sum_{i=1}^{n}(-1)^i n_{\mathfrak{q}\circ\delta_i^{n-1}} \\ &= d_{n-1}^{\mathfrak{X}}(\mathrm{n})_{\mathfrak{q}}\end{aligned} \qquad 4.8.26;$$

but, if $\mathfrak{q}(0) \in \mathfrak{Y}$ then we get $m_{\mathfrak{q}} = 0 = d_{n-1}^{\mathfrak{X}}(\mathrm{n})_{\mathfrak{q}}$; hence, we finally get $\mathrm{m} = d_{n-1}^{\mathfrak{X}}(\mathrm{n})$ which proves our claim. We are done.

## 5. The natural $\mathcal{F}$-basic $P \times P$-sets

5.1. Recall that a *basic* $P\times P$-*set* [9, 21,4] is a finite nonempty $P\times P$-set $\Omega$ such that $\{1\}\times P$ acts *freely* on $\Omega$, we have

$$\Omega^\circ \cong \Omega \quad \text{and} \quad |\Omega|/|P| \not\equiv 0 \bmod p \qquad 5.1.1,$$

where we denote by $\Omega^\circ$ the $P\times P$-set obtained by exchanging both factors, and, for any subgroup $Q$ of $P$ and any group homomorphism $\varphi: Q \to P$ such that $\Omega$ contains a $P\times P$-subset isomorphic to $(P\times P)/\Delta_\varphi(Q)$, we have a $Q\times P$-isomorphism

$$\mathrm{Res}_{\varphi\times \mathrm{id}_P}(\Omega) \cong \mathrm{Res}_{\iota_Q^P\times\mathrm{id}_P}(\Omega) \qquad 5.1.2$$

where, for any $\varphi,\varphi' \in \mathcal{F}(P,Q)$, we set

$$\Delta_{\varphi,\varphi'}(Q) = \{(\varphi(u),\varphi'(u))\}_{u\in Q} \quad \text{and} \quad \Delta_\varphi(Q) = \Delta_{\mathrm{id}_Q,\varphi}(Q) \qquad 5.1.3$$

and, as above, denote by $\iota_Q^P$ the corresponding inclusion map.

5.2. Then, for any pair of subgroups $Q$ and $R$ of $P$, denoting by $\mathcal{F}^\Omega(Q,R)$ the set of group homomorphisms $\varphi: R\to P$ such that

$$\varphi(R)\subset Q \quad \text{and} \quad \mathrm{Res}_{\varphi\times\mathrm{id}_P}(\Omega) \cong \mathrm{Res}_{\iota_R^P\times\mathrm{id}_P}(\Omega) \qquad 5.2.1,$$

it follows from [9, Proposition 21.9] that $\mathcal{F}^\Omega$ is a *Frobenius $P$-category*; we say that $\Omega$ is a *$\mathcal{F}$-basic $P\times P$-set* whenever $\mathcal{F}^\Omega = \mathcal{F}$ ; on the other hand, it follows from [9, Proposition 21.12] that the Frobenius $P$-category $\mathcal{F}$ admits an $\mathcal{F}$-basic $P\times P$-set.

5.3. More generally, we say that a $P\times P$-set $\Omega^{\mathfrak{X}}$ is *$\mathcal{F}^{\mathfrak{X}}$-basic* if either is empty or it fulfills condition 5.1.1 and the statement

5.3.1 *The stabilizer of any element of $\Omega^{\mathfrak{X}}$ coincides with $\Delta_{\psi,\psi'}(R)$ for some $R\in\mathfrak{X}$ and suitable $\psi,\psi'\in\mathcal{F}(P,R)$ , and we have*

$$\big|(\Omega^{\mathfrak{X}})^{\Delta_{\varphi,\varphi'}(Q)}\big| = \big|(\Omega^{\mathfrak{X}})^{\Delta(Q)}\big|$$

*for any $Q\in\mathfrak{X}$ and any $\varphi,\varphi'\in\mathcal{F}(P,Q)$ .*

Recall that, according to [9, Proposition 21.12], for any $\mathcal{F}^{\mathfrak{X}}$-basic $P\times P$-set $\Omega^{\mathfrak{X}}$ there is an $\mathcal{F}$-basic $P\times P$-set $\Omega$ containing $\Omega^{\mathfrak{X}}$ and fulfilling

$$\Omega^{\Delta_\varphi(Q)} = (\Omega^{\mathfrak{X}})^{\Delta_\varphi(Q)} \qquad 5.3.2$$

for any $Q\in\mathfrak{X}$ and any $\varphi\in\mathcal{F}(P,Q)$ .

**Proposition 5.4.** *Assume that any element of $\mathfrak{X}$ is $\mathcal{F}$-selfcentralizing. Then, the $P\times P$-set*

$$\Omega^{\mathfrak{X}} = \bigsqcup_{Q}\bigsqcup_{\tilde\varphi}(P\times P)/\Delta_\varphi(Q) \qquad 5.4.1,$$

*where $Q$ runs over a set of representatives for the $P$-conjugacy classes in $\mathfrak{X}$ and $\tilde\varphi$ runs over a set of representatives for the set of $\tilde{\mathcal{F}}_P(Q)$-orbits in $\tilde{\mathcal{F}}(P,Q)_{\tilde\iota_Q^P}$ , is an $\mathcal{F}^{\mathfrak{X}}$-basic $P\times P$-set which for any $Q\in\mathfrak{X}$ fulfills*

$$|(\Omega^{\mathfrak{X}})^{\Delta(Q)}| = |Z(Q)| \qquad 5.4.2.$$

**Proof:** Since we clearly have

$$(\Omega^{\mathfrak{X}})^\circ \cong \Omega^{\mathfrak{X}} \quad \text{and} \quad |\Omega^{\mathfrak{X}}/P| \equiv |\tilde{\mathcal{F}}(P)| \bmod p \qquad 5.4.3,$$

it suffices to check that, for any $R\in\mathfrak{X}$ and any $\psi\in\mathcal{F}(P,R)$ , we have

$$|(\Omega^{\mathfrak{X}})^{\Delta_\psi(R)}| = |Z(R)| \qquad 5.4.4;$$

but, for any subgroup $Q$ of $P$ and any $\varphi\in\mathcal{F}(P,R)$ , $\Delta_\psi(R)$ fixes the class of $(u,v)\in P\times P$ in $(P\times P)/\Delta_\varphi(Q)$ if and only if it is contained in $\Delta_\varphi(Q)^{(u,v)}$ or, equivalently, we have

$$uRu^{-1}\subset Q \quad \text{and} \quad \varphi(uwu^{-1}) = v\psi(w)v^{-1} \text{ for any } w\in R \qquad 5.4.5,$$

which amounts to saying that the following $\tilde{\mathcal{F}}$-diagram is commutative

$$\begin{tikzcd} P \arrow[d, equal] & & P \arrow[d, equal] \\ P & Q \arrow[ul, "\tilde{\imath}_Q^P"'] \arrow[ur, "\tilde{\varphi}"] & P \\ & R \arrow[ul, "\tilde{\imath}_R^P"] \arrow[u, "\tilde{\kappa}_{Q,R}(u)"] \arrow[ur, "\tilde{\psi}"'] & \end{tikzcd} \qquad 5.4.6$$

where $\kappa_{Q,R}(u) : R \to Q$ is the group homomorphism determined by the conjugation by $u$.

Since $\tilde{\varphi}$ belongs to $\tilde{\mathcal{F}}(P,Q)_{\tilde{\imath}_Q^P}$, it follows from Proposition 4.5 that the pair $(\tilde{\imath}_R^P, \tilde{\psi})$ determines the isomorphism class of the $(\tilde{\mathcal{F}}^\circ)_R$-object

$$\tilde{\kappa}_{Q,R}(u) : R \longrightarrow Q \qquad 5.4.7;$$

that is to say, if $(u', v') \in P \times P$ is another element such that $\Delta_\varphi(Q)^{(u',v')}$ contains $\Delta_\psi(R)$, we have $u' = su$ for some $s \in Q$ and therefore we get

$$\psi(w) = v'^{-1}\varphi(suwu^{-1}s^{-1})v' = \varphi(uwu^{-1})^{\varphi(s)^{-1}v'} \qquad 5.4.8$$

for any $w \in R$; at this point, it follows from [9, Proposition 4.6] that, for a suitable $z \in Z(R)$, we have $\varphi(s)^{-1}v' = vz$, which proves our claim.

5.5. Moreover, we say that an $\mathcal{F}$-basic $P \times P$-set $\Omega$ is *thick outside of* $\mathfrak{X}$ if the multiplicity of the indecomposable $P \times P$-set $(P \times P)/\Delta_\varphi(Q)$ is at least two for any subgroup $Q$ of $P$ outside of $\mathfrak{X}$ and any $\varphi \in \mathcal{F}(P,Q)$. Let us call *natural* any $\mathcal{F}$-basic $P \times P$-set $\Omega$ which is thick outside of the set of $\mathcal{F}$-selfcentralizing subgroups of $P$ and, for any $\mathcal{F}$-selfcentralizing subgroup $Q$ of $P$ and any $\varphi \in \mathcal{F}(P,Q)$, it fulfills

$$|\Omega^{\Delta_\varphi(Q)}| = |Z(Q)| \qquad 5.5.1;$$

then, the existence of *natural $\mathcal{F}$-basic $P \times P$-sets* follows from Proposition 5.4 and from [9, Proposition 21.12].

5.6. Let $\Omega$ be an $\mathcal{F}$-basic $P \times P$-set and $Q$ a subgroup of $P$; it follows from our definition in 5.2 that any orbit of $\mathrm{Res}_{Q\times P}(\Omega)$ is isomorphic to the quotient set $(Q \times P)/\Delta_\eta(T)$ (cf. 21.3) for some subgroup $T$ of $Q$ and some $\eta \in \mathcal{F}(P,T)$; note that the isomorphism class of this $Q \times P$-set $(Q \times P)/\Delta_\eta(T)$ only depends on the conjugacy class of $T$ in $Q$ and on the class $\tilde{\eta}$ of $\eta$ in $\tilde{\mathcal{F}}(P,T)$; moreover, it is quite clear that $\bar{N}_{Q\times P}\big(\Delta_\eta(T)\big)$ acts regularly on $\big((Q \times P)/\Delta_\eta(T)\big)^{\Delta_\eta(T)}$ and that we have a group isomorphism

$$\mathrm{Aut}\big((Q \times P)/\Delta_\eta(T)\big) \cong \bar{N}_{Q\times P}\big(\Delta_\eta(T)\big) \qquad 5.6.1$$

**Proposition 5.7.** *Let $\Omega$ be a natural $\mathcal{F}$-basic $P\times P$-set, $Q$ and $T$ $\mathcal{F}$-self-centralizing subgroups of $P$ such that $T\subset Q$, and $\eta$ an element of $\mathcal{F}(P,T)$. Then, the multiplicity of $(Q\times P)/\Delta_\eta(T)$ in $\mathrm{Res}_{Q\times P}(\Omega)$ is at most one, and it is one if and only if $\tilde\eta$ belongs to $\tilde{\mathcal{F}}(P,T)_{\tilde\iota_T^Q}$. Moreover, in this case we have*

$$\mathrm{Aut}\big((Q\times P)/\Delta_\eta(T)\big)\cong Z(T) \qquad 5.7.1.$$

**Proof:** According to our definition, we have

$$|\Omega^{\Delta_\eta(T)}| = |Z(T)| \qquad 5.7.2;$$

hence, if the multiplicity of $(Q\times P)/\Delta_\eta(T)$ in $\mathrm{Res}_{Q\times P}(\Omega)$ is not zero, then it is one and we have (cf. 5.6)

$$|\bar N_{Q\times P}\big(\Delta_\eta(T)\big)|\le |Z(T)| \qquad 5.7.3$$

which forces isomorphism 5.7.1; finally, since $N_{Q\times P}\big(\Delta_\eta(T)\big)$ *covers* the intersection $\mathcal{F}_Q(T)\cap {}^{\eta^*}\mathcal{F}_P\big(\eta(T)\big)$, in this case it follows from 4.4 that $\tilde\eta$ belongs to $\tilde{\mathcal{F}}(P,T)_{\tilde\iota_T^Q}$.

## 6. Construction of $\mathcal{F}$-localities from $\mathcal{F}$-basic $P\times P$-sets

6.1. Let $\Omega$ be an $\mathcal{F}$-basic $P\times P$-set and denote by $G$ the group of automorphisms of $\mathrm{Res}_{\{1\}\times P}(\Omega)$; it is clear that we have an injective map from $P\times\{1\}$ into $G$ and we identify this image with the $p$-group $P$, so that from now on $P$ is contained in $G$ and acts freely on $\Omega$. Recall that, for any pair of subgroups $Q$ and $R$ of $P$, we have (cf. 5.2)

$$T_G(R,Q)/C_G(R)\cong \mathcal{F}(Q,R) \qquad 6.1.1.$$

6.2. Let $Q$ be a subgroup of $P$; clearly, the centralizer $C_G(Q)$ coincides with the group of automorphisms of $\mathrm{Res}_{Q\times P}(\Omega)$ and therefore, denoting by $\mathfrak{O}_Q$ a set of representatives for the isomorphism classes of the set of $Q\times P$-orbits of $\Omega$, by $k_O$ the multiplicity of $O\in\mathfrak{O}_Q$ in $\Omega$ and by $\mathfrak{S}_{k_O}$ the $k_O$-*symmetric* group, it is easily checked that we have a *canonical* $\widetilde{\mathfrak{Gr}}$-isomorphism [9, 22.5.1]

$$\tilde\omega_Q : C_G(Q)\cong \prod_{O\in\mathfrak{O}_Q}\mathrm{Aut}(O)\wr\mathfrak{S}_{k_O} \qquad 6.2.1.$$

More precisely, as in [9, Proposition 22.11], for any subgroup $R$ of $Q$ we have a commutative $\widetilde{\mathfrak{Gr}}$-diagram

$$\begin{array}{ccc} C_G(Q) & \longrightarrow & C_G(R)\\ \uparrow & & \uparrow\\ \prod_{O\in\mathfrak{O}_Q}\mathfrak{S}_{k_O} & \longrightarrow & \prod_{O'\in\mathfrak{O}_R}\mathfrak{S}_{k_{O'}}\end{array} \qquad 6.2.2$$

where the bottom homomorphism depends on $\mathrm{Res}_{R\times P}(O)$ for any $O\in\mathfrak{O}_Q$.

6.3. As in [9, Proposition 22.7], let us denote by $\mathfrak{S}_G(Q)$ the minimal normal subgroup of $C_G(Q)$ containing $(\omega_Q)^{-1}\big(\prod_{O\in\mathfrak{O}_Q}\mathfrak{S}_{k_O}\big)$ for a representative $\omega_Q$ of $\tilde\omega_Q$ ; then, denoting by $\mathfrak{O}'_Q$ the subset of $O\in\mathfrak{O}_Q$ with multiplicity one and by $\mathfrak{ab}\big(\mathrm{Aut}(O)\big)$ the maximal Abelian quotient of $\mathrm{Aut}(O)$ , it follows from [9, Lemma 22.8] that

$$C_G(Q)/\mathfrak{S}_G(Q)\cong\prod_{O\in\mathfrak{O}'_Q}\mathrm{Aut}(O)\times\prod_{O\in\mathfrak{O}_Q-\mathfrak{O}'_Q}\mathfrak{ab}\big(\mathrm{Aut}(O)\big)\qquad 6.3.1.$$

Moreover, although in [9, Chap. 22] we assume that $\Omega$ is *thick outside of* $\emptyset$ , it is easily checked that the elementary arguments in [9, Proposition 22.11] still prove that, for any subgroup $R$ of $Q$ , we have

$$\mathfrak{S}_G(Q)\subset\mathfrak{S}_G(R)\qquad 6.3.2.$$

First of all, let us recall the definition of the *basic $\mathcal{F}$-locality* [9, Proposition 22.12].

**Propsition 6.4.** *If $\Omega$ is thick outside of $\emptyset$ then the correspondence mapping any pair of subgroups $Q$ and $R$ of $P$ on the quotient set*

$$\mathcal{L}^{\mathrm{b}}(Q,R)=T_G(R,Q)/\mathfrak{S}_G(R)\qquad 6.4.1,$$

*endowed with the natural maps*

$$\tau^{\mathrm{b}}_{Q,R}:\mathcal{T}_P(Q,R)\to\mathcal{L}^{\mathrm{b}}(Q,R)\quad\textit{and}\quad\pi^{\mathrm{b}}_{Q,R}:\mathcal{L}^{\mathrm{b}}(Q,R)\to\mathcal{F}(Q,R)\qquad 6.4.2,$$

*defines a $p$-coherent $\mathcal{F}$-locality $(\tau^{\mathrm{b}},\mathcal{L}^{\mathrm{b}},\pi^{\mathrm{b}})$ which does not depend on the choice of the $\mathcal{F}$-basic $P\times P$-set thick outside of $\emptyset$ .*

6.5. Here, we are interested in the analogous construction starting with a *natural* $\mathcal{F}$-basic set. Till the end of this section, assume that $\Omega$ is a natural $\mathcal{F}$-basic set; then, denoting by $\mathfrak{O}^{\mathrm{sc}}_Q$ the subset of $Q\times P$-orbits $O\in\mathfrak{O}_Q$ such that the stabilizers come from $\mathcal{F}$-selcentralizing subgroups of $P$ , it follows from isomorphism 6.3.1 and from Proposition 5.7 that

$$C_G(Q)/\mathfrak{S}_G(Q)\cong\prod_{T\in\tilde{\mathcal{S}}_Q}\;\prod_{\tilde\eta\in\tilde{\mathcal{F}}(P,T)_{\iota_T^Q}}Z(T)\times\prod_{O\in\mathfrak{O}_Q-\mathfrak{O}^{\mathrm{sc}}_Q}\mathfrak{ab}\big(\mathrm{Aut}(O)\big)\qquad 6.5.1$$

where $\tilde{\mathcal{S}}_Q$ denotes a set of representatives for the set of $Q$-conjugacy classes of $\mathcal{F}$-selfcentralizing subgroups of $Q$ . The following result introduces the *natural $\mathcal{F}$-locality*.

**Proposition 6.6.** *If $\Omega$ is natural then the correspondence mapping any pair of subgroups $Q$ and $R$ of $P$ on the quotient set*

$$\mathcal{L}^{\mathrm{n}}(Q,R) = T_G(R,Q)/\mathfrak{S}_G(R) \qquad 6.6.1,$$

*endowed with the natural maps*

$$\tau^{\mathrm{n}}_{Q,R} : \mathcal{T}_P(Q,R) \to \mathcal{L}^{\mathrm{n}}(Q,R) \quad \textit{and} \quad \pi^{\mathrm{n}}_{Q,R} : \mathcal{L}^{\mathrm{n}}(Q,R) \to \mathcal{F}(Q,R) \qquad 6.6.2,$$

*defines a p-coherent $\mathcal{F}$-locality $(\tau^{\mathrm{n}}, \mathcal{L}^{\mathrm{n}}, \pi^{\mathrm{n}})$. This $\mathcal{F}$-locality does not depend on the choice of the natural $\mathcal{F}$-basic $P\times P$-set and we have a canonical surjective functor of $\mathcal{F}$-localities*

$$(\tau^{\mathrm{b}}, \mathcal{L}^{\mathrm{b}}, \pi^{\mathrm{b}}) \longrightarrow (\tau^{\mathrm{n}}, \mathcal{L}^{\mathrm{n}}, \pi^{\mathrm{n}}) \qquad 6.6.3.$$

**Proof:** From inclusion 6.3.2 it is not difficult to check that, for any triple of subgroups $Q$, $R$ and $T$ of $P$, the product in $G$ induces a map

$$\mathcal{L}^{\mathrm{n}}(Q,R) \times \mathcal{L}^{\mathrm{n}}(R,T) \longrightarrow \mathcal{L}^{\mathrm{n}}(Q,T) \qquad 6.6.4\,;$$

then, it is quite clear that these maps determine a *composition* in the correspondence $\mathcal{L}^{\mathrm{n}}$ above and that the natural maps in 6.6.2 define structural functors

$$\tau^{\mathrm{n}} : \mathcal{T}_P \longrightarrow \mathcal{L}^{\mathrm{n}} \quad \text{and} \quad \pi^{\mathrm{n}} : \mathcal{L}^{\mathrm{n}} \longrightarrow \mathcal{F} \qquad 6.6.5;$$

moreover, the *divisibility* and the *coherence* of $\mathcal{L}^{\mathrm{n}}$ (cf. 2.8) are easy consequences of the fact that $G$ is a group.

On the other hand, for another choice of a natural $\mathcal{F}$-basic $P\times P$-set $\Omega'$, we already know that we can embed their *disjoint union* $\Omega \sqcup \Omega'$ in a third $\mathcal{F}$-basic $P\times P$-set $\Omega''$ containing the *same* isomorphism classes of indecomposable $P\times P$-sets [9, 21.5]; thus, setting

$$G' = \mathrm{Aut}_{\{1\}\times P}(\Omega') \quad \text{and} \quad G'' = \mathrm{Aut}_{\{1\}\times P}(\Omega'') \qquad 6.6.6$$

and denoting by $(G'')_{\Omega,\Omega'}$ the stabilizer in $G''$ of the images of $\Omega$ and $\Omega'$, we have canonical *surjective* group homomorphisms

$$G \longleftarrow (G'')_{\Omega,\Omega'} \longrightarrow G' \qquad 6.6.7$$

mapping $P \subset (G'')_{\Omega,\Omega'}$ onto both $P \subset G$ and $P \subset G'$.

More precisely, for any pair of subgroups $Q$ and $R$ of $P$, it is easily checked that we still have *surjective* maps

$$T_G(R,Q) \longleftarrow T_{(G'')_{\Omega,\Omega'}}(R,Q) \longrightarrow T_{G'}(R,Q) \qquad 6.6.8;$$

then, denoting by $\mathfrak{S}_{G'}(R)$ and $\mathfrak{S}_{G''}(R)$ the corresponding normal subgroups of $C_{G'}(R)$ and $C_{G''}(R)$ defined above, it is easily checked that we get

$$T_{G''}(R,Q) = T_{(G'')_{\Omega,\Omega'}}(R,Q)\cdot\mathfrak{S}_{G''}(R) \qquad 6.6.9$$

and that this equality, together with the surjective maps 6.6.8, induce bijections

$$T_G(R,Q)/\mathfrak{S}_G(R) \cong T_{G''}(R,Q)/\mathfrak{S}_{G''}(R) \cong T_{G'}(R,Q)/\mathfrak{S}_{G'}(R) \qquad 6.6.10$$

which are clearly compatible with the corresponding natural maps 6.6.2. Consequently, the three $\mathcal{F}$-localities we obtain are mutually equivalent.

Analogously, we can embed the *disjoint union* of $\Omega$ and of an $\mathcal{F}$-basic $P\times P$-set $\hat{\Omega}'$ thick outside of $\emptyset$ in a third $\mathcal{F}$-basic $P\times P$-set $\hat{\Omega}''$ [9, 21.5], which will be necessarily thick outside of $\emptyset$ ; then, setting

$$\hat{G}' = \mathrm{Aut}_{\{1\}\times P}(\hat{\Omega}') \quad \text{and} \quad \hat{G}'' = \mathrm{Aut}_{\{1\}\times P}(\hat{\Omega}'') \qquad 6.6.11$$

for any pair of subgroups $Q$ and $R$ of $P$, the argument above respectively supplies surjections

$$\begin{array}{ccc} T_G(R,Q)/\mathfrak{S}_G(R) & \longleftarrow & T_{\hat{G}''}(R,Q)/\mathfrak{S}_{\hat{G}''}(R) \\ \| & & \wr\| \\ \mathcal{L}^{\mathrm{n}}(Q,R) & & \mathcal{L}^{\mathrm{b}}(Q,R) \end{array} \qquad 6.6.11$$

which are clearly compatible with the corresponding natural maps 6.6.2. This proves the last statement.

6.7. More generally, let us denote by $\mathcal{L}^{\mathrm{n},\mathfrak{X}}$ the *full* $\mathcal{F}^{\mathfrak{X}}$-subcategory of $\mathcal{L}^{\mathrm{n}}$ over $\mathfrak{X}$ (cf. 2.7). Assuming that any group in $\mathfrak{X}$ is $\mathcal{F}$-selfcentralizing, we are actually interested in the following quotient of $\mathcal{L}^{\mathrm{n},\mathfrak{X}}$ ; for any $Q\in\mathfrak{X}$ denote by $\tilde{\mathcal{S}}_Q^{\mathfrak{X}}$ the subset of $T\in\tilde{\mathcal{S}}_Q$ belonging to $\mathfrak{X}$ and by $\mathfrak{S}_G^{\mathfrak{X}}(Q)$ the subgroup of $C_G(Q)$ fulfilling

$$\mathfrak{S}_G^{\mathfrak{X}}(Q)/\mathfrak{S}_G(Q) \cong \prod_{T\in\tilde{\mathcal{S}}_Q-\tilde{\mathcal{S}}_Q^{\mathfrak{X}}}\prod_{\tilde{\eta}} Z(T) \times \prod_{O\in\mathfrak{O}_Q-\mathfrak{O}_Q^{\mathrm{sc}}} \mathfrak{ab}\big(\mathrm{Aut}(O)\big) \qquad 6.7.1;$$

where $\tilde{\eta}$ runs over a set of representatives for the set of $\mathcal{F}_Q(T)$-orbits in $\tilde{\mathcal{F}}(P,T)_{\tilde{\iota}_T^Q}$ [9, 23.7]; then, it is easily checked from diagram 6.2.2 that, for any $R\in\mathfrak{X}$ contained in $Q$, we still have

$$\mathfrak{S}_G^{\mathfrak{X}}(Q) \subset \mathfrak{S}_G^{\mathfrak{X}}(R) \qquad 6.7.2.$$

Consequently, we obtain a new *p-coherent* $\mathcal{F}^{\mathfrak{X}}$*-locality* $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{X}}$ as the *quotient* of $\mathcal{L}^{\mathrm{n},\mathfrak{X}}$ defined by

$$\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{X}}(Q,R) = T_G(R,Q)/\mathfrak{S}_G^{\mathfrak{X}}(R) \qquad 6.7.3$$

for any pair of subgroups $Q$ and $R$ in $\mathfrak{X}$, together with the induced functors

$$\bar{\tau}^{\mathrm{n},\mathfrak{X}} : \mathcal{T}_P^{\mathfrak{X}} \to \bar{\mathcal{L}}^{\mathrm{n},\mathfrak{X}} \quad \text{and} \quad \bar{\pi}^{\mathrm{n},\mathfrak{X}} : \bar{\mathcal{L}}^{\mathrm{n},\mathfrak{X}} \to \mathcal{F}^{\mathfrak{X}} \qquad 6.7.4;$$

note that $\bar{\tau}^{\mathrm{n},\mathfrak{X}}$ is a *faithful* functor.

