Local Dynamics Near Unstable Branches of NLS Solitons

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Abstract

Consider $\{\phi_\omega\}_{\omega \in \mathcal{I}}$, a branch of unstable solitons of NLS whose linearized operators have one pair of simple real eigenvalues in addition to the zero eigenvalue. Under radial symmetry and standard assumptions, solutions to initial data from a neighbourhood of the branch either converge to a soliton, or exit a larger neighbourhood of the branch transversally. The qualitative dynamic near a branch of unstable solitons is irrespective of whether blowup eventually occurs, which has practical implications for the description of blowup of NLS with supercritical nonlinearity.

1 Introduction

1.1 Nonlinear Schrödinger equation and solitons

Let us consider the nonlinear Schrödinger equation (NLS) with covariant general nonlinearity $g(u) := f(|u|^2)u$,

$$i\partial_t u + \Delta u - V_0(x)u + f(|u|^2)u = 0, \tag{1.1}$$

where $u : (t,x) \in \mathbb{R} \times \mathbb{R}^N \mapsto u(t,x) \in \mathbb{C}$, $N \geq 1$, and $u(0,\cdot) = u_0 \in H^1(\mathbb{R}^N)$. We assume that f is real-valued, the C^1 nonlinearity $g(u)$ is C^2 for $u \neq 0$, and H^1-subcritical in the sense that:

$$g(0) = 0, \qquad g'(0) = 0 \qquad \text{and} \qquad \left|g''(u)\right| \lesssim |u|^{m_1-2} + |u|^{m_2-2} \quad \text{for } u \neq 0, \tag{1.2}$$

with $1 < m_1 \leq m_2 < m_{\max}$; $m_{\max} = \frac{N+2}{N-2}$ if $N \geq 3$, and $m_{\max} = +\infty$ if $N = 1, 2$. V_0 is either zero, or a smooth localized potential such that a range of assumptions hold; see Section 1.2. These equations are Hamiltonian, with the following conserved quantities:

$$E(u_0) \quad = \quad E(u) \quad := \quad \frac{1}{2}\int \left(|\nabla u|^2 + V_0 |u|^2\right) - \int G(u) \quad \text{(energy)},$$

$$M(u_0) \quad = \quad M(u) \quad := \quad \frac{1}{2}\int |u|^2 \qquad\qquad\qquad \text{(mass)},$$

where $G(s) = \int_0^s g(\bar{s})d\bar{s}$. For data in the *energy space* $H^1(\mathbb{R}^N)$, equation (1.1) is locally well posed with a blowup alternative: there exists $T_{\max} > 0$ such that

$$u(t) \in C\left([0, T_{\max}), H^1\right),$$

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and either $T_{\max} = \infty$ or $\lim_{t \to T_{\max}} \|u(t)\|_{H^1} = \infty$. The modern proof is due to Kato [20].

Another well known property of (1.1) is the existence of solitary wave (or *soliton*) solutions of the form $u(t, x) = \phi_\omega(x)e^{i\omega t}$. Berestycki and Lions [4] proved that for $V_0 = 0$ there exists a solution $\phi_\omega > 0$ of

$$\Delta \phi_\omega - \omega \phi_\omega + f(|\phi_\omega|^2)\phi_\omega = 0, \tag{1.3}$$

provided $\omega > 0$ and there exists $u_1 > 0$ such that $G(u_1) > \frac{\omega}{2}u_1^2$. Note that this further assumption on the nonlinearity g is independent of N. The positive solution of (1.3) is unique and radially symmetric up to translations. See Cazenave's book [8] and the references therein, particularly McLeod [28]. For $V_0 \neq 0$, see Rose and Weinstein [33] and articles that cite it.

We are interested in the dynamics and stability of these objects. Because of rotation invariance of equation (1.1), the proper notion is *orbital stability*: we say that ϕ_{ω_0} is orbitally stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|u_0 - \phi_{\omega_0}\|_{H^1} \leq \delta \implies \sup_{t>0}\left(\inf_{s\in\mathbb{R}}\left\|u(t) - \phi_{\omega_0}e^{is}\right\|_{H^1}\right) \leq \varepsilon.$$

Note that, when the soliton family has the spatial translation symmetry, the solution may move but remain close to the soliton family. In that case, the above concept of orbital stability is not adequate, and the nullspace of the linearized operator (to be discussed in Section 2.1) will have extra elements ($\nabla\phi_\omega$). One possible remedy is to enlarge the "orbit" to contain all translated solitons. Another way, which we choose in this paper, is to restrict ourselves to the cases with no translation. For example, we may assume the perturbation is either radial or even. Alternatively, we may also assume the potential V_0 is nonradial and hence the soliton family has no translation symmetry.

We now assume, as is usually the case, that we have a whole *branch* of solitons, that is a family $(\phi_\omega)_{\omega \in \mathcal{I}}$ of solitary waves, with $\phi_\omega > 0$ radial, $\mathcal{I} = (\underline{\omega}, \overline{\omega})$, and a C^2 map $\omega \in \mathcal{I} \mapsto \phi_\omega \in H^1(\mathbb{R}^N)$. Under these assumptions and some generic spectral conditions, for $\omega_0 \in \mathcal{I}$, we have:

(i) ϕ_{ω_0} is orbitally stable if $\frac{d}{d\omega}\int |\phi_\omega|^2 \big|_{\omega=\omega_0} > 0$;

(ii) ϕ_{ω_0} is orbitally unstable if $\frac{d}{d\omega}\int |\phi_\omega|^2 \big|_{\omega=\omega_0} < 0$.

The part (i) is due to Cazenave-Lions [9] and Weinstein [40], by variational and energy methods. The part (ii) is due to Shatah and Strauss [35]. See also [17]. We are interested in *unstable* branches of solitons.

In the case of a pure power *focusing* nonlinearity, $f(s) = +s^{(m-1)/2}$ and $V_0 = 0$, there exists a profile ϕ_ω for all $\omega > 0$. Moreover, there is scaling invariance, $\phi_\omega(x) = \omega^{\frac{1}{m-1}}\phi_1(\sqrt{\omega}x)$, by which we can compute $\|\phi_\omega\|_{L^2} = \omega^{\frac{1}{m-1}-\frac{N}{4}}\|\phi_1\|_{L^2}$. We let $m_c = \frac{4}{N} + 1$ and note that ϕ_ω is stable for all $\omega > 0$ when $1 < m < m_c$ (L^2 subcritical case), and ϕ_ω is unstable for all $\omega > 0$ when $m_c < m < m_{\max}$ (L^2 supercritical case).

