THE RATIONALITY OF THE MODULI SPACES OF TRIGONAL CURVES

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Abstract. The moduli spaces of trigonal curves are proven to be rational when the genus is divisible by 4.

1. Introduction

A smooth projective curve is called *trigonal* if it carries a free g^1_3. When the curve has genus ≥ 5, such a pencil is unique if it exists. The object of our study is the moduli space $\mathcal{T}_g$ of trigonal curves of genus $g \geq 5$. This space has been proven to be rational when $g \equiv 2$ (4) by Shepherd-Barron [9], and when g is odd in [7]. In the present article we prove that $\mathcal{T}_g$ is rational in the left case $g \equiv 0$ (4), completing the following.

Theorem . The moduli space $\mathcal{T}_g$ of trigonal curves of genus g is rational for every $g \geq 5$.

This can be seen as an analogue of the rationality of the moduli spaces of hyperelliptic curves due to Katsylo and Bogomolov [5], [2].

Note that $\mathcal{T}_g$ is regarded as a sublocus of the moduli space $\mathcal{M}_g$ of genus g curves. When g is large enough, it seems that $\mathcal{T}_g$ has maximal dimension among the known rational subvarieties of $\mathcal{M}_g$. It would be interesting whether the tetragonal (and pentagonal) locus is rational as well. It is unirational by Arbarello-Cornalba [1], but at present known to be rational only in genus 7 ([3]). In another direction, Castorena and Ciliberto [4] shows that for $g \geq 23$, $\mathcal{T}_g$ has larger dimension than any other locus in $\mathcal{M}_g$ obtained from a linear system on a surface.

We approach our problem from invariant theory for $\mathrm{SL}_2 \times \mathrm{SL}_2$. Let $V_{a,b} = H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b))$ be the space of bi-forms of bidegree (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$, which is an irreducible representation of $\mathrm{SL}_2 \times \mathrm{SL}_2$. It is classically known that a general trigonal curve C of genus $g = 4N$ is canonically embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth curve of bidegree $(3, 2N+1)$. This is based on the fact that the canonical model of C lies on a unique rational normal scroll which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. As a consequence, we have a natural birational equivalence

$$\mathcal{T}_{4N} \sim \mathbb{P}V_{3,2N+1}/\mathrm{SL}_2 \times \mathrm{SL}_2. \qquad (1.1)$$

Hence the problem is restated as follows.

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Theorem 1.1. *The quotient* $\mathbb{P}V_{3,b}/\mathrm{SL}_2 \times \mathrm{SL}_2$ *is rational for every odd* $b \geq 5$*.*

To prove this, we adopt the traditional and computational method of *double bundle* ([2], [10]) as follows. By examining the Clebsch-Gordan formula for $\mathrm{SL}_2 \times \mathrm{SL}_2$, we take a suitable $\mathrm{SL}_2 \times \mathrm{SL}_2$-bilinear mapping (bi-transvectant)

$$T : V_{3,b} \times V_{a',b'} \to V_{a'',b''} \tag{1.2}$$

such that $\dim V_{a',b'} > \dim V_{a'',b''}$. Putting $c = \dim V_{a',b'} - \dim V_{a'',b''}$, this induces the rational map to the Grassmannian

$$V_{3,b} \dashrightarrow G(c, V_{a',b'}), \qquad v \mapsto \mathrm{Ker}(T(v, \cdot)). \tag{1.3}$$

We shall find a bi-transvectant for which (1.3) is well-defined and dominant. In that case, (1.3) makes $V_{3,b}$ birationally an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle over $G(c, V_{a',b'})$. Utilizing this bundle structure and taking care of -1 scalar action, we reduce the rationality of $\mathbb{P}V_{3,b}/\mathrm{SL}_2 \times \mathrm{SL}_2$ to a stable rationality of $G(c, V_{a',b'})/\mathrm{SL}_2 \times \mathrm{SL}_2$, which in turn can be shown in a more or less standard way.

The point for this proof is to choose the bi-transvectant T carefully so that (i) a', b', c are odd (to care -1 scalar action) and that (ii) c is small (for $V_{3,b}$ to have larger dimension than $G(c, V_{a',b'})$). For that, we will provide T according to the congruence of b modulo 5, based on some easy calculation in elementary number theory. Then the bulk of proof is devoted to the check of non-degeneracy of (1.3), which is facilitated by keeping c small but is still rather laborious.

The rest of the article is as follows. In §2.1 we recall bi-transvectants. We explain the method of double bundle in §2.2. In §3 we prepare some stable rationality results in advance, to which the rationality of $\mathbb{P}V_{3,b}/\mathrm{SL}_2 \times \mathrm{SL}_2$ will be eventually reduced. Then we prove Theorem 1.1 in §4.

We work over the complex numbers. The Grassmannian $G(a, V)$ parametrizes a-dimensional linear subspaces of the vector space V. We shall use the notation $([x, y], [X, Y])$ for the bi-homogeneous coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus elements of $V_{a,b}$ will be expressed as

$$\sum_i F_i(x, y) G_i(X, Y), \tag{1.4}$$

where F_i, G_i are binary forms of degree a, b respectively.

2. Bi-transvectant

2.1. **Bi-transvectant.** Let V_d denote the SL_2-representation $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. Let $e \leq d$. According to the Clebsch-Gordan decomposition

$$V_d \otimes V_e = \bigoplus_{r=0}^{e} V_{d+e-2r}, \tag{2.1}$$

there exists a unique (up to constant) SL_2-bilinear mapping

$$T^{(r)} : V_d \times V_e \to V_{d+e-2r}, \tag{2.2}$$

which is called the *r-th transvectant*. For two binary forms $F(X,Y) \in V_d$ and $G(X,Y) \in V_e$, we have the well-known explicit formula (cf. [8])

$$T^{(r)}(F,G) = \frac{(d-r)!}{d!}\frac{(e-r)!}{e!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\frac{\partial^r F}{\partial X^{r-i}\partial Y^i}\frac{\partial^r G}{\partial X^i \partial Y^{r-i}}. \tag{2.3}$$

We will need this formula when $r = e$ and $r = e-1$.