## 7. Construction of $\mathcal{F}^{\mathfrak{X}}$-basic $P \times P$-sets from perfect $\mathcal{F}^{\mathfrak{X}}$-localities

7.1. Assume that any group in $\mathfrak{X}$ is $\mathcal{F}$-selfcentralizing. As in [9, A4.10], let us consider the *additive cover* $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$ of $\tilde{\mathcal{F}}^{\mathfrak{X}}$, namely the category where the objects are the finite sequences $\{Q_i\}_{i\in I}$ — denoted by $Q = \bigoplus_{i\in I} Q_i$ — of subgroups $Q_i$ in $\mathfrak{X}$, and where a morphism from another object $R = \bigoplus_{j\in J} R_j$ to $Q = \bigoplus_{i\in I} Q_i$ is a pair $(\tilde\alpha, f)$ formed by a map $f : J \to I$ and by a family $\tilde\alpha = \{\tilde\alpha_j\}_{j\in J}$ of $\tilde{\mathcal{F}}^{\mathfrak{X}}$-morphisms $\tilde\alpha_j : R_j \to Q_{f(j)}$. The composition of $(\tilde\alpha, f)$ with another $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$-morphism

$$(\tilde\beta, g) : T = \bigoplus_{\ell\in L} T_\ell \longrightarrow R = \bigoplus_{j\in J} R_j \qquad 7.1.1,$$

formed by a map $g : L \to J$ and by a family $\tilde\beta = \{\tilde\beta_\ell\}_{\ell\in L}$, is the pair formed by $f \circ g$ and by the family $\{\tilde\alpha_{g(\ell)} \circ \tilde\beta_\ell\}_{\ell\in L}$ of composed morphisms

$$\tilde\alpha_{g(\ell)} \circ \tilde\beta_\ell : T_\ell \longrightarrow R_{g(\ell)} \longrightarrow Q_{(f\circ g)(\ell)} \qquad 7.1.2.$$

7.2. As in [9, Chap. 6], Proposition 4.5 allows us to define a *distributive direct product* in $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$. First of all, if $R$ and $T$ are two subgroups in $\mathfrak{X}$, we consider the set $\tilde{\mathfrak{T}}^{\mathfrak{X}}_{R,T}$ of *triples* $(\tilde\alpha, Q, \tilde\beta)$ where $Q \in \mathfrak{X}$ and we have (cf. 4.3.1)

$$\tilde\alpha \in \tilde{\mathcal{F}}(R, Q)_{\tilde\beta} \quad \text{and} \quad \tilde\beta \in \tilde{\mathcal{F}}(T, Q)_{\tilde\alpha} \qquad 7.2.1;$$

we say that two triples $(\tilde\alpha, Q, \tilde\beta)$ and $(\tilde\alpha', Q', \tilde\beta')$ are *equivalent* if there is an $\tilde{\mathcal{F}}$-isomorphism $\tilde\theta : Q \cong Q'$ fulfilling

$$\tilde\alpha' \circ \tilde\theta = \tilde\alpha \quad \text{and} \quad \tilde\beta' \circ \tilde\theta = \tilde\beta \qquad 7.2.2;$$

then, $\tilde\theta$ is unique since, assuming that the triples coincide and choosing $\alpha \in \tilde\alpha$, $\beta \in \tilde\beta$ and $\theta \in \tilde\theta$, it is easily checked that $\theta$ belongs to (cf. 4.4.1)

$${}^{\alpha^*}\mathcal{F}_R\big(\alpha(Q)\big) \cap {}^{\beta^*}\mathcal{F}_T\big(\beta(Q)\big) = \mathcal{F}_Q(Q) \qquad 7.2.3.$$

7.3. Denoting by $\tilde{\tilde{\mathfrak{T}}}^{\mathfrak{X}}_{R,T}$ a set of representatives for the set of equivalence classes in $\tilde{\mathfrak{T}}^{\mathfrak{X}}_{R,T}$, we call $\tilde{\mathcal{F}}^{\mathfrak{X}}$-*intersection* of $R$ and $T$ the $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$-object

$$R \cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}} T = \bigoplus_{(\tilde\alpha, Q, \tilde\beta)\in \tilde{\tilde{\mathfrak{T}}}^{\mathfrak{X}}_{R,T}} Q \qquad 7.3.1;$$

note that, if we choose another set of representatives, then the uniqueness of the isomorphism above yields a unique $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$-isomorphism between both $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$-objects; in particular, with the notation in 6.7 above, we have

$$R \cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}} T \cong \bigoplus_{Q\in \tilde{\mathcal{S}}^{\mathfrak{X}}_R} \bigoplus_{\tilde\gamma} Q \qquad 7.3.2.$$

where $\tilde\gamma$ runs over a set of representatives for the set of $\mathcal{F}_R(Q)$-orbits in $\tilde{\mathcal{F}}(T,Q)_{\tilde\iota_Q^R}$. Finally, if $R=\bigoplus_{j\in J}R_j$ and $T=\bigoplus_{\ell\in L}T_\ell$ are two $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$-objects, we define

$$R\cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}}T=\bigoplus_{(j,\ell)\in J\times L}R_j\cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}}T_\ell \qquad 7.3.3.$$

Although in [9, Chap. 6] we consider the set of *all* the $\mathcal{F}$-selfcentralizing subgroups of $P$, the same arguments there show that the $\tilde{\mathcal{F}}^{\mathfrak{X}}$-*intersection* defines a *distributive direct product* in $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$.

7.4. Analogously, the existence of a *perfect* $\mathcal{F}^{\mathfrak{X}}$-*locality* $\mathcal{P}^{\mathfrak{X}}$ actually determines a *distributive direct product* in the *additive cover* $\mathfrak{ac}(\mathcal{P}^{\mathfrak{X}})$ of $\mathcal{P}^{\mathfrak{X}}$ and then a suitable $\mathcal{F}^{\mathfrak{X}}$-*basic* $P\times P$-*set*; this fact is already proved in [9, Chap. 24] whenever $\mathfrak{X}$ is the set of *all* the $\mathcal{F}$-selfcentralizing subgroups of $P$; although the same arguments apply to the general case, we partially recall them. The starting point is the following result which admits the same proof as in [9, Proposition 24.2].

**Lemma 7.5.** *Any $\mathcal{P}^{\mathfrak{X}}$-morphism $x:R\to Q$ is a monomorphism and an epimorphism.*

7.6. For any triple of subgroups $Q$, $R$ and $T$ in $\mathfrak{X}$, as in 4.3 above any morphism $x\in\mathcal{P}^{\mathfrak{X}}(T,Q)$ induces an injective map from $\mathcal{P}^{\mathfrak{X}}(T,R)$ to $\mathcal{P}^{\mathfrak{X}}(T,Q)$ and then, as in 4.3.1, we set

$$\mathcal{P}^{\mathfrak{X}}(T,Q)_x=\mathcal{P}^{\mathfrak{X}}(T,Q)-\bigcup_{z'}\mathcal{P}^{\mathfrak{X}}(T,Q')\cdot z' \qquad 7.6.1$$

where $z'$ runs over the set of $\mathcal{P}^{\mathfrak{X}}$-*nonisomorphisms* $z':Q\to Q'$ from $Q$ or, equivalently, the set of *nonfinal objects* in the category $\big((\mathcal{P}^{\mathfrak{X}})^\circ\big)_Q$ (cf. 2.1) fulfilling $x'.z'=x$ for some $x'\in\mathcal{P}^{\mathfrak{X}}(R,Q')$; then, $x'$ is uniquely determined by this equality and we simply say that $z'$ *divides* $x$ setting $x'=x/z'$. Note that the existence of $x'$ for some $z'\in\mathcal{P}^{\mathfrak{X}}(Q',Q)$ is equivalent to the existence of a subgroup of $R$ which is $\mathcal{F}$-isomorphic to $Q'$ and contains $\big(\pi_{R,Q'}(x)\big)(Q)$; thus, it is quite clear that

7.6.2 *$\mathcal{P}^{\mathfrak{X}}(T,Q)_x$ is the converse image of $\tilde{\mathcal{F}}^{\mathfrak{X}}(T,Q)_{\widetilde{\pi_{R,Q'}(x)}}$ in $\mathcal{P}^{\mathfrak{X}}(T,Q)$.*

**Proposition 7.7.** *For any triple of elements $Q$, $R$ and $T$ in $\mathfrak{X}$, and any $x\in\mathcal{P}^{\mathfrak{X}}(R,Q)$, we have*

$$\mathcal{P}^{\mathfrak{X}}(T,Q)=\bigsqcup_{z'}\mathcal{P}^{\mathfrak{X}}(T,Q')_{x/z'}\cdot z' \qquad 7.7.1$$

*where $z':Q\to Q'$ runs over a set of representatives for the isomorphism classes of $\big((\mathcal{P}^{\mathfrak{X}})^\circ\big)_Q$-objects dividing $x$.*

7.8. As above, if $R$ and $T$ are two subgroups in $\mathfrak{X}$, we consider the set $\mathfrak{T}^{\mathfrak{x}}_{R,T}$ of $\mathcal{P}^{\mathfrak{x}}$-triples $(x,Q,y)$ where $Q\in\mathfrak{X}$ and moreover $x$ and $y$ respectively belong to $\mathcal{P}^{\mathfrak{x}}(R,Q)_y$ and to $\mathcal{P}^{\mathfrak{x}}(T,Q)_x$ or, equivalently, setting $\alpha=\pi^{\mathfrak{x}}_{R,Q}(x)$ and $\beta=\pi^{\mathfrak{x}}_{T,Q}(y)$ we have (cf. 4.4.1)

$$^{\alpha^*}\mathcal{F}_R\big(\alpha(Q)\big)\cap{}^{\beta^*}\mathcal{F}_T\big(\beta(Q)\big)=\mathcal{F}_Q(Q) \qquad 7.8.1;$$

note that, for any $v\in R$ and any $w\in T$, the $\mathcal{P}^{\mathfrak{x}}$-triple

$$v{\cdot}(x,Q,y){\cdot}w^{-1}=\big(\tau^{\mathfrak{x}}_R(v){\cdot}x,Q,\tau^{\mathfrak{x}}_T(w){\cdot}y\big) \qquad 7.8.2$$

still belongs to $\mathfrak{T}^{\mathfrak{x}}_{R,T}$ and the quotient set $(R\times T)\backslash\mathfrak{T}^{\mathfrak{x}}_{R,T}$ clearly coincides with $\tilde{\mathfrak{T}}^{\mathfrak{x}}_{R,T}$.

7.9. Similarly, we say that two $\mathcal{P}^{\mathfrak{x}}$-triples $(x,Q,y)$ and $(x',Q',y')$ are *equivalent* if there exists a $\mathcal{P}^{\mathfrak{x}}$-isomorphism $z:Q\cong Q'$ fulfilling

$$x'{\cdot}z=x\quad\text{and}\quad y'{\cdot}z=y \qquad 7.9.1;$$

since $\mathcal{P}^{\mathfrak{x}}$ is divisible, such a $\mathcal{P}^{\mathfrak{x}}$-isomorphism $z$ is *unique*; in particular, in any equivalent class we may find a unique element fulfilling $Q\subset R$ and $x=\tau^{\mathfrak{x}}_{R,Q}(1)$. Consequently, for any $Q\in\mathfrak{X}$ denoting by $\mathcal{S}^{\mathfrak{x}}_Q$ the set of subgroups of $Q$ belonging to $\mathfrak{X}$, in $\mathfrak{ac}(\mathcal{P}^{\mathfrak{x}})$ we can define

$$R\cap^{\mathcal{P}^{\mathfrak{x}}}T=\bigoplus_{Q\in\mathcal{S}^{\mathfrak{x}}_Q}\ \bigoplus_{y\in\mathcal{P}^{\mathfrak{x}}(T,Q)_{\tau^{\mathfrak{x}}_{R,Q}(1)}} Q \qquad 7.9.2$$

and we clearly have canonical $\mathfrak{ac}(\mathcal{P}^{\mathfrak{x}})$-morphisms

$$R\longleftarrow R\cap^{\mathcal{P}^{\mathfrak{x}}}T\longrightarrow T \qquad 7.9.3$$

respectively determined by $\tau^{\mathfrak{x}}_{R,Q}(1)$ and $y$. Note that, for any choice of a set of representatives for the set of equivalence classes in $\mathfrak{T}^{\mathfrak{x}}_{R,T}$, we get an isomorphic object and a *unique* $\mathfrak{ac}(\mathcal{P}^{\mathfrak{x}})$-isomorphism, which is compatible with the canonical morphisms. Once again, we get following result [9, Proposition 24.8].

**Proposition 7.10.** *The category $\mathfrak{ac}(\mathcal{P}^{\mathfrak{x}})$ admits a distributive direct product mapping any pair of elements $R$ and $T$ of $\mathfrak{X}$ on their $\mathcal{P}^{\mathfrak{x}}$-intersection $R\cap^{\mathcal{P}^{\mathfrak{x}}}T$.*

7.11. Here, we are particularly interested in the $\mathcal{P}^{\mathfrak{x}}$-intersection of $P$ with itself; more explicitly, denoting by $\Omega^{\mathfrak{x}}$ the set of pairs $(Q,y)$ formed by $Q\in\mathfrak{X}$ and $y\in\mathcal{P}^{\mathfrak{x}}(P,Q)_{\tau^{\mathfrak{x}}_{P,Q}(1)}$, we have

$$P\cap^{\mathcal{P}^{\mathfrak{x}}}P=\bigoplus_{(Q,y)\in\Omega^{\mathfrak{x}}}Q \qquad 7.11.1;$$

moreover, since $P\times P$ acts on the set $\mathfrak{T}^{\mathfrak{x}}_{P,P}$ (cf. 7.8.2) preserving the equivalence classes, this group acts on $\Omega^{\mathfrak{x}}$ and it is easily checked that [9, 24.9]

7.11.2 $\quad$ *$(u,v)\in P\times P$ maps $(Q,y)\in\Omega^{\mathfrak{x}}$ on* $\big(Q^{u^{-1}},\tau^{\mathfrak{x}}_{P}(v){\cdot}y{\cdot}\tau^{\mathfrak{x}}_{Q,Q^{u^{-1}}}(u^{-1})\big)$.

In particular, $\{1\}\times P$ acts *freely* on $\Omega^{\mathfrak{x}}$. On the other hand, it is clear that the map sending a $\mathcal{P}^{\mathfrak{x}}$-triple $(x,Q,y)\in\mathfrak{T}^{\mathfrak{x}}_{P,P}$ to $(y,Q,x)$ induces a $P\times P$-set isomorphism $\Omega^{\mathfrak{x}}\cong(\Omega^{\mathfrak{x}})^{\circ}$. The point is that, from [9, Proposition 24.10 and Corollary 24.11], and from Proposition 5.4 above, we can give a complete description of $\Omega^{\mathfrak{x}}$.

**Proposition 7.12.** *With the notation above, the stabilizer of $(Q,y)\in\Omega^{\mathfrak{x}}$ in $P\times P$ coincides with $\Delta_{\pi^{\mathfrak{x}}_{P,Q}(y)}(Q)$. In particular, we have a $P\times P$-set isomorphism*

$$\Omega^{\mathfrak{x}}\cong\bigsqcup_{Q}\bigsqcup_{\tilde\varphi}(P\times P)/\Delta_{\varphi}(Q) \qquad 7.12.1$$

*where $Q$ runs over a set of representatives for the set of $P$-conjugacy classes in $\mathfrak{X}$, and $\tilde\varphi$ runs over a set of representatives for the set of $\tilde{\mathcal{F}}_P(Q)$-orbits in $\tilde{\mathcal{F}}(P,Q)_{\tilde\iota^P_Q}$.*

7.13. Consequently, we may assume that $\Omega^{\mathfrak{x}}$ is contained in a *natural $\mathcal{F}$-basic $P\times P$-set* $\Omega$ (cf. 5.5) and our purpose is to show that the perfect $\mathcal{F}^{\mathfrak{x}}$-locality $\mathcal{P}^{\mathfrak{x}}$ is contained in the quotient $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{x}}$ above, of the *natural $\mathcal{F}^{\mathfrak{x}}$-locality* $\mathcal{L}^{\mathrm{n},\mathfrak{x}}$ (cf. 6.7). First of all, it follows from Proposition 7.10 that for any $Q\in\mathfrak{X}$ the inclusion $Q\subset P$ determines an $\mathfrak{ac}(\mathcal{P}^{\mathfrak{x}})$-morphism

$$\tau^{\mathfrak{x}}_{P,Q}(1)\cap^{\mathcal{P}^{\mathfrak{x}}}\tau^{\mathfrak{x}}_{P}(1):Q\cap^{\mathcal{P}^{\mathfrak{x}}}P\longrightarrow P\cap^{\mathcal{P}^{\mathfrak{x}}}P \qquad 7.13.1;$$

actually, according to 7.9.2 and denoting by $\Omega^{\mathfrak{x}}_Q$ the set of pairs $(T,z)$ formed by a subgroup $T$ in $\mathfrak{X}$ contained in $Q$ and by an element $z$ of $\mathcal{P}^{\mathfrak{x}}(P,T)_{\tau^{\mathfrak{x}}_{Q,T}(1)}$, we have

$$Q\cap^{\mathcal{P}^{\mathfrak{x}}}P=\bigoplus_{(T,z)\in\Omega^{\mathfrak{x}}_Q}T \qquad 7.13.2,$$

the group $Q\times P$ acts on $\Omega_Q^{\mathfrak{X}}$ , and the $\mathfrak{ac}(\mathcal{P}^{\mathfrak{X}})$-morphism 7.13.1 determines a $Q\times P$-set homomorphism

$$f_Q^{\mathfrak{X}} : \Omega_Q^{\mathfrak{X}} \longrightarrow \mathrm{Res}_{Q\times P}(\Omega^{\mathfrak{X}}) \subset \mathrm{Res}_{Q\times P}(\Omega) \qquad 7.13.3.$$

From the arguments in [9, Proposition 24.15] we get the following result.

**Proposition 7.14.** *For any $Q\in\mathfrak{X}$, the map $f_Q^{\mathfrak{X}}:\Omega_Q^{\mathfrak{X}}\to\Omega^{\mathfrak{X}}$ sends an element $(T,z)\in\Omega_Q^{\mathfrak{X}}$ to $(R,y)\in\Omega^{\mathfrak{X}}$ if and only if we have $T=Q\cap R$ and $z=y{\cdot}\tau^{\mathfrak{X}}_{R,T}(1)$ . In particular, this map is injective.*

7.15. Thus, according to this proposition, the image of $\Omega_Q^{\mathfrak{X}}$ in the natural $\mathcal{F}$-basic $P\times P$-set $\Omega$ coincides with the union of all the $Q\times P$-orbits isomorphic to $(Q\times P)/\Delta_\eta(T)$ for some $T\in\mathfrak{X}$ . On the other hand, for any $\mathcal{P}^{\mathfrak{X}}$-isomorphism $x:Q\cong Q'$ , it follows again from Proposition 7.10 that we have an $\mathfrak{ac}(\mathcal{P}^{\mathfrak{X}})$-isomorphism

$$x\cap^{\mathcal{P}^{\mathfrak{X}}}\tau_P^{\mathfrak{X}}(1) : Q\cap^{\mathcal{P}^{\mathfrak{X}}}P\cong Q'\cap^{\mathcal{P}^{\mathfrak{X}}}P \qquad 7.15.1$$

and therefore we get a bijection between the sets of indices $\Omega_Q^{\mathfrak{X}}$ and $\Omega_{Q'}^{\mathfrak{X}}$ , which is compatible *via* $\pi^{\mathfrak{X}}_{Q',Q}(x)$ with the respective actions of $Q\times P$ and $Q'\times P$ ; that is to say, we get $Q\times P$-set isomorphism

$$f_x^{\mathfrak{X}} : \Omega_Q^{\mathfrak{X}} \cong \mathrm{Res}_{\pi_{Q',Q}(x)\times \mathrm{id}_P}(\Omega_{Q'}^{\mathfrak{X}}) \qquad 7.15.2.$$

**Proposition 7.16.** *For any $\mathcal{P}^{\mathfrak{X}}$-isomorphism $x:Q\cong Q'$ , the $Q\times P$-set isomorphism*

$$f_x^{\mathfrak{X}} : \Omega_Q^{\mathfrak{X}} \cong \mathrm{Res}_{\pi^{\mathfrak{X}}_{Q',Q}(x)\times \mathrm{id}_P}(\Omega_{Q'}^{\mathfrak{X}}) \qquad 7.16.1$$

*can be extended to an element $f_x$ of $T_G(Q,Q')$ and the image of $f_x$ in $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{X}}(Q',Q)$ is uniquely determined by $x$ .*

**Proof:** Since the $Q\times P$-sets $\mathrm{Res}_{Q\times P}(\Omega)$ and $\mathrm{Res}_{\pi^{\mathfrak{X}}_{Q',Q}(x)\times\mathrm{id}_P}\big(\mathrm{Res}_{Q'\times P}(\Omega)\big)$ are isomorphic (cf. 5.1.2), and the $Q\times P$- and $Q'\times P$-set homomorphisms

$$f_Q^{\mathfrak{X}} : \Omega_Q^{\mathfrak{X}} \longrightarrow \mathrm{Res}_{Q\times P}(\Omega) \quad \text{and} \quad f_{Q'}^{\mathfrak{X}} : \Omega_{Q'}^{\mathfrak{X}} \longrightarrow \mathrm{Res}_{Q'\times P}(\Omega) \qquad 7.16.2$$

are injective (cf. Proposition 7.14), identifying $\Omega_Q^{\mathfrak{X}}$ and $\Omega_{Q'}^{\mathfrak{X}}$ with their images in $\Omega$ , $f_x^{\mathfrak{X}}$ can be extended to a $Q\times P$-set isomorphism

$$f_x : \mathrm{Res}_{Q\times P}(\Omega) \cong \mathrm{Res}_{\pi^{\mathfrak{X}}_{Q',Q}(x)\times\mathrm{id}_P}\big(\mathrm{Res}_{Q'\times P}(\Omega)\big) \qquad 7.16.3;$$

that is to say, we get an element $f_x$ of $T_G(Q,Q')$ (cf. 6.1).

Then, we claim that the image of $f_x$ in $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{x}}(Q',Q)$ (cf. 6.6.1) is independent of our choices; indeed, for another choice $g_x \in T_G(Q,Q')$ fulfilling the above conditions, the composed map $(f_x)^{-1}\circ g_x$ belongs to $C_G(Q)$ and induces the *identity map* on $\Omega_Q^{\mathfrak{x}}$; but, we have $C_G(Q) = \mathrm{Aut}_{Q\times P}(\Omega)$ and, considering the obvious decomposition $\Omega = \Omega_Q^{\mathfrak{x}} \sqcup (\Omega - \Omega_Q^{\mathfrak{x}})$, it follows from 7.15 that

$$\mathrm{Aut}_{Q\times P}(\Omega) \cong \mathrm{Aut}_{Q\times P}(\Omega_Q^{\mathfrak{x}}) \times \mathrm{Aut}_{Q\times P}(\Omega - \Omega_Q^{\mathfrak{x}}) \qquad 7.16.4;$$

moreover, it is easily checked that $\mathfrak{S}_G^{\mathfrak{x}}(Q)$ contains $\mathrm{Aut}_{Q\times P}(\Omega - \Omega_Q^{\mathfrak{x}})$; hence, $(f_x)^{-1}\circ g_x$ belongs to $\mathfrak{S}_G^{\mathfrak{x}}(Q)$ and therefore it has a trivial image in $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{x}}(Q)$, so that $f_x$ and $g_x$ have the same image in $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{x}}(Q',Q)$ (cf. 6.7.3). We are done.

**Corollary 7.17.** *There is a faithful functor $\lambda^{\mathfrak{x}} : \mathcal{P}^{\mathfrak{x}} \to \bar{\mathcal{L}}^{\mathrm{n},\mathfrak{x}}$ which is compatible with the structural functors, and sends any $\mathcal{P}^{\mathfrak{x}}$-isomorphism $x : Q \cong Q'$ to the image of $f_x$ in $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{x}}(Q',Q)$.*

**Proof:** Let us denote by $\lambda^{\mathfrak{x}}(x)$ the image of $f_x$ in $\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{x}}(Q',Q)$; first of all, let $x' : Q' \cong Q''$ be a second $\mathcal{P}^{\mathfrak{x}}$-isomorphism; it is clear that the automorphism $\mathrm{Res}_{\pi^{\mathfrak{x}}_{Q',Q}(x)\times \mathrm{id}_P}(f_{x'}) \circ f_x$ of $\mathrm{Res}_{Q\times P}(\Omega)$ extends $\mathrm{Res}_{\pi^{\mathfrak{x}}_{Q',Q}(x)\times \mathrm{id}_P}(f^{\mathfrak{x}}_{x'}) \circ f^{\mathfrak{x}}_x$; consequently, by the proposition above, we get

$$\lambda^{\mathfrak{x}}(x'{\cdot}x) = \lambda^{\mathfrak{x}}(x'){\cdot}\lambda^{\mathfrak{x}}(x) \qquad 7.17.1.$$

On the other hand, by the *divisibility* of $\mathcal{P}^{\mathfrak{x}}$, any $\mathcal{P}^{\mathfrak{x}}$-morphism $z : T \to Q$ is the composition of $\tau^{\mathfrak{x}}_{Q,T'}(1)$ with a $\mathcal{P}^{\mathfrak{x}}$-isomorphism $z_* : T \cong T'$ where we set $T' = \big(\pi^{\mathfrak{x}}_{Q,T}(z)\big)(T)$; then, we simply define

$$\lambda^{\mathfrak{x}}(z) = \bar{\tau}^{\mathrm{n},\mathfrak{x}}_{Q,T'}(1){\cdot}\lambda^{\mathfrak{x}}(z_*) \qquad 7.17.2.$$

Now, in order to prove that this correspondence defines a functor, it suffices to show that, for any $\mathcal{P}^{\mathfrak{x}}$-isomorphism $x : Q \cong Q'$ and any subgroup $R$ of $Q$, setting $R' = \big(\pi^{\mathfrak{x}}_{Q',Q}(x)\big)(R)$ and denoting by $y : R \cong R'$ the $\mathcal{P}^{\mathfrak{x}}$-isomorphism induced by $x$ (cf. 2.8), we still have

$$\lambda^{\mathfrak{x}}(x){\cdot}\bar{\tau}^{\mathrm{n},\mathfrak{x}}_{Q,R}(1) = \bar{\tau}^{\mathrm{n},\mathfrak{x}}_{Q',R'}(1){\cdot}\lambda^{\mathfrak{x}}(y) \qquad 7.17.3.$$

But, it is quite clear that the commutative $\mathfrak{ac}(\mathcal{P}^{\mathfrak{x}})$-diagram (cf. Proposition 7.10)

$$\begin{array}{ccc} R \cap^{\mathcal{P}^{\mathfrak{x}}} P & \xrightarrow{\tau^{\mathfrak{x}}_{Q,R}(1) \cap^{\mathcal{P}^{\mathfrak{x}}} \tau^{\mathfrak{x}}_P(1)} & Q \cap^{\mathcal{P}^{\mathfrak{x}}} P \\ {\scriptstyle y \cap^{\mathcal{P}^{\mathfrak{x}}} \tau^{\mathfrak{x}}_P(1)} \downarrow & & \downarrow {\scriptstyle x \cap^{\mathcal{P}^{\mathfrak{x}}} \tau^{\mathfrak{x}}_P(1)} \\ R' \cap^{\mathcal{P}^{\mathfrak{x}}} P & \xrightarrow{\tau^{\mathfrak{x}}_{Q',R'}(1) \cap^{\mathcal{P}^{\mathfrak{x}}} \tau^{\mathfrak{x}}_P(1)} & Q' \cap^{\mathcal{P}^{\mathfrak{x}}} P \end{array} \qquad 7.17.4$$

determines a commutative diagram of $R\times P$-sets (cf. 7.13)

$$\begin{array}{ccc}\Omega_R^{\mathfrak{X}} & \longrightarrow & \mathrm{Res}^{Q\times P}_{R\times P}(\Omega_Q^{\mathfrak{X}})\\ f_y^{\mathfrak{X}}\big\downarrow & & \big\downarrow \mathrm{Res}^{Q\times P}_{R\times P}(f_x^{\mathfrak{X}})\\ \mathrm{Res}_{\pi_y\times \mathrm{id}_P}(\Omega_{R'}^{\mathfrak{X}}) & \longrightarrow & \mathrm{Res}^{Q\times P}_{R\times P}\big(\mathrm{Res}_{\pi_x\times\mathrm{id}_P}(\Omega_{Q'}^{\mathfrak{X}})\big)\end{array} \qquad 7.17.5.$$

Consequently, the element $f_x$ of $T_G(Q,Q')$ extending $f_x^{\mathfrak{X}}$ also extends $f_y^{\mathfrak{X}}$ and we can choose $f_y = f_x$. Moreover, since $\bar\tau^{\mathrm{n},\mathfrak{X}}$ is *faithful* (cf. 6.7), it is easily checked that $\lambda^{\mathfrak{X}}$ induces an injective group homomorphism $\mathcal{P}^{\mathfrak{X}}(Q)\to\bar{\mathcal{L}}^{\mathrm{n},\mathfrak{X}}(Q)$ for any $Q\in\mathfrak{X}$ and therefore this functor is faithful too. We are done.