When the solitary wave ϕ_{ω_0} is stable, it is likely that a nearby solution will relax to some ϕ_{ω_+} as time goes to infinity. The frequency ω_+ is close to ω_0 but most likely different. In addition, the convergence happens only locally since there is radiation going to infinity with unvanishing L^2-mass. Thus one also considers a local concept of stability of the

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branch: we say that the branch $(\phi_\omega)_{\omega \in \mathcal{I}}$ is *asymptotically stable* if, for any u_0 in a suitable neighbourhood,

$$\|u(t) - \phi_{\omega(t)} e^{i\theta(t)}\|_{L^2_{loc}} \longrightarrow 0, \quad \text{as} \quad t \to \infty,$$

for some continuous functions $\omega(t) \in \mathcal{I}$ and $\theta(t) \in \mathbb{R}$. Asymptotic stability is the main stability concept used in the statement of our main theorem.

This paper is organized as follows. Section 1.2 will itemize our assumptions in terms of the nonlinearity and in terms of the evolution operator $\mathbf{JL}$ linearized about ϕ_ω. We then state our main result. Section 1.3 will discuss the context for our theorem and relevant literature. The linearized operator $\mathbf{JL}$ will be properly introduced in Section 2.1, followed by the proof of our main result.

1.2   Assumptions & Main Result

1. Solitons exist for $\omega \in \mathcal{I}$, and, moreover, the profiles ϕ_ω and their derivatives $\partial_\omega^a \partial_x^b \phi_\omega$ have exponential decay. We assume the map $\omega \mapsto \phi_\omega \in H^1$ is C^2.

2. For all $\omega \in \mathcal{I}$, the linearized operator $\mathbf{JL}$ has eigenvalue zero with multiplicity two. The discrete spectrum contains exactly two simple real eigenvalues, $e_+ > 0 > e_-$, with corresponding eigenfunctions Y_+ and Y_- with exponential decay. The purely imaginary continuous spectrum is bounded away from zero. There is no embedded eigenvalue or resonance in the continuous spectrum or its endpoints.

   As unstable solitons are typically characterized by $\langle \partial_\omega \phi_\omega, \phi_\omega \rangle < 0$, we assume that $\langle \partial_\omega \phi_\omega, \phi_\omega \rangle \neq 0$ with a uniform bound on $\mathcal{I}$.

3. Nonlinearity $g(u) := f(|u|^2)u$ is sufficiently strong. In addition to (1.2), we require

$$m_1 > 1 + \frac{2}{N}\left(1 + \sigma_p\right), \tag{1.4}$$

$$m_1 > 1 + \frac{2}{N}\left(1 + \frac{2}{m_0 + 1}\right), \tag{1.5}$$

   where $m_0 \equiv \min\{2, m_1\}$.

   Here, we use p to denote $m_2 + 1 \equiv p < p_{\max} \equiv m_{\max} + 1$, and the standard notation $\sigma_r = N\left(\frac{1}{2} - \frac{1}{r}\right)$ and $\frac{1}{r} + \frac{1}{r'} = 1$ for any exponent r. See equation (2.24), below, for a related fixed constant $q \in [p, p_{\max})$ determined by both m_1 and m_2. The choice of q will require $N \geq 2$.

   Condition (1.4) implies $m_1 > \frac{1}{2} + \frac{1}{N} + \frac{\sqrt{N^2 + 12N + 4}}{2N}$, the Strauss exponent. However, we do not approach this value, as condition (1.5) implies either $m_1 > 1 + \frac{2}{N}\left(\frac{5}{3}\right)$ or $m_1 > \frac{1}{N} + \frac{\sqrt{N^2 + 6N + 1}}{N}$, which are more restrictive. See figures 1 and 2. We note that conditions (1.4) and (1.5) admit the entire range of L^2-critical and supercritical exponents, $1 + \frac{4}{N} \leq m_1 \leq m_2 < \frac{N+2}{N-2}$.

   Relations (1.4) and (1.5) are required to complete bootstrap estimates of the L^p and L^q norms, respectively. The arithmetic consequences of (1.4) and (1.5) are discussed in Section 2.7.