The e-th transvectant $T^{(e)}\colon V_d \times V_e \to V_{d-e}$ is especially called the *apolar covariant*. By (2.3), $T^{(e)}(F,G)$ is calculated by applying the differential polynomial $(d!)^{-1}(d-e)!G(-\partial_Y, \partial_X)$ to $F(X,Y)$. In particular, we have

$$T^{(e)}(X^iY^{d-i}, X^{e-j}Y^j) = \begin{cases} (-1)^{e-j}\binom{d}{i}^{-1}\binom{d-e}{i-j}X^{i-j}Y^{(d-e)-(i-j)}, & j \le i,\ e-j \le d-i, \\ 0, & \text{otherwise.}\end{cases}$$

For the $(e-1)$-th transvectant $T^{(e-1)}\colon V_d \times V_e \to V_{d-e+2}$, we have

$$T^{(e-1)}(\cdot, X^{e-j}Y^j) = (-1)^{e-j}\frac{1}{e}\frac{(d-e+1)!}{d!}\left\{jY\partial_X^{j-1}\partial_Y^{e-j} - (e-j)X\partial_X^j\partial_Y^{e-j-1}\right\},$$

where $\partial_X^{-1} = \partial_Y^{-1} = 0$ by convention. Therefore

$$T^{(e-1)}(X^iY^{d-i}, X^{e-j}Y^j) = \begin{cases} AX^{i-j+1}Y^{(d-i)-(e-j)+1}, & j \le i+1,\ e-j \le d-i+1, \\ 0, & \text{otherwise,}\end{cases}$$

where

$$A = (-1)^{e-j}\binom{d}{i}^{-1}\binom{d-e+2}{i-j+1}\frac{j(d+2)-(i+1)e}{e(d-e+2)}.$$

We stress in particular that

Lemma 2.1. *Let* $0 \le j \le i+1$ *and* $0 \le e-j \le d-i+1$. *The bilinear map*

$$T^{(e-1)}: \mathbb{C}X^iY^{d-i} \times \mathbb{C}X^{e-j}Y^j \to \mathbb{C}X^{i-j+1}Y^{(d-e+2)-(i-j+1)} \tag{2.4}$$

is non-degenerate if and only if $j(d+2) \neq (i+1)e$. *This is always the case when* $d+2$ *is coprime to* e.

Now we consider $\mathrm{SL}_2 \times \mathrm{SL}_2$-representations. The space $V_{a,b} = H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b))$ is the tensor representation $V_a \boxtimes V_b$. Substituting (2.1) into

$$V_{a,b} \otimes V_{a',b'} = (V_a \otimes V_{a'}) \boxtimes (V_b \otimes V_{b'}), \tag{2.5}$$

we obtain the Clebsch-Gordan decomposition for $\mathrm{SL}_2 \times \mathrm{SL}_2$,

$$V_{a,b} \otimes V_{a',b'} = \bigoplus_{r,s} V_{a+a'-2r, b+b'-2s}, \tag{2.6}$$

where $0 \le r \le \min\{a,a'\}$ and $0 \le s \le \min\{b,b'\}$. To each irreducible summand $V_{a+a'-2r,b+b'-2s}$ is associated the (r,s)*-th bi-transvectant*

$$T^{(r,s)} : V_{a,b} \times V_{a',b'} \to V_{a+a'-2r,b+b'-2s}. \tag{2.7}$$

This $\mathrm{SL}_2 \times \mathrm{SL}_2$-bilinear mapping is calculated from the above transvectants by

$$T^{(r,s)}(F \boxtimes G,\ F' \boxtimes G') = T^{(r)}(F,F') \boxtimes T^{(s)}(G,G'), \tag{2.8}$$

where $F \in V_a$, $G \in V_b$, $F' \in V_{a'}$, and $G' \in V_{b'}$.

2.2. **The method of double bundle.** In §4, we will use the method of double bundle ([2]) and its generalization ([10]). We here give some account in the present situation. Suppose we have a bi-transvectant

$$T = T^{(r,s)} : V_{a,b} \times V_{a',b'} \to V_{a'',b''} \tag{2.9}$$

such that $c = \dim V_{a',b'} - \dim V_{a'',b''}$ is positive and that $\dim V_{a,b} > c \cdot \dim V_{a'',b''}$. Then we consider the $\mathrm{SL}_2 \times \mathrm{SL}_2$-equivariant rational map

$$\varphi : V_{a,b} \dashrightarrow G(c, V_{a',b'}), \qquad v \mapsto \mathrm{Ker}(T(v,\cdot)). \tag{2.10}$$

We assume (hope) that

(♣) *φ is well-defined and dominant.*

If this holds, then $V_{a,b}$ becomes birational to the unique component $\mathcal{E}$ of the incidence

$$\mathcal{X} = \{(v, P) \in V_{a,b} \times G(c, V_{a',b'}),\ \ T(v, P) \equiv 0\} \tag{2.11}$$

that dominates $G(c, V_{a',b'})$. Indeed, the first projection $\pi \colon \mathcal{X} \to V_{a,b}$ is isomorphic over the domain U of regularity of φ, and then the dominance of φ implies that $\pi^{-1}(U)$ is contained in $\mathcal{E}$. Since $\mathcal{E}$ is (generically) a sub vector bundle of $V_{a,b} \times G(c, V_{a',b'})$ preserved under the $\mathrm{SL}_2 \times \mathrm{SL}_2$-action, it is an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle over $G(c, V_{a',b'})$. In this situation one might try to apply the no-name lemma to $\mathcal{E} \sim V_{a,b}$, taking care of the scalar action of $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$.

The non-degeneracy requirement (♣) may be checked as follows.

Lemma 2.2 (cf. [2])**.** *The condition (♣) is satisfied if and only if there exists $(v, w_1, \cdots, w_c) \in V_{a,b} \times (V_{a',b'})^c$ such that*
(i) $w_1, \cdots, w_c \in V_{a',b'}$ are linearly independent,
(ii) $T(v, w_i) = 0$ for every w_i,
(iii) the map $T(v,\cdot) : V_{a',b'} \to V_{a'',b''}$ is surjective, and
(iv) the map $(T(\cdot, w_1), \cdots, T(\cdot, w_c)) : V_{a,b} \to V_{a'',b''}^{\oplus c}$ is surjective.