## 8. Existence and uniqueness of the perfect $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality

8.1. Let $P$, $\mathcal{F}$ and $\mathfrak{X}$ be as above and denote by $\hat{\mathfrak{X}}$ the subset of subgroups $Q$ in $\mathfrak{X}$ which are $\mathcal{F}$-selfcentralizing. As in §2 above, consider a $p$-coherent $\mathcal{F}^{\mathfrak{X}}$-locality $\mathcal{L}^{\mathfrak{X}}$ with the structural functors

$$\tau^{\mathfrak{X}} : \mathcal{T}_P^{\mathfrak{X}} \longrightarrow \mathcal{L}^{\mathfrak{X}} \quad \text{and} \quad \pi^{\mathfrak{X}} : \mathcal{L}^{\mathfrak{X}} \longrightarrow \mathcal{F}^{\mathfrak{X}} \qquad 8.1.1;$$

in the notation, we replace $\mathfrak{X}$ by $\hat{\mathfrak{X}}$ for the corresponding restrictions. Let us denote by $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}$ the *quotient* $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality of $\mathcal{L}^{\hat{\mathfrak{X}}}$ defined by

$$\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}(Q,R) = \mathcal{L}^{\hat{\mathfrak{X}}}(Q,R)\big/\tau^{\hat{\mathfrak{X}}}_R\big(Z(R)\big)\cdot\Phi\big(\mathrm{Ker}(\pi^{\hat{\mathfrak{X}}}_R)\big) \qquad 8.1.2$$

for any $Q,R\in\hat{\mathfrak{X}}$, which is easily checked to be *p-coherent*, and denote by $\bar\tau^{\hat{\mathfrak{X}}}$ and $\bar\pi^{\hat{\mathfrak{X}}}$ its structural functors.

**Lemma 8.2.** *With the notation and hypothesis above, we have a contravariant functor*

$$\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}} : \tilde{\mathcal{F}}^{\hat{\mathfrak{X}}} \longrightarrow k\text{-}\mathfrak{mod} \qquad 8.2.1$$

*mapping any $Q$ subgroup in $\hat{\mathfrak{X}}$ on $\mathrm{Ker}(\bar\pi^{\hat{\mathfrak{X}}}_Q)$ and any $\tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$-morphism $\tilde\varphi : R\to Q$ on the $k$-linear map sending $\bar u\in\mathrm{Ker}(\bar\pi^{\hat{\mathfrak{X}}}_Q)$ to the unique element $\bar v\in\mathrm{Ker}(\bar\pi^{\hat{\mathfrak{X}}}_R)$ fulfilling $\bar x\cdot\bar v = \bar u\cdot\bar x$ for some $\bar x\in\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}(Q,R)$ lifting $\tilde\varphi$.*

**Proof:** Setting $\varphi = \bar\pi^{\hat{\mathfrak{X}}}_{Q,R}(\bar x)$ and $\alpha = \bar\pi^{\hat{\mathfrak{X}}}_Q(\bar u)$, clearly $\alpha\big(\varphi(R)\big) = \varphi(R)$ and therefore the existence and the uniqueness of $\bar v$ follow from the *divisibility* of $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}$; moreover, for another lifting $\bar x'$ of $\tilde\varphi$, we have $\bar x' = \bar x\cdot\bar\tau^{\hat{\mathfrak{X}}}_R(z)$ for some $z\in R$; but, since $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}$ is *coherent*, it follows from [9, Proposition 17.10] that $\bar\tau^{\hat{\mathfrak{X}}}_R(z)$ centralizes $\mathrm{Ker}(\bar\pi^{\hat{\mathfrak{X}}}_R)$; hence, we get

$$\bar x'\cdot\bar v = \bar x\cdot\bar\tau_R(z)\cdot\bar v = \bar x\cdot\bar v\cdot\bar\tau_R(z) = \bar u\cdot\bar x' \qquad 8.2.2;$$

at this point, the functoriality of $\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}$ is easily checked.

8.3. Since $\bar\tau^{\mathfrak{X}}_Q\big(Z(Q)\big) = \{1\}$ for any $Q \in \hat{\mathfrak{X}}$, it follows from 3.14 that, for any pair of subgroups $Q$ and $R$ in $\hat{\mathfrak{X}}$, and any $\mathcal{F}$-morphism $\varphi\colon R \to Q$, we can choose a lifting $\bar x_\varphi \in \bar{\mathcal{L}}^{\hat{\mathfrak{X}}}(Q,R)$ in such a way that we have

$$\bar x_{\rho\circ\varphi\circ\sigma} = \rho{\cdot}\bar x_\varphi{\cdot}\sigma \tag{8.3.1}$$

for any $\rho \in \mathcal{F}(Q)$ and any $\sigma \in \mathcal{F}(R)$; this choice and Theorem 4.8 lead to the following result.

**Theorem 8.4.** *If $\mathcal{L}^{\hat{\mathfrak{X}}}$ is a p-coherent $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality such that the structural functor $\tau^{\hat{\mathfrak{X}}}$ is faithful then a minimal $\mathcal{F}^{\hat{\mathfrak{X}}}$-sublocality $\mathcal{P}^{\hat{\mathfrak{X}}}$ of $\mathcal{L}^{\hat{\mathfrak{X}}}$ is perfect.*

**Proof:** Actually, arguing by induction, we already may assume that $\mathcal{L}^{\hat{\mathfrak{X}}}$ has no proper $\mathcal{F}^{\hat{\mathfrak{X}}}$-sublocalities.

For any triple of subgroups $Q$, $R$ and $T$ in $\hat{\mathfrak{X}}$, and any pair of $\mathcal{F}$-morphisms $\psi\colon T \to R$ and $\varphi\colon R \to Q$, since $\bar x_\varphi{\cdot}\bar x_\psi$ and $\bar x_{\varphi\circ\psi}$ have the same image in $\mathcal{F}(Q,T)$, the divisibility of $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}$ guarantees the existence and the uniqueness of $\bar m_{\varphi,\psi} \in \bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(T)$ fulfilling

$$\bar x_\varphi{\cdot}\bar x_\psi = \bar x_{\varphi\circ\psi}{\cdot}\bar m_{\varphi,\psi} \tag{8.4.1;}$$

that is to say, we have a correspondence mapping any $\mathcal{F}^{\hat{\mathfrak{X}}}$-*chain* $\mathfrak{q}\colon\Delta_2 \to \mathcal{F}^{\hat{\mathfrak{X}}}$ on $\bar m_{\mathfrak{q}(0\bullet1),\mathfrak{q}(1\bullet2)}$ and we claim that this correspondence is *stable* (cf. 4.2).

Indeed, for any $\mathcal{F}$-isomorphisms $\sigma\colon Q \cong Q'$, $\rho\colon R \cong R'$ and $\omega\colon T \cong T'$, setting $\varphi' = \sigma\circ\varphi\circ\rho^{-1}$ and $\psi' = \rho\circ\psi\circ\omega^{-1}$, we get

$$\begin{aligned}\bar x_{\varphi'\circ\psi'}{\cdot}\bar m_{\varphi',\psi'} &= \bar x_{\varphi'}{\cdot}\bar x_{\psi'} = (\sigma{\cdot}\bar x_\varphi{\cdot}\rho^{-1}){\cdot}(\rho{\cdot}\bar x_\psi{\cdot}\omega^{-1})\\ &= \sigma{\cdot}(\bar x_{\varphi\circ\psi}{\cdot}\bar m_{\varphi,\psi}){\cdot}\omega^{-1} = (\sigma{\cdot}\bar x_{\varphi\circ\psi}{\cdot}\omega^{-1}){\cdot}(\omega{\cdot}\bar m_{\varphi,\psi}{\cdot}\omega^{-1})\\ &= \bar x_{\varphi'\circ\psi'}{\cdot}\big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde\omega)\big)(\bar m_{\varphi,\psi})\end{aligned} \tag{8.4.2}$$

and therefore we obtain $\bar m_{\varphi',\psi'} = \big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde\omega)\big)(\bar m_{\varphi,\psi})$. In particular note that, since $\varphi\circ\kappa^{\hat{\mathfrak{X}}}_R(w) = \kappa^{\hat{\mathfrak{X}}}_Q\big(\varphi(w)\big)\circ\varphi$ for any $w \in R$, we obtain

$$\bar m_{\kappa^{\hat{\mathfrak{X}}}_Q(u)\circ\varphi,\kappa^{\hat{\mathfrak{X}}}_R(v)\circ\psi} = \bar m_{\varphi,\psi} \tag{8.4.3,}$$

for any $u \in Q$ and any $v \in R$, proving that $\bar m_{\varphi,\psi}$ only depends on the classes $\tilde\varphi \in \tilde{\mathcal{F}}(Q,R)$ of $\varphi$ and $\tilde\psi \in \tilde{\mathcal{F}}(R,T)$ of $\psi$; that is to say, the above correspondence factorizes throughout the set of $\tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$-*chains* $\tilde{\mathfrak{q}}\colon\Delta_2 \to \tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$.

Moreover, for a third $\mathcal{F}^{\hat{\mathfrak{X}}}$-morphism $\eta\colon U \to T$, it is clear that

$$\begin{aligned}\bar x_\varphi{\cdot}\bar x_\psi{\cdot}\bar x_\eta &= \bar x_{\varphi\circ\psi}{\cdot}\bar m_{\varphi,\psi}{\cdot}\bar x_\eta = \bar x_{\varphi\circ\psi}{\cdot}\bar x_\eta{\cdot}\big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde\eta)\big)(\bar m_{\varphi,\psi})\\ &= \bar x_{\varphi\circ\psi\circ\eta}{\cdot}\bar m_{\varphi\circ\psi,\eta}{\cdot}\big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde\eta)\big)(\bar m_{\varphi,\psi})\end{aligned} \tag{8.4.4}$$

and, similarly, we have

$$\bar{x}_\varphi\cdot\bar{x}_\psi\cdot\bar{x}_\eta = \bar{x}_{\varphi\circ\psi\circ\eta}\cdot\bar{m}_{\varphi,\psi\circ\eta}\cdot\bar{m}_{\psi,\eta} \qquad 8.4.5;$$

hence, in the $k$-vector space $\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(T)$ we get the 2-*cycle* condition

$$\bar{m}_{\varphi\circ\psi,\eta} + \big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde{\eta})\big)(\bar{m}_{\varphi,\psi}) = \bar{m}_{\varphi,\psi\circ\eta} + \bar{m}_{\psi,\eta} \qquad 8.4.6.$$

In conclusion, the above correspondence determines a *stable* $\bar{m}^{\hat{\mathfrak{X}}}$*-valued* 2-*cocycle* over $\tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$, and then it follows from Theorem 4.8 that, for any $\mathcal{F}^{\hat{\mathfrak{X}}}$-morphism $\tilde{\varphi}: R \to Q$, there exists an element $\bar{\ell}_{\tilde{\varphi}} \in \bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(R)$ in such a way that in $\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(R)$ we have

$$\bar{m}_{\varphi,\psi} = \big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde{\psi})\big)(\bar{\ell}_{\tilde{\varphi}}) - (\bar{\ell}_{\tilde{\varphi}\circ\tilde{\psi}}) + \bar{\ell}_{\tilde{\psi}} \qquad 8.4.7;$$

then, the image in $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}(Q,T)$ of equality 8.4.1 becomes

$$\big(\bar{x}_\varphi\cdot(\bar{\ell}_{\tilde{\varphi}})^{-1}\big)\cdot\big(\bar{x}_\psi\cdot(\bar{\ell}_{\tilde{\psi}})^{-1}\big) = \bar{x}_{\varphi\circ\psi}\cdot(\bar{\ell}_{\tilde{\varphi}\circ\tilde{\psi}})^{-1} \qquad 8.4.8.$$

Thus, we get a functorial *section*

$$\bar{\sigma} : \mathcal{F}^{\hat{\mathfrak{X}}} \longrightarrow \bar{\mathcal{L}}^{\hat{\mathfrak{X}}} \qquad 8.4.9$$

of the structural functor $\bar{\pi}^{\hat{\mathfrak{X}}}$; then, the converse image in $\mathcal{L}^{\hat{\mathfrak{X}}}$ of the image of this section yields a $\mathcal{F}^{\hat{\mathfrak{X}}}$-sublocality $\mathcal{L}^{\hat{\mathfrak{X}}}$ which, by minimality, has to coincide with $\mathcal{L}^{\hat{\mathfrak{X}}}$; hence, $\bar{\sigma}$ is an isomorphism and this fact forces

$$\mathrm{Ker}(\pi^{\hat{\mathfrak{X}}}_Q) = \tau^{\hat{\mathfrak{X}}}_Q\big(Z(Q)\big)\cdot\Phi\big(\mathrm{Ker}(\pi^{\hat{\mathfrak{X}}}_Q)\big) \qquad 8.4.10$$

for any $Q \in \hat{\mathfrak{X}}$, so that we obtain $\mathrm{Ker}(\pi^{\hat{\mathfrak{X}}}_Q) = \tau^{\hat{\mathfrak{X}}}_Q\big(Z(Q)\big)$; hence, it follows from 2.11 and Theorem 2.10 that $\mathcal{L}^{\hat{\mathfrak{X}}}$ is a *perfect* $\mathcal{F}^{\hat{\mathfrak{X}}}$*-locality*.

8.5. Given a family $z = \{z_Q\}_{Q\in\mathfrak{X}}$ of elements $z_Q \in \mathrm{Ker}(\pi^{\mathfrak{X}}_Q)$, we can define a *bijective functor* $\kappa_z : \mathcal{L}^{\mathfrak{X}} \cong \mathcal{L}^{\mathfrak{X}}$ which is the identity map over $\mathfrak{X}$ and sends any $\mathcal{L}^{\mathfrak{X}}$-morphism $x: R \to Q$ to $z_Q\cdot x\cdot(z_R)^{-1}$; let us call *inner* $\mathcal{F}^{\mathfrak{X}}$*-automorphisms* of $\mathcal{L}^{\mathfrak{X}}$ this kind of functors.

**Theorem 8.6.** *With the notation and the hypothesis above, if* $\mathcal{P}^{\hat{\mathfrak{X}}}$ *and* $\mathcal{P}'^{\hat{\mathfrak{X}}}$ *are minimal* $\mathcal{F}^{\hat{\mathfrak{X}}}$*-sublocalities of* $\mathcal{L}^{\hat{\mathfrak{X}}}$ *then there exists an inner* $\mathcal{F}^{\mathfrak{X}}$*-automorphism* $\kappa_z$ *of* $\mathcal{L}^{\mathfrak{X}}$ *such that* $\mathcal{P}'^{\hat{\mathfrak{X}}} = \kappa_z(\mathcal{P}^{\hat{\mathfrak{X}}})$.

**Proof:** Once again, arguing by induction, we may assume that any proper $\mathcal{F}^{\hat{\mathfrak{X}}}$-sublocality of $\mathcal{L}^{\hat{\mathfrak{X}}}$ containing $\mathcal{P}'^{\hat{\mathfrak{X}}}$ does not contain $\kappa_z(\mathcal{P}^{\hat{\mathfrak{X}}})$ for any inner $\mathcal{F}^{\mathfrak{X}}$-automorphism $\kappa_z$ of $\mathcal{L}^{\mathfrak{X}}$. Denote by

$$\mathfrak{i}^{\hat{\mathfrak{X}}} : \mathcal{P}^{\hat{\mathfrak{X}}} \longrightarrow \mathcal{L}^{\hat{\mathfrak{X}}} \quad \text{and} \quad \mathfrak{i}'^{\hat{\mathfrak{X}}} : \mathcal{P}'^{\hat{\mathfrak{X}}} \longrightarrow \mathcal{L}^{\hat{\mathfrak{X}}} \tag{8.6.1}$$

the functors determined by the inclusions; actually, it follows from Theorem 8.4 that they induce two functorial *sections*

$$\bar{\varpi} : \mathcal{F}^{\hat{\mathfrak{X}}} \cong \bar{\mathcal{P}}^{\hat{\mathfrak{X}}} \longrightarrow \bar{\mathcal{L}}^{\hat{\mathfrak{X}}} \quad \text{and} \quad \bar{\varpi}' : \mathcal{F}^{\hat{\mathfrak{X}}} \cong \bar{\mathcal{P}}'^{\hat{\mathfrak{X}}} \longrightarrow \bar{\mathcal{L}}^{\hat{\mathfrak{X}}} \tag{8.6.2}$$

of the structural functor $\bar{\pi}^{\hat{\mathfrak{X}}}$; in particular, for any pair of subgroups $Q$ and $R$ in $\hat{\mathfrak{X}}$, and any $\mathcal{F}^{\hat{\mathfrak{X}}}$-morphism $\varphi : R \to Q$ there is a unique element $\bar{m}_\varphi$ in $\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(R)$ such that

$$\bar{\varpi}'(\varphi) = \bar{\varpi}(\varphi)\cdot\bar{m}_\varphi \tag{8.6.3.}$$

On the other hand, it follows from Proposition 3.9 that there are unique natural maps

$$\lambda_{\mathcal{P}^{\hat{\mathfrak{X}}}} : \mathfrak{loc}_{\mathcal{F}^{\hat{\mathfrak{X}}}} \longrightarrow \mathfrak{loc}_{\mathcal{P}^{\hat{\mathfrak{X}}}} \quad \text{and} \quad \lambda_{\mathcal{P}'^{\hat{\mathfrak{X}}}} : \mathfrak{loc}_{\mathcal{F}^{\hat{\mathfrak{X}}}} \longrightarrow \mathfrak{loc}_{\mathcal{P}'^{\hat{\mathfrak{X}}}} \tag{8.6.4}$$

such that

$$\mathfrak{lv} * \lambda_{\mathcal{P}^{\hat{\mathfrak{X}}}} = \mathrm{id}_{\mathfrak{aut}_{\mathcal{F}^{\hat{\mathfrak{X}}}}} = \mathfrak{lv} * \lambda_{\mathcal{P}'^{\hat{\mathfrak{X}}}} \tag{8.6.5}$$

and that, for any $\mathcal{F}^{\hat{\mathfrak{X}}}$-chain $\mathfrak{q} : \Delta_n \to \mathcal{F}^{\hat{\mathfrak{X}}}$ fully normalized in $\mathcal{F}$, we have

$$(\lambda_{\mathcal{P}^{\hat{\mathfrak{X}}}})_{(\mathfrak{q},\Delta_n)} \circ \tau_{\mathfrak{q}}^{\hat{\mathfrak{X}}} = \tau_{\hat{\mathfrak{q}}}^{\hat{\mathfrak{X}}} \quad \text{and} \quad (\lambda_{\mathcal{P}'^{\hat{\mathfrak{X}}}})_{(\mathfrak{q},\Delta_n)} \circ \tau_{\mathfrak{q}}^{\hat{\mathfrak{X}}} = \tau_{\hat{\mathfrak{q}}'}^{\hat{\mathfrak{X}}} \tag{8.6.6}$$

where $\hat{\mathfrak{q}} : \Delta_n \to \mathcal{P}^{\hat{\mathfrak{X}}}$ and $\hat{\mathfrak{q}}' : \Delta_n \to \mathcal{P}'^{\hat{\mathfrak{X}}}$ are functors lifting $\mathfrak{q}$; but, it is quite clear that the functors $\mathfrak{i}^{\hat{\mathfrak{X}}}$ and $\mathfrak{i}'^{\hat{\mathfrak{X}}}$ determine natural maps (cf. 3.8.2)

$$\overline{\mathfrak{loc}}_{\mathfrak{i}^{\hat{\mathfrak{X}}}} : \mathfrak{loc}_{\mathcal{P}^{\hat{\mathfrak{X}}}} \longrightarrow \mathfrak{loc}_{\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}} \quad \text{and} \quad \overline{\mathfrak{loc}}_{\mathfrak{i}'^{\hat{\mathfrak{X}}}} : \mathfrak{loc}_{\mathcal{P}'^{\hat{\mathfrak{X}}}} \longrightarrow \mathfrak{loc}_{\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}} \tag{8.6.7;}$$

then, it is easily checked that the compositions $\overline{\mathfrak{loc}}_{\mathfrak{i}^{\hat{\mathfrak{X}}}} \circ \lambda_{\mathcal{P}^{\hat{\mathfrak{X}}}}$ and $\overline{\mathfrak{loc}}_{\mathfrak{i}'^{\hat{\mathfrak{X}}}} \circ \lambda_{\mathcal{P}'^{\hat{\mathfrak{X}}}}$ from $\mathfrak{loc}_{\mathcal{F}^{\hat{\mathfrak{X}}}}$ to $\mathfrak{loc}_{\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}}$ fulfill the corresponding conditions 8.6.5 and 8.6.6 above and therefore, once again from Proposition 3.9 they coincide.

In particular, considering representatives for the corresponding $\widetilde{\mathfrak{Loc}}$-morphisms (cf. 3.4), for any $Q \in \hat{\mathfrak{X}}$ there exists $\bar{z}_Q \in \bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(Q) \subset \bar{\mathcal{L}}^{\hat{\mathfrak{X}}}(Q)$ fulfilling

$$\bar{\mathcal{P}}'^{\hat{\mathfrak{X}}}(Q) = \bar{\mathcal{P}}^{\hat{\mathfrak{X}}}(Q)^{\bar{z}_Q} \tag{8.6.8.}$$

That is to say, up to replace $\mathcal{P}^{\hat{\mathfrak{X}}}$ by its image *via* a suitable *inner* $\mathcal{F}^{\hat{\mathfrak{X}}}$*-automorphism* of $\mathcal{L}^{\hat{\mathfrak{X}}}$, in 8.6.2 above we may assume that, for any $Q \in \hat{\mathfrak{X}}$, in $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}(Q)$ we have

$$\bar{\varpi}'\big(\mathcal{F}(Q)\big) = \bar{\varpi}\big(\mathcal{F}(Q)\big) \qquad 8.6.9.$$

In this situation, we claim that the correspondence in 8.6.3 above mapping any $\mathcal{F}^{\hat{\mathfrak{X}}}$*-chain* $\mathfrak{r}:\Delta_1 \to \mathcal{F}^{\hat{\mathfrak{X}}}$ on $\bar{m}_{\mathfrak{r}(0)}$ is *stable* (cf. 4.2); indeed, for any $\mathcal{F}$-isomorphisms $\sigma : Q \cong Q'$ and $\rho : R \cong R'$, setting $\varphi' = \sigma\circ\varphi\circ\rho^{-1}$ we have

$$\begin{aligned}\bar{\varpi}(\sigma)\cdot\bar{\varpi}(\varphi)\cdot\bar{\varpi}(\rho)^{-1}\cdot\bar{m}_{\varphi'} &= \bar{\varpi}(\varphi')\cdot\bar{m}_{\varphi'} = \bar{\varpi}'(\varphi')\\ &= \bar{\varpi}'(\sigma)\cdot\bar{\varpi}'(\varphi)\cdot\bar{\varpi}'(\rho)^{-1}\\ &= \bar{\varpi}'(\sigma)\cdot(\bar{\varpi}(\varphi)\cdot\bar{m}_\varphi)\cdot\bar{\varpi}'(\rho)^{-1}\end{aligned} \qquad 8.6.10;$$

but, since $\bar{\varpi}'\big(\mathcal{F}(Q)\big) = \bar{\varpi}\big(\mathcal{F}(Q)\big)$, the element $\bar{\varpi}(\sigma)^{-1}\cdot\bar{\varpi}'(\sigma)$ belongs to $\bar{\varpi}\big(\mathcal{F}(Q)\big)$ and therefore isomorphism 3.11.2 and equality 8.6.3 implies that $\bar{\varpi}'(\sigma) = \bar{\varpi}(\sigma)$; similarly, we get $\bar{\varpi}'(\rho) = \bar{\varpi}(\rho)$ and thus equality 8.6.10 forces

$$\bar{\varpi}(\rho)^{-1}\cdot\bar{m}_{\varphi'} = \bar{m}_\varphi\cdot\bar{\varpi}(\rho)^{-1} \qquad 8.6.11,$$

proving the *stability* of the above correspondence. In particular, for any $u \in Q$, note that we obtain $\bar{m}_{\kappa_Q^{\hat{\mathfrak{X}}}(u)\circ\varphi} = \bar{m}_\varphi$ proving that $\bar{m}_\varphi$ only depends on the class $\tilde{\varphi} \in \tilde{\mathcal{F}}(Q,R)$ of $\varphi$; thus, the above correspondence factorizes throughout the set of $\tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$*-chains* $\tilde{\mathfrak{r}}:\Delta_1 \to \tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$.