4. There is an adequate dispersive estimate, uniformly in ω:

$$\sup_{\omega \in \mathcal{I}} \|e^{t\mathbf{JL}} \mathcal{P}_c \xi\|_{L^r} \lesssim t^{-\sigma_r} \|\xi\|_{L^{r'}}, \tag{1.6}$$
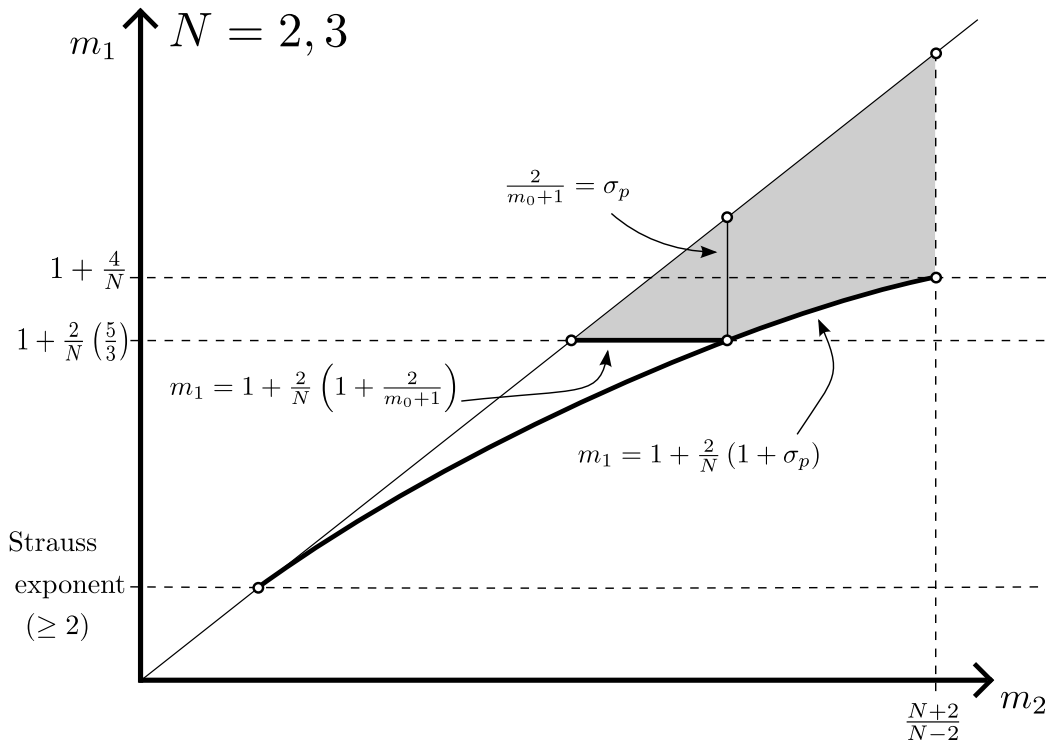
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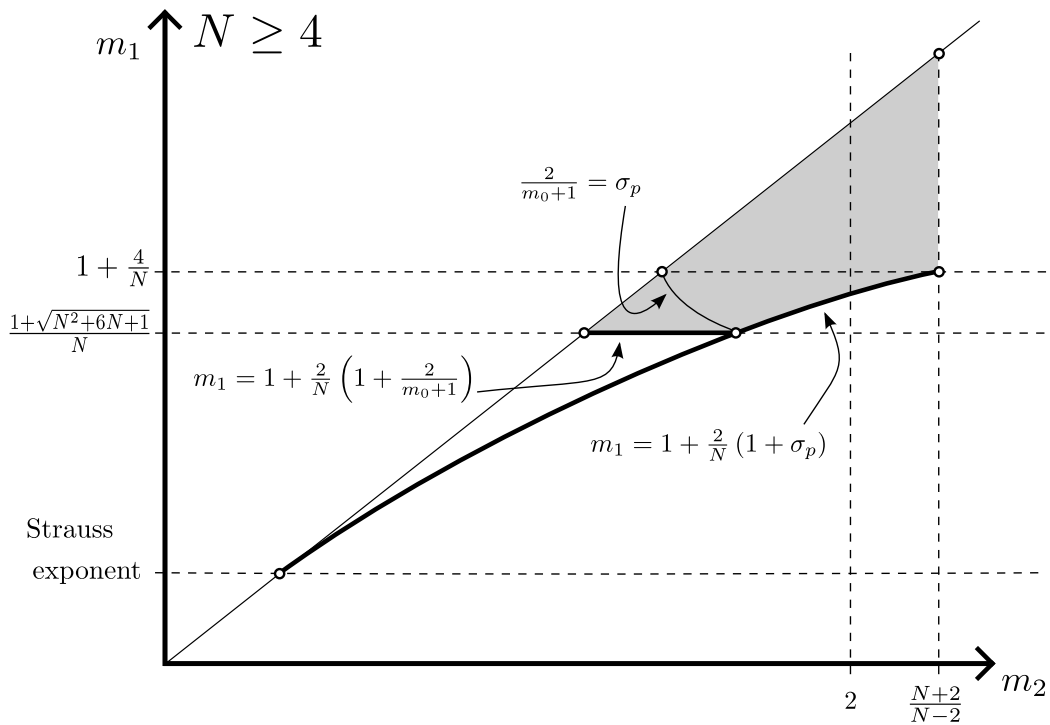Figure 1: Shaded region illustrates conditions (1.4) and (1.5) for $N = 2, 3$.

Figure 2: Shaded region illustrates conditions (1.4) and (1.5) for $N \geq 4$.

where $\mathcal{P}_c$ is the continuous spectral projection with respect to the linearized operator **JL** (depends on ω). See (2.10) below for details. We only require estimate (1.6) for $r = p, q$, which excludes endpoint cases.

For $V_0 = 0$, (1.6) is a consequence of the spectral assumption. For $N > 2$, see Cuccagna [11, Corollary 2.2], built on the earlier works of Yajima [41, 42]. For $N = 2$, see Cuccagna and Tarulli [13].

The class of non-trivial potentials V_0 for which (1.6) holds is unknown.

All assumptions are known to be true for the monic cubic focusing equation in three dimensions. The lack of embedded eigenvalues is due to a numerically assisted proof by Marzuola and Simpson [27]. All assumptions are expected to be true for any monic L^2-supercritical and energy-subcritical equation, and for their perturbations. Therefore, after rescaling, the assumptions should hold for sums of two monic nonlinearities in certain ranges of ω. Indeed, the spectral assumption is partially known for the cubic-quintic nonlinearity in exactly such a situation. See Asad and Simpson [2]. See Hundertmark and Lee [19] regarding the decay of Y_+, Y_-.

Theorem 1.1. *Let $1 < N < \infty$ and suppose the above assumptions are valid for ω in an interval $\mathcal{I}$ with uniform estimates. There exist positive constants $\overline{\alpha_0} = \overline{\alpha_0}(f, \mathcal{I})$ and $C = C(f, \mathcal{I})$ such that the following holds.*

For any $\alpha_0 \in (0, \overline{\alpha_0}]$, consider initial data u_0 for which there exist $\theta_0 \in \mathbb{R}$ and $\omega_0 \in \mathcal{I}$ with $\mathrm{dist}(\omega_0, \partial\mathcal{I}) > C\alpha_0$, such that:

$$0 < \alpha \equiv \|u_0 - \phi_{\omega_0} e^{i\theta_0}\|_{H^1 \cap L^1} < \alpha_0.$$

Either:

1. **(escape case)** *there exist $\omega_+ \in \mathcal{I}$, $|\omega_0 - \omega_+| \leq C\alpha_0$, $\theta_+ \in [0, 2\pi]$ and finite time $T_{exit} > 0$, such that,*

$$\|u(T_{exit}) - \phi_{\omega_+} e^{i\theta_+}\|_{L^2_{loc}} = \inf_{\omega \in \mathcal{I}, \theta \in \mathbb{R}} \|u(T_{exit}) - \phi_\omega e^{i\theta}\|_{L^2_{loc}} \geq 2\alpha_0, \quad or,$$

2. **(convergence case)** *there exist $\omega_+ \in \mathcal{I}$, $|\omega_0 - \omega_+| \leq C\alpha$, $t \mapsto \theta(t)$ continuous and $\mu = \mu(f) > 0$, such that,*

$$\|u(t) - \phi_{\omega_+} e^{i\theta(t)}\|_{L^p} \leq C\alpha \langle t \rangle^{-\mu} \quad \text{for all } t > 0.$$

Recall the standard notation $\langle t \rangle = \sqrt{1 + t^2}$ for $t \in \mathbb{R}$, and that $p = m_2 + 1$ is the largest exponent in the potential energy. Since $L^p \hookrightarrow L^2_{loc}$, the convergence case is an asymptotic stability result. Let us emphasize that the estimates must be uniform over $\mathcal{I}$. In particular, the real eigenvalues e_+, e_- and $\langle \partial_\omega \phi_\omega, \phi_\omega \rangle$ are uniformly bounded away from zero. The restriction $\mathrm{dist}(\omega_0, \partial\mathcal{I}) > C\alpha_0$ ensures that both $\omega_+ \in \mathcal{I}$ and $\omega(t) \in \mathcal{I}$, where $\omega(t)$ will be defined by (2.12).