Proof. Let $P \in G(c, V_{a',b'})$ be the span of $w_1, \cdots, w_c$. The conditions (ii) and (iii) mean that v is contained in the domain U of regularity of φ with $\varphi(v) = P$, whence $U \neq \emptyset$. Then (iv) implies that the fiber of the morphism $\varphi \colon U \to G(c, V_{a',b'})$ over P has the expected dimension $\dim V_{a,b} - \dim G(c, V_{a',b'})$. Hence $\varphi(U)$ has dimension $\geq \dim G(c, V_{a',b'})$, and so φ is dominant. □

3. Some stable rationality

We set $\overline{G} = \mathrm{SL}_2 \times \mathrm{SL}_2 / (-1, -1)$. When $a, b > 0$ are odd, the element $(-1, -1)$ of $\mathrm{SL}_2 \times \mathrm{SL}_2$ acts on $V_{a,b}$ trivially so that $\overline{G}$ acts on $V_{a,b}$. This linear $\overline{G}$-action is almost free if $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts on $\mathbb{P}V_{a,b}$ almost freely, that is, general bidegree (a, b) curves on $\mathbb{P}^1 \times \mathbb{P}^1$ have no non-trivial stabilizer.

Lemma 3.1. *The group $\overline{G}$ acts on $V_{1,1}^{\oplus 3}$ almost freely with the quotient $V_{1,1}^{\oplus 3} / \mathrm{SL}_2 \times \mathrm{SL}_2$ rational.*

Proof. The first assertion follows from the almost freeness of the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-action on $(\mathbb{P}V_{1,1})^3$. For the second assertion, we first note that

$$V_{1,1}^{\oplus 3}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim (V_{1,1}^{\oplus 3}/\mathrm{GL}_2 \times \mathrm{GL}_2) \times \mathbb{C}^\times. \tag{3.1}$$

The group $\mathrm{GL}_2 \times \mathrm{GL}_2$ acts on $V_{1,1}$ almost transitively with the stabilizer of a general point isomorphic to GL_2 (identify $V_{1,1}$ with $\mathrm{Hom}(V_1, V_1)$). Hence, applying the slice method to the first projection $V_{1,1}^{\oplus 3} \to V_{1,1}$, we obtain

$$V_{1,1}^{\oplus 3}/\mathrm{GL}_2 \times \mathrm{GL}_2 \sim V_{1,1}^{\oplus 2}/\mathrm{GL}_2, \tag{3.2}$$

where GL_2 acts on $V_{1,1}^{\oplus 2}$ linearly in the right hand side. Then the quotient $V_{1,1}^{\oplus 2}/\mathrm{GL}_2$ is rational by the result of Katsylo [6]. □

A variety X is called *stably rational of level N* if $X \times \mathbb{P}^N$ is rational. In §4, the proof of Theorem 1.1 will be finally reduced to the following stable rationality results.

Corollary 3.2. *Let $n > 0$ be an odd number. Then $\mathbb{P}V_{1,n}/\mathrm{SL}_2 \times \mathrm{SL}_2$ and $\mathbb{P}V_{3,n}/\mathrm{SL}_2 \times \mathrm{SL}_2$ are stably rational of level* 13*.*

Proof. We treat the case of $V_{1,n}$. For dimensional reason we may assume $n > 3$. Then the group $\overline{G}$ acts on $V_{1,n}$ almost freely. Hence we may apply the no-name lemma to both projections $V_{1,1}^{\oplus 3} \oplus V_{1,n} \to V_{1,n}$ and $V_{1,1}^{\oplus 3} \oplus V_{1,n} \to V_{1,1}^{\oplus 3}$ to see that

$$(V_{1,n}/\mathrm{SL}_2 \times \mathrm{SL}_2) \times \mathbb{C}^{12} \sim (V_{1,1}^{\oplus 3}/\mathrm{SL}_2 \times \mathrm{SL}_2) \times \mathbb{C}^{2n+2}. \tag{3.3}$$

By Lemma 3.1, $V_{1,n}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is stably rational of level 12. Since $V_{1,n}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is birational to $\mathbb{C}^\times \times (\mathbb{P}V_{1,n}/\mathrm{SL}_2 \times \mathrm{SL}_2)$, our assertion is proved. The case of $V_{3,n}$ is similar. □

Proposition 3.3. *When $n > 1$ is odd, $G(3, V_{3,n})/\mathrm{SL}_2 \times \mathrm{SL}_2$ is stably rational of level* 2*.*

Proof. Let $\mathcal{F} \to G(3, V_{3,n})$ be the universal sub vector bundle of rank 3, on which $\mathrm{SL}_2 \times \mathrm{SL}_2$ acts equivariantly. The elements $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ act on $\mathcal{F}$ by multiplication by -1. Since $\mathcal{F}$ has odd rank, they act on the line bundle $\det\mathcal{F}$ also by -1. Hence the bundle $\mathcal{F}' = \mathcal{F} \otimes \det\mathcal{F}$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized. Note that $\mathbb{P}\mathcal{F}$ is canonically identified with $\mathbb{P}\mathcal{F}'$. Since $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts on $G(3, V_{3,n})$ almost freely, we can apply the no-name lemma to $\mathcal{F}'$ to see that

$$\mathbb{P}\mathcal{F}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{F}'/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^2 \times (G(3, V_{3,n})/\mathrm{SL}_2 \times \mathrm{SL}_2). \tag{3.4}$$

Thus it suffices to show that $\mathbb{P}\mathcal{F}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational.

Regarding $\mathbb{P}\mathcal{F}$ as an incidence in $G(3, V_{3,n}) \times \mathbb{P}V_{3,n}$, we have second projection $\mathbb{P}\mathcal{F} \to \mathbb{P}V_{3,n}$. Its fiber over $\mathbb{C}l \in \mathbb{P}V_{3,n}$ is the sub Grassmannian in $G(3, V_{3,n})$ of 3-planes containing $\mathbb{C}l$, and hence identified with $G(2, V_{3,n}/\mathbb{C}l)$. Therefore, if $\mathcal{G} \to \mathbb{P}V_{3,n}$ is the universal quotient bundle of rank $\dim V_{3,n} - 1$, then $\mathbb{P}\mathcal{F}$ is identified with the relative Grassmannian $G(2, \mathcal{G})$. The elements $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ act on $\mathcal{G}$ by multiplication by -1, and also on $\mathcal{O}_{\mathbb{P}V_{3,n}}(1)$ by -1. Thus the bundle $\mathcal{G}' = \mathcal{G} \otimes \mathcal{O}_{\mathbb{P}V_{3,n}}(1)$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized, and $G(2, \mathcal{G})$ is canonically isomorphic to $G(2, \mathcal{G}')$. Since $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts on $\mathbb{P}V_{3,n}$ almost freely, we can use the no-name lemma to trivialize the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-bundle $\mathcal{G}'$ locally in the Zariski topology. Hence we have

$$G(2, \mathcal{G}')/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim G(2, \mathbb{C}^{4n+3}) \times (\mathbb{P}V_{3,n}/\mathrm{SL}_2 \times \mathrm{SL}_2). \tag{3.5}$$

Now our assertion follows from Corollary 3.2. □

We also treat $G(3, V_{3,1})$.