Moreover, for a second $\mathcal{F}^{\hat{\mathfrak{X}}}$-morphism $\psi : T \to R$, we have (cf. 8.2.2)

$$\begin{aligned}\bar{\varpi}(\varphi)\cdot\bar{\varpi}(\psi)\cdot\bar{m}_{\varphi\circ\psi} &= \bar{\varpi}(\varphi\circ\psi)\cdot\bar{m}_{\varphi\circ\psi} = \bar{\varpi}'(\varphi\circ\psi)\\ &= \bar{\varpi}'(\varphi)\cdot\bar{\varpi}'(\psi) = \bar{\varpi}(\varphi)\cdot\bar{m}_\varphi\cdot\bar{\varpi}(\psi)\cdot\bar{m}_\psi\\ &= \bar{\varpi}(\varphi)\cdot\bar{\varpi}(\psi)\cdot\big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde{\psi})\big)(\bar{m}_\varphi)\cdot\bar{m}_\psi\end{aligned} \qquad 8.6.12$$

and therefore in the $k$-vector space $\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(T)$ we get the 1-*cocycle* condition

$$\bar{m}_{\varphi\circ\psi} = \big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde{\psi})\big)(\bar{m}_\varphi) + \bar{m}_\psi \qquad 8.6.13.$$

In conclusion, the above correspondence determines a *stable* $\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}$*-valued* 1-*cocycle* over $\tilde{\mathcal{F}}^{\hat{\mathfrak{X}}}$ and then it follows from Theorem 4.8 that for any $Q$-subgroup in $\mathfrak{X}$ there exists $\bar{z}_Q \in \bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(Q)$ in such a way that in $\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(R)$ we have

$$\bar{m}_\varphi = \big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde{\varphi})\big)(\bar{z}_Q) - \bar{z}_R \qquad 8.6.14.$$

Consequently, the image in $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}(Q,T)$ of equality 8.6.3 becomes

$$\bar{\varpi}'(\varphi) = \bar{\varpi}(\varphi)\cdot\big(\bar{\mathfrak{m}}^{\hat{\mathfrak{X}}}(\tilde{\varphi})\big)(\bar{z}_Q)\cdot(\bar{z}_R)^{-1} = \bar{z}_Q\cdot\bar{\varpi}(\varphi)\cdot(\bar{z}_R)^{-1} \qquad 8.6.15;$$

that is to say, lifting $\bar{z}_Q$ to $z_Q \in \mathrm{Ker}(\pi_Q^{\hat{\mathfrak{X}}})$ for any $Q \in \hat{\mathfrak{X}}$ and considering the family $z = \{z_Q\}_{Q\in\hat{\mathfrak{X}}}$, equalities 8.6.15 show that the converse image in $\mathcal{L}^{\hat{\mathfrak{X}}}$ of the image of the functor $\bar{\varpi}'$ in $\bar{\mathcal{L}}^{\hat{\mathfrak{X}}}$ contains $\mathcal{P}'^{\hat{\mathfrak{X}}}$ and $\kappa_z(\mathcal{P}^{\hat{\mathfrak{X}}})$; hence, by minimality, this converse image is equal to $\mathcal{L}^{\hat{\mathfrak{X}}}$ and therefore $\bar{\varpi}'$ is an isomorphism which once again forces

$$\mathrm{Ker}(\pi_Q^{\hat{\mathfrak{X}}}) = \tau_Q^{\hat{\mathfrak{X}}}\big(Z(Q)\big)\cdot\Phi\big(\mathrm{Ker}(\pi_Q^{\hat{\mathfrak{X}}})\big) \qquad 8.6.16$$

for any $Q \in \hat{\mathfrak{X}}$, so that we obtain $\mathrm{Ker}(\pi_Q^{\hat{\mathfrak{X}}}) = \tau_Q^{\hat{\mathfrak{X}}}\big(Z(Q)\big)$; consequently, we get $\mathcal{P}^{\hat{\mathfrak{X}}} = \mathcal{L}^{\hat{\mathfrak{X}}} = \mathcal{P}'^{\hat{\mathfrak{X}}}$. We are done.

**Corollary 8.7.** *There exists a perfect $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality $\mathcal{P}^{\hat{\mathfrak{X}}}$, unique up to isomorphisms.*

**Proof:** Considering the quotient $\bar{\mathcal{L}}^{\mathrm{n}\hat{\mathfrak{X}}}$ (cf. 6.7) of the *natural $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality* $\mathcal{L}^{\mathrm{n},\hat{\mathfrak{X}}}$, we know that $\bar{\tau}^{\mathrm{n},\mathfrak{X}}$ is a *faithful* functor and therefore it suffices to apply Theorem 8.4 to get the existence of a perfect $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality $\mathcal{P}^{\hat{\mathfrak{X}}} \subset \mathcal{L}^{\mathrm{n},\hat{\mathfrak{X}}}$. On the other hand, if follows from Corollary 7.17 that any perfect $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality $\mathcal{P}'^{\hat{\mathfrak{X}}}$ is contained in $\bar{\mathcal{L}}^{\mathrm{n},\hat{\mathfrak{X}}}$ and therefore the uniqueness follows from Theorem 8.6.

## 9. Existence, uniqueness, universality and functoriality of $\mathcal{P}^{\mathfrak{X}}$

9.1. It follows from Corollary 8.7 that there exists a perfect $\mathcal{F}^{\mathrm{sc}}$-locality $\mathcal{P}^{\mathrm{sc}}$, unique up to isomorphisms, and therefore it follows from [9, Chap. 20]† that there also exists a *perfect $\mathcal{F}$-locality $\mathcal{P}$, unique up to isomorphisms*; in particular, there exists a perfect $\mathcal{F}^{\mathfrak{X}}$-locality $\mathcal{P}^{\mathfrak{X}}$, but its possible uniqueness has to be discussed; more generally, the announced *universality* has to be discussed in all the cases.

**Theorem 9.2.** *There exists a perfect $\mathcal{F}^{\mathfrak{X}}$-locality $\mathcal{P}^{\mathfrak{X}}$, unique up to isomorphisms. Moreover, for any p-coherent $\mathcal{F}^{\mathfrak{X}}$-locality $\mathcal{L}^{\mathfrak{X}}$, there exists a functor $\mathfrak{h}^{\mathfrak{X}} : \mathcal{P}^{\mathfrak{X}} \to \mathcal{L}^{\mathfrak{X}}$, compatible with the structural functors, unique up to inner $\mathcal{F}^{\mathfrak{X}}$-automorphism of $\mathcal{L}^{\mathfrak{X}}$.*

**Proof:** As we mention above, it follows from [9, Theorem 20.24] that there exists a perfect $\mathcal{F}$-locality $\mathcal{P}$ and therefore a perfect $\mathcal{F}^{\mathfrak{X}}$-locality $\mathcal{P}^{\mathfrak{X}}$; denote

† The argument in [9, 20.16] has been scratched; below we develop the correct argument.

by $\breve{\tau}^{\mathfrak{X}}$ and $\breve{\pi}^{\mathfrak{X}}$ the corresponding structural functors. Note that the uniqueness of $\mathcal{P}^{\mathfrak{X}}$ will follow from the last statement applied to another *perfect* $\mathcal{F}^{\mathfrak{X}}$*-locality* $\mathcal{P}'^{\mathfrak{X}}$ (cf. 3.5.1).

Let $\mathcal{L}^{\mathfrak{X}}$ be a *p-coherent* $\mathcal{F}^{\mathfrak{X}}$*-locality* and denote by $\tau^{\mathfrak{X}}$ and $\pi^{\mathfrak{X}}$ its structural functors; restricting everything to $\hat{\mathfrak{X}}$ (cf. 8.1), consider the $p$-coherent $\mathcal{F}^{\hat{\mathfrak{X}}}$-locality defined by the *pull-back*

$$\begin{array}{ccccc} & & \mathcal{F}^{\hat{\mathfrak{X}}} & & \\ & \nearrow & & \nwarrow {\scriptstyle \pi^{\hat{\mathfrak{X}}}} & \\ \mathcal{P}^{\hat{\mathfrak{X}}} & & & & \mathcal{L}^{\hat{\mathfrak{X}}} \\ & {\scriptstyle \alpha^{\hat{\mathfrak{X}}}} \nwarrow & & \nearrow {\scriptstyle \beta^{\hat{\mathfrak{X}}}} & \\ & & \mathcal{P}^{\hat{\mathfrak{X}}} \times_{\mathcal{F}^{\hat{\mathfrak{X}}}} \mathcal{L}^{\hat{\mathfrak{X}}} & & \end{array} \qquad 9.2.1;$$

it is easily checked that the structural functor

$$\hat{\tau}^{\hat{\mathfrak{X}}} : \mathcal{T}_P^{\hat{\mathfrak{X}}} \longrightarrow \mathcal{P}^{\hat{\mathfrak{X}}} \times_{\mathcal{F}^{\hat{\mathfrak{X}}}} \mathcal{L}^{\hat{\mathfrak{X}}} \qquad 9.2.2$$

is *faithful* and therefore it follows form Theorem 8.4 that the bottom left-hand functor in the diagram above

$$\alpha^{\hat{\mathfrak{X}}} : \mathcal{P}^{\hat{\mathfrak{X}}} \times_{\mathcal{F}^{\hat{\mathfrak{X}}}} \mathcal{L}^{\hat{\mathfrak{X}}} \longrightarrow \mathcal{P}^{\hat{\mathfrak{X}}} \qquad 9.2.3$$

admits a functorial *section* $\sigma^{\hat{\mathfrak{X}}} : \mathcal{P}^{\hat{\mathfrak{X}}} \to \mathcal{P}^{\hat{\mathfrak{X}}} \times_{\mathcal{F}^{\hat{\mathfrak{X}}}} \mathcal{L}^{\hat{\mathfrak{X}}}$ .

Thus, we get a functor

$$\mathfrak{h}^{\hat{\mathfrak{X}}} = \beta^{\hat{\mathfrak{X}}} \circ \sigma^{\hat{\mathfrak{X}}} : \mathcal{P}^{\hat{\mathfrak{X}}} \longrightarrow \mathcal{L}^{\hat{\mathfrak{X}}} \qquad 9.2.4$$

which is easily checked to be compatible with the structural functors; moreover, any such a functor $\mathfrak{h}'^{\hat{\mathfrak{X}}} : \mathcal{P}^{\hat{\mathfrak{X}}} \to \mathcal{L}^{\hat{\mathfrak{X}}}$ determines, in the *pull-back* 9.2.1, a new functorial *section*

$$\sigma'^{\hat{\mathfrak{X}}} : \mathcal{P}^{\hat{\mathfrak{X}}} \longrightarrow \mathcal{P}^{\hat{\mathfrak{X}}} \times_{\mathcal{F}^{\hat{\mathfrak{X}}}} \mathcal{L}^{\hat{\mathfrak{X}}} \qquad 9.2.5$$

which fulfills $\mathfrak{h}'^{\hat{\mathfrak{X}}} = \beta^{\hat{\mathfrak{X}}} \circ \sigma'^{\hat{\mathfrak{X}}}$ ; then, it follows from Theorem 8.6 that there is an *inner* $\mathcal{F}^{\mathfrak{X}}$*-automorphism* $\kappa_{\hat{z}}$ of $\mathcal{P}^{\hat{\mathfrak{X}}} \times_{\mathcal{F}^{\hat{\mathfrak{X}}}} \mathcal{L}^{\hat{\mathfrak{X}}}$ fulfilling

$$\sigma'^{\hat{\mathfrak{X}}} (\mathcal{P}^{\hat{\mathfrak{X}}}) = \kappa_{\hat{z}}\big(\sigma^{\hat{\mathfrak{X}}} (\mathcal{P}^{\hat{\mathfrak{X}}})\big) \qquad 9.2.6\,.$$

In particular, for any $Q \in \hat{\mathfrak{X}}$ fully normalized in $\mathcal{F}$, the functors $\sigma'^{\hat{\mathfrak{X}}}$ and $\kappa_z \circ \sigma^{\hat{\mathfrak{X}}}$ determine a group automorphism

$$L_{\mathcal{F}}(Q) = \mathcal{P}^{\hat{\mathfrak{X}}}(Q) \cong \mathcal{P}^{\hat{\mathfrak{X}}}(Q) = L_{\mathcal{F}}(Q) \qquad 9.2.7$$

which is compatible with the structural group homomorphisms

$$\tau_Q^{\hat{\mathfrak{X}}} : N_P(Q) \longrightarrow L_{\mathcal{F}}(Q) \quad \text{and} \quad \pi_Q^{\hat{\mathfrak{X}}} : L_{\mathcal{F}}(Q) \longrightarrow \mathcal{F}(Q) \qquad 9.2.8;$$

hence, it follows from Theorem 2.10 that this automorphism coincides with the conjugation by some element $z_Q \in \tau_Q^{\hat{\mathfrak{X}}}\big(Z(Q)\big)$, and this fact remains true for any $Q \in \hat{\mathfrak{X}}$.

Consequently, it follows from [9, Corollary 5.14] suitably translated to $\mathcal{P}^{\hat{\mathfrak{X}}}$ that, setting $z = \{z_Q\}_{Q\in\hat{\mathfrak{X}}}$, the self-equivalence of $\mathcal{P}^{\hat{\mathfrak{X}}}$ determined by ${\sigma'}^{\hat{\mathfrak{X}}}$ and $\kappa_z \circ \sigma^{\hat{\mathfrak{X}}}$ coincides with the *inner* $\mathcal{F}^{\mathfrak{X}}$*-automorphism* $\kappa_z$ of $\mathcal{P}^{\hat{\mathfrak{X}}}$; that is to say, up to modifying our choice of $\hat{z}$, we may assume that ${\sigma'}^{\hat{\mathfrak{X}}} = \kappa_z \circ \sigma^{\hat{\mathfrak{X}}}$; in this case, since ${\mathfrak{h}'}^{\hat{\mathfrak{X}}} = \beta^{\hat{\mathfrak{X}}} \circ {\sigma'}^{\hat{\mathfrak{X}}}$, we still have

$$ {\mathfrak{h}'}^{\hat{\mathfrak{X}}} = \kappa_{\beta^{\hat{\mathfrak{X}}}(z)} \circ \mathfrak{h}^{\hat{\mathfrak{X}}} \qquad 9.2.9 $$

for an obvious definition of $\beta^{\hat{\mathfrak{X}}}(z)$.

At this point, it suffices to prove that the functor $\mathfrak{h}^{\hat{\mathfrak{X}}}$ can be extended to a unique functor $\mathfrak{h}^{\mathfrak{X}} : \mathcal{P}^{\mathfrak{X}} \to \mathcal{L}^{\mathfrak{X}}$, compatible with the corresponding structural functors. Our proof follows the same pattern as the proof in [9, Chap. 20] of the existence of $\mathcal{P}$ from the existence of $\mathcal{P}^{^{\rm sc}}$, and we borrow our notation and arguments there.

Let $Q$ and $Q'$ be $\mathcal{F}$-isomorphic subgroups in $\mathfrak{X}$, $R \in \mathfrak{X}$ a subgroup of $Q$ and $R' \in \mathfrak{X}$ a subgroup of $Q'$, and let us assume that the set $\mathcal{F}(Q',Q)_{R',R}$ of $\varphi \in \mathcal{F}(Q',Q)$ fulfilling $\varphi(R) = R'$ is not empty; since $\mathcal{F}$ is divisible, there is a unique *restriction map*

$$ \mathfrak{r}^{Q',Q}_{R',R} : \mathcal{F}(Q',Q)_{R',R} \longrightarrow \mathcal{F}(R',R) \qquad 9.2.10 $$

sending $\varphi \in \mathcal{F}(Q',Q)_{R',R}$ to $\psi \in \mathcal{F}(R',R)$ such that $\iota^{Q'}_{R'} \circ \psi = \varphi \circ \iota^Q_R$; similarly, since $\mathcal{P}^{\mathfrak{X}}$ and $\mathcal{L}^{\mathfrak{X}}$ are also divisible, we can define the *restriction maps*

$$ \begin{aligned} \mathfrak{s}^{Q',Q}_{R',R} &: \mathcal{P}^{\mathfrak{X}}(Q',Q)_{R',R} \longrightarrow \mathcal{P}^{\mathfrak{X}}(R',R) \\ \mathfrak{t}^{Q',Q}_{R',R} &: \mathcal{L}^{\mathfrak{X}}(Q',Q)_{R',R} \longrightarrow \mathcal{L}^{\mathfrak{X}}(R',R) \end{aligned} \qquad 9.2.11 $$

where we replace $\iota^Q_R$ and $\iota^{Q'}_{R'}$ by the corresponding images of 1 *via* the structural functors.

If all these groups are $\mathcal{F}$-selfcentralizing, it is clear that $\mathfrak{h}^{\hat{\mathfrak{X}}}$ determines a commutative diagram

$$ \begin{array}{ccc} \mathcal{P}^{\mathfrak{X}}(Q',Q)_{R',R} & \longrightarrow & \mathcal{P}^{\mathfrak{X}}(R',R) \\ \downarrow & & \downarrow \\ \mathcal{L}^{\mathfrak{X}}(Q',Q)_{R',R} & \longrightarrow & \mathcal{L}^{\mathfrak{X}}(R',R) \end{array} \qquad 9.2.12; $$

conversely, in order to get the announced functor $\mathfrak{h}^{\mathfrak{X}}$, it is easily checked that it suffices to define such maps

$$\mathfrak{h}^{\mathfrak{X}}_{Q',Q} : \mathcal{P}^{\mathfrak{X}}(Q',Q) \longrightarrow \mathcal{L}^{\mathfrak{X}}(Q',Q) \qquad 9.2.13$$

for all the elements of $\mathfrak{X}$, in such a way that they are compatible with both compositions and both structural functors, and that the corresponding diagrams above are commutative.

In the general case, let $T$ and $T'$ be $\mathcal{F}$-isomorphic subgroups in $\hat{\mathfrak{X}}$ respectively containing and normalizing $Q$ and $Q'$; then, we claim that the map

$$\mathcal{P}^{\mathfrak{X}}(T',T)_{Q',Q} \longrightarrow \mathcal{L}^{\mathfrak{X}}(T',T)_{Q',Q} \qquad 9.2.14$$

determined by $\mathfrak{h}^{\hat{\mathfrak{X}}}$ composed with $\mathfrak{t}^{T',T}_{Q',Q}$ factorizes through the image of $\mathfrak{s}^{T',T}_{Q',Q}$; indeed, if $x,y \in \mathcal{P}^{\mathfrak{X}}(T',T)_{Q',Q}$ have the same image in $\mathcal{P}^{\mathfrak{X}}(Q',Q)$ then we have $y = x{\cdot}z$ for some $z$ in the kernel $K_{\mathcal{P}^{\mathfrak{X}}}$ of the group homomorphism

$$\mathcal{P}^{\mathfrak{X}}(T)_Q \longrightarrow \mathcal{P}^{\mathfrak{X}}(Q) \qquad 9.2.15$$

and it suffices to prove that the group homomorphism $\mathcal{P}^{\mathfrak{X}}(T)_Q \to \mathcal{L}^{\mathfrak{X}}(T)_Q$ determined by $\mathfrak{h}^{\hat{\mathfrak{X}}}$ sends $K_{\mathcal{P}^{\mathfrak{X}}}$ to the kernel $K_{\mathcal{L}^{\mathfrak{X}}}$ of the group homomorphism

$$\mathcal{L}^{\mathfrak{X}}(T)_Q \longrightarrow \mathcal{L}^{\mathfrak{X}}(Q) \qquad 9.2.16.$$

Respectively denoting by $C_{\mathcal{P}^{\mathfrak{X}}(T)}(Q)$ and $C_{\mathcal{L}^{\mathfrak{X}}(T)}(Q)$ the kernels of the obvious group homomorphisms

$$\mathcal{P}^{\mathfrak{X}}(T)_Q \longrightarrow \mathcal{F}(Q) \quad \text{and} \quad \mathcal{L}^{\mathfrak{X}}(T)_Q \longrightarrow \mathcal{F}(Q) \qquad 9.2.17,$$

it is clear that homomorphisms 9.2.15 and 9.2.16 induce group homomorphisms

$$C_{\mathcal{P}^{\mathfrak{X}}(T)}(Q) \longrightarrow \mathrm{Ker}(\breve{\pi}^{\mathfrak{X}}_Q) \quad \text{and} \quad C_{\mathcal{L}^{\mathfrak{X}}(T)}(Q) \longrightarrow \mathrm{Ker}(\pi^{\mathfrak{X}}_Q) \qquad 9.2.18$$

and that $\mathfrak{h}^{\hat{\mathfrak{X}}}$ also determines a group homomorphism

$$C_{\mathcal{P}^{\mathfrak{X}}(T)}(Q) \longrightarrow C_{\mathcal{L}^{\mathfrak{X}}(T)}(Q) \qquad 9.2.19.$$

But, up to $\mathcal{F}$-isomorphisms, we may assume that $Q$ is fully normalized in $\mathcal{F}$ [9, Proposition 2.7]; in this case, since $\mathcal{P}^{\mathfrak{X}}$ is *perfect* and $\mathcal{L}^{\mathfrak{X}}$ is *p-coherent*, $\mathrm{Ker}(\pi^{\mathfrak{X}}_Q)$ is a $p$-group and we have (cf. 2.8.2)

$$\mathrm{Ker}(\breve{\pi}^{\mathfrak{X}}_Q) \cong C_P(Q)/H_{C_{\mathcal{F}}(Q)} \quad \text{and} \quad H_{C_{\mathcal{F}}(Q)} \subset \mathrm{Ker}(\tau^{\mathfrak{X}}_Q) \qquad 9.2.20\,.$$

Consequently, we get

$$\mathbb{O}^p(K_{\mathcal{P}^{\mathfrak{X}}}) = \mathbb{O}^p\big(C_{\mathcal{P}^{\mathfrak{X}}(T)}(Q)\big) \quad \text{and} \quad \mathbb{O}^p(K_{\mathcal{L}^{\mathfrak{X}}}) = \mathbb{O}^p\big(C_{\mathcal{L}^{\mathfrak{X}}(T)}(Q)\big) \qquad 9.2.21$$

and, always from 9.2.20, it is easily checked that $\mathfrak{h}^{\hat{\mathfrak{X}}}$ sends a Sylow $p$-subgroup of $K_{\mathcal{P}^{\mathfrak{X}}}$ to a Sylow $p$-subgroup of $K_{\mathcal{L}^{\mathfrak{X}}}$, proving our claim.

Assuming that $Q$ and $Q'$ are fully centralized in $\mathcal{F}$ and choosing

$$T = Q{\cdot}C_P(Q) \quad \text{and} \quad T = Q{\cdot}C_P(Q) \qquad 9.2.22,$$

we already know that the *restriction map*

$$\mathfrak{r}^{T',T}_{Q',Q} : \mathcal{F}(T',T)_{Q',Q} \longrightarrow \mathcal{F}(Q',Q) \qquad 9.2.23$$

is surjective (cf. condition 2.2.3), so that the corresponding maps $\mathfrak{s}^{T',T}_{Q',Q}$ and $\mathfrak{t}^{T',T}_{Q',Q}$ are surjective too; consequently, from the remark above we get a commutative diagram

$$\begin{array}{ccc} \mathcal{P}^{\mathfrak{X}}(T',T)_{Q',Q} & \xrightarrow{\mathfrak{h}^{\hat{\mathfrak{X}}}_{T',T}} & \mathcal{L}^{\mathfrak{X}}(T',T)_{Q',Q} \\ \mathfrak{s}^{T',T}_{Q',Q} \downarrow & & \downarrow \mathfrak{t}^{T',T}_{Q',Q} \\ \mathcal{P}^{\mathfrak{X}}(Q',Q) & \xrightarrow{\mathfrak{h}^{\mathfrak{X}}_{Q',Q}} & \mathcal{L}^{\mathfrak{X}}(Q',Q) \end{array} \qquad 9.2.24$$

which defines the bottom map $\mathfrak{h}^{\mathfrak{X}}_{Q',Q}$.

These maps are compatible with the structural functors; this is clear for $\check\pi^{\mathfrak{X}}_{Q',Q}$ and $\pi^{\mathfrak{X}}_{Q',Q}$; moreover, if $\mathcal{T}_P(Q',Q)$ is not empty and $u \in \mathcal{T}_P(Q',Q)$ then $u$ still belongs to $\mathcal{T}_P(T',T)$ and therefore we get

$$\begin{aligned} \mathfrak{h}^{\mathfrak{X}}_{Q',Q}\big(\check\tau^{\mathfrak{X}}_{Q',Q}(u)\big) &= \mathfrak{t}^{T',T}_{Q',Q}\Big(\mathfrak{h}^{\hat{\mathfrak{X}}}_{T',T}\big(\check\tau^{\mathfrak{X}}_{T',T}(u)\big)\Big) \\ &= \mathfrak{t}^{T',T}_{Q',Q}\big(\tau^{\mathfrak{X}}_{T',T}(u)\big) = \tau^{\mathfrak{X}}_{Q',Q}(u) \end{aligned} \qquad 9.2.25.$$

On the other hand, since the top map in diagram 9.2.24 is defined by a functor, for a third $Q'' \in \mathfrak{X}$ fully centralized in $\mathcal{F}$ and isomorphic to $Q$ and $Q'$, setting $T'' = Q''{\cdot}C_P(Q'')$ the compositions in $\mathcal{P}^{\mathfrak{X}}$ and $\mathcal{L}^{\mathfrak{X}}$ supply the commutative diagram

$$\begin{array}{ccc} \mathcal{P}^{\mathfrak{X}}(T'',T')_{Q'',Q'} \times \mathcal{P}^{\mathfrak{X}}(T',T)_{Q',Q} & \longrightarrow & \mathcal{P}^{\mathfrak{X}}(T'',T)_{Q'',Q} \\ \downarrow & & \downarrow \\ \mathcal{L}^{\mathfrak{X}}(T'',T')_{Q'',Q'} \times \mathcal{L}^{\mathfrak{X}}(T',T)_{Q',Q} & \longrightarrow & \mathcal{L}^{\mathfrak{X}}(T'',T)_{Q'',Q} \end{array} \qquad 9.2.26$$

which forces the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{P}^{\mathfrak{X}}(Q'',Q')\times\mathcal{P}^{\mathfrak{X}}(Q',Q) & \longrightarrow & \mathcal{P}^{\mathfrak{X}}(Q'',Q) \\ {\scriptstyle \mathfrak{h}^{\mathfrak{X}}_{Q'',Q'}\times\mathfrak{h}^{\mathfrak{X}}_{Q',Q}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{h}^{\mathfrak{X}}_{Q'',Q}} \\ \mathcal{L}^{\mathfrak{X}}(Q'',Q')\times\mathcal{L}^{\mathfrak{X}}(Q',Q) & \longrightarrow & \mathcal{L}^{\mathfrak{X}}(Q'',Q) \end{array} \qquad 9.2.27.$$

Actually, we claim that diagram 9.2.24 remains true for any choice of $\mathcal{F}$-isomorphic subgroups $T$ and $T'$ in $\hat{\mathfrak{X}}$ containing and normalizing $Q$ and $Q'$; indeed, consider $x\in\mathcal{P}^{\mathfrak{X}}(Q',Q)$ and set $\alpha=\check{\pi}^{\mathfrak{X}}_{Q',Q}(x)$; it is clear that

$$\mathcal{F}_T(Q)\subset\mathcal{F}_P(Q)\cap{}^{\alpha^*}\mathcal{F}_P(Q') \quad\text{and}\quad \mathcal{F}_{T'}(Q')\subset{}^{\alpha_*}\mathcal{F}_P(Q)\cap\mathcal{F}_P(Q') \qquad 9.2.28;$$

denoting by $N_\alpha$ and $N'_\alpha$ the respective converse images of these intersections in $N_P(Q)$ and $N_P(Q')$, we know that $\alpha$ can be extended to an $\mathcal{F}$-isomorphism $\hat\alpha: N_\alpha\cong N'_\alpha$ (cf. condition 2.2.3) and therefore we can find an element $\hat x$ in $\mathcal{P}^{\mathfrak{X}}(N'_\alpha,N_\alpha)_{Q',Q}$ lifting $x$; then, since $\hat\alpha$ clearly maps $\hat Q=Q{\cdot}C_P(Q)$ onto $\hat Q'=Q'{\cdot}C_P(Q')$, we have

$$\begin{aligned} \mathfrak{h}^{\mathfrak{X}}_{Q',Q}\big(\mathfrak{s}^{N'_\alpha,N_\alpha}_{Q',Q}(\hat x)\big) &= \mathfrak{h}^{\mathfrak{X}}_{Q',Q}\Big(\mathfrak{s}^{\hat Q',\hat Q}_{Q',Q}\big(\mathfrak{s}^{N'_\alpha,N_\alpha}_{\hat Q',\hat Q}(\hat x)\big)\Big) \\ &= \mathfrak{t}^{\hat Q',\hat Q}_{Q',Q}\Big(\mathfrak{h}^{\hat{\mathfrak{X}}}_{\hat Q',\hat Q}\big(\mathfrak{s}^{N'_\alpha,N_\alpha}_{\hat Q',\hat Q}(\hat x)\big)\Big) = \mathfrak{t}^{N'_\alpha,N_\alpha}_{Q',Q}\big(\mathfrak{h}^{\hat{\mathfrak{X}}}_{N'_\alpha,N_\alpha}(\hat x)\big) \end{aligned} \qquad 9.2.29\,.$$

But, if $y$ is an element of $\mathcal{P}^{\mathfrak{X}}(T',T)_{Q',Q}$ lifting $x$ then for some $z\in C_P(Q)$ we have (cf. 9.2.20)

$$y=\mathfrak{s}^{N'_\alpha,N_\alpha}_{T',T}(\hat x){\cdot}\check\tau^{\mathfrak{X}}_{T',T}(z)=\mathfrak{s}^{N'_\alpha,N_\alpha}_{T',T}\big(\hat x{\cdot}\check\tau^{\mathfrak{X}}_{N'_\alpha,N_\alpha}(z)\big) \qquad 9.2.30;$$

hence, we get

$$\begin{aligned} \mathfrak{h}^{\mathfrak{X}}_{Q',Q}\big(\mathfrak{s}^{T',T}_{Q',Q}(y)\big) &= \mathfrak{h}^{\mathfrak{X}}_{Q',Q}\Big(\mathfrak{s}^{N'_\alpha,N_\alpha}_{Q',Q}\big(\hat x{\cdot}\check\tau^{\mathfrak{X}}_{N'_\alpha,N_\alpha}(z)\big)\Big) \\ &= \mathfrak{t}^{N'_\alpha,N_\alpha}_{Q',Q}\Big(\mathfrak{h}^{\hat{\mathfrak{X}}}_{N'_\alpha,N_\alpha}\big(\hat x{\cdot}\check\tau^{\mathfrak{X}}_{N'_\alpha,N_\alpha}(z)\big)\Big) = \mathfrak{t}^{T',T}_{Q',Q}\big(\mathfrak{h}^{\hat{\mathfrak{X}}}_{T',T}(y)\big) \end{aligned} \qquad 9.2.31,$$

which proves our claim.