1.3   Context and Importance

Asymptotic stability of orbitally stable solitary waves is well studied and has a vast, growing literature, initiated by Soffer-Weinstein [36, 37] and Buslaev-Perelman [6].

For unstable solitary waves, the general result of Shatah-Strauss [35] exhibits solutions which are initially arbitrarily close to the solitary waves but leave their neighbourhood in finite times. Also see Comech-Pelinovsky [10], who give a similar result when ω_0 is the borderline between stable and unstable branches. There are also results showing the existence of stable manifolds, solutions which converge to the unstable solitary wave, see [39, 26, 34]. Solutions on a stable manifold are necessarily nongeneric. Indeed, there are few results addressing *all* solutions with initial data in a neighbourhood of unstable solitary waves. There are three exceptions:

1. Small solitary waves obtained from a linear potential.

   A complete description of asymptotic behaviour is known in some cases. See [38, 32].

2. Solitary waves of the pure-power L^2-critical nonlinearity, $g(u) = |u|^{\frac{4}{N}} u$.

   There exist disjoint open subsets $\mathcal{K}^{\pm} \subset H^1$ such that the solitary waves belong to $\overline{\mathcal{K}^+} \cap \overline{\mathcal{K}^-}$. In some cases, we know that solutions in $\mathcal{K}^+$ scatter. See Killip, Tao and Visan [22] and papers that refer to it. On the other hand, solutions in $\mathcal{K}^-$ blowup in finite time, as proved by Merle and Raphaël [29] following a couple decades of careful asymptotic arguments. These blowup solutions are precisely described in terms of a soliton profile and a tracking error. The tracking error is arbitrarily small, and converges, in L^2. Indeed, it converges in H^1 outside any ball of fixed radius around the blowup point. The primary growth of H^1 norm is captured by the soliton profile $\phi_{\omega(t)}$, for which $\|\phi_{\omega(t)}\|_{\dot{H}^1} \to \infty$ as $\omega(t) \to \infty$.

   The L^2-critical nonlinearity is a degenerate case with physical relevance. The related literature is very large.

3. Solitary waves of the $\dot{H}^{\frac{1}{2}}$-critical nonlinearity, $g(u) = |u|^{\frac{4}{N-1}} u$.

   There exist disjoint open subsets $\mathcal{K}^{\pm} \subset H^1$ such that the solitary waves belong to $\overline{\mathcal{K}^+} \cap \overline{\mathcal{K}^-}$. Moreover, solutions in $\mathcal{K}^+$ scatter as $t \to \infty$ and radially symmetric solutions in $\mathcal{K}^-$ blowup in finite time. See the combined works of Duyckaerts, Holmer and Roudenko [18, 14, 16], expanding on the energy-critical argument of Kenig and Merle [21, 15]. For general energy-subcritical pure-power nonlinearities, see Akahori and Nawa [1].

   The nature of the blowup dynamic is unclear. Numerical results suggest a stable (almost) self-similar blowup regime. It is then natural to ask if there are solutions that blowup by following the soliton manifold, as in the L^2-critical case.

   For slightly L^2-supercritical nonlinearities, $g(u) = |u|^{\frac{4}{N}+\delta} u$, Merle, Raphaël and Szeftel [31] have shown that the L^2-critical blowup regime survives as sets of initial data $\mathcal{O}$, open in H^1, for which blowup occurs and can be described in terms of a member of the soliton family and a tracking error. At blowup time, the tracking error converges in all subcritical[1]norms to a fixed residue which is outside the critical space. This agrees with the more general result of Merle and Raphaël [30] that the critical norm of radially symmetric blowup solutions is unbounded. Moreover, there is a universal lower bound for the size of the residue in L^2_{loc}. Should blowup with a soliton profile occur in any other L^2-supercritical problems, Theorem 1.1 suggests there should be a similar L^2_{loc} lower bound on any residue.

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[1]For a pure-power nonlinearity, $g(u) = |u|^{p-2} u$, the critical norm is $L^{\frac{N}{2}(p-2)}$.

Should equation (1.1) lead to blowup, it may be structurally perturbed by a vanishing multiple of $u\,|u|^{m_2-1+}$ to be globally wellposed. As a result, Theorem 1.1 shows that the qualitative dynamic near a branch of unstable solitons is universal, irrespective of whether blowup eventually occurs. We do not exclude the possibility of solutions that blowup with a soliton profile, following an unstable branch of solitons at some distance. Should such solutions exist, Theorem 1.1 suggests that the blowup dynamic is only observed once they lie outside a particular neighbourhood of the manifold.

While we are concerned with large solitons, our approach will be similar to the small-soliton case. Kirr and Mızrak [25, 23, 24] consider the nonlinear Schrödinger equation with potential in dimensions two to five and detail the convergence to a center manifold of small stable solitons. Since they do not require (1.5), their work admits a larger range of nonlinearities down to the Strauss exponent. The technique in all three papers is focused on time-dependent linear operators, which we avoid, and is strictly limited to small solitons. Recent work of Beceanu [3] may offer a new route to soliton stability results in the H^1-setting. We do not know if Theorem 1.1 is optimal.

Our approach recovers the asymptotic stability of large solitary waves with no non-zero eigenvalue, for radial perturbations:

Proposition 1.2. *Let $1 < N < \infty$, suppose that for $\omega \in \mathcal{I}$ the linearized operator $\mathbf{JL}$ has eigenvalue zero with multiplicity two, no other discrete spectra, and that all other assumptions are valid with uniform estimates. Then the result of Theorem 1.1 holds. Moreover, only the convergence case occurs.*

For small solitons, potentials V_0 with linearized operators as in Proposition 1.2 exist. Proposition 1.2 then recovers a range of nonlinearities covered by Kirr, Mızrak and Zarnescu.