Proposition 3.4. *The quotient $G(3, V_{3,1})/\mathrm{SL}_2 \times \mathrm{SL}_2$ is stably rational of level* 5*.*

Proof. As before, let $\mathcal{F}$ be the universal sub bundle over $G(3, V_{3,1})$. Using the no-name lemma for the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized bundle $\mathcal{F}^{\oplus 2} \otimes \det\mathcal{F}$, we obtain

$$\mathbb{P}(\mathcal{F}^{\oplus 2})/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^5 \times (G(3, V_{3,1})/\mathrm{SL}_2 \times \mathrm{SL}_2). \tag{3.6}$$

On the other hand, we have a natural $\mathrm{SL}_2 \times \mathrm{SL}_2$-equivariant morphism

$$\mathbb{P}(\mathcal{F}^{\oplus 2}) \to \mathbb{P}(V_{3,1}^{\oplus 2}), \qquad (P, \mathbb{C}(v_1, v_2)) \mapsto \mathbb{C}(v_1, v_2), \tag{3.7}$$

where $v_1, v_2 \in V_{3,1}$ are vectors contained in the 3-plane P. This is birationally the projectivization of a quotient bundle $\mathcal{G}$ of $V_{3,1} \times \mathbb{P}(V_{3,1}^{\oplus 2})$. Applying the no-name lemma to the $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized bundle $\mathcal{G} \otimes \mathcal{O}_{\mathbb{P}(V_{3,1}^{\oplus 2})}(1)$, we have

$$\mathbb{P}\mathcal{G}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^5 \times (\mathbb{P}(V_{3,1}^{\oplus 2})/\mathrm{SL}_2 \times \mathrm{SL}_2).$$

Thus it suffices to prove that $\mathbb{P}(V_{3,1}^{\oplus 2})/\mathrm{SL}_2 \times \mathrm{SL}_2$ is stably rational of level 5.

Consider the representation $W = V_{1,1} \oplus V_{3,1}^{\oplus 2}$. We apply the no-name lemma to both projections $\mathbb{P}W \dashrightarrow \mathbb{P}(V_{3,1}^{\oplus 2})$ and $\mathbb{P}W \dashrightarrow \mathbb{P}(V_{1,1} \oplus V_{3,1})$ to see that

$$\mathbb{C}^4 \times (\mathbb{P}(V_{3,1}^{\oplus 2})/\mathrm{SL}_2 \times \mathrm{SL}_2) \sim \mathbb{C}^8 \times (\mathbb{P}(V_{1,1} \oplus V_{3,1})/\mathrm{SL}_2 \times \mathrm{SL}_2). \tag{3.8}$$

Using the slice method for the projection $V_{1,1} \oplus V_{3,1} \to V_{1,1}$, we then have

$$(V_{1,1} \oplus V_{3,1})/\mathrm{GL}_2 \times \mathrm{GL}_2 \sim V_{3,1}/\mathrm{GL}_2. \tag{3.9}$$

Finally, $V_{3,1}/\mathrm{GL}_2$ is rational by Katsylo [6]. □

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by using the method of double bundle as explained in §2.2. We provide the bi-transvectants according to the congruence of b modulo 5, based on dimensional calculation for the representations involved. The exceptional case $b = 7$ requires a separate treatment.

4.1. **The case $b \equiv 0$ (5).** Let $n > 0$ be an odd number. We consider the bi-apolar covariant

$$T = T^{(3,n)} : V_{3,5n} \times V_{3,n} \to V_{0,4n}. \tag{4.1}$$

Since $\dim V_{3,n} = 4n + 4$ and $\dim V_{0,4n} = 4n + 1$, we obtain a rational map

$$V_{3,5n} \dashrightarrow G(3, V_{3,n}) \tag{4.2}$$

as in (2.10). We take vectors $v \in V_{3,5n}$, $\vec{w} = (w_1, w_2, w_3) \in (V_{3,n})^3$ by

$$v = \binom{5n}{n} X^n Y^{4n} x^3 + 3\binom{5n}{2n} X^{2n} Y^{3n} x^2 y + 3\binom{5n}{2n} X^{3n} Y^{2n} x y^2 + \binom{5n}{n} X^{4n} Y^n y^3,$$

$$\begin{aligned} w_1 &= Y^n x^3 - X^n x^2 y, \\ w_2 &= Y^n x^2 y - X^n x y^2, \\ w_3 &= Y^n x y^2 - X^n y^3. \end{aligned}$$

Lemma 4.1. *The vectors* $(v, \vec{w}) \in V_{3,5n} \times (V_{3,n})^3$ *satisfy the conditions in Lemma 2.2.*

Proof. The three vectors $w_1, w_2, w_3 \in V_{3,n}$ are apparently linearly independent. That $T(v, w_i) = 0$ is checked by using the formulae in §2.1. The map $T(v, \cdot) \colon V_{3,n} \to V_{0,4n}$ is surjective because

$$T(v, V_n x^3) = \mathbb{C}\langle X^{4n}, \cdots, X^{3n} Y^n \rangle, \quad T(v, V_n x^2 y) = \mathbb{C}\langle X^{3n} Y^n, \cdots, X^{2n} Y^{2n} \rangle,$$

$$T(v, V_n x y^2) = \mathbb{C}\langle X^{2n} Y^{2n}, \cdots, X^n Y^{3n} \rangle, \quad T(v, V_n y^3) = \mathbb{C}\langle X^n Y^{3n}, \cdots, Y^{4n} \rangle.$$

To see the surjectivity of $T(\cdot, \vec{w}) \colon V_{3,5n} \to V_{0,4n}^{\oplus 3}$, we note that

$$T(V_{5n} x^3 \oplus V_{5n} y^3, \vec{w}) = (V_{0,4n}, 0, V_{0,4n}) \subset V_{0,4n}^{\oplus 3}.$$