In particular, if $R\in\mathfrak{X}$ is a normal subgroup of $Q$ and $R'\in\mathfrak{X}$ a normal subgroup of $Q'$, both fully normalized in $\mathcal{F}$, the argument above proves that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{P}^{\mathfrak{X}}(\hat Q',\hat Q)_{R',R} & \xrightarrow{\mathfrak{h}^{\hat{\mathfrak{X}}}_{\hat Q',\hat Q}} & \mathcal{L}^{\mathfrak{X}}(\hat Q',\hat Q)_{R',R} \\ {\scriptstyle \mathfrak{s}^{\hat Q',\hat Q}_{R',R}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{t}^{\hat Q',\hat Q}_{R',R}} \\ \mathcal{P}^{\mathfrak{X}}(R',R) & \xrightarrow{\mathfrak{h}^{\mathfrak{X}}_{R',R}} & \mathcal{L}^{\mathfrak{X}}(R',R) \end{array} \qquad 9.2.32;$$

but, with evident notation, in the following commutative diagram

$$\begin{array}{ccc}\mathcal{P}^{\mathfrak{x}}(\hat{Q}',\hat{Q})_{Q',R',Q,R} & \xrightarrow{\mathfrak{h}^{\hat{\mathfrak{x}}}_{\hat{Q}',\hat{Q}}} & \mathcal{L}^{\mathfrak{x}}(\hat{Q}',\hat{Q})_{Q',R',Q,R}\\ {\scriptstyle \mathfrak{s}^{\hat{Q}',\hat{Q}}_{Q',Q}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{t}^{\hat{Q}',\hat{Q}}_{Q',Q}}\\ \mathcal{P}^{\mathfrak{x}}(Q',Q)_{R',R} & \xrightarrow{\mathfrak{h}^{\mathfrak{x}}_{Q',Q}} & \mathcal{L}^{\mathfrak{x}}(Q',Q)_{R',R}\end{array} \qquad 9.2.33$$

the left-hand vertical arrow is surjective; hence, in this situation we finally obtain the commutative diagram announced in 9.2.12

$$\begin{array}{ccc}\mathcal{P}^{\mathfrak{x}}(Q',Q)_{R',R} & \xrightarrow{\mathfrak{h}^{\mathfrak{x}}_{Q',Q}} & \mathcal{L}^{\mathfrak{x}}(Q',Q)_{R',R}\\ {\scriptstyle \mathfrak{s}^{Q',Q}_{R',R}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{t}^{Q',\hat{Q}}_{R',R}}\\ \mathcal{P}^{\mathfrak{x}}(R',R) & \xrightarrow{\mathfrak{h}^{\mathfrak{x}}_{R',R}} & \mathcal{L}^{\mathfrak{x}}(R',R)\end{array} \qquad 9.2.34.$$

We are ready to define the map $\mathfrak{h}^{\mathfrak{x}}_{Q',Q}$ for *any* pair of $\mathcal{F}$-isomorphic subgroups $Q$ and $Q'$ in $\mathfrak{X}$; we proceed by induction on $|P:Q|$ and, obviously, our definition will extend the previous ones; thus, we may assume that $N_P(Q)\neq Q$. It follows from [9, Corollary 2.21] that there is an $\mathcal{F}$-morphism $\nu: N_P(Q)\to P$ such that $\nu\big(N_P(Q)\big)$ and $\nu(Q)$ are both fully centralized in $\mathcal{F}$; so, let us consider the nonempty set $\mathfrak{N}(Q)$ of pairs $(N,s)$ formed by a subgroup $N$ of $P$ which strictly contains and normalizes $Q$, and by $s\in\mathcal{P}^{\mathfrak{x}}\big(\rho(N),N\big)$ lifting an $\mathcal{F}$-morphism $\rho: N\to P$ such that $\rho(N)$ and $\rho(Q)$ are both fully centralized in $\mathcal{F}$; note that, according to our induction hypothesis, we may assume that the map $\mathfrak{h}^{\mathfrak{x}}_{\rho(N),N}$ is already defined and then $t=\mathfrak{h}^{\mathfrak{x}}_{\rho(N),N}(s)$ makes sense and belongs to $\mathcal{L}^{\mathfrak{x}}\big(\rho(N),N\big)$; moreover, we respectively denote by $s_Q$ and $t_Q$ the corresponding elements of $\mathcal{P}^{\mathfrak{x}}\big(\rho(Q),Q\big)$ and $\mathcal{L}^{\mathfrak{x}}\big(\rho(Q),Q\big)$.

For another pair $(\bar{N},\bar{s})$ in $\mathfrak{N}(Q)$, denoting by $\bar{\rho}:\bar{N}\to P$ the $\mathcal{F}$-morphism determined by $\bar{s}$, setting $\bar{\bar{N}}=\langle N,\bar{N}\rangle$ and considering a new $\mathcal{F}$-morphism $\bar{\bar{\rho}}:\bar{\bar{N}}\to P$ such that $\bar{\bar{\rho}}(\bar{\bar{N}})$ and $\bar{\bar{\rho}}(Q)$ are both fully centralized in $\mathcal{F}$, we can obtain a third pair $(\bar{\bar{N}},\bar{\bar{s}})$ in $\mathfrak{N}(Q)$; then, the elements $\bar{\bar{s}}\cdot\check{\tau}^{\mathfrak{x}}_{\bar{\bar{N}},N}(1)\cdot s^{-1}$ and $\bar{\bar{s}}\cdot\check{\tau}^{\mathfrak{x}}_{\bar{\bar{N}},\bar{N}}(1)\cdot\bar{s}^{-1}$ respectively belong to $\mathcal{P}^{\mathfrak{x}}\big(\bar{\bar{\rho}}(\bar{\bar{N}}),\rho(N)\big)$ and to $\mathcal{P}^{\mathfrak{x}}\big(\bar{\bar{\rho}}(\bar{\bar{N}}),\bar{\rho}(\bar{N})\big)$; in particular, since $\rho(Q)$, $\bar{\rho}(Q)$ and $\bar{\bar{\rho}}(Q)$ are fully centralized in $\mathcal{F}$, the maps $\mathfrak{h}^{\mathfrak{x}}_{\bar{\bar{\rho}}(Q),\rho(Q)}$, $\mathfrak{h}^{\mathfrak{x}}_{\bar{\bar{\rho}}(Q),\bar{\rho}(Q)}$ and $\mathfrak{h}^{\mathfrak{x}}_{\bar{\rho}(Q),\rho(Q)}$ are already defined and, considering the element

$$n_{\bar{s},s}=\bar{s}_Q\cdot(s_Q)^{-1}\in\mathcal{P}^{\mathfrak{x}}\big(\bar{\rho}(Q),\rho(Q)\big) \qquad 9.2.34,$$

we clearly have

$$\mathfrak{h}^{\mathfrak{X}}_{\bar\rho(Q),\rho(Q)}(n_{\bar s,s}) = \bar t_Q\cdot(t_Q)^{-1} = m_{\bar t,t} \in \mathcal{L}^{\mathfrak{X}}\big(\bar\rho(Q),\rho(Q)\big) \qquad 9.2.36.$$

Note that all these elements fulfill a transitive property as in [9, 20.16.2].†

At this point, choosing pairs $(N,s)$ in $\mathfrak{N}(Q)$ and $(N',s')$ in $\mathfrak{N}(Q')$, and denoting by $\rho\colon N\to P$ and $\rho'\colon N'\to P$ the $\mathcal{F}$-morphisms respectively determined by $s$ and $s'$, and by $t$ and $t'$ the respective images of $s$ and $s'$ *via* $\mathfrak{h}^{\mathfrak{X}}_{\rho(N),N}$ and $\mathfrak{h}^{\mathfrak{X}}_{\rho'(N'),N'}$, the map $\mathfrak{h}^{\mathfrak{X}}_{\rho'(Q'),\rho(Q)}$ is already defined and then we define the map

$$\mathfrak{h}^{\mathfrak{X}}_{Q',Q} : \mathcal{P}^{\mathfrak{X}}(Q',Q) \longrightarrow \mathcal{L}^{\mathfrak{X}}(Q',Q) \qquad 9.2.37$$

sending $x\in\mathcal{P}^{\mathfrak{X}}(Q',Q)$ to

$$\mathfrak{h}^{\mathfrak{X}}_{Q',Q}(x) = (t'_{Q'})^{-1}\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho'(Q'),\rho(Q)}\big(s'_{Q'}\cdot x\cdot(s_Q)^{-1}\big)\cdot t_Q \qquad 9.2.38.$$

We claim that this element does not depend on our choice; indeed, respectively replacing $(N,s)$, $(N',s')$, $\rho$ and $\rho'$ by $(\bar N,\bar s)$, $(\bar N',\bar s')$, $\bar\rho$ and $\bar\rho'$ we get the element $(\bar t'_{Q'})^{-1}\cdot\mathfrak{h}^{\mathfrak{X}}_{\bar\rho'(Q'),\bar\rho(Q)}\big(\bar s'_{Q'}\cdot x\cdot(\bar s_Q)^{-1}\big)\cdot\bar t_Q$ and therefore it suffices to prove the equality (cf. 9.2.36)

$$m_{\bar t',t'}\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho'(Q'),\rho(Q)}\big(s'_{Q'}\cdot x\cdot(s_Q)^{-1}\big) = \mathfrak{h}^{\mathfrak{X}}_{\bar\rho'(Q'),\bar\rho(Q)}\big(\bar s'_{Q'}\cdot x\cdot(\bar s_Q)^{-1}\big)\cdot m_{\bar t,t} \qquad 9.2.39;$$

but, from the commutativity of diagram 9.2.27 and equality 9.2.36 we get

$$\begin{aligned} m_{\bar t',t'}\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho'(Q'),\rho(Q)}\big(s'_{Q'}\cdot x\cdot(s_Q)^{-1}\big) &= \mathfrak{h}^{\mathfrak{X}}_{\bar\rho'(Q'),\rho(Q)}\big(\bar s'_{Q'}\cdot x\cdot(s_Q)^{-1}\big)\\ \mathfrak{h}^{\mathfrak{X}}_{\bar\rho'(Q'),\bar\rho(Q)}\big(\bar s'_{Q'}\cdot x\cdot(\bar s_Q)^{-1}\big)\cdot m_{\bar t,t} &= \mathfrak{h}^{\mathfrak{X}}_{\bar\rho'(Q'),\rho(Q)}\big(\bar s'_{Q'}\cdot x\cdot(s_Q)^{-1}\big) \end{aligned} \qquad 9.2.40,$$

which proves the claim.

For a third subgroup $Q''$ in $\mathfrak{X}$ isomorphic to $Q$ and $Q'$, we claim that the corresponding diagram 9.2.27 defined by both compositions is also commutative; indeed, choosing a pair $(N'',s'')$ in $\mathfrak{N}(Q'')$ and denoting by $\rho''\colon N''\to P$ the $\mathcal{F}$-morphism determined by $s''$, for any $x'\in\mathcal{P}^{\mathfrak{X}}(Q'',Q')$ we have

$$\mathfrak{h}^{\mathfrak{X}}_{Q'',Q'}(x') = (t''_{Q''})^{-1}\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho''(Q''),\rho'(Q')}\big(s''_{Q''}\cdot x'\cdot(s'_{Q'})^{-1}\big)\cdot t'_{Q'} \qquad 9.2.41$$

and therefore we get

$$\begin{aligned} &t''_{Q''}\cdot\mathfrak{h}^{\mathfrak{X}}_{Q'',Q'}(x')\cdot\mathfrak{h}^{\mathfrak{X}}_{Q',Q}(x)\cdot t_Q\\ &= \mathfrak{h}^{\mathfrak{X}}_{\rho''(Q''),\rho'(Q')}\big(s''_{Q''}\cdot x'\cdot(s'_{Q'})^{-1}\big)\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho'(Q'),\rho(Q)}\big(s'_{Q'}\cdot x\cdot(s_Q)^{-1}\big)\\ &= \mathfrak{h}^{\mathfrak{X}}_{\rho''(Q''),\rho(Q)}\big(s''_{Q''}\cdot(x'\cdot x)\cdot(s_Q)^{-1}\big) = t''_{Q''}\cdot\mathfrak{h}^{\mathfrak{X}}_{Q'',Q'}(x'\cdot x)\cdot t_Q \end{aligned} \qquad 9.2.42,$$

which proves the claim.

† The argument above provides the right way to obtain the elements $g_{\bar n,n}$ in [9, 20.16.1].

Finally, it remains to prove that the commutativity of diagram 9.2.12 holds in the general case. Let $T$ and $T'$ be $\mathcal{F}$-isomorphic subgroups in $\hat{\mathfrak{X}}$ respectively containing $Q$ and $Q'$, and fulfilling $\mathcal{F}(T',T)_{Q',Q} \neq \emptyset$; assume that $Q \neq T$ and set $N = N_T(Q)$ and $N' = N_{T'}(Q')$; since we are arguing by induction on $|P{:}Q|$, we may assume that we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{P}^{\mathfrak{X}}(T',T)_{Q',Q} & \xrightarrow{\mathfrak{h}^{\mathfrak{X}}_{T',T}} & \mathcal{L}^{\mathfrak{X}}(T',T)_{Q',Q} \\ \mathfrak{s}^{T',T}_{N',N} \downarrow & & \downarrow \mathfrak{t}^{T',T}_{N',N} \\ \mathcal{P}^{\mathfrak{X}}(N',N)_{Q',Q} & \xrightarrow{\mathfrak{h}^{\mathfrak{X}}_{N',N}} & \mathcal{L}^{\mathfrak{X}}(N',N)_{Q',Q} \end{array} \qquad 9.2.43$$

and therefore it suffices to prove the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{P}^{\mathfrak{X}}(N',N)_{Q',Q} & \xrightarrow{\mathfrak{h}^{\hat{\mathfrak{X}}}_{N',N}} & \mathcal{L}^{\mathfrak{X}}(N',N)_{Q',Q} \\ \mathfrak{s}^{N',N}_{Q',Q} \downarrow & & \downarrow \mathfrak{t}^{N',N}_{Q',Q} \\ \mathcal{P}^{\mathfrak{X}}(Q',Q) & \xrightarrow{\mathfrak{h}^{\mathfrak{X}}_{Q',Q}} & \mathcal{L}^{\mathfrak{X}}(Q',Q) \end{array} \qquad 9.2.44.$$

As above, we can choose pairs $(N,s)$ in $\mathfrak{N}(Q)$ and $(N',s')$ in $\mathfrak{N}(Q')$; let us denote by $\rho : N \to P$ and $\rho' : N' \to P$ the $\mathcal{F}$-morphisms respectively determined by $s$ and $s'$, and by $t$ and $t'$ the respective images of $s$ and $s'$ *via* $\mathfrak{h}^{\mathfrak{X}}_{\rho(N),N}$ and $\mathfrak{h}^{\mathfrak{X}}_{\rho'(N'),N'}$; then, for any $y \in \mathcal{P}^{\mathfrak{X}}(N',N)_{Q',Q}$ it follows from definition 9.2.38 and from the commutativity of diagram 9.2.24 that we have

$$\begin{aligned} \mathfrak{h}^{\mathfrak{X}}_{Q',Q}\big(\mathfrak{s}^{N',N}_{Q',Q}(y)\big) &= (t'_{Q'})^{-1}\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho'(Q'),\rho(Q)}\big(s'_{Q'}\cdot\mathfrak{s}^{N',N}_{Q',Q}(y)\cdot(s_Q)^{-1}\big)\cdot t_Q \\ &= (t'_{Q'})^{-1}\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho'(Q'),\rho(Q)}\big(\mathfrak{s}^{\rho'(N'),\rho(N)}_{\rho'(Q'),\rho(Q)}(s'\cdot y\cdot s^{-1})\big)\cdot t_Q \\ &= (t'_{Q'})^{-1}\cdot\mathfrak{t}^{\rho'(N'),\rho(N)}_{\rho'(Q'),\rho(Q)}\big(\mathfrak{h}^{\mathfrak{X}}_{\rho'(N'),\rho(N)}(s'\cdot y\cdot s^{-1})\big)\cdot t_Q \qquad 9.2.45, \\ &= \mathfrak{t}^{N',N}_{Q',Q}\big(t'^{-1}\cdot\mathfrak{h}^{\mathfrak{X}}_{\rho'(N'),\rho(N)}(s'\cdot y\cdot s^{-1})\cdot t\big) \\ &= \mathfrak{t}^{N',N}_{Q',Q}\big(\mathfrak{h}^{\mathfrak{X}}_{N',N}(y)\big) \end{aligned}$$

proving the commutativity of diagram 9.2.44. The compatibility of the functor $\mathfrak{h}^{\mathfrak{X}}$ with the structural functors is easily checked. We are done.

9.3. It remains to dicuss the *functoriality* of the *perfect $\mathcal{F}$-locality* $\mathcal{P}$; as a matter of fact, assuming its existence we already prove in [9, Theorem 17.18] the existence of all the possible *perfect quotients* $\bar{\mathcal{P}}$ of $\mathcal{P}$, which presently simplifies our work. Let us recall the construction of $\bar{\mathcal{P}}$; let $U$ be an $\mathcal{F}$*-stable* subgroup of $P$ (cf. 2.5), set $\bar{P} = P/U$ and denote by $\bar{\mathcal{F}}$ the *quotient* Frobenius $\bar{P}$-category $\mathcal{F}/U$ [9, Proposition 12.3]; for any subgroup $Q$ of $P$, denote by $\bar{Q}$ the image of $Q$ in $\bar{P}$ and by $U_{\mathcal{F}}(Q)$ the kernel of the canonical group homomorphism $\mathcal{F}(Q) \to \bar{\mathcal{F}}(\bar{Q})$ ; moreover, if $Q$ is fully normalized in $\mathcal{F}$, for short we set

$$P^{^Q} = N_P^{U_\mathcal{F}(Q)}(Q) \quad \text{and} \quad \mathcal{F}^{^Q} = N_\mathcal{F}^{U_\mathcal{F}(Q)}(Q) \qquad 9.3.1,$$

so that $\mathcal{F}^{^Q}$ is a Frobenius $P^{^Q}$-category; in the group $\mathcal{P}(Q)$ we define (cf. 2.4)

$$U_\mathcal{P}(Q) = \mathbb{O}^p\Big(\pi_{_Q}^{-1}\big(U_\mathcal{F}(Q)\big)\Big)\cdot\tau_{_Q}\big(N_U(Q)\cdot H_{\mathcal{F}^Q}\big) \qquad 9.3.2;$$

actually, *via* $\mathcal{P}$-isomorphisms we can extend the definition of $U_\mathcal{P}(Q)$ to any subgroup $Q$ of $P$. Then, $\bar{\mathcal{P}}$ is the *perfect* $\bar{\mathcal{F}}$*-locality* fulfilling [9, 17.15-17]

$$\bar{\mathcal{P}}(\bar{Q},\bar{R}) = \mathcal{P}(Q,R)\big/U_\mathcal{P}(R) \qquad 9.3.3$$

for any pair of subgroups $Q$ and $R$ of $P$.

9.4. Let $P'$ be a second finite $p$-group, $\mathcal{F}'$ a Frobenius $P'$-category and $\mathcal{P}'$ the corresponding *perfect* $\mathcal{F}'$*-locality*, and denote by

$$\tau' : \mathcal{T}_{P'} \longrightarrow \mathcal{P}' \quad \text{and} \quad \pi' : \mathcal{P}' \longrightarrow \mathcal{F}' \qquad 9.4.1$$

the structural functors; let $\alpha : P \to P'$ be an $(\mathcal{F},\mathcal{F}')$-functorial group homomorphism [9, 12.1]; recall that we have a so-called *Frobenius functor* $\mathfrak{f}_\alpha : \mathcal{F} \to \mathcal{F}'$ [9, 12.1] and denote by $\mathfrak{t}_\alpha : \mathcal{T}_P \to \mathcal{T}_{P'}$ the functor induced by $\alpha$.

**Theorem 9.5.** *With the notation above, there is a functor* $\mathfrak{g}_\alpha : \mathcal{P} \to \mathcal{P}'$, *unique up to inner* $\mathcal{F}'$*-automorphisms of* $\mathcal{P}'$, *fulfilling*

$$\tau' \circ \mathfrak{t}_\alpha = \mathfrak{g}_\alpha \circ \tau \quad \textit{and} \quad \pi' \circ \mathfrak{g}_\alpha = \mathfrak{f}_\alpha \circ \pi \qquad 9.5.1.$$

*Moreover, if* $P''$ *is a third finite* $p$*-group,* $\mathcal{F}''$ *a Frobenius* $P''$*-category,* $\mathcal{P}''$ *the perfect* $\mathcal{F}''$*-locality and* $\alpha' : P' \to P''$ *an* $(\mathcal{F}',\mathcal{F}'')$*-functorial group homomorphism, then the functors* $\mathfrak{g}_{\alpha'} \circ \mathfrak{g}_\alpha$ *and* $\mathfrak{g}_{\alpha'\circ\alpha}$ *from* $\mathcal{P}$ *to* $\mathcal{P}''$ *coincide up to inner* $\mathcal{F}''$*-automorphisms of* $\mathcal{P}''$.

**Proof:** As we mention above, if $\alpha$ is surjective then the existence of $\mathfrak{g}_\alpha$ follows from [9, Theorem 17.18].

Assume that $\alpha$ is injective and consider the $\mathcal{F}$-locality $\mathcal{L}_\alpha$ defined by the *pull-back*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\mathfrak{f}_\alpha} & \mathcal{F}' \\ \uparrow & & \uparrow \pi' \\ \mathcal{L}_\alpha & \longrightarrow & \mathcal{P}' \end{array} \qquad 9.5.2;$$

that is to say, for any pair of subgroups $Q$ and $R$ of $P$, setting $Q' = \alpha(Q)$ and $R' = \alpha(R)$ we have the *pull-back*

$$\begin{array}{ccc} \mathcal{F}(Q,R) & \longrightarrow & \mathcal{F}'(Q',R') \\ \uparrow & & \uparrow \\ \mathcal{L}_\alpha(Q,R) & \longrightarrow & \mathcal{P}'(Q',R') \end{array} \qquad 9.5.3;$$

then, since $\alpha$ is injective, the *divisibility* and the *p-coherence* of $\mathcal{P}'$ forces the the *divisibility* and the *p-coherence* of $\mathcal{L}_\alpha$.

Consequently, it follows form Theorem 9.2 that we have a functor

$$\mathfrak{h}_\alpha : \mathcal{P} \longrightarrow \mathcal{L}_\alpha \qquad 9.5.4$$

which is compatible with the structural functors and unique up to inner $\mathcal{F}'$-automorphisms of $\mathcal{L}_\alpha$ ; then, the composition of $\mathfrak{h}_\alpha$ with the bottom functor in diagram 9.5.2 is a functor $\mathfrak{g}_\alpha : \mathcal{P} \to \mathcal{P}'$ clearly compatible with the corresponding structural functors. Conversely, for any functor $\mathfrak{g}'_\alpha : \mathcal{P} \to \mathcal{P}'$ compatible with the structural functors, the *pull-back* 9.5.2 clearly determines a functor $\mathfrak{h}'_\alpha : \mathcal{P} \to \mathcal{L}_\alpha$ compatible with the structural functors, and it suffices to apply the uniqueness of $\mathfrak{h}_\alpha$ .

Once again, from [9, Theorem 17.18] the last statement is easily checked whenever $\alpha$ and $\alpha'$ are both surjective. If $\alpha$ and $\alpha'$ are both injective then the last statement follows from the following commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathfrak{f}_\alpha"] & \mathcal{F}' \arrow[r, "\mathfrak{f}_{\alpha'}"] & \mathcal{F}'' \\
\mathcal{L}_{\alpha'\circ\alpha} \arrow[u] \arrow[r] & \mathcal{L}_{\alpha'} \arrow[u] \arrow[r] & \mathcal{P}'' \arrow[u, "\pi''"'] \\
\mathcal{L}_\alpha \arrow[u] \arrow[r] & \mathcal{P}' \arrow[u, "\mathfrak{h}_{\alpha'}"'] \arrow[ur] & \\
\mathcal{P} \arrow[u, "\mathfrak{h}_\alpha"'] \arrow[ur] & &
\end{tikzcd} \qquad 9.5.5$$

where all the possible rectangles and squares are *pull-back*.

Now, in order to discuss the general case, it suffices in the situation above to consider an $\mathcal{F}'$-stable subgroup $U'$ of $P'$ and then, setting $U = \alpha^{-1}(U')$ which is clearly an $\mathcal{F}$-stable subgroup of $P$ , to prove the commutativity, up to inner $\mathcal{F}'$-automorphisms of $\mathcal{P}'$ , of the following diagram

$$\begin{tikzcd}
\bar{\mathcal{P}} \arrow[r, "\mathfrak{g}_{\bar\alpha}"] & \bar{\mathcal{P}}' \\
\mathcal{P} \arrow[u] \arrow[r, "\mathfrak{g}_\alpha"] & \mathcal{P}' \arrow[u]
\end{tikzcd} \qquad 9.5.6$$

where, setting $\bar{P} = P/U$ and $\bar{P}' = P'/U'$ , $\bar\alpha : \bar{P} \to \bar{P}'$ is the *injective* group homomorphism induced by $\alpha$ and, denoting by $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}'$ the respective *quotients* $\mathcal{F}/U$ and $\mathcal{F}'/U'$ [9, Proposition 12.3], $\bar{\mathcal{P}}$ and $\bar{\mathcal{P}}'$ are the respective *perfect* $\bar{\mathcal{F}}$*- and* $\bar{\mathcal{F}}'$*-localities*.