2 Decomposition and Algebraic Relations

First, in Section 2.1, we properly introduce the linearized operator $\mathbf{JL}$. Second, in Section 2.2, we decompose the solutions $u(t)$. This will allow us to phrase the bootstrap argument for Theorem 1.1, explained in Section 2.3. We then introduce particular tools in preparation for the following chapters. We state the dynamic equations of the modulation parameters in Section 2.4, the tracking-error equation in Section 2.5, and a Buslaev-Perelman decomposition and estimate of the continuous spectral projection operator of $\mathbf{JL}$ in Section 2.6. Finally, in Section 2.7, we introduce decay-rate constants and verify associated arithmetic.

2.1 Linearized Operator

Expand the potential around ϕ,

$$
\begin{aligned}
g(\phi + \varepsilon) &= g(\phi) + Dg(\phi)\varepsilon + \mathcal{N}(\varepsilon) \\
&= \phi f(|\phi|^2) + \left(\varepsilon f(|\phi|^2) + \phi f'(|\phi|^2)(\phi\overline{\varepsilon} + \overline{\phi}\varepsilon) \right) + \mathcal{N}(\varepsilon).
\end{aligned}
\tag{2.1}
$$

Immediately, we recognize the linearized operator

$$
\mathcal{L}(\varepsilon) = -\Delta\varepsilon + V_0\varepsilon + \omega\varepsilon - \left(\varepsilon f(|\phi|^2) + \phi f'(|\phi|^2)(\phi\overline{\varepsilon} + \overline{\phi}\varepsilon) \right)
$$

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around ϕ, which satisfies $i\partial_t \varepsilon = \mathcal{L}(\varepsilon) + \mathcal{O}(\varepsilon^2)$. Equation (1.2) allows us to estimate the nonlinear term (e.g. [24, equation (22)]),

$$|\mathcal{N}(\varepsilon)| \lesssim A_1 |\phi|^{m_1-2} |\varepsilon|^2 + A_2 |\phi|^{m_2-2} |\varepsilon|^2 + |\varepsilon|^{m_1} + |\varepsilon|^{m_2}, \qquad (2.2)$$

where $A_j = 0$ if $m_j \leq 2$.

Remark 2.1 (Complex-valued Functions as Vectors). Consider complex-valued functions $u = u_1 + iu_2$ and $v = v_1 + iv_2$, which we write as $\mathbb{R}^2$-valued functions: $[\![u]\!] = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $[\![v]\!] = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. The correct inner product is

$$\left\langle \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle = \int (u_1 \overline{v_1} + u_2 \overline{v_2}) \, dx.$$

We include the complex conjugate of v since we will later consider $\mathbb{C}^2$-valued functions. We denote the symplectic operator by

$$\mathbf{J} = \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix}.$$

Write $\varepsilon = \varepsilon_1 + i\varepsilon_2$, and represent ε as $[\![\varepsilon]\!] = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$. Then,

$$[\![\mathcal{L}(\varepsilon)]\!] = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \equiv \mathbf{L} [\![\varepsilon]\!] \equiv ((-\Delta + \omega)\mathbf{I} + \mathbf{V}) [\![\varepsilon]\!],$$

where

$$L_+ = -\Delta + V_0 + \omega - \left(f(\phi^2) + 2\phi^2 f'(\phi^2) \right), \quad L_- = -\Delta + V_0 + \omega - f(\phi^2).$$

The term $\mathbf{I}$ denotes the identity, and the potential term $\mathbf{V}$ has localized support and is written separately in anticipation of equation (2.19). Note that $\mathbf{L}$ is self-adjoint, and that the linearization of (1.1) near ϕ will feature $\mathbf{JL}$. The kernel of $\mathbf{JL}$ can be found by inspection: $L_-\phi = 0$ and $L_+\partial_\omega\phi = -\phi$. In vector notation:

$$\mathbf{JL} \begin{bmatrix} \partial_\omega\phi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \phi \end{bmatrix}, \quad \mathbf{JL} \begin{bmatrix} 0 \\ \phi \end{bmatrix} = 0.$$

Let $Y_+ = Y_{re} + iY_{im}$ be the eigenfunction associated with e_+:

$$\mathbf{JL} [\![Y_+]\!] = e_+ [\![Y_+]\!] \quad \Rightarrow \quad L_-Y_{im} = e_+Y_{re}, \quad L_+Y_{re} = -e_+Y_{im}. \qquad (2.3)$$

Let Y_- be an eigenfunction associated with e_-. From (2.3), one can verify $Y_- = \overline{Y_+}$ and $e_- = -e_+$. Since L_- is non-negative, and $L_-Y_{im} = e_+Y_{re}$, we note that $\langle Y_{re}, Y_{im} \rangle = \frac{1}{e_+} \langle L_-Y_{im}, Y_{im} \rangle > 0$. Without loss of generality we specify $2 \langle Y_{re}, Y_{im} \rangle = +1$. This describes the discrete eigenfunctions of $\mathbf{JL}$. Note that

$$\left\langle [\![Y_\pm]\!], \mathbf{J} \begin{bmatrix} 0 \\ \phi \end{bmatrix} \right\rangle = 0 \quad \text{and} \quad \left\langle [\![Y_\pm]\!], \mathbf{J} \begin{bmatrix} \partial_\omega\phi \\ 0 \end{bmatrix} \right\rangle = 0. \qquad (2.4)$$

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2.2 Orthogonality Conditions

We assume the decomposition

$$u(t) = \left(\phi_{\omega(t)} + a(t)\partial_\omega \phi_{\omega(t)} + b_+(t)Y_+ + b_-(t)Y_- + \eta(t) \right) e^{i\theta(t)}, \tag{2.5}$$

where the modulation parameters $\omega(t), \theta(t), a(t), b_+(t), b_-(t) \in \mathbb{R}$ are continuous functions determined by enforcing orthogonality conditions[2]. To determine the parameters:

$$\left\langle [\![\eta]\!], \mathbf{J} \begin{bmatrix} 0 \\ \phi \end{bmatrix} \right\rangle = 0 \tag{2.6}$$

$$\left\langle [\![\eta]\!], \mathbf{J} \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} \right\rangle = 0 \tag{2.7}$$

$$\langle [\![\eta]\!], \mathbf{J} [\![Y_+]\!] \rangle = 0 \tag{2.8}$$

$$\langle [\![\eta]\!], \mathbf{J} [\![Y_-]\!] \rangle = 0 \tag{2.9}$$

Parameters $\omega(t)$ and $a(t)$ are not independent; see the proof of Lemma 2.2, below. We choose to fix $a(t) = 0$. As a consequence, the linearized operator $\mathbf{JL}$, the eigenfunctions Y_+, Y_-, and their associated eigenvalues, are all themselves functions of time through $\omega(t)$. To simplify notation, this dependence is usually omitted, as in (2.5). When we consider a fixed operator, associated with some fixed value $\omega(T)$, we will refer to the associated linearized operator as $\mathbf{L}_T$ and the eigenvalues as $e_{\pm,\omega(T)}$.