Since $T(V_{5n} x^2 y, w_2) = V_{0,4n}$, then $(0, V_{0,4n}, 0) \subset V_{0,4n}^{\oplus 3}$ is also contained in the image of $T(\cdot, \vec{w})$. □

By this lemma, we may apply the double bundle method so that via (4.2), $V_{3,5n}$ becomes birationally an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle $\mathcal{E}$ over $G(3, V_{3,n})$. Note that $\mathcal{E}$ is a subbundle of $V_{3,5n} \times G(3, V_{3,n})$. Since both 3 and $5n$ are odd, the elements $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ act on $\mathcal{E}$ by multiplication by -1. On the other hand, $(\pm 1, \mp 1)$ also act by -1 on the universal sub bundle $\mathcal{F}$ over $G(3, V_{3,n})$. Since $\mathcal{F}$ has odd rank 3, then the bundle $\mathcal{E}' = \mathcal{E} \otimes \det \mathcal{F}$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized. We thus have

$$\mathbb{P}V_{3,5n}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{E}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{E}'/\mathrm{SL}_2 \times \mathrm{SL}_2. \tag{4.3}$$

The group $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts on $G(3, V_{3,n})$ almost freely. Therefore we can use the no-name lemma for $\mathcal{E}'$ to obtain

$$\mathbb{P}V_{3,5n}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^{8n} \times (G(3, V_{3,n})/\mathrm{SL}_2 \times \mathrm{SL}_2). \tag{4.4}$$

Comparing this with Proposition 3.3, we see that $\mathbb{P}V_{3,5n}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational for $n > 1$. When $n = 1$, $\mathbb{P}V_{3,5}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational by Proposition 3.4.

4.2. **The case** $b \equiv 1$ (5)**.** Let $n > 0$ be an even number. We consider the bi-apolar covariant

$$T = T^{(1,3n+1)} : V_{3,5n+1} \times V_{1,3n+1} \to V_{2,2n}. \tag{4.5}$$

Since $\dim V_{1,3n+1} = 6n + 4$ and $\dim V_{2,2n} = 6n + 3$, this defines a rational map

$$V_{3,5n+1} \dashrightarrow \mathbb{P}V_{1,3n+1} \tag{4.6}$$

as in (2.10). In order to show that this determines a double bundle, we take the following vectors of $V_{3,5n+1}$ and $V_{1,3n+1}$:

$$\begin{aligned} v &= \binom{5n+1}{2n} X^{3n+1}Y^{2n}x^3 + 3\binom{5n+1}{n} X^{4n+1}Y^n x^2 y \\ &\quad +3\binom{5n+1}{2n} X^{2n}Y^{3n+1}xy^2 + \binom{5n+1}{n} X^n Y^{4n+1} y^3, \\ w &= (X^{3n+1} - Y^{3n+1})x - (X^nY^{2n+1} - X^{2n+1}Y^n)y. \end{aligned}$$

Lemma 4.2. *The vectors* $(v, w) \in V_{3,5n+1} \times V_{1,3n+1}$ *meet the conditions in Lemma 2.2.*

Proof. One calculates that $T(v, w) = 0$ using the formulae in §2.1. Conversely, suppose we have a vector $w' = G_+(X, Y)x + G_-(X, Y)y$ in $V_{1,3n+1}$ with $T(v, w') = 0$. Then we have

$$\begin{aligned} T^{(3n+1)}(X^nY^{4n+1}, G_+) &= b_0T^{(3n+1)}(X^{2n}Y^{3n+1}, G_-), \\ T^{(3n+1)}(X^{2n}Y^{3n+1}, G_+) &= b_1T^{(3n+1)}(X^{4n+1}Y^n, G_-), \\ T^{(3n+1)}(X^{4n+1}Y^n, G_+) &= b_2T^{(3n+1)}(X^{3n+1}Y^{2n}, G_-). \end{aligned}$$

for suitable constants b_j. Expanding $G_\pm = \sum_i \alpha_i^\pm X^{3n+1-i}Y^i$, we obtain

$$\begin{aligned} \alpha_i^+ &= c_{1i}\alpha_{i+n}^- \ (0 \le i \le n), & \alpha_i^- &= 0 \ (0 \le i \le n-1), \\ \alpha_i^+ &= c_{2i}\alpha_{i+2n+1}^- \ (0 \le i \le n), & \alpha_i^+ &= 0 \ (n+1 \le i \le 2n), \\ \alpha_{i+n}^+ &= c_{3i}\alpha_i^- \ (n+1 \le i \le 2n+1), & \alpha_i^- &= 0 \ (2n+2 \le i \le 3n+1), \end{aligned}$$

for some fixed constants c_*. This reduces to the relations

$$\alpha_0^+ = d_1\alpha_{3n+1}^+ = d_2\alpha_n^- = d_3\alpha_{2n+1}^-$$

where d_j are appropriate constants, and $\alpha_i^\pm = 0$ for other i. Hence the map $T(v, \cdot)\colon V_{1,3n+1} \to V_{2,2n}$ has 1-dimensional kernel, and so is surjective. We also see that the map $T(\cdot, w)\colon V_{3,5n+1} \to V_{2,2n}$ is surjective, noticing that

$$T(V_{5n+1}y^3, w) = V_{2n}y^2, \qquad T(V_{5n+1}x^3, w) = V_{2n}x^2,$$

$$T(V_{5n+1}xy^2, (X^{3n+1} - Y^{3n+1})x) = V_{2n}xy.$$

□

Thus we can use the method of double bundle to see that via (4.6), $V_{3,5n+1}$ becomes birational to an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle $\mathcal{E}$ over $\mathbb{P}V_{1,3n+1}$. As before, the elements $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ act by multiplication by -1 on both $\mathcal{E}$ (which is a subbundle of $V_{3,5n+1} \times \mathbb{P}V_{1,3n+1}$) and $\mathcal{O}_{\mathbb{P}V_{1,3n+1}}(1)$. Hence the bundle $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}V_{1,3n+1}}(1)$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized. The group $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts almost freely on $\mathbb{P}V_{1,3n+1}$. By the no-name lemma for $\mathcal{E}'$, we obtain

$$\text{(4.7)} \quad \mathbb{P}V_{3,5n+1}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{E}'/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^{14n+4} \times (\mathbb{P}V_{1,3n+1}/\mathrm{SL}_2 \times \mathrm{SL}_2).$$

Therefore $\mathbb{P}V_{3,5n+1}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational by Corollary 3.2.