From 9.3 above, we already know that, for any pair of subgroups $Q$ and $R$ of $P$ , setting $Q' = \alpha(Q)$ and $R' = \alpha(R)$ we have

$$\bar{\mathcal{P}}(\bar{Q}, \bar{R}) = \mathcal{P}(Q,R)/U_{\mathcal{P}}(R) \quad \text{and} \quad \bar{\mathcal{P}}'(\bar{Q}', \bar{R}') = \mathcal{P}'(Q',R')/U_{\mathcal{P}'}(R') \quad 9.5.7;$$

hence, it suffices to prove that the functor $\mathfrak{g}_\alpha$ sends $U_{\mathcal{P}}(R)$ to $U_{\mathcal{P}'}(R')$ ; indeed, in this case $\mathfrak{g}_\alpha$ induces a functor $\bar{\mathfrak{g}}_\alpha : \bar{\mathcal{P}} \to \bar{\mathcal{P}}'$ which is easily checked to be compatible with the structural functors; then, the uniqueness proved above shows that $\bar{\mathfrak{g}}_\alpha$ coincides with $\mathfrak{g}_{\bar\alpha}$ up to inner $\mathcal{F}'$-automorphisms of $\mathcal{P}'$ .

But, the commutative diagram of group homomorphisms

$$\begin{array}{ccc} \bar{\mathcal{F}}(\bar{R}) & \longrightarrow & \bar{\mathcal{F}}'(\bar{R}') \\ \uparrow & & \uparrow \\ \mathcal{F}(R) & \longrightarrow & \mathcal{F}'(R') \end{array} \qquad 9.5.8$$

already proves that the functor $\mathfrak{f}_\alpha$ sends $U_{\mathcal{F}}(R)$ to $U_{\mathcal{F}'}(R')$ and therefore the functor $\mathfrak{g}_\alpha$ sends $\mathbb{O}^p\big(\pi_{_R}^{-1}\big(U_{\mathcal{F}}(R)\big)\big)$ to $\mathbb{O}^p\big(\pi_{_{R'}}^{-1}\big(U_{\mathcal{F}'}(R')\big)\big)$ ; moreover, it is clear that $\alpha\big(N_U(R)\big) \subset N_{U'}(R')$ . Finally, assuming that $R$ is fully normalized in $\mathcal{F}$ and choosing a $\mathcal{P}'$-isomorphism $y' : R' \cong \hat{R}'$ such that $\hat{R}'$ is fully normalized in $\mathcal{F}'$ , it follows from condition 2.2.3 that there is an $\mathcal{F}'$-morphism

$$\xi : \alpha(P^{^R}) \longrightarrow {P'}^{\hat{R}'} \qquad 9.5.9$$

extending $\pi'_{_{\hat{R}',R'}}(y)$ ; then, it is easily checked that the composition $\zeta$ of $\xi$ with the restriction of $\alpha$ to $P^{^R}$ is $(\mathcal{F}^{^R}, {\mathcal{F}'}^{\hat{R}'})$*-functorial* [9, 12.1]; hence, we get a *Frobenius functor* $\mathfrak{f}_\zeta : \mathcal{F}^{^R} \to {\mathcal{F}'}^{\hat{R}'}$ and therefore we have

$$\zeta(H_{\mathcal{F}^R}) \subset H_{{\mathcal{F}'}^{\hat{R}'}} \qquad 9.5.10.$$

Consequently, considering the suitable $\mathcal{P}'$-isomorphisms, we obtain that $\mathfrak{g}_\alpha$ sends $U_{\mathcal{P}}(R)$ to $U_{\mathcal{P}'}(R')$ . We are done.

## 10. Vanishing cohomology

10.1. As we mention in 1.5 above, the existence of the *perfect* $\mathcal{F}^{^{\rm sc}}$*-locality* $\mathcal{P}^{^{\rm sc}}$ allows us to give a direct proof of Oliver's result in [4] on vanishing cohomology. More generally, assume that all the subgroups in $\mathfrak{X}$ are $\mathcal{F}$-selfcentralizing and denote by $\mathcal{O}$ a complete discrete valuation ring of characteristic zero lifting $k$, by $\mathcal{O}\text{-}\mathfrak{mod}$ the category of finitely generated $\mathcal{O}$-modules and by $\tilde{\mathfrak{m}}^{^\mathfrak{X}} : \tilde{\mathcal{F}}^{^\mathfrak{X}} \to \mathcal{O}\text{-}\mathfrak{mod}$ a *contravariant* functor; we will give a *direct* proof of that for any $n \ge 2$ we have

$$\mathbb{H}^n(\tilde{\mathcal{F}}^{^\mathfrak{X}}, \tilde{\mathfrak{m}}^{^\mathfrak{X}}) = \{0\} \qquad 10.1.1,$$

together with a suitable description of $\mathbb{H}^1(\tilde{\mathcal{F}}^{^\mathfrak{X}}, \tilde{\mathfrak{m}}^{^\mathfrak{X}})$ .

10.2. Denote by $\mathcal{P}^{^\mathfrak{X}}$ the *perfect* $\mathcal{F}^{^\mathfrak{X}}$*-locality* and by $\tau^\mathfrak{X}$ and $\pi^\mathfrak{X}$ the structural functors; note that the *exterior quotient* $\tilde{\mathcal{P}}^{^\mathfrak{X}}$ of $\mathcal{P}^{^\mathfrak{X}}$ (cf. 2.1) coincides with $\tilde{\mathcal{F}}^{\mathfrak{X}}$, and let us denote by $\mathfrak{m}^\mathfrak{X} : \mathcal{P}^{^\mathfrak{X}} \to \mathcal{O}\text{-}\mathfrak{mod}$ the *contravariant* functor induced by $\tilde{\mathfrak{m}}^{^\mathfrak{X}}$. Consider the *additive covers* $\mathfrak{ac}(\mathcal{P}^{^\mathfrak{X}})$ of $\mathcal{P}^{^\mathfrak{X}}$ and $\mathfrak{ac}(\tilde{\mathcal{F}}^{^\mathfrak{X}})$ of $\tilde{\mathcal{F}}^{^\mathfrak{X}}$ (cf. 7.1) and denote by

$$\mathfrak{j}^{\mathcal{P}^{\mathfrak{X}}} : \mathcal{P}^{^\mathfrak{X}} \longrightarrow \mathfrak{ac}(\mathcal{P}^{^\mathfrak{X}}) \quad \text{and} \quad \mathfrak{j}^{\tilde{\mathcal{F}}^{\mathfrak{X}}} : \tilde{\mathcal{F}}^{^\mathfrak{X}} \longrightarrow \mathfrak{ac}(\tilde{\mathcal{F}}^{^\mathfrak{X}}) \qquad 10.2.1$$

the canonical functors mapping any $Q \in \mathfrak{X}$ on the $\mathfrak{ac}(\mathcal{P}^{^\mathfrak{X}})$- and $\mathfrak{ac}(\tilde{\mathcal{F}}^{^\mathfrak{X}})$-object

$\bigoplus_{\{\emptyset\}} Q$ that we still denote by $Q$. Note that $\mathfrak{m}^{\mathfrak{X}}$ and $\tilde{\mathfrak{m}}^{\mathfrak{X}}$ can be *additively* extended to *contravariant* functors

$$\hat{\mathfrak{m}}^{\mathfrak{X}} : \mathfrak{ac}(\mathcal{P}^{\mathfrak{X}}) \longrightarrow \mathcal{O}\text{-}\mathfrak{mod} \quad \text{and} \quad \hat{\tilde{\mathfrak{m}}}^{\mathfrak{X}} : \mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}}) \longrightarrow \mathcal{O}\text{-}\mathfrak{mod} \qquad 10.2.2$$

mapping $\bigoplus_{i\in I} Q_i$ on $\prod_{i\in I} \mathfrak{m}^{\mathfrak{X}}(Q_i)$.

10.3. On the other hand, we need to consider the functors (cf. Proposition 7.10)

$$\mathfrak{int}_P^{\mathcal{P}^{\mathfrak{X}}} : \mathcal{P}^{\mathfrak{X}} \longrightarrow \mathfrak{ac}(\mathcal{P}^{\mathfrak{X}}) \quad \text{and} \quad \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{X}}} : \tilde{\mathcal{F}}^{\mathfrak{X}} \longrightarrow \mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}}) \qquad 10.3.1$$

mapping any $Q \in \mathfrak{X}$ on the respective $\mathcal{P}^{\mathfrak{X}}$- and $\tilde{\mathcal{F}}^{\mathfrak{X}}$-*intersections* $Q \cap^{\mathcal{P}^{\mathfrak{X}}} P$ and $Q \cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}} P$, and any $\mathcal{P}^{\mathfrak{X}}$- and $\tilde{\mathcal{F}}^{\mathfrak{X}}$-morphisms $x : R \to Q$ and $\tilde{\varphi} : R \to Q$ on the corresponding $\mathfrak{ac}(\mathcal{P}^{\mathfrak{X}})$- and $\mathfrak{ac}(\tilde{\mathcal{F}}^{\mathfrak{X}})$-morphisms

$$\begin{aligned} x \cap^{\mathcal{P}^{\mathfrak{X}}} \tau_P^{\mathfrak{X}}(1) &: R \cap^{\mathcal{P}^{\mathfrak{X}}} P \longrightarrow Q \cap^{\mathcal{P}^{\mathfrak{X}}} P \\ \tilde{\varphi} \cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}} \widetilde{\mathrm{id}}_P &: R \cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}} P \longrightarrow Q \cap^{\tilde{\mathcal{F}}^{\mathfrak{X}}} P \end{aligned} \qquad 10.3.2;$$

note that we have obvious *natural maps* (cf. 7.9.3)

$$\omega^{\mathcal{P}^{\mathfrak{X}}} : \mathfrak{int}_P^{\mathcal{P}^{\mathfrak{X}}} \longrightarrow \mathfrak{j}^{\mathcal{P}^{\mathfrak{X}}} \quad \text{and} \quad \omega^{\tilde{\mathcal{F}}^{\mathfrak{X}}} : \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{X}}} \longrightarrow \mathfrak{j}^{\tilde{\mathcal{F}}^{\mathfrak{X}}} \qquad 10.3.3;$$

moreover, any $u \in P$ determines a *natural automorphism* of $\mathfrak{int}_P^{\mathcal{P}^{\mathfrak{X}}}$ mapping any $Q \in \mathfrak{X}$ on $\tau_Q^{\mathfrak{X}}(1) \cap^{\mathcal{P}^{\mathfrak{X}}} \tau_P^{\mathfrak{X}}(u^{-1})$.

10.4. Explicitly, denoting by $\Omega$ a *natural $\mathcal{F}$-basic $P \times P$-set* (cf. 5.5) and by $\Omega_Q^{\mathfrak{X}} \subset \Omega$ the $Q \times P$-subset of elements $\omega \in \Omega$ such that the projection $Q_\omega$ in $Q$ of the stabilizer of $\omega$ in $Q \times P$ belongs to $\mathfrak{X}$, recall that we have (cf. 7.9.2 and Proposition 7.14)

$$Q \cap^{\mathcal{P}^{\mathfrak{X}}} P = \bigoplus_{\omega \in \Omega_Q^{\mathfrak{X}}} Q_\omega \qquad 10.4.1.$$

Actually, choosing a set of representatives $\Gamma_Q \subset \Omega_Q^{\mathfrak{X}}$ for the set of $Q \times P$-orbits in $\Omega_Q^{\mathfrak{X}}$, and denoting by $\omega^{\Gamma_Q}$ the representative of the $Q \times P$-class of $\omega \in \Omega_Q^{\mathfrak{X}}$, we have $\mathfrak{ac}(\mathcal{P}^{\mathfrak{X}})$-isomorphisms

$$Q \cap^{\mathcal{P}^{\mathfrak{X}}} P \cong \bigoplus_{\omega \in \Omega_Q^{\mathfrak{X}}} Q_{\omega^{\Gamma_Q}} \qquad 10.4.2;$$

but, note that the action of $Q \times P$ on $\Omega_Q^{\mathfrak{X}}$ determines a unique $\mathcal{O}$-module isomorphism $\mathfrak{m}^{\mathfrak{X}}(Q_{\omega^{\Gamma_Q}}) \cong \mathfrak{m}^{\mathfrak{X}}(Q_\omega)$; hence, up to identification, we may write

$$\hat{\mathfrak{m}}^{\mathfrak{X}}(Q \cap^{\mathcal{P}^{\mathfrak{X}}} P) = \prod_{\omega \in \Omega_Q^{\mathfrak{X}}} \mathfrak{m}^{\mathfrak{X}}(Q_{\omega^{\Gamma_Q}}) \qquad 10.4.3;$$

through this isomorphism, the actions of $u \in P$ and $v \in Q$ on $\hat{\mathfrak{m}}(Q \cap^{\mathcal{P}^{\mathfrak{x}}} P)$ are just given by the permutation of the indices.

10.5. Note that we actually have

$$\hat{\mathfrak{m}}^{\mathfrak{x}}(Q \cap^{\mathcal{P}^{\mathfrak{x}}} P)^{Q\times P} \cong \hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}}(Q \cap^{\tilde{\mathcal{F}}^{\mathfrak{x}}} P) \qquad 10.5.1$$

since the quotient set $(Q \times P)\backslash \mathfrak{T}_{Q,P}$ coincides with $\tilde{\mathfrak{T}}_{Q,P}$ (cf. 7.8) and, once again, the stabilizer of $\omega \in \Omega_Q^{\mathfrak{x}}$ in $Q \times P$ acts trivially on $\mathfrak{m}^{\mathfrak{x}}(Q_\omega)$ (cf. 7.8.1). Clearly, we have a *contravariant* functor $(\hat{\mathfrak{m}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\mathcal{P}^{\mathfrak{x}}})^P$ from $\mathcal{P}^{\mathfrak{x}}$ to $\mathcal{O}$-$\mathfrak{mod}$ mapping any $Q \in \mathfrak{X}$ on $\hat{\mathfrak{m}}^{\mathfrak{x}}(Q \cap^{\mathcal{P}^{\mathfrak{x}}} P)^P$, and it follows from [9, 14.21] that it determines a new *contravariant* functor $\mathfrak{h}^0\big((\hat{\mathfrak{m}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\mathcal{P}^{\mathfrak{x}}})^P\big)$ from $\mathcal{P}^{\mathfrak{x}}$ to $\mathcal{O}$-$\mathfrak{mod}$, mapping $Q \in \mathfrak{X}$ on $\hat{\mathfrak{m}}^{\mathfrak{x}}(Q \cap^{\mathcal{P}^{\mathfrak{x}}} P)^{Q\times P}$, which factorizes through the *exterior quotient*

$$\tilde{\mathfrak{h}}^0\big((\hat{\mathfrak{m}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\mathcal{P}^{\mathfrak{x}}})^P\big) : \tilde{\mathcal{P}}^{\mathfrak{x}} = \tilde{\mathcal{F}}^{\mathfrak{x}} \longrightarrow \mathcal{O}\text{-}\mathfrak{mod} \qquad 10.5.2;$$

hence, from the naturality of isomorphism 10.5.1 we get a *natural isomorphism*

$$\tilde{\mathfrak{h}}^0\big((\hat{\mathfrak{m}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\mathcal{P}^{\mathfrak{x}}})^P\big) \cong \hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{x}}} \qquad 10.5.3.$$

10.6. Moreover, the natural map $\omega^{\tilde{\mathcal{F}}^{\mathfrak{x}}}$ in 10.3.3 above determines an *injective natural map*

$$\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} * \omega^{\tilde{\mathcal{F}}^{\mathfrak{x}}} : \tilde{\mathfrak{m}}^{\mathfrak{x}} \longrightarrow \hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{x}}} \qquad 10.6.1$$

and therefore, up to identification, we get the exact sequence of *contravariant* functors

$$0 \longrightarrow \tilde{\mathfrak{m}}^{\mathfrak{x}} \longrightarrow \hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{x}}} \longrightarrow (\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{x}}})/\tilde{\mathfrak{m}}^{\mathfrak{x}} \longrightarrow 0 \qquad 10.6.2;$$

thus, in order to prove equality 10.1.1, it suffices to show that the $n$-cohomology groups of the middle and the right-hand members vanish for $n \geq 1$.

10.7. For the middle term, according to isomorphism 10.5.3 we may replace $\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{x}}}$ by $\tilde{\mathfrak{l}}^{\mathfrak{x}} = \tilde{\mathfrak{h}}^0\big((\hat{\mathfrak{m}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\mathcal{P}^{\mathfrak{x}}})^P\big)$; this *contravariant* functor is nothing but the factorization of $\mathfrak{l}^{\mathfrak{x}} = \mathfrak{h}^0\big((\hat{\mathfrak{m}}^{\mathfrak{x}} \circ \mathfrak{int}_P^{\mathcal{P}^{\mathfrak{x}}})^P\big)$ through the *exterior quotient* $\tilde{\mathcal{P}}^{\mathfrak{x}}$ of $\mathcal{P}^{\mathfrak{x}}$ and, denoting by $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$ the subcategory of $\mathcal{P}^{\mathfrak{x}}$ formed by all the objects and all the "inner isomorphisms" — in some sense, the "kernel" of the canonical functor $\mathcal{P}^{\mathfrak{x}} \to \tilde{\mathcal{P}}^{\mathfrak{x}}$ — we claim that the $n$-cohomology group of the *exterior quotient* $\tilde{\mathcal{P}}^{\mathfrak{x}}$ over $\tilde{\mathfrak{l}}^{\mathfrak{x}}$ coincides with the $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$-*stable* $n$-cohomology group of $\mathcal{P}^{\mathfrak{x}}$ over $\mathfrak{l}^{\mathfrak{x}}$ [9, A3.18]; we actually will prove that the last one is zero for $n \geq 1$.

10.8. Explicity, recall that we set (cf. 4.2.2)

$$\begin{aligned}\mathbb{C}^n(\tilde{\mathcal{P}}^{\mathfrak{x}},\tilde{\mathfrak{l}}^{\mathfrak{x}}) &= \prod_{\tilde{\mathfrak{q}}\in\mathfrak{Fct}(\Delta_n,\tilde{\mathcal{P}}^{\mathfrak{x}})} \hat{\mathfrak{m}}^{\mathfrak{x}}\big(\tilde{\mathfrak{q}}(0)\cap^{\mathcal{P}^{\mathfrak{x}}} P\big)^{\tilde{\mathfrak{q}}(0)\times P} \\ \mathbb{C}^n(\mathcal{P}^{\mathfrak{x}},\mathfrak{l}^{\mathfrak{x}}) &= \prod_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\mathcal{P}^{\mathfrak{x}})} \hat{\mathfrak{m}}^{\mathfrak{x}}\big(\mathfrak{q}(0)\cap^{\mathcal{P}^{\mathfrak{x}}} P\big)^{\mathfrak{q}(0)\times P}\end{aligned} \qquad 10.8.1;$$

we say that an element $m = (m_{\mathfrak{q}})_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\mathcal{P}^{\mathfrak{x}})}$ of $\mathbb{C}^n(\mathcal{P}^{\mathfrak{x}},\mathfrak{l}^{\mathfrak{x}})$ is $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$-*stable* if it fulfills [9, A3.17]

$$m_{\mathfrak{q}} = \big(\mathfrak{l}^{\mathfrak{x}}(\nu_0)\big)(m_{\bar{\mathfrak{q}}}) \qquad 10.8.2$$

for any *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$-*isomorphism* $\nu:\mathfrak{q}\cong\bar{\mathfrak{q}}$ between two $\mathcal{P}^{\mathfrak{x}}$-*chains*

$$\mathfrak{q}:\Delta_n \longrightarrow \mathcal{P}^{\mathfrak{x}} \quad \text{and} \quad \bar{\mathfrak{q}}:\Delta_n \longrightarrow \mathcal{P}^{\mathfrak{x}} \qquad 10.8.3$$

and then we denote by $\mathbb{C}^n_{\mathcal{I}(\mathcal{P}^{\mathfrak{x}})}(\mathcal{P}^{\mathfrak{x}},\mathfrak{l}^{\mathfrak{x}})$ the $\mathcal{O}$-submodule of $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$-*stable elements* of $\mathbb{C}^n(\mathcal{P}^{\mathfrak{x}},\mathfrak{l}^{\mathfrak{x}})$; note that, for any $i\in\Delta_n$, we have $\mathfrak{q}(i)=\bar{\mathfrak{q}}(i)$ and $\nu_i = \tau^{\mathfrak{x}}_{\mathfrak{q}(i)}(v_i)$ for some $v_i\in\mathfrak{q}(i)$; in particular, $\mathfrak{l}^{\mathfrak{x}}(\nu_0)$ is the conjugation by an element of $\mathfrak{q}(0)$. Now, it is easily checked that the homomorphism

$$\mathbb{C}^n(\tilde{\mathcal{P}}^{\mathfrak{x}},\tilde{\mathfrak{l}}^{\mathfrak{x}}) \longrightarrow \mathbb{C}^n(\mathcal{P}^{\mathfrak{x}},\mathfrak{l}^{\mathfrak{x}}) \qquad 10.8.4$$

determined by the canonical map $\mathfrak{Fct}(\Delta_n,\mathcal{P}^{\mathfrak{x}})\to\mathfrak{Fct}(\Delta_n,\tilde{\mathcal{P}}^{\mathfrak{x}})$ induces an isomorphism $\mathbb{C}^n(\tilde{\mathcal{P}}^{\mathfrak{x}},\tilde{\mathfrak{l}}^{\mathfrak{x}})\cong\mathbb{C}^n_{\mathcal{I}(\mathcal{P}^{\mathfrak{x}})}(\mathcal{P}^{\mathfrak{x}},\mathfrak{l}^{\mathfrak{x}})$ and therefore for any $n\in\mathbb{N}$ we have [9, A3.18]

$$\mathbb{H}^n(\tilde{\mathcal{P}}^{\mathfrak{x}},\tilde{\mathfrak{l}}^{\mathfrak{x}}) = \mathbb{H}^n_{\mathcal{I}(\mathcal{P}^{\mathfrak{x}})}(\mathcal{P}^{\mathfrak{x}},\mathfrak{l}^{\mathfrak{x}}) \qquad 10.8.5.$$

Actually, this does not depend on the nature of the functor $\tilde{\mathfrak{l}}^{\mathfrak{x}}$.

10.9. In order to prove that right member vanish for $n\ge 1$, for any $\mathcal{P}^{\mathfrak{x}}$-*chain* $\mathfrak{q}:\Delta_n\to\mathcal{P}^{\mathfrak{x}}$ we have to consider the set $\mathcal{V}_{\mathfrak{q}}$ of triples $(\mu',\mathfrak{q}',x')$ formed by a $\mathcal{P}^{\mathfrak{x}}$-*chain* $\mathfrak{q}':\Delta_n\to\mathcal{P}^{\mathfrak{x}}$, by a natural map $\mu':\mathfrak{q}'\to\mathfrak{q}$ and by and element $x'\in\mathcal{P}^{\mathfrak{x}}\big(P,\mathfrak{q}'(n)\big)$ in such a way that $\mathfrak{q}'(i)$ belongs to $\Gamma_{\mathfrak{q}'(i)}$ and that $x'_i = x'\cdot\mathfrak{q}'(i\bullet n)$ belongs to $\mathcal{P}^{\mathfrak{x}}\big(P,q'(i)\big)_{\mu'_i}$ for any $i\in\Delta_n$; similarly as above, we say that two such triples $(\mu',\mathfrak{q}',x')$ and $(\mu'',\mathfrak{q}'',x'')$ are *equivalent* if there is a *natural isomorphism* $\theta:\mathfrak{q}'\cong\mathfrak{q}''$ fulfilling

$$\mu''\circ\theta=\mu' \quad \text{and} \quad x''\cdot\theta_n = x' \qquad 10.9.1;$$

in this case, note that $\theta$ is actually a *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$-*isomorphism* (cf. 7.8.1); moreover, denoting by $\iota_{\mathfrak{q}'}:\mathfrak{q}'\cong\mathfrak{q}'$ the *identity natural map*, it is obvious that the triple $(\iota_{\mathfrak{q}'},\mathfrak{q}',x')$ belongs to $\mathcal{V}_{\mathfrak{q}'}$.

10.10. It is clear that any element $u \in P$ acts on $\mathcal{V}_{\mathfrak{q}}$ sending $(\mu', \mathfrak{q}', x')$ to $(\mu', \mathfrak{q}', \tau_P^{\mathfrak{x}}(u)\cdot x')$ and therefore it acts on the set of *equivalence classes*; similarly, any *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism* $\alpha : \mathfrak{q} \cong \bar{\mathfrak{q}}$ maps $\mathcal{V}_{\mathfrak{q}}$ on $\mathcal{V}_{\bar{\mathfrak{q}}}$ sending $(\mu', \mathfrak{q}', x')$ to $(\alpha \circ \mu', \mathfrak{q}', x')$ since we have $\bar{\mathfrak{q}}(i) = \mathfrak{q}(i)$ for any $i \in \Delta_n$ ; once again, $\alpha$ acts on the set of *equivalence classes*; in particular, any element $v \in \mathfrak{q}(0)$ acts on $\mathcal{V}_{\mathfrak{q}}$ preserving the *equivalence classes*, since it defines a *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism* $\tau_v^{\mathfrak{q}} : \mathfrak{q} \cong \mathfrak{q}$ sending $i \in \Delta_n$ to $\tau_{\mathfrak{q}(i)}^{\mathfrak{x}}\big(\mathfrak{q}(0 \bullet i)(v)\big)$. Let us denote by $\check{\mathcal{V}}_{\mathfrak{q}}$ a set of representatives for the set of *equivalent classes* in $\mathcal{V}_{\mathfrak{q}}$ ; then, it follows from 7.9 and Proposition 7.10 that, for any $\mathcal{P}^{\mathfrak{x}}$*-chain* $\mathfrak{q} : \Delta_n \to \mathcal{P}^{\mathfrak{x}}$ , an element $(\mu', \mathfrak{q}', x')$ in $\check{\mathcal{V}}_{\mathfrak{q}}$ is determined by $(\mu'_0, \mathfrak{q}'(0), x'_0)$ and therefore it is easily checked from 10.4.1 that we have

$$\mathfrak{q}(0) \cap^{\mathcal{P}^{\mathfrak{x}}} P = \bigoplus_{(\mu', \mathfrak{q}', x') \in \check{\mathcal{V}}_{\mathfrak{q}}} \mathfrak{q}'(0) \qquad 10.10.1;$$

in particular, according to 10.4.3, we get

$$\hat{\mathfrak{m}}^{\mathfrak{x}}\big(\mathfrak{q}(0) \cap^{\mathcal{P}^{\mathfrak{x}}} P\big) = \prod_{(\mu', \mathfrak{q}', x') \in \check{\mathcal{V}}_{\mathfrak{q}}} \mathfrak{m}^{\mathfrak{x}}\big(\mathfrak{q}'(0)\big) \qquad 10.10.2$$

and this decomposition does not depend on the choice of $\check{\mathcal{V}}_{\mathfrak{q}}$ ; we denote by $m_{(\mu', \mathfrak{q}', x')} \in \mathfrak{m}^{\mathfrak{x}}\big(\mathfrak{q}'(0)\big)$ the corresponding component of any element $m$ in $\hat{\mathfrak{m}}^{\mathfrak{x}}\big(\mathfrak{q}(0) \cap^{\mathcal{P}^{\mathfrak{x}}} P\big)$ ; note that $\mathfrak{q}(0) \times P$ acts on $\hat{\mathfrak{m}}^{\mathfrak{x}}\big(\mathfrak{q}(0) \cap^{\mathcal{P}^{\mathfrak{x}}} P\big)$ by permuting the indices *via* its action on the set of *equivalent classes*.