The chosen orthogonality conditions are with the eigenfunctions of the adjoint of $\mathbf{JL}$. These conditions will allow an easy derivation of the dynamical equations in Section 2.4. Indeed, η is the projection onto the continuous spectrum of $\mathbf{JL}$,

$$\eta = \mathcal{P}_c \left(e^{-i\theta(t)} u(t) \right), \tag{2.10}$$

whenever the orthogonality conditions uniquely determine η.

To see that this is the case, let X_0 denote the $\mathbf{JL}$-invariant subspace associated with the generalized kernel, and X_1, X_c the subspaces associated with eigenvalues $e_\pm$ and the continuous spectrum respectively, so that $L^2 = X_0 \oplus X_1 \oplus X_c$. Let $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_c = \mathrm{Id} - \mathcal{P}_0 - \mathcal{P}_1$ denote the projection operators onto X_0, X_1 and X_c respectively. Explicitly,

$$\begin{aligned} \mathcal{P}_0 [\![f]\!] &= \frac{1}{\frac{1}{2}\partial_\omega \|\phi\|_{L^2}^2} \left(\left\langle [\![f]\!], \mathbf{J} \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} \right\rangle \begin{bmatrix} 0 \\ \phi \end{bmatrix} - \left\langle [\![f]\!], \mathbf{J} \begin{bmatrix} 0 \\ \phi \end{bmatrix} \right\rangle \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} \right), \\ \mathcal{P}_1 [\![f]\!] &= \frac{\langle [\![f]\!], [\![Y_+]\!] \rangle}{2 \langle Y_{re}, Y_{im} \rangle} \mathbf{J} [\![Y_-]\!] - \frac{\langle [\![f]\!], [\![Y_-]\!] \rangle}{2 \langle Y_{re}, Y_{im} \rangle} \mathbf{J} [\![Y_+]\!], \end{aligned} \tag{2.11}$$

and we note that $\mathcal{P}_c [\![f]\!]$ satisfies the orthogonality conditions.

Lemma 2.2 (Ability to Modulate). *Fix $\omega_1 \in \mathcal{I}$, $\theta_1 \in \mathbb{R}$ and Banach space L^r, $1 \le r < \infty$. Then, for u in some neighbourhood of $\phi_{\omega_1} e^{i\theta_1}$, there exists a Lipschitz map $u \mapsto (\theta, \omega, b_+, b_-)$ such that, under decomposition (2.5), orthogonality conditions (2.6)-(2.9) are satisfied. The radius of the neighbourhood is of the order $\mathrm{dist}(\omega_1, \partial\mathcal{I})$ as $\mathrm{dist}(\omega_1, \partial\mathcal{I}) \to 0^+$.*

---

[2]By way of comparison with Buslaev-Sulem, note that the coefficients $b_\pm(t)$ and the components of $[\![Y_\pm]\!]$ are real-valued. In the case of stability and eigenvalues $\pm i\lambda$, the second component of the eigenfunction in vector form is purely imaginary (corresponding to an entirely real-valued eigenfunction), and the appropriate decomposition is $z(t) [\![\psi]\!] + \overline{z(t)} \, [\![\overline{\psi}]\!]$.

For our application we take $r = p = m_2 + 1$. In particular, $|\omega_0 - \omega(0)| \lesssim \alpha$ and the modulation parameters are continuous in time.

Proof. Define the map $\rho : (\theta, \omega, u) \in \mathbb{R} \times \mathbb{R} \times L^r \rightarrow \rho(\theta, \omega, u) \in \mathbb{R}^2$ by

$$\rho(\theta, \omega, u) = \left(\left\langle [\![\varepsilon]\!], \mathbf{J} \begin{bmatrix} 0 \\ \phi_\omega \end{bmatrix} \right\rangle, \left\langle [\![\varepsilon]\!], \mathbf{J} \begin{bmatrix} \partial_\omega \phi_\omega \\ 0 \end{bmatrix} \right\rangle \right) \quad \text{where} \quad \varepsilon = ue^{-i\theta} - \phi_\omega.$$

At $\left(\theta_1, \omega_1, \phi_{\omega_1} e^{i\theta_1} \right)$, $\rho = 0$, and the Jacobian with respect to (θ, ω) is

$$\begin{bmatrix} 0 & -\frac{1}{2} \partial_\omega \|\phi\|^2_{L^2} \big|_{\omega = \omega_1} \\ \frac{1}{2} \partial_\omega \|\phi\|^2_{L^2} \big|_{\omega = \omega_1} & 0 \end{bmatrix}.$$

The Jacobian is nonsingular and its inverse is uniformly bounded over $\omega_1 \in \mathcal{I}$ by assumption. We may apply the implicit function theorem on Banach spaces (e.g. Berger [5]) to solve for $(\theta(u), \omega(u))$ such that $\rho(\theta(u), \omega(u), u) = 0$. We then define η, b_+, and b_- by (2.10) and (2.11). $\qquad\square$

2.3 Critical Time T_{crit} & Proof Strategy

For all data under consideration, there exists $T_{dec} > 0$ such that we may decompose the solution in the manner of (2.5) for $t \in [0, T_{dec})$,

$$u(t) = \left(\phi_{\omega(t)} + b_+(t) Y_+ + b_-(t) Y_- + \eta(t) \right) e^{i\theta(t)}. \tag{2.12}$$

Define a new time scale $T_{crit} \leq T_{dec}$ in terms of persistent good control of $b_+(t)$,

$$T_{crit} = \sup_{T < T_{dec}} \left\{ \forall t \in [0, T], \quad |b_+(t)| < \alpha \langle t \rangle^{-1} \right\}. \tag{2.13}$$

Our proof proceeds on two paths, depending on T_{crit}:

1. $T_{crit} = T_{dec}$

   This case is considered in Section 3. We prove that $T_{crit} = +\infty$ and that the solution converges to the soliton family.