4.3. **The case $b \equiv 2$ (5).** Let $n > 0$ be an odd number. We use the bi-apolar covariant

$$T = T^{(3,n)} : V_{3,5n+2} \times V_{3,n} \to V_{0,4n+2}. \tag{4.8}$$

Since $\dim V_{3,n} = 4n+4$ and $\dim V_{0,4n+2} = 4n+3$, we obtain as in (2.10) a rational map

$$V_{3,5n+2} \dashrightarrow \mathbb{P}V_{3,n}. \tag{4.9}$$

To see that this defines a double bundle, we take vectors in $V_{3,5n+2}$ and $V_{3,n}$ by

$$v = X^n Y^{4n+2} x^3 + X^{2n+1} Y^{3n+1} x^2 y + X^{3n+1} Y^{2n+1} x y^2 + X^{4n+2} Y^n y^3,$$
$$w = Y^n x^2 y - X^n x y^2.$$

Lemma 4.3. *The vectors* $(v, w) \in V_{3,5n+2} \times V_{3,n}$ *satisfy the conditions in Lemma 2.2.*

Proof. It is immediate to check that $T(v, w) = 0$. The map $T(v, \cdot) \colon V_{3,n} \to V_{0,4n+2}$ is surjective because

$$\begin{aligned} T(v, V_n x^3) &= \mathbb{C}\langle X^{4n+2}, \cdots, X^{3n+2} Y^n \rangle, \\ T(v, V_n x^2 y) &= \mathbb{C}\langle X^{3n+1} Y^{n+1}, \cdots, X^{2n+1} Y^{2n+1} \rangle, \\ T(v, V_n x y^2) &= \mathbb{C}\langle X^{2n+1} Y^{2n+1}, \cdots, X^{n+1} Y^{3n+1} \rangle, \\ T(v, V_n y^3) &= \mathbb{C}\langle X^n Y^{3n+2}, \cdots, Y^{4n+2} \rangle. \end{aligned}$$

On the other hand, we have $T(V_{5n+2} x y^2, w) = V_{0,4n+2}$ so that the map $T(\cdot, w) : V_{3,5n+2} \to V_{0,4n+2}$ is also surjective. □

This lemma enables the application of the method of double bundle. Therefore the map (4.9) makes $V_{3,5n+2}$ birationally an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle $\mathcal{E}$ over $\mathbb{P}V_{3,n}$. The elements $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ act on $\mathcal{E}$ by multiplication by -1. Since both 3 and n are odd, $(\pm 1, \mp 1)$ also act by -1 on $\mathcal{O}_{\mathbb{P}V_{3,n}}(1)$. Thus the bundle $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}V_{3,n}}(1)$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized. When $n > 1$, $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ acts on $\mathbb{P}V_{3,n}$ almost freely. Then by the no-name lemma we have

$$\mathbb{P}V_{3,5n+2}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{E}'/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^{16n+8} \times (\mathbb{P}V_{3,n}/\mathrm{SL}_2 \times \mathrm{SL}_2). \tag{4.10}$$

By Corollary 3.2, we see that $\mathbb{P}V_{3,5n+2}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational for $n > 1$.

This argument does not work for the case $n = 1$ because a general point of $\mathbb{P}V_{3,1}$ has the Klein 4-group as its stabilizer. We treat this case in §4.6.

4.4. **The case $b \equiv 3$ (5).** Let $n > 0$ be an even number. We consider the $(3, n)$-th bi-transvectant

$$T = T^{(3,n)} : V_{3,5n+3} \times V_{3,n+1} \to V_{0,4n+4}. \tag{4.11}$$

Since $\dim V_{3,n+1} = 4n+8$ and $\dim V_{0,4n+4} = 4n+5$, this induces a rational map as in (2.10),

$$V_{3,5n+3} \dashrightarrow G(3, V_{3,n+1}). \tag{4.12}$$

In order to apply the method of double fibration, we take the following vectors of $V_{3,5n+3}$ and $V_{3,n+1}$ according to the congruence of n modulo 5:

(1) When $n \not\equiv 4 \bmod 5$, we set

$$\begin{aligned} v &= \binom{5n+3}{n} X^n Y^{4n+3} x^3 + \binom{5n+3}{2n+1} X^{2n+1} Y^{3n+2} x^2 y \\ &\quad + \binom{5n+3}{2n+1} X^{3n+2} Y^{2n+1} x y^2 + \binom{5n+3}{n} X^{4n+3} Y^n y^3, \\ w_1 &= X^{n+1} y^3 + Y^{n+1} x y^2, \\ w_2 &= X^{n+1} x y^2 + Y^{n+1} x^2 y, \\ w_3 &= X^{n+1} x^2 y + Y^{n+1} x^3. \end{aligned}$$

(2) When $n \equiv 4 \bmod 5$, we denote $n = 2m$ (remember n is even) and set

$$\begin{aligned} v &= \left\{ \frac{7m+3}{m+1} \frac{5m+2}{3m+2} \binom{5n+3}{m} X^m Y^{9m+3} + X^{9m+5} Y^{m-2} \right\} x^3 \\ &\quad + 3 \frac{5m+2}{3m+2} \binom{5n+3}{3m+1} X^{3m+1} Y^{7m+2} x^2 y + 3 \binom{5n+3}{5m+2} X^{5m+2} Y^{5m+1} x y^2 \\ &\quad + \frac{5m+3}{3m+1} \binom{5n+3}{7m+3} X^{7m+3} Y^{3m} y^3, \end{aligned}$$

and use the same w_i as above.