10.11. As in 4.7 above, for any triple $(\mu', \mathfrak{q}', x') \in \mathcal{V}_{\mathfrak{q}}$ and any $\ell \in \Delta_n$ , let us denote by

$$\mathfrak{h}_\ell^n(\mu', x') : \Delta_{n+1} \longrightarrow \mathcal{P}^{\mathfrak{x}} \qquad 10.11.1$$

the functor which coincides with $\mathfrak{q}'$ over $\Delta_\ell$ , maps $i \in \Delta_{n+1} - \Delta_\ell$ on $\mathfrak{q}(i-1)$ , maps $i \bullet i+1$ on $\mathfrak{q}(i-1 \bullet i)$ if $i \le n$ , and maps $\ell \bullet \ell{+}1$ on $\mu_\ell : \mathfrak{q}'(\ell) \to \mathfrak{q}(\ell)$ [9, Lemma A4.2]; moreover, denote by

$$\mathfrak{h}_{n+1}^n(\mu', x') : \Delta_{n+1} \longrightarrow \mathcal{P}^{\mathfrak{x}} \qquad 10.11.2$$

the $\mathcal{P}^{\mathfrak{x}}$-chain extending $q'$ and mapping $n+1$ on $P$ and $n \bullet n+1$ on $x'$ . Note that if $(\mu'', \mathfrak{q}'', x'') \in \mathcal{V}_{\mathfrak{q}}$ is a triple equivalent to $(\mu', \mathfrak{q}', x')$ then, according to 10.9, we have a *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism* $\theta : \mathfrak{r}' \cong \mathfrak{r}''$ and therefore, for any $\ell \in \Delta_{n+1}$ , we get the *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism*

$$\bar{\mathfrak{h}}_\ell^n(\theta) : \mathfrak{h}_\ell^n(\mu', x') \cong \mathfrak{h}_\ell^n(\mu'', x'') \qquad 10.11.3$$

sending $i \in \Delta_\ell$ to $\theta_i$ or on $\tau_P^{\mathfrak{x}}(1)$ if $i = \ell = n+1$ , and $i \in \Delta_{n+1} - \Delta_\ell$ to $\tau_{\mathfrak{q}(i-1)}^{\mathfrak{x}}(1)$ . Moreover, for any *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism* $\alpha : \mathfrak{q} \cong \bar{\mathfrak{q}}$ to another $\mathcal{P}^{\mathfrak{x}}$-chain $\bar{\mathfrak{q}}$ , and any $\ell \in \Delta_{n+1}$ , we get the *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism*

$$\mathfrak{h}_\ell^n(\alpha) : h_\ell^n(\mu', x') \cong h_\ell^n(\alpha \circ \mu', x') \qquad 10.11.4$$

sending $i \in \Delta_\ell$ to $\tau^{\mathfrak{x}}_{\mathfrak{q}'(i)}(1)$ or to $\tau^{\mathfrak{x}}_P(1)$ if $i = \ell = n+1$, and $i \in \Delta_{n+1} - \Delta_\ell$ to $\alpha_{i-1}$; similarly, for any $u \in P$ we have $h^n_\ell(\mu', \tau^{\mathfrak{x}}_P(u)\cdot x') = h^n_\ell(\mu', x')$ if $\ell \in \Delta_n$, and a *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism*

$$\mathfrak{h}^n_u : h^n_{n+1}(\mu', x') \cong h^n_{n+1}(\mu', \tau^{\mathfrak{x}}_P(u)\cdot x') \qquad 10.11.5$$

sending $i \in \Delta_n$ to $\tau^{\mathfrak{x}}_{\mathfrak{q}'(i)}(1)$ and $n+1$ to $\tau^{\mathfrak{x}}_P(u)$.

**Theorem 10.12.** *With the notation above, for any $n \geq 1$ we have*

$$\mathbb{H}^n\big(\tilde{\mathcal{F}}^{\mathfrak{x}}, \hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}^{\tilde{\mathcal{F}}^{\mathfrak{x}}}_P\big) = \{0\} \qquad 10.12.1.$$

*Moreover, we have* $\mathbb{H}^0\big(\tilde{\mathcal{F}}^{\mathfrak{x}}, \hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}^{\tilde{\mathcal{F}}^{\mathfrak{x}}}_P\big) \cong \tilde{\mathfrak{m}}^{\mathfrak{x}}(P)$.

**Proof:** First of all, let us prove the last isomorphism; we already know that

$$\mathbb{H}^0\big(\tilde{\mathcal{F}}^{\mathfrak{x}}, \hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}^{\tilde{\mathcal{F}}^{\mathfrak{x}}}_P\big) \cong \lim_{\longleftarrow}\big(\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}} \circ \mathfrak{int}^{\tilde{\mathcal{F}}^{\mathfrak{x}}}_P\big) \qquad 10.12.2;$$

but, an element of this inverse limit has the form $\mathrm{m} = (m_Q)_{Q\in\mathfrak{X}}$ for elements $m_Q$ belonging to

$$\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}}(Q \cap^{\tilde{\mathcal{F}}^{\mathfrak{x}}} P) = \prod_{(\tilde{\imath}^Q_{Q'}, Q', \tilde{\theta}')} \tilde{\mathfrak{m}}^{\mathfrak{x}}(Q') \qquad 10.12.3,$$

where $Q' \in \mathfrak{X}$ runs over a set of representatives of the set of $Q$-conjugacy classes of subgroups of $Q$ such that $\tilde{\mathcal{F}}^{\mathfrak{x}}(P, Q')_{\tilde{\imath}^Q_{Q'}} \neq \emptyset$, and $\tilde{\theta}'$ over a set of representatives for the set of $\tilde{\mathcal{F}}^{\mathfrak{x}}_Q(Q')$-orbits in $\tilde{\mathcal{F}}^{\mathfrak{x}}(P, Q')_{\tilde{\imath}^Q_{Q'}}$, in such a way that, for any $\tilde{\mathcal{F}}^{\mathfrak{x}}$-morphism $\tilde{\varphi} : R \to Q$, the group homomorphism $\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}}(\tilde{\varphi} \cap^{\tilde{\mathcal{F}}^{\mathfrak{x}}} \widetilde{\mathrm{id}}_P)$ maps $m_Q$ on $m_R$.

In particular, denoting by $(m_Q)_{(\tilde{\imath}^Q_{Q'}, Q', \tilde{\theta}')}$ the corresponding component of $m_Q$ in $\tilde{\mathfrak{m}}^{\mathfrak{x}}(Q')$, we necessarily have

$$(m_Q)_{(\tilde{\imath}^Q_{Q'}, Q', \tilde{\theta}')} = (m_{Q'})_{(\widetilde{\mathrm{id}}_{Q'}, Q', \tilde{\theta}')} \qquad 10.12.4;$$

moreover, it is easily checked that the group homomorphism $\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}}(\tilde{\theta}' \cap^{\tilde{\mathcal{F}}^{\mathfrak{x}}} \widetilde{\mathrm{id}}_P)$ sends $(m_P)_{(\widetilde{\mathrm{id}}_P, P, \widetilde{\mathrm{id}}_P)}$ to $(m_{Q'})_{(\widetilde{\mathrm{id}}_{Q'}, Q', \tilde{\theta}')}$. Conversely, for any $m \in \tilde{\mathfrak{m}}^{\mathfrak{x}}(P)$, it suffices to consider the element $m_Q$ in $\hat{\tilde{\mathfrak{m}}}^{\mathfrak{x}}(Q \cap^{\tilde{\mathcal{F}}^{\mathfrak{x}}} P)$ defied by (cf. 10.12.3)

$$(m_Q)_{(\tilde{\imath}^Q_{Q'}, Q', \tilde{\theta}')} = \big(\tilde{\mathfrak{m}}^{\mathfrak{x}}(\tilde{\theta}')\big)(m) \qquad 10.12.5$$

to get an element $\mathrm{m} = (m_Q)_{Q\in\mathfrak{X}}$ in the inverse limit. We are done.

For $n \geq 1$, let $\mathrm{m} = (m_{\mathfrak{q}})_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\mathcal{P}^{\mathfrak{X}})}$ be an $\mathcal{I}(\mathcal{P}^{\mathfrak{X}})$*-stable* $\mathfrak{l}^{\mathfrak{X}}$*-valued* $n$*-cocycle*; that is to say, $m_{\mathfrak{q}}$ belongs to $\hat{\mathfrak{m}}^{\mathfrak{X}}\big(\mathfrak{q}(0)\cap^{\mathcal{P}^{\mathfrak{X}}}P\big)^{\mathfrak{q}(0)\times P}$ and, denoting by $d_n^{\mathfrak{X}}$ the corresponding *differential map*, we have

$$d_n^{\mathfrak{X}}(\mathrm{m}) = 0 \quad \text{and} \quad \big(\hat{\mathfrak{m}}^{\mathfrak{X}}(\alpha_0 \cap^{\mathcal{P}^{\mathfrak{X}}} \tau_P^{\mathfrak{X}}(1)\big)(m_{\bar{\mathfrak{q}}}) = m_{\mathfrak{q}} \qquad 10.12.6$$

for any *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{X}})$*-isomorphism* $\alpha : \mathfrak{q} \cong \bar{\mathfrak{q}}$. In particular, for any triple $(\mu', \mathfrak{q}', x') \in \mathcal{V}_{\mathfrak{q}}$, since $v \in \mathfrak{q}(0)$ fixes $m_{\mathfrak{q}}$ we have (cf. 10.10)

$$(m_{\mathfrak{q}})_{(\tau_v^{\mathfrak{q}}\circ\mu',\mathfrak{q}',x')} = (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} \qquad 10.12.7;$$

similarly, since $u \in P$ fixes $m_{\mathfrak{q}}$, we also have (cf. 10.10)

$$(m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',\tau_P^{\mathfrak{X}}(u)\cdot x')} = (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} \qquad 10.12.8.$$

At this point, for any $\mathcal{P}^{\mathfrak{X}}$*-chain* $\mathfrak{q} : \Delta_n \to \mathcal{P}^{\mathfrak{X}}$, any triple $(\mu', \mathfrak{q}', x')$ in $\mathcal{V}_{\mathfrak{q}}$ and any $\ell \in \Delta_{n+1}$, consider the component of $d_n^{\mathfrak{X}}(\mathrm{m})$ on the $\mathcal{P}^{\mathfrak{X}}$*-chain*

$$\mathfrak{h}_\ell^n(\mu', x') : \Delta_{n+1} \longrightarrow \mathcal{P}^{\mathfrak{X}} \qquad 10.12.9;$$

since $d_n^{\mathfrak{X}}(\mathrm{m}) = 0$, we get the following equalities

$$\begin{aligned} 0 &= \big(\mathfrak{l}^{\mathfrak{X}}(\mu_0)\big)(m_{\mathfrak{q}}) + \sum_{i=1}^{n+1} (-1)^i m_{\mathfrak{h}_0^n(\mu',x')\circ\delta_i^n} \\ 0 &= \big(\mathfrak{l}^{\mathfrak{X}}(\mathfrak{q}'(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n}) + \sum_{i=1}^{n+1} (-1)^i m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n} \end{aligned} \qquad 10.12.10$$

for any $1 \leq \ell \leq n+1$, and then the theorem follows from the following lemma.

**Lemma 10.13.** *For* $n \geq 1$, *let* $\mathrm{m} = (m_{\mathfrak{q}})_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\mathcal{P}^{\mathfrak{X}})}$ *be an* $\mathcal{I}(\mathcal{P}^{\mathfrak{X}})$*-stable element of* $\mathbb{C}^n(\mathcal{P}^{\mathfrak{X}}, \mathfrak{l}^{\mathfrak{X}})$ *such that, for a* $\mathcal{P}^{\mathfrak{X}}$*-chain* $\mathfrak{q} : \Delta_n \to \mathcal{P}^{\mathfrak{X}}$, *any triple* $(\mu', \mathfrak{q}', x')$ *in* $\mathcal{V}_{\mathfrak{q}}$ *and any* $1 \leq \ell \leq n+1$, *we have*

$$\begin{aligned} 0 &= \big(\mathfrak{l}^{\mathfrak{X}}(\mu_0)\big)(m_{\mathfrak{q}}) + \sum_{i=1}^{n+1} (-1)^i m_{\mathfrak{h}_0^n(\mu',x')\circ\delta_i^n} \\ 0 &= \big(\mathfrak{l}^{\mathfrak{X}}(\mathfrak{q}'(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n}) + \sum_{i=1}^{n+1} (-1)^i m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n} \end{aligned} \qquad 10.13.1.$$

*For any* $\mathcal{P}^{\mathfrak{X}}$*-chain* $\mathfrak{r} : \Delta_{n-1} \to \mathcal{P}^{\mathfrak{X}}$ *consider* $n_{\mathfrak{r}} \in \hat{\mathfrak{m}}^{\mathfrak{X}}\big(\mathfrak{r}(0)\cap^{\mathcal{P}^{\mathfrak{X}}}P\big)$ *defined by*

$$(n_{\mathfrak{r}})_{(\nu',\mathfrak{r}',y')} = \Big(\sum_{\ell=0}^{n} (-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\nu',y')}\Big)_{(\iota_{\mathfrak{r}'},\mathfrak{r}',y')} \qquad 10.13.2$$

*for any* $(\nu', \mathfrak{r}', y') \in \check{\mathcal{V}}_{\mathfrak{r}}$. *Then* $\mathrm{n} = (n_{\mathfrak{r}})_{\mathfrak{r}\in\mathfrak{Fct}(\Delta_{n-1},\mathcal{P}^{\mathfrak{X}})}$ *is an* $\mathcal{I}(\mathcal{P}^{\mathfrak{X}})$*-stable element of* $\mathbb{C}^{n-1}(\mathcal{P}^{\mathfrak{X}}, \mathfrak{l}^{\mathfrak{X}})$ *and we have* $d_{n-1}^{\mathfrak{X}}(\mathrm{n})_{\mathfrak{q}} = m_{\mathfrak{q}}$.

**Proof:** According to 10.9 and 10.10, our definition of $n_{\mathfrak{r}}$ makes sense since for any $\ell \in \Delta_n$ we have

$$m_{\mathfrak{h}_\ell^{n-1}(\nu',y')} \in \hat{\mathfrak{m}}^{\mathfrak{x}}\big(\mathfrak{r}'(0) \cap^{\mathcal{P}^{\mathfrak{x}}} P\big) \quad \text{and} \quad (\iota_{\mathfrak{r}'}, \mathfrak{r}', y') \in \mathcal{V}_{\mathfrak{r}'} \qquad 10.13.3.$$

Moreover, the definition of $n_{\mathfrak{r}}$ does not depend on the choice of $\check{\mathcal{V}}_{\mathfrak{r}}$; indeed, if $(\nu'', \mathfrak{r}'', y'')$ is a triple equivalent to $(\nu', \mathfrak{r}', y')$ then, according to 10.9 and 10.11, we have *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphisms*

$$\eta : \mathfrak{r}' \cong \mathfrak{r}'' \quad \text{and} \quad \bar{\mathfrak{h}}_\ell^{n-1}(\eta) : \mathfrak{h}_\ell^{n-1}(\nu', y') \cong \mathfrak{h}_\ell^{n-1}(\nu'', y'') \qquad 10.13.4$$

for any $\ell \in \Delta_n$ ; hence, on the one hand $(\iota_{\mathfrak{r}''}, \mathfrak{r}'', y'')$ is equivalent to $(\iota_{\mathfrak{r}'}, \mathfrak{r}', y')$ and, on the other hand, since m is $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-stable*, for any $\ell \in \Delta_n$ we have

$$m_{\mathfrak{h}_\ell^{n-1}(\nu',y')} = m_{\mathfrak{h}_\ell^{n-1}(\nu'',y'')} \qquad 10.13.5.$$

We claim that n is an $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$-stable element of $\mathbb{C}^{n-1}(\mathcal{P}^{\mathfrak{x}}, \mathfrak{l}^{\mathfrak{x}})$ ; indeed, for any *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism* $\beta : \mathfrak{r} \cong \bar{\mathfrak{r}}$, it follows from 10.11 above that, for any $\ell \in \Delta_n$ , we have a *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-isomorphism*

$$\mathfrak{h}_\ell^{n-1}(\beta) : h_\ell^{n-1}(\nu', y') \cong h_\ell^{n-1}(\beta \circ \nu', y') \qquad 10.13.6$$

and therefore, since m is $\mathcal{I}(\mathcal{P}^{\mathfrak{x}})$*-stable*, we get $m_{\mathfrak{h}_\ell^{n-1}(\nu',y')} = m_{\mathfrak{h}_\ell^{n-1}(\beta\circ\nu',y')}$ ; hence, we obtain

$$(n_{\mathfrak{r}})_{(\nu',\mathfrak{r}',y')} = \Big(\sum_{\ell=0}^{n} (-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\beta\circ\nu',y')}\Big)_{(\iota_{\mathfrak{r}'},\mathfrak{r}',y')} = (n_{\bar{\mathfrak{r}}})_{(\beta\circ\nu',\mathfrak{r}',y')} \qquad 10.13.7,$$

proving our claim; in particular, note that $\mathfrak{r}(0)$ fixes $n_{\mathfrak{r}}$ . Similarly, we claim that $u \in P$ fixes $n_{\mathfrak{r}}$ for any $\mathfrak{r} \in \mathfrak{Fct}(\Delta_{n-1}, \mathcal{P}^{\mathfrak{x}})$ ; indeed, since $u \in P$ fixes $m_{\mathfrak{q}}$ , we also get (cf. 10.10)

$$\begin{aligned}
(n_{\mathfrak{r}})_{(\nu',\mathfrak{r}',\tau_P^{\mathfrak{x}}(u)\cdot y')} &= \Big(\sum_{\ell=0}^{n} (-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\nu')}\Big)_{(\iota_{\mathfrak{r}'},\mathfrak{r}',\tau_P^{\mathfrak{x}}(u)\cdot y')} \\
&= \sum_{\ell=0}^{n} (-1)^\ell (m_{\mathfrak{h}_\ell^{n-1}(\nu')})_{(\iota_{\mathfrak{r}'},\mathfrak{r}',\tau_P^{\mathfrak{x}}(u)\cdot y')} \qquad 10.13.8; \\
&= \sum_{\ell=0}^{n} (-1)^\ell (m_{\mathfrak{h}_\ell^{n-1}(\nu')})_{(\iota_{\mathfrak{r}'},\mathfrak{r}',y')} = (n_{\mathfrak{r}})_{(\nu',\mathfrak{r}',\tau_P^{\mathfrak{x}}(u)\cdot y')}
\end{aligned}$$

thus, $n_{\mathfrak{r}}$ belongs to $\hat{\mathfrak{m}}\big(\mathfrak{r}(0) \cap^{\mathcal{P}^{\mathfrak{x}}} P\big)^{\mathfrak{r}(0)\times P}$.

Denoting by $d_{n-1}^{\mathfrak{x}}(\mathrm{n})_{\mathfrak{q}}$ the component of $d_{n-1}^{\mathfrak{x}}(\mathrm{n})$ on $\mathfrak{q}$ , it remains to prove that $d_{n-1}^{\mathfrak{x}}(\mathrm{n})_{\mathfrak{q}} = m_{\mathfrak{q}}$ or, equivalently, that for any triple $(\mu', \mathfrak{q}', x')$ in $\mathcal{V}_{\mathfrak{q}}$ we have

$$\big(d_{n-1}^{\mathfrak{x}}(\mathrm{n})_{\mathfrak{q}}\big)_{(\mu',\mathfrak{q}',x')} = (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} \qquad 10.13.9.$$

In equalities 10.13.1, the alternating sum of all the first terms of the right-hand members yields

$$a_0 = \big(\mathfrak{l}^{\mathfrak{x}}(\mu_0)\big)(m_{\mathfrak{q}}) + \sum_{\ell=1}^{n+1}(-1)^{\ell}\big(\mathfrak{l}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n}) \qquad 10.13.10;$$

but, it follows from [9, Lemma A4.2] that, for any $1 \le \ell \le n+1$, we have

$$\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n = \mathfrak{h}_{\ell-1}^{n-1}(\mu' * \delta_0^n, x') \qquad 10.13.11;$$

hence, in $\hat{\mathfrak{m}}^{\mathfrak{x}}\big(\mathfrak{q}'(0)\cap^{\mathcal{P}^{\mathfrak{x}}}P\big)$ we get

$$a_0 = \big(\mathfrak{l}^{\mathfrak{x}}(\mu_0)\big)(m_{\mathfrak{q}}) - \sum_{\ell=0}^{n}(-1)^{\ell}\big(\mathfrak{l}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^{n-1}(\mu'*\delta_0^n,x')}) \qquad 10.13.12.$$

Let us consider the $(\iota_{\mathfrak{q}'},\mathfrak{q}',x')$-component of this element (cf. 10.10); on the one hand, by the very definitions we have

$$\Big(\big(\mathfrak{l}^{\mathfrak{x}}(\mu_0)\big)(m_{\mathfrak{q}})\Big)_{(\iota_{\mathfrak{q}'},\mathfrak{q}',x')} = (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} \qquad 10.13.13;$$

on the other hand, for any $\ell \in \Delta_n$, it is easily checked that

$$\begin{aligned}\Big(\big(\mathfrak{l}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^{n-1}(\mu'*\delta_0^n,x')})\Big)_{(\iota_{\mathfrak{q}'},\mathfrak{q}',x')}&\\ = \big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)\big((m_{\mathfrak{h}_\ell^{n-1}(\mu'*\delta_0^n,x')})_{(\iota_{\mathfrak{q}'\circ\delta_0^n},\mathfrak{q}'\circ\delta_0^n,x')}\big)&\end{aligned} \qquad 10.13.14;$$

consequently, according to the definition of $n_{\mathfrak{q}\circ\delta_0^n}$, we get

$$\begin{aligned}(a_0)_{(\iota_{\mathfrak{q}'},\mathfrak{q}',x')} &= (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} -\\ &- \big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)\big((\sum_{\ell=0}^{n}(-1)^{\ell} m_{\mathfrak{h}_\ell^{n-1}(\mu'*\delta_0^n,x')})_{(\iota_{\mathfrak{q}'\circ\delta_0^n},\mathfrak{q}'\circ\delta_0^n,x')}\big)\\ &= (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} - \big(\mathfrak{m}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)\big((n_{\mathfrak{q}\circ\delta_0^n})_{(\mu'*\delta_0^n,\mathfrak{q}'\circ\delta_0^n,x')}\big)\\ &= (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} - \Big(\big(\mathfrak{l}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)(n_{\mathfrak{q}\circ\delta_0^n})\Big)_{(\mu',\mathfrak{q}',x')}\end{aligned} \qquad 10.13.15.$$

Moreover, for any $1 \le i \le n+1$, the alternating sum of all the $i$-terms of the right-hand members in 10.13.1 yields

$$a_i = \sum_{\ell=0}^{n+1}(-1)^{\ell+i} m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n} \qquad 10.13.16;$$

more precisely, it follows from [9, Lemma A4.2] that the terms $i-1$ and $i$ of this sum cancel each other and therefore, setting

$$\begin{aligned} a'_i &= \sum_{\ell=0}^{i-2}(-1)^{\ell+i} m_{\mathfrak{h}^n_\ell(\mu',x')\circ\delta^n_i} \\ a''_i &= \sum_{\ell=i+1}^{n+1}(-1)^{\ell+i} m_{\mathfrak{h}^n_\ell(\mu',x')\circ\delta^n_i} \end{aligned} \qquad 10.13.17$$

we get $a_i = a'_i + a''_i$. Then, always from [9, Lemma A4.2], for any $1 \le i \le n-1$ we obtain

$$\begin{aligned} a'_{i+1} &= -\sum_{\ell=0}^{i-1}(-1)^{\ell+i} m_{\mathfrak{h}^{n-1}_\ell(\mu' * \delta^{n-1}_i, x')} \\ a''_i &= -\sum_{\ell=i}^{n}(-1)^{\ell+i} m_{\mathfrak{h}^{n-1}_\ell(\mu' * \delta^{n-1}_i, x')} \end{aligned} \qquad 10.13.18$$

and therefore we finally have

$$a'_{i+1} + a''_i = -(-1)^i \sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}^{n-1}_\ell(\mu' * \delta^{n-1}_i, x')} \qquad 10.13.19.$$

As above, according to the definition of $n_{\mathfrak{q}\circ\delta^{n-1}_i}$, the $(\iota_{\mathfrak{q}'}, \mathfrak{q}', x')$-component of $a'_{i+1} + a''_i$ yields

$$\begin{aligned} &(a'_{i+1} + a''_i)_{(\iota_{\mathfrak{q}'}, \mathfrak{q}', x')} \\ &\quad = -(-1)^i \Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}^{n-1}_\ell(\mu' * \delta^{n-1}_i, x')}\Big)_{(\iota_{\mathfrak{q}'}, \mathfrak{q}', x')} \\ &\quad = -(-1)^i \Big(\sum_{\ell=0}^{n}(-1)^\ell m_{\mathfrak{h}^{n-1}_\ell(\mu' * \delta^{n-1}_i, x')}\Big)_{(\iota_{\mathfrak{q}'\circ\delta^{n-1}_i}, \mathfrak{q}'\circ\delta^{n-1}_i, x')} \\ &\quad = -(-1)^i (n_{\mathfrak{q}\circ\delta^{n-1}_i})_{(\mu' * \delta^{n-1}_i, \mathfrak{q}'\circ\delta^{n-1}_i, x')} \\ &\quad = -(-1)^i (n_{\mathfrak{q}\circ\delta^{n-1}_i})_{(\mu', \mathfrak{q}', x')} \end{aligned} \qquad 10.13.20$$

since the *equivalence classes* of $(\iota_{\mathfrak{q}'}, \mathfrak{q}', x')$ and $(\iota_{\mathfrak{q}'\circ\delta^{n-1}_i}, \mathfrak{q}'\circ\delta^{n-1}_i, x')$ are determined by the triple (cf. 10.10)

$$\big((\iota_{\mathfrak{q}'})_0, \mathfrak{q}'(0), x'_0\big) = \big((\iota_{\mathfrak{q}'\circ\delta^{n-1}_i})_0, (\mathfrak{q}'\circ\delta^{n-1}_i)(0), x'_0\big) \qquad 10.13.21,$$

and the same happens with $(\mu' * \delta^{n-1}_i, \mathfrak{q}'\circ\delta^{n-1}_i, x')$ and $(\mu', \mathfrak{q}', x')$, namely

$$\big((\mu' * \delta^{n-1}_i)_0, (\mathfrak{q}'\circ\delta^{n-1}_i)(0), x'_0\big) = (\mu'_0, \mathfrak{q}'(0), x'_0) \qquad 10.13.22.$$

Finally, we have $a'_1 = 0$ , $a''_n = -m_{\mathfrak{h}_n^{n-1}(\mu'*\delta_n^{n-1},x'_{n-1})}$ (cf. 10.9) and

$$a_{n+1} = -(-1)^n \sum_{\ell=0}^{n-1} (-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\mu'*\delta_n^{n-1},x'_{n-1})} \qquad 10.13.23.$$

Once again, according to the definition of $n_{\mathfrak{q}\circ\delta_n^{n-1}}$ , the $(\iota_{\mathfrak{q}'},\mathfrak{q}',x')$-component of $a_{n+1} + a''_n$ yields

$$\begin{aligned}
&(a_{n+1} + a''_n)_{(\iota_{\mathfrak{q}'},\mathfrak{q}',x')} \\
&= -(-1)^i \Big( \sum_{\ell=0}^{n} (-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\mu'*\delta_n^{n-1},x'_{n-1})} \Big)_{(\iota_{\mathfrak{q}'},\mathfrak{q}',x')} \\
&= -(-1)^i \Big( \sum_{\ell=0}^{n} (-1)^\ell m_{\mathfrak{h}_\ell^{n-1}(\mu'*\delta_n^{n-1},x'_{n-1})} \Big)_{(\iota_{\mathfrak{q}'\circ\delta_n^{n-1}},\mathfrak{q}'\circ\delta_n^{n-1},x'_{n-1})} \qquad 10.13.24 \\
&= -(-1)^i (n_{\mathfrak{q}\circ\delta_n^{n-1}})_{(\mu'*\delta_n^{n-1},\mathfrak{q}'\circ\delta_n^{n-1},x'_{n-1})} \\
&= -(-1)^i (n_{\mathfrak{q}\circ\delta_n^{n-1}})_{(\mu',\mathfrak{q}',x')}
\end{aligned}$$

since the *equivalence classes* of $(\iota_{\mathfrak{q}'},\mathfrak{q}',x')$ and $(\iota_{\mathfrak{q}'\circ\delta_n^{n-1}},\mathfrak{q}'\circ\delta_n^{n-1},x'_{n-1})$ are determined by the same triple, and the same happens with the *equivalence classes* of $(\mu'*\delta_n^{n-1},\mathfrak{q}'\circ\delta_n^{n-1},x'_{n-1})$ and $(\mu',\mathfrak{q}',x')$ .