2. $T_{crit} < T_{dec}$

   This case is considered in Section 4. We prove that the growth of $b_+(t)$ cannot be controlled, and that exit from the neighbourhood of the soliton family occurs at T_{exit}, with $T_{crit} < T_{exit} < T_{dec}$.

2.4 Dynamic Equations

Substitute (2.12) into (1.1) and use the expansion (2.1):

$$i(\dot{\theta} - \omega)(\phi + \varepsilon) + \dot{\omega}\partial_\omega \phi + \partial_t \varepsilon = -i\left(\mathcal{L}\varepsilon + \mathcal{N}(\varepsilon)\right),$$

where $\varepsilon = b_+ Y_+ + b_- Y_- + \eta$, and $\mathcal{N}$ was defined in (2.1). In vector form,

$$\begin{aligned} \partial_t \left([\![\eta]\!] + b_+ [\![Y_+]\!] + b_- [\![Y_-]\!] \right) + \dot{\omega} \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} + (\dot{\theta} - \omega) \begin{bmatrix} 0 \\ \phi \end{bmatrix} \\ = \mathbf{JL} [\![\varepsilon]\!] + (\dot{\theta} - \omega)\mathbf{J} [\![\varepsilon]\!] + \mathbf{J}N(\varepsilon), \end{aligned} \tag{2.14}$$

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where we use the notation $N(\varepsilon) = [\![\mathcal{N}(\varepsilon)]\!]$ and write $\partial_t [\![\varepsilon]\!]$ in full to emphasize its terms will be handled differently. Take the product of (2.14) by $\mathbf{J}\begin{bmatrix} 0 \\ \phi \end{bmatrix}$ and use (2.6) to integrate $\left\langle \partial_t [\![\eta]\!], \mathbf{J}\begin{bmatrix} 0 \\ \phi \end{bmatrix} \right\rangle$ by parts in time. We get

$$\dot\omega \langle \partial_\omega \phi, \phi \rangle = \dot\omega \left\langle [\![\varepsilon]\!], \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} \right\rangle + (\dot\theta - \omega) \left\langle [\![\varepsilon]\!], \begin{bmatrix} 0 \\ \phi \end{bmatrix} \right\rangle + \left\langle N(\varepsilon), \begin{bmatrix} 0 \\ \phi \end{bmatrix} \right\rangle. \tag{2.15}$$

Similarly, with $-\mathbf{J}\begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix}$ and using (2.7), we get

$$(\dot\theta - \omega) \langle \partial_\omega \phi, \phi \rangle = \dot\omega \left\langle [\![\varepsilon]\!], \begin{bmatrix} 0 \\ \partial_\omega^2 \phi \end{bmatrix} \right\rangle - (\dot\theta - \omega) \left\langle [\![\varepsilon]\!], \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} \right\rangle - \left\langle N(\varepsilon), \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} \right\rangle. \tag{2.16}$$

Now with $\mathbf{J}[\![Y_+]\!]$ and using (2.8) to remove the terms in $\partial_t \eta$ and $\mathbf{L}\eta$, we obtain (recall that $2\langle Y_{re}, Y_{im} \rangle = 1$):

$$\dot b_- = e_- b_-$$
$$+ \frac{1}{2\langle Y_{re}, Y_{im} \rangle} \begin{pmatrix} \dot\omega \left(\langle [\![\eta]\!], \partial_\omega \mathbf{J}[\![Y_+]\!] \rangle - \langle b_+ \partial_\omega [\![Y_+]\!] + b_- \partial_\omega [\![Y_-]\!], \mathbf{J}[\![Y_+]\!] \rangle \right) \\ + \left\langle \left(\dot\theta - \omega\right)[\![\varepsilon]\!] + N(\varepsilon), [\![Y_+]\!] \right\rangle \end{pmatrix}. \tag{2.17}$$

Finally, with $\mathbf{J}[\![Y_-]\!]$ and using (2.9) to remove the terms in $\partial_t \eta$ and $\mathbf{L}\eta$, we get:

$$\dot b_+ = e_+ b_+$$
$$- \frac{1}{2\langle Y_{re}, Y_{im} \rangle} \begin{pmatrix} \dot\omega \left(\langle [\![\eta]\!], \partial_\omega \mathbf{J}[\![Y_-]\!] \rangle - \langle b_+ \partial_\omega [\![Y_+]\!] + b_- \partial_\omega [\![Y_-]\!], \mathbf{J}[\![Y_-]\!] \rangle \right) \\ + \left\langle \left(\dot\theta - \omega\right)[\![\varepsilon]\!] + N(\varepsilon), [\![Y_-]\!] \right\rangle \end{pmatrix}. \tag{2.18}$$

2.5 Tracking-Error Equation

It will prove more straightforward to estimate the tracking-error η in terms of a fixed operator. Let $\mathcal{P}_{c,T}$ denote the projection onto the continuous spectrum of the operator with some fixed $\omega(T)$.[3] Denote $\widetilde\eta = \mathcal{P}_{c,T}\eta$. We do **not** change our choice of decomposition.

We first isolate linear terms in η in (2.14), so that

$$\partial_t [\![\eta]\!] = \mathbf{J}\mathbf{L}[\![\eta]\!] + (\dot\theta - \omega)\mathbf{J}[\![\eta]\!] + B_0 + \mathbf{J}N, \quad \text{where}$$
$$B_0 = -\dot\omega \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} - (\dot\theta - \omega)\begin{bmatrix} 0 \\ \phi \end{bmatrix} + \left\{ \mathbf{J}\mathbf{L} - \partial_t + (\dot\theta - \omega)\mathbf{J} \right\} (b_+ [\![Y_+]\!] + b_- [\![Y_-]\!]).$$

Note that $\mathbf{V} \equiv \mathbf{L} - (-\Delta + V_0 + \omega)\mathbf{I}$ is a localized potential. We further rewrite

$$\partial_t [\![\eta]\!] = \mathbf{J}\mathbf{L}_T [\![\eta]\!] + (\dot\theta - \omega_T)\mathbf{J}[\![\eta]\!] + \mathbf{J}(\mathbf{V} - \mathbf{V}_T)[\![\eta]\!] + B_0 + \mathbf{J}N.$$

Applying $\mathcal{P}_{c,T}$ to all terms, we get

$$\partial_t [\![\widetilde\eta]\!] = \mathbf{J}\mathbf{L}_T [\![\widetilde\eta]\!] + \left(\dot\theta - \omega_T\right)\mathcal{P}_{c,T}\mathbf{J}[\![\widetilde\eta]\!]$$
$$+ \mathcal{P}_{c,T}\left(-\left(\dot\theta - \omega_T\right)\mathbf{J}(\mathcal{P}_{c,T} - \mathcal{P}_c)[\![\eta]\!] + \mathbf{J}(\mathbf{V} - \mathbf{V}_T)[\![\eta]\!] + B + \mathbf{J}N \right), \tag{2.19}$$

---

[3]The convergence of $\omega(t)$ as $t \to T_{crit}$ will be established by Lemma 3.1.