Lemma 4.4. *The vectors* $(v, w_1, w_2, w_3) \in V_{3,5n+3} \times (V_{3,n+1})^3$ *meet the conditions in Lemma 2.2.*

Proof. The linear independence of w_1, w_2, w_3 is apparent. It is not difficult to check that $T(v, w_i) = 0$ for every i, by using the formulae in §2.1. When $n \not\equiv 4 \bmod 5$, we have no $0 \leq j \leq n+1$ with $j(5n+5) = (i+1)(n+1)$ for $i = n, 2n+1, 3n+2, 4n+3$. Hence by Lemma 2.1, for those i the bilinear map

$$T^{(n)} : \mathbb{C}X^i Y^{5n+3-i} \times \mathbb{C}X^{n+1-j} Y^j \to \mathbb{C}X^{i-j+1} Y^{4n+3-i+j} \tag{4.13}$$

is non-degenerate for any j, as far as the indices are non-negative. It follows that

$$\begin{aligned} T(v, V_{n+1} x^3) &= \mathbb{C}\langle X^{4n+4}, \cdots, X^{3n+3} Y^{n+1} \rangle, \\ T(v, V_{n+1} x^2 y) &= \mathbb{C}\langle X^{3n+3} Y^{n+1}, \cdots, X^{2n+2} Y^{2n+2} \rangle, \\ T(v, V_{n+1} x y^2) &= \mathbb{C}\langle X^{2n+2} Y^{2n+2}, \cdots, X^{n+1} Y^{3n+3} \rangle, \\ T(v, V_{n+1} y^3) &= \mathbb{C}\langle X^{n+1} Y^{3n+3}, \cdots, Y^{4n+4} \rangle, \end{aligned}$$

whence the map $T(v, \cdot) : V_{3,n+1} \to V_{0,4n+4}$ is surjective. We leave it to the reader to check similar surjectivity when $n \equiv 4$ (5). In that case, since $m \equiv 2$ (5), we have no j with $j(5n+5) = (i+1)(n+1)$ for $i = m + k(n+1)$, $0 \leq k \leq 3$, and $i = 9m+5$. Hence for those i the map (4.13) is non-degenerate for any relevant j, again by Lemma 2.1.

To see that

$$T(\cdot, \vec{w}) = (T(\cdot, w_1), T(\cdot, w_2), T(\cdot, w_3)) : V_{3,5n+3} \to V_{0,4n+4}^{\oplus 3}$$

is surjective (regardless of $[n] \in \mathbb{Z}/5$), we note that the bilinear maps

$$T^{(n)}(\cdot, X^{n+1}) : \mathbb{C}X^i Y^{5n+3-i} \to \mathbb{C}X^{i+1} Y^{4n+3-i}$$

$$T^{(n)}(\cdot, Y^{n+1}) : \mathbb{C}X^iY^{5n+3-i} \to \mathbb{C}X^{i-n}Y^{5n+4-i}$$

are non-degenerate whenever the indices are non-negative. It follows that

$$T(V_{5n+3}x^3, \vec{w}) = (\langle X^{4n+4}, \cdots, XY^{4n+3}\rangle, 0, 0),$$

$$T(\langle X^nY^{4n+3}x^2y, X^{2n+1}Y^{3n+2}xy^2, X^{3n+2}Y^{2n+1}y^3\rangle, \vec{w}) \supset (\mathbb{C}Y^{4n+4}, 0, 0),$$

so that $(V_{0,4n+4}, 0, 0) \subset V_{0,4n+4}^{\oplus 3}$ is contained in the image of $T(\cdot, \vec{w})$. Similarly, we see that $(0, 0, V_{0,4n+4}) \subset V_{0,4n+4}^{\oplus 3}$ is contained in the image too. Finally, since $T(\cdot, w_2)$ maps the space $V_{5n+3}x^2y \oplus V_{5n+3}xy^2$ onto $V_{0,4n+4}$, we find using the above results that $(0, V_{0,4n+4}, 0)$ is also contained in the image. □

Thus, by the method of double bundle for (4.12), $V_{3,5n+3}$ is birationally an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle $\mathcal{E}$ over $G(3, V_{3,n+1})$. The elements $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ act on $\mathcal{E}$ by multiplication by -1. Let $\mathcal{F}$ be the universal sub bundle over $G(3, V_{3,n+1})$. On $\det\mathcal{F}$ the elements $(\pm 1, \mp 1)$ act also by -1 because both 3 and $n + 1$ are odd and $\mathcal{F}$ has odd rank. Therefore $\mathcal{E}' = \mathcal{E} \otimes \det\mathcal{F}$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized. By the no-name lemma for $\mathcal{E}'$, we then see that

$$\mathbb{P}V_{3,5n+3}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{E}'/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^{8n} \times (G(3, V_{3,n+1})/\mathrm{SL}_2 \times \mathrm{SL}_2).$$

By Proposition 3.3, $\mathbb{P}V_{3,5n+3}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational.

4.5. **The case $b \equiv 4$ (5).** Let $n > 0$ be an odd number. We use the $(1, 3n + 3)$-th bi-transvectant

$$T = T^{(1,3n+3)} : V_{3,5n+4} \times V_{1,3n+4} \to V_{2,2n+2}. \tag{4.14}$$

Since $\dim V_{1,3n+4} = 6n + 10$ and $\dim V_{2,2n+2} = 6n + 9$, we obtain a rational map

$$V_{3,5n+4} \dashrightarrow \mathbb{P}V_{1,3n+4} \tag{4.15}$$

as in (2.10). In order to check that this defines a double bundle, we take the following vectors of $V_{3,5n+4}$ and $V_{1,3n+4}$:

$$\begin{aligned} v &= \frac{3n+4}{n+2}\frac{3n+4}{n+1}\binom{5n+4}{2n+1}X^{3n+3}Y^{2n+1}x^3 + 3\frac{3n+4}{n+1}\binom{5n+4}{n}X^{4n+4}Y^nx^2y \\ &\quad -3\binom{5n+4}{2n+1}X^{2n+1}Y^{3n+3}xy^2 - \frac{n+2}{3n+4}\binom{5n+4}{n}X^nY^{4n+4}y^3, \\ w &= (X^{3n+4} + Y^{3n+4})x + (X^{2n+3}Y^{n+1} + X^{n+1}Y^{2n+3})y. \end{aligned}$$

Lemma 4.5. *The vectors $(v, w) \in V_{3,5n+4} \times V_{1,3n+4}$ meet the conditions in Lemma 2.2.*