In conclusion, from equalities 10.13.15, 10.13.20 and 10.13.24 we obtain

$$\begin{aligned}
&(m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} = (m_{\mathfrak{q}})_{(\mu',\mathfrak{q}',x')} - \sum_{i=0}^{n+1} (a_0)_{(\iota_{\mathfrak{q}'},\mathfrak{q}',x')} \\
&= \Big( \big(\mathfrak{l}^{\mathfrak{X}}(\mathfrak{q}'(0\bullet 1))\big)(n_{\mathfrak{q}\circ\delta_0^n}) \Big)_{(\mu',\mathfrak{q}',x')} + \sum_{i=1}^{n} (-1)^i (n_{\mathfrak{q}\circ\delta_i^{n-1}})_{(\mu',\mathfrak{q}',x')} \qquad 10.13.25. \\
&= \big( d_{n-1}^{\mathfrak{X}}(\mathrm{n})_{\mathfrak{q}} \big)_{(\mu',\mathfrak{q}',x')}
\end{aligned}$$

We are done.

**Theorem 10.14.** *With the notation above, assume that the functor* $\tilde{\mathfrak{m}}^{\mathfrak{X}}$ *sends the* $\tilde{\mathcal{F}}^{\mathrm{sc}}$*-morphisms to injective* $\mathcal{O}$*-module homomorphisms. Then, for any* $n \geq 1$ *we have*

$$\mathbb{H}^n\big(\tilde{\mathcal{F}}^{\mathfrak{X}}, (\hat{\tilde{\mathfrak{m}}}^{\mathfrak{X}} \circ \mathfrak{int}_P^{\tilde{\mathcal{F}}^{\mathfrak{X}}})/\tilde{\mathfrak{m}}^{\mathfrak{X}}\big) = \{0\} \qquad 10.14.1.$$

**Proof:** Since $\tilde{\mathcal{F}}(P)$ is a $p'$-group (cf. 2.2.2), if $\mathfrak{X} = \{P\}$ then we clearly have

$$\mathbb{H}^n\big(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}/\tilde{\mathfrak{m}}^{\mathfrak{X}}\big) = \mathbb{H}^n\big(\tilde{\mathcal{F}}(P), (\tilde{\mathfrak{l}}^{\mathfrak{X}}/\tilde{\mathfrak{m}}^{\mathfrak{X}})(P)\big) = \{0\} \qquad 10.14.2.$$

Assuming that $\mathfrak{X} \neq \{P\}$ , we argue by induction on $|\mathfrak{X}|$ and, setting

$$\mathfrak{X} = \mathfrak{Y} \sqcup \{\theta(U) \mid \theta \in \mathcal{F}(P,U)\} \qquad 10.14.3,$$

for a minimal element $U \in \mathfrak{X}$, we may assume that for any $n \leq 1$ we have

$$\mathbb{H}^n(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}/\tilde{\mathfrak{m}}^{\mathfrak{Y}}) = \{0\} \tag{10.14.4}$$

where $\tilde{\mathfrak{l}}^{\mathfrak{Y}}$ and $\tilde{\mathfrak{m}}^{\mathfrak{Y}}$ denote the respective restrictions of $\tilde{\mathfrak{l}}^{\mathfrak{X}}$ and $\tilde{\mathfrak{m}}^{\mathfrak{X}}$ to $\tilde{\mathcal{F}}^{\mathfrak{Y}}$.

That is to say, considering the commutative diagram (cf. 10.8)

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{C}^0(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}/\tilde{\mathfrak{m}}^{\mathfrak{Y}}) & \xrightarrow{\tilde{d}_0^{\mathfrak{Y}}} & \mathbb{C}^1(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}/\tilde{\mathfrak{m}}^{\mathfrak{Y}}) & \xrightarrow{\tilde{d}_1^{\mathfrak{Y}}} & \mathbb{C}^2(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}/\tilde{\mathfrak{m}}^{\mathfrak{Y}}) & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{C}^0(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}/\tilde{\mathfrak{m}}^{\mathfrak{X}}) & \xrightarrow{\tilde{d}_0^{\mathfrak{X}}} & \mathbb{C}^1(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}/\tilde{\mathfrak{m}}^{\mathfrak{X}}) & \xrightarrow{\tilde{d}_1^{\mathfrak{X}}} & \mathbb{C}^2(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}/\tilde{\mathfrak{m}}^{\mathfrak{X}}) & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\bar{\mathbb{K}}^0 & \longrightarrow & \bar{\mathbb{K}}^1 & \longrightarrow & \bar{\mathbb{K}}^2 & & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & 0 & &
\end{array} \tag{10.14.5}$$

where the vertical sequences are exact, the induction hypothesis guaranties that the top sequence is also exact and therefore, in order to prove that the middle sequence is exact, it suffices to prove that the bottom sequence is so. In particular, we may assume that $\tilde{\mathfrak{m}}^{\mathfrak{X}}(U) \neq \{0\}$.

But, considering the new commutative diagram (cf. 10.8)

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\tilde{\mathfrak{m}}^{\mathfrak{X}}(P) & \longrightarrow & \mathbb{C}^0(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}) & \xrightarrow{d_0^{\mathfrak{Y}}} & \mathbb{C}^1(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}) & \xrightarrow{d_1^{\mathfrak{Y}}} & \mathbb{C}^2(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}) & \dots \\
\| & & \uparrow & & \uparrow & & \uparrow & & \\
\tilde{\mathfrak{m}}^{\mathfrak{X}}(P) & \longrightarrow & \mathbb{C}^0(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}) & \xrightarrow{d_0^{\mathfrak{X}}} & \mathbb{C}^1(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}) & \xrightarrow{d_1^{\mathfrak{X}}} & \mathbb{C}^2(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}) & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathbb{K}^0 & \longrightarrow & \mathbb{K}^1 & \longrightarrow & \mathbb{K}^2 & & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{10.14.6.}$$

where again the vertical sequences are exact, Theorem 10.12 implies that the top and the middle sequences are exact, and then it is easily checked that the bottom sequence is exact too.

Moreover, it is quite clear that the surjective $\mathcal{O}$-module homomorphisms

$$\mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}) \to \mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{Y}}, \tilde{\mathfrak{l}}^{\mathfrak{Y}}/\tilde{\mathfrak{m}}^{\mathfrak{Y}}) \quad \text{and} \quad \mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}) \to \mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}/\tilde{\mathfrak{m}}^{\mathfrak{X}}) \tag{10.14.7}$$

for any $n \in \mathbb{N}$, determine a commutative diagram

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \bar{\mathbb{K}}^0 & \xrightarrow{\bar{d}_0^{\mathfrak{X}}} & \bar{\mathbb{K}}^1 & \xrightarrow{\bar{d}_1^{\mathfrak{X}}} & \bar{\mathbb{K}}^2 & \xrightarrow{\bar{d}_2^{\mathfrak{X}}} & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathbb{K}^0 & \xrightarrow{d_0^{\mathfrak{X}}} & \mathbb{K}^1 & \xrightarrow{d_1^{\mathfrak{X}}} & \mathbb{K}^2 & \xrightarrow{d_2^{\mathfrak{X}}} & \dots \end{array} \qquad 10.14.8$$

where the vertical sequences are exact. Consequently, since the bottom sequence is exact, it suffices to prove that any element in the intersection $\bar{\mathbb{K}}^n \cap \mathrm{Ker}(\bar{d}_n^{\mathfrak{X}})$ can be lifted to $\mathbb{K}^n \cap \mathrm{Ker}(d_n^{\mathfrak{X}})$ for any $n \in \mathbb{N}$.

Recall that, for any $n \in \mathbb{N}$, we have $\mathcal{O}$-module isomorphisms (cf. 10.8)

$$\begin{gathered} \mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}) \cong \mathbb{C}^n_{\mathcal{I}(\mathcal{P}^{\mathfrak{X}})}(\mathcal{P}^{\mathfrak{X}}, \mathfrak{l}^{\mathfrak{X}}) \\ \mathbb{C}^n(\tilde{\mathcal{F}}^{\mathfrak{X}}, \tilde{\mathfrak{l}}^{\mathfrak{X}}/\tilde{\mathfrak{m}}^{\mathfrak{X}}) \cong \mathbb{C}^n_{\mathcal{I}(\mathcal{P}^{\mathfrak{X}})}(\mathcal{P}^{\mathfrak{X}}, \mathfrak{l}^{\mathfrak{X}}/\mathfrak{m}^{\mathfrak{X}}) \end{gathered} \qquad 10.14.9;$$

hence, an element $\mathrm{m} = (m_{\mathfrak{q}})_{\mathfrak{q} \in \mathfrak{Fct}(\Delta_n, \mathcal{P}^{\mathfrak{X}})}$ in $\mathbb{C}^n(\mathcal{P}^{\mathfrak{X}}, \mathfrak{l}^{\mathfrak{X}})$ belongs to $\mathbb{K}^n$ if and only it fulfills

$$m_{\mathfrak{q}'} = \big(\mathfrak{l}^{\mathfrak{X}}(\alpha_0)\big)(m_{\mathfrak{q}}) \qquad 10.14.10$$

for any *natural* $\mathcal{I}(\mathcal{P}^{\mathfrak{X}})$-isomorphism $\alpha\colon \mathfrak{q} \cong \mathfrak{q}'$ between two $\mathcal{P}^{\mathfrak{X}}$-*chains* $\mathfrak{q}$ and $\mathfrak{q}'$ (cf. 10.8) and, for any $\mathcal{P}^{\mathfrak{Y}}$-*chain* $\mathfrak{r}$, $m_{\mathfrak{r}}$ belongs to the kernel of the canonical map

$$\hat{\mathfrak{m}}^{\mathfrak{X}}\big(\mathfrak{r}(0) \cap^{\mathcal{P}^{\mathfrak{X}}} P\big)^{\mathfrak{r}(0)\times P} \longrightarrow \hat{\mathfrak{m}}^{\mathfrak{Y}}\big(\mathfrak{r}(0) \cap^{\mathcal{P}^{\mathfrak{Y}}} P\big)^{\mathfrak{r}(0)\times P} \qquad 10.14.11,$$

which, denoting by $\mathcal{U}_{\mathfrak{r}}$ a set of representatives for the set of *equivalent classes* of triples $(t, U, s)$ such that $t \in \mathcal{P}^{\mathfrak{X}}(\mathfrak{r}(0), U)_s$ and $s \in \mathcal{P}^{\mathfrak{X}}(P, U)_t$, is equal to

$$\mathbb{K}_{\mathfrak{r}(0)} = \Big(\prod_{(t,U,s)\in\mathcal{U}_{\mathfrak{r}}} \mathfrak{m}^{\mathfrak{X}}(U)\Big)^{\mathfrak{r}(0)\times P} \qquad 10.14.12;$$

note that, identifying $\mathfrak{m}^{\mathfrak{X}}\big(\mathfrak{r}(0)\big)$ with its "diagonal" image in $\hat{\mathfrak{m}}^{\mathfrak{X}}\big(\mathfrak{r}(0) \cap^{\mathcal{P}^{\mathfrak{X}}} P\big)$, we have

$$\mathbb{K}_{\mathfrak{r}(0)} \cap \mathfrak{m}^{\mathfrak{X}}\big(\mathfrak{r}(0)\big) = \{0\} \qquad 10.14.13;$$

moreover, we may assume that $\mathcal{U}_U = \big\{\big(\tau_U^{\mathfrak{X}}(1), U, s\big)\big\}_{s \in \mathcal{P}^{\mathfrak{X}}(P,U)}$ and, in particular, we get $K_U \neq \{0\}$.

Let $\bar{\mathrm{m}} = (\bar{m}_{\mathfrak{q}})_{\mathfrak{q} \in \mathfrak{Fct}(\Delta_n, \mathcal{P}^{\mathfrak{X}})}$ be an element of $\bar{\mathbb{K}}^n$ such that $\bar{d}_n^{\mathfrak{X}}(\bar{\mathrm{m}}) = 0$. First of all, for any $\mathfrak{q} \in \mathfrak{Fct}(\Delta_n, \mathcal{P}^{\mathfrak{X}})$ assume that we have $\bar{m}_{\mathfrak{q}} \neq 0$ only if $\mathfrak{q}(0) \in \mathfrak{Y}$; in this case, in order to lift $\bar{\mathrm{m}}$ to $\mathbb{K}^n \cap \mathrm{Ker}(d_n^{\mathfrak{X}})$, it suffices to consider $\mathrm{m} = (m_{\mathfrak q})_{\mathfrak q \in \mathfrak{Fct}(\Delta_n,\mathcal P^{\mathfrak X})}$ where $m_{\mathfrak q} \neq 0$ only if $\mathfrak q(0) \in \mathfrak Y$ and then $m_{\mathfrak q}$ is the *unique* element in $\mathbb K_{\mathfrak q(0)}$ lifting $\bar m_{\mathfrak q}$ (cf. 10.14.13); indeed, for any $\mathcal P^{\mathfrak X}$-*chain* $\mathfrak t\colon \Delta_{n+1} \to \mathcal P^{\mathfrak X}$ we obviously have

$$d_n^{\mathfrak X}(\mathrm m)_{\mathfrak t} = \big(\mathfrak l^{\mathfrak X}(\mathfrak t(0\bullet 1))\big)(m_{\mathfrak t\circ\delta_0^n}) + \sum_{i=0}^{n+1}(-1)^i m_{\mathfrak t\circ\delta_i^n} \qquad 10.14.14;$$

if $\mathfrak t(0)$ belongs to $\mathfrak Y$ then $d_n^{\mathfrak X}(\mathrm m)_{\mathfrak t}$ is the *unique* element lifting $\bar d_n^{\mathfrak X}(\bar{\mathrm m})_{\mathfrak t} = 0$ (cf. 10.14.13), so that we get $d_n^{\mathfrak X}(\mathrm m)_{\mathfrak t} = 0$; otherwise, we have $m_{\mathfrak t\circ\delta_i^n} = 0$ for any $1 \le i \le n+1$ and therefore we get

$$0 = \big((\mathfrak l^{\mathfrak X}/\mathfrak m^{\mathfrak X})(\mathfrak t(0\bullet 1))\big)(\bar m_{\mathfrak t\circ\delta_0^n}) \qquad 10.14.15;$$

if $\mathfrak t(1)$ does not belong to $\mathfrak Y$ then we have $m_{\mathfrak t\circ\delta_0^n} = 0$ since $(\mathfrak t\circ\delta_0^n)(0) \notin \mathfrak Y$; but, if $\mathfrak t(1)$ belongs to $\mathfrak Y$ and we have $m_{\mathfrak t\circ\delta_0^n} \neq 0$ then, since $\tilde{\mathfrak m}^{\mathfrak X}$ sends the $\tilde{\mathcal F}^{\mathrm{sc}}$-morphisms to *injective* $\mathcal O$-module homomorphisms and we have

$$\hat{\mathfrak m}^{\mathfrak X}\big(\mathfrak t(1)\cap^{\mathcal P^{\mathfrak X}} P\big)^{\mathfrak t(1)\times P} = \hat{\mathfrak m}^{\mathfrak Y}\big(\mathfrak t(1)\cap^{\mathcal P^{\mathfrak Y}} P\big)^{\mathfrak t(1)\times P} \times \mathbb K_{\mathfrak t(1)} \qquad 10.14.16,$$

the element $\big(\mathfrak l^{\mathfrak X}(\mathfrak t(0\bullet 1))\big)(m_{\mathfrak t\circ\delta_0^n})$ does not belong to the "diagonal" image of $\mathfrak m^{\mathfrak X}\big(\mathfrak t(0)\big)$ in $\hat{\mathfrak m}^{\mathfrak X}\big(\mathfrak t(0)\cap^{\mathcal P^{\mathfrak X}} P\big)$; thus, we also have $m_{\mathfrak t\circ\delta_0^n} = 0$ and therefore we still get $d_n^{\mathfrak X}(\mathrm m)_{\mathfrak t} = 0$.

Now, we may assume that there is $\mathfrak q \in \mathfrak{Fct}(\Delta_n,\mathcal P^{\mathfrak X})$ such that $\bar m_{\mathfrak q} \neq 0$ and that $\mathfrak q(0)$ does not belong to $\mathfrak Y$; then, we argue by induction on the cardinal of the set of *natural* $\mathcal I(\mathcal P^{\mathfrak X})$-*isomorphism classes* of this set. Let us choose a *minimal* element $\mathfrak q_\circ$ of this set; that is to say, we assume that, for any $\mathfrak q \in \mathfrak{Fct}(\Delta_n,\mathcal P^{\mathfrak X})$ admitting a natural map $\mathfrak q \to \mathfrak q_\circ$ which is *not* an isomorphism, we have $\bar m_{\mathfrak q} = 0$. At this point, for any $\mathfrak q \in \mathfrak{Fct}(\Delta_n,\mathcal P^{\mathfrak X})$ such that $\mathfrak q(0)$ does not belong to $\mathfrak Y$, we define a *section*

$$\sigma_{\mathfrak q} : \hat{\mathfrak m}^{\mathfrak X}\big(\mathfrak q(0)\cap^{\mathcal P^{\mathfrak X}} P)^{\mathfrak q(0)\times P} \longrightarrow \mathfrak m^{\mathfrak X}\big(\mathfrak q(0)\big) \qquad 10.14.17$$

of the *diagonal map* (cf. 10.4.3)

$$\mathfrak m^{\mathfrak X}\big(\mathfrak q(0)\big) \longrightarrow \hat{\mathfrak m}^{\mathfrak X}\big(\mathfrak q(0)\cap^{\mathcal P^{\mathfrak X}} P) = \prod_{t\in\mathcal P^{\mathfrak X}(P,\mathfrak q(0))} \mathfrak m^{\mathfrak X}\big(\mathfrak q(0)\big) \qquad 10.14.18$$

sending $z = (z_t)_{t\in\mathcal P^{\mathfrak X}(P,\mathfrak q(0))}$ to the element (cf. Proposition 4.5))

$$\sigma_{\mathfrak q}(z) = |\tilde{\mathcal F}(P,\mathfrak q(1))|^{-1}\cdot\sum_t z_t \qquad 10.14.19$$

where $t$ runs over a set of representatives for the set of $P$-orbits in the set $\mathcal{P}^{\mathfrak{x}}\big(P,\mathfrak{q}(1)\big)\circ\mathfrak{q}(0\bullet 1)$; then, we consider $\mathrm{m} = (m_{\mathfrak{q}})_{\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\mathcal{P}^{\mathfrak{x}})}$ where $m_{\mathfrak{q}}$ lifts $\bar{m}_{\mathfrak{q}}$ and moreover it belongs to $\mathrm{Ker}(\sigma_{\mathfrak{q}})$ whenever $\mathfrak{q}(0)$ does not belong to $\mathfrak{Y}$; note that, since the intersection of $\mathrm{Ker}(\sigma_{\mathfrak{q}})$ with the image of the *diagonal map* is trivial, m is *uniquely* determined.

At this point, for any $\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\mathcal{P}^{\mathfrak{x}})$ admitting a natural map $\mathfrak{q}\to\mathfrak{q}_\circ$, any triple $(\mu',\mathfrak{q}',x')$ in $X_{\mathfrak{q}}$ and any $\ell\in\Delta_{n+1}$, consider the *component* of $\bar{d}_n^{\mathfrak{x}}(\bar{\mathrm{m}})$ on the $\mathcal{P}^{\mathfrak{x}}$*-chain* $\mathfrak{h}_\ell^n(\mu',x')\colon\Delta_{n+1}\to\mathcal{P}^{\mathfrak{x}}$; since $\bar{d}_n^{\mathfrak{x}}(\bar{\mathrm{m}}) = 0$, we get the following equalities

$$\begin{aligned}
0 &= \big((\mathfrak{l}^{\mathfrak{x}}/\mathfrak{m}^{\mathfrak{x}})(\mu'_0)\big)(\bar{m}_{\mathfrak{q}}) + \sum_{i=1}^{n+1}(-1)^i\bar{m}_{\mathfrak{h}_0^n(\mu',x')\circ\delta_i^n}\\
0 &= \big((\mathfrak{l}^{\mathfrak{x}}/\mathfrak{m}^{\mathfrak{x}})(\mathfrak{q}(0\bullet 1))\big)(\bar{m}_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n}) + \sum_{i=1}^{n+1}(-1)^i\bar{m}_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n}
\end{aligned} \qquad 10.14.20;$$

for any $1\le\ell\le n$.

But, since for any $\ell\in\Delta_{n+1}$ and any $1\le i\le n+1$ we have

$$\big(\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n\big)(0) = \mathfrak{q}(0) \qquad 10.14.21,$$

and $\mathfrak{q}(0)$ does not belong to $\mathfrak{Y}$, it follows from our choice that $m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n}$ belongs to $\mathrm{Ker}(\sigma_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n})$; if $\ell\ge 1$ or $i=1$, we have a group homomorphism

$$\mathfrak{q}'(1)\longrightarrow\big(\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n\big)(1) \qquad 10.14.22$$

and therefore $\mathrm{Ker}(\sigma_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n})$ is contained in $\mathrm{Ker}(\sigma_{\mathfrak{q}'})$; thus, the only case where $\mathrm{Ker}(\sigma_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n})$ is not contained in $\mathrm{Ker}(\sigma_{\mathfrak{q}'})$ we have $\ell=0$, $1<i$ and $\mathfrak{q}'(1)\not\cong\mathfrak{q}'(0)$, which forces $\mathfrak{q}(1)\not\cong\mathfrak{q}(0)$, and in this situation we have an evident natural map

$$\mathfrak{h}_0^n(\mu',x')\circ\delta_i^n\longrightarrow\mathfrak{q} \qquad 10.14.23$$

which is *not* an isomorphism, so that $m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n} = 0$; hence, in all the cases, $m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n}$ belongs to $\mathrm{Ker}(\sigma_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n})$. Moreover, if $\ell\ne 0$ then we have

$$\big(\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n\big)(0) = \mathfrak{q}'(1) \qquad 10.14.24$$

and this group either belongs to $\mathfrak{Y}$ which, according to our definition of $\sigma_{\mathfrak{q}'}$, implies that $\big(\mathfrak{l}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n})$ belongs to $\mathrm{Ker}(\sigma_{\mathfrak{q}'})$, or $\mathfrak{q}'(0\bullet 1)$ is a $\mathcal{P}^{\mathfrak{x}}$-isomorphism which implies that $\big(\mathfrak{l}^{\mathfrak{x}}(\mathfrak{q}'(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n})$ still belongs to $\mathrm{Ker}(\sigma_{\mathfrak{q}'})$; finally, since $\mu'_0$ has to be a $\mathcal{P}^{\mathfrak{x}}$-isomorphism, the element $\big(\mathfrak{l}^{\mathfrak{x}}(\mu'_0)\big)(m_{\mathfrak{q}})$ also belongs to $\mathrm{Ker}(\sigma_{\mathfrak{q}'})$.

Consequently, since $\mathrm{Ker}(\sigma_{\mathfrak{q}'})$ is a complement for the image of $\mathfrak{m}^x\big(\mathfrak{q}'(0)\big)$ in $\hat{\mathfrak{m}}^x\big(\mathfrak{q}'(0)\cap^{\mathcal{P}^x} P\big)^{\mathfrak{q}'(0)\times P}$, we obtain the equalities

$$0 = \big(\mathfrak{l}^x(\mu'_0)\big)(m_{\mathfrak{q}}) + \sum_{i=1}^{n+1}(-1)^i m_{\mathfrak{h}_0^n(\mu',x')\circ\delta_i^n}$$
$$0 = \big(\mathfrak{l}^x(\mathfrak{q}(0\bullet 1))\big)(m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_0^n}) + \sum_{i=1}^{n+1}(-1)^i m_{\mathfrak{h}_\ell^n(\mu',x')\circ\delta_i^n} \qquad 10.14.25$$

for any $1\le \ell\le n$. Now, it is quite clear from Lemma 10.13 that, for a suitable $\mathcal{I}(\mathcal{P}^x)$-stable element $\mathrm{n}$ of $\mathbb{C}^{n-1}(\mathcal{P}^x,\mathfrak{l}^x)$ and for any $\mathfrak{q}\in\mathfrak{Fct}(\Delta_n,\mathcal{P}^x)$ admitting a natural map $\mu:\mathfrak{q}\to\mathfrak{q}_\circ$, we have $d_{n-1}^x(\mathrm{n})_{\mathfrak{q}} = m_{\mathfrak{q}}$, so that $\bar{d}_{n-1}^x(\bar{\mathrm{n}})_{\mathfrak{q}} = 0$ if $\mu$ is not an isomorphism; hence, it suffices to apply the induction hypothesis to $\bar{\mathrm{m}} - \bar{d}_{n-1}^x(\bar{\mathrm{n}})$. We are done.

**Corollary 10.15.** *With the notation above, for any contravariant functor $\tilde{\mathfrak{m}}^x:\tilde{\mathcal{F}}^x\to\mathcal{O}\text{-}\mathfrak{mod}$ sending the $\tilde{\mathcal{F}}^{\mathrm{sc}}$-morphisms to injective $\mathcal{O}$-module homomorphisms, we have*

$$\mathbb{H}^1(\tilde{\mathcal{F}}^x,\tilde{\mathfrak{m}}^x)\cong \lim_{\longleftarrow}\,(\tilde{\mathfrak{l}}^x/\tilde{\mathfrak{m}}^x)\Big/\overline{\tilde{\mathfrak{m}}^x(P)} \quad\text{and}\quad \mathbb{H}^n(\tilde{\mathcal{F}}^x,\tilde{\mathfrak{m}}^x) = \{0\} \qquad 10.15.1$$

*for any $n\ge 2$, where $\overline{\tilde{\mathfrak{m}}^x(P)}$ denotes the image of $\tilde{\mathfrak{m}}^x(P)$.*

**Proof:** It suffices to apply Theorems 10.12 and 10.14 to the *long exact sequence* associated with the short exact sequence of *contravariant* functors 10.6.2.

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