Proof. We leave it to the reader to check that $T(v, w) = 0$. To show that the map $T(v, \cdot)\colon V_{1,3n+4} \to V_{2,2n+2}$ is surjective, we first note that $5n + 6$ and $3n + 4$ are coprime by the Euclidean algorithm. By Lemma 2.1, the bilinear map

$$T^{(3n+3)} : \mathbb{C}X^iY^{5n+4-i} \times \mathbb{C}X^{3n+4-j}Y^j \to \mathbb{C}X^{i-j+1}Y^{2n+1-i+j}$$

is non-degenerate whenever the indices are non-negative. Now suppose a vector $w' = G_+(X,Y)x + G_-(X,Y)y$ in $V_{1,3n+4}$ satisfies $T(v,w') = 0$. This is rewritten as

$$\begin{aligned} T^{(3n+3)}(X^{3n+3}Y^{2n+1}, G_-) &= b_0 T^{(3n+3)}(X^{4n+4}Y^n, G_+), \\ T^{(3n+3)}(X^{4n+4}Y^n, G_-) &= b_1 T^{(3n+3)}(X^{2n+1}Y^{3n+3}, G_+), \\ T^{(3n+3)}(X^{2n+1}Y^{3n+3}, G_-) &= b_2 T^{(3n+3)}(X^nY^{4n+4}, G_+), \end{aligned}$$

for some constants b_j. Expanding $G_\pm(X,Y) = \sum_{j=0}^{3n+4} \alpha_j^\pm X^{3n+4-j}Y^j$, we obtain the relation

$$\begin{aligned} \alpha_{j+n+1}^+ &= c_{1j}\alpha_j^- \quad (n+2 \le j \le 2n+3), & \alpha_j^- &= 0 \quad (2n+4 \le j \le 3n+4), \\ \alpha_j^+ &= c_{2j}\alpha_{j+2n+3}^- \quad (0 \le j \le n+1), & \alpha_j^+ &= 0 \quad (n+2 \le j \le 2n+2), \\ \alpha_j^+ &= c_{3j}\alpha_{j+n+1}^- \quad (0 \le j \le n+1), & \alpha_j^- &= 0 \quad (0 \le j \le n), \end{aligned}$$

where c_* are suitable non-zero constants. This is reduced to the relations

$$\alpha_0^+ = d_1\alpha_{n+1}^- = d_2\alpha_{2n+3}^- = d_3\alpha_{3n+4}^+$$

for some constants d_j, and $\alpha_i^\pm = 0$ for other i. Therefore the map $T(v,\cdot)\colon V_{1,3n+4} \to V_{2,2n+2}$ has 1-dimensional kernel.

On the other hand, the surjectivity of the map $T(\cdot,w)\colon V_{3,5n+4} \to V_{2,2n+2}$ follows by noticing that

$$\begin{gathered} T(V_{5n+4}x^3, w) = V_{2n+2}x^2, \qquad T(V_{5n+4}y^3, w) = V_{2n+2}y^2, \\ T(V_{5n+4}xy^2, (X^{3n+4} + Y^{3n+4})x) = V_{2n+2}xy. \end{gathered}$$

□

This lemma assures that, via (4.15), $V_{3,5n+4}$ becomes birationally an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle $\mathcal{E}$ over $\mathbb{P}V_{1,3n+4}$. Since n is odd, the elements $(\pm 1, \mp 1) \in \mathrm{SL}_2 \times \mathrm{SL}_2$ act by multiplication by -1 on both $\mathcal{E}$ and $\mathcal{O}_{\mathbb{P}V_{1,3n+4}}(1)$. Thus $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}V_{1,3n+4}}(1)$ is $\mathrm{PGL}_2 \times \mathrm{PGL}_2$-linearized. Using the no-name lemma for $\mathcal{E}'$, we have

$$\mathbb{P}V_{3,5n+4}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{E}'/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^{14n+10} \times (\mathbb{P}V_{1,3n+4}/\mathrm{SL}_2 \times \mathrm{SL}_2).$$

Then $\mathbb{P}V_{3,5n+4}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational by Corollary 3.2.

4.6. **The case $b = 7$.** We treat $V_{3,7}$ which is excluded from §4.3. We use the $(2,3)$-th bi-transvectant

$$T = T^{(2,3)} : V_{3,7} \times V_{3,3} \to V_{2,4}, \tag{4.16}$$

which defines a rational map

$$V_{3,7} \dashrightarrow \mathbb{P}V_{3,3} \tag{4.17}$$

as in (2.10). We choose the following vectors of $V_{3,7}$ and $V_{3,3}$:

$$\begin{aligned} v &= \binom{7}{3}X^3Y^4x^3 - 9Y^7x^2y + \binom{7}{1}X^6Yxy^2 + \binom{7}{3}X^4Y^3y^3, \\ w &= Y^3x^3 + X^3xy^2 + (XY^2 + Y^3)y^3. \end{aligned}$$

Lemma 4.6. *The vectors $(v,w) \in V_{3,7} \times V_{3,3}$ satisfy the conditions in Lemma 2.2.*

Proof. We leave it to the reader to check that $T(v,w) = 0$ and that w spans the kernel of the map $T(v,\cdot) : V_{3,3} \to V_{2,4}$ (cf. Proofs 4.2 and 4.5). We shall show that $T(\cdot,w)\colon V_{3,7} \to V_{2,4}$ is surjective. First note that the bilinear map

$$T^{(2)} : \mathbb{C}x^i y^{3-i} \times \mathbb{C}x^{3-j}y^j \to \mathbb{C}x^{i-j+1}y^{j-i+1}$$

is non-degenerate whenever the indices are non-negative, for 3 and 5 are coprime (Lemma 2.1). Then we have

$$T(V_7y^3, w) = T(V_7y^3, Y^3x^3) = V_4xy.$$

Since $T^{(3)}(V_7, X^3) = V_4$, we have $T(V_7x^3, w) \subset V_4x^2 \oplus V_4xy$ with surjective projection $T(V_7x^3, w) \to V_4x^2$. Therefore V_4x^2 is also contained in the image of $T(\cdot, w)$. Finally, since $T(V_7xy^2, X^3xy^2) = V_4y^2$, the space V_4y^2 is contained in the image too. □

Thus we may apply the double bundle method to see that (4.17) makes $V_{3,7}$ birational to an $\mathrm{SL}_2 \times \mathrm{SL}_2$-linearized vector bundle $\mathcal{E}$ over $\mathbb{P}V_{3,3}$. As before, after twisting $\mathcal{E}$ by $\mathcal{O}_{\mathbb{P}V_{3,3}}(1)$, we use the no-name lemma to see that

$$\mathbb{P}V_{3,7}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}\mathcal{E}/\mathrm{SL}_2 \times \mathrm{SL}_2 \sim \mathbb{P}^{16} \times (\mathbb{P}V_{3,3}/\mathrm{SL}_2 \times \mathrm{SL}_2). \tag{4.18}$$

Then $\mathbb{P}V_{3,7}/\mathrm{SL}_2 \times \mathrm{SL}_2$ is rational by Corollary 3.2.

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