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where $\mathcal{P}_{c,T}B = \mathcal{P}_{c,T}B_0$ and

$$B = - (\mathcal{P}_{c,T} - \mathcal{P}_c) \left(\dot{\omega} \begin{bmatrix} \partial_\omega \phi \\ 0 \end{bmatrix} + \left(\dot{\theta} - \omega \right) \begin{bmatrix} 0 \\ \phi \end{bmatrix} \right)$$

$$- (\mathcal{P}_{c,T} - \mathcal{P}_c) \left((\dot{b}_+ - e_+ b_+) [\![Y_+]\!] + (\dot{b}_- - e_- b_-) [\![Y_-]\!] \right)$$

$$- \dot{\omega} (b_+ \partial_\omega [\![Y_+]\!] + b_- \partial_\omega [\![Y_-]\!]) + \left(\dot{\theta} - \omega \right) \mathbf{J} (b_+ [\![Y_+]\!] + b_- [\![Y_-]\!]).$$

Observe from (2.11) that both $\mathbf{V} - \mathbf{V}_T$ and $\mathcal{P}_{c,T} - \mathcal{P}_c$ are localized potentials, of the order $|\omega - \omega_T|$, depending on $\mathcal{I}$ and provided $|\omega - \omega_T|$ is sufficiently small. Also note that B has localized spatial support, and

$$B = \mathcal{O} \left((|\omega - \omega_T| + |b_+| + |b_-|)(|\dot{\omega}| + \left| \dot{\theta} - \omega \right| + \left| \dot{b}_+ - e_+ b_+ \right| + \left| \dot{b}_- - e_- b_- \right|) \right). \qquad (2.20)$$

2.6 Buslaev-Perelman Estimate

Later we will assume the tracking-error $\widetilde{\eta}$ is small, and then the evolution given by (2.19) should be essentially linear. To handle the leading order correction in $(\dot{\theta} - \omega)$, we use a technique introduced by Buslaev and Perelman [6], later discussed by Buslaev and Sulem [7]. For $N \geq 3$, the proof is given by Cuccagna [12]. For $N = 2$, the proof is claimed by Cuccagna and Tarulli [13]. Let $\mathcal{P}_+$ and $\mathcal{P}_-$ denote the spectral projection operators onto the positive and negative continuous spectrum. That is, $\mathcal{P}_{c,T} = \mathcal{P}_+ + \mathcal{P}_-$.

Proposition 2.3.

$$- \mathcal{P}_{c,T} \mathbf{J} = \mathcal{P}_{c,T} \mathbf{J}^{-1} = i\mathcal{P}_+ - i\mathcal{P}_- + \mathbf{K}, \qquad (2.21)$$

where $\mathbf{K}$ is a localizing operator, bounded $L^q \mapsto L^{r'}$, for any pair $1 < r' \leq 2 \leq q < \infty$.

Remark 2.4. To motivate (2.21), note that an equation of the form $\partial_t [\![f]\!] = \mathbf{J} \mathbf{L}_T [\![f]\!]$ may also be written in the form

$$i\partial_t \begin{bmatrix} f \\ \overline{f} \end{bmatrix} = \begin{bmatrix} -\Delta + \omega - V_1 & V_2 \\ V_3 & \Delta - \omega + V_4 \end{bmatrix} \begin{bmatrix} f \\ \overline{f} \end{bmatrix},$$

where the potentials V_j are due to both V_0 and the nonlinearity. We may view this matrix operator as a perturbation of $\begin{bmatrix} -\Delta + \omega + V_0 & 0 \\ 0 & \Delta - \omega - V_0 \end{bmatrix}$, for which (2.21) is an identity when $\mathbf{K} = 0$.

Let us decompose $[\![\widetilde{\eta}]\!] = \mathcal{P}_{c,T} [\![\eta]\!]$ according to $\mathcal{P}_\pm$, and incorporate the accumulated error in tracking the phase,

$$\widetilde{\eta}_\pm = (\mathcal{P}_\pm [\![\widetilde{\eta}]\!]) e^{\mp i(\theta(t) - t\omega_T)} \equiv (\mathcal{P}_\pm [\![\widetilde{\eta}]\!]) e^{\pm i\Theta(t)}.$$

The evolution of $\widetilde{\eta}_\pm$ follows from (2.19),

$$\partial_t \widetilde{\eta}_\pm = \mathbf{J} \mathbf{L}_T \widetilde{\eta}_\pm + \mathcal{P}_\pm e^{\pm i\Theta(t)} \left(\left(\dot{\theta} - \omega_T \right) \mathbf{K} [\![\widetilde{\eta}]\!] + \mathbf{J}(\mathbf{V} - \mathbf{V}_T) [\![\widetilde{\eta}]\!] + B + \mathbf{J} N(\varepsilon) \right), \qquad (2.22)$$

where we have abused notation to absorb $(\dot{\theta} - \omega_T)(\mathcal{P}_{c,T} - \mathcal{P}_c)$ into $\mathbf{V} - \mathbf{V}_T$. Equation (2.22) will be our means to establish L^p and L^q estimates for $\widetilde{\eta}$ through the equivalent Duhamel formulation:

$$\widetilde{\eta}_\pm = e^{t\mathbf{J}\mathbf{L}_T} \widetilde{\eta}_\pm(0)$$

$$+ \int_0^t e^{(t-s)\mathbf{J}\mathbf{L}_T} \mathcal{P}_\pm e^{\pm i\Theta(s)} \left(\left(\dot{\theta} - \omega \right) \mathbf{K} [\![\widetilde{\eta}]\!] + \mathbf{J}(\mathbf{V} - \mathbf{V}_T) [\![\widetilde{\eta}]\!] + B + \mathbf{J} N(\varepsilon) \right) (s) \, ds. \qquad (2.23)$$

2.7   Arithmetic

For later reference, we collect here an assortment of arithmetic facts. Due to (1.4) and (1.5), there exists $\delta > 0$ sufficiently small such that we may define:

$$\sigma_q = \begin{cases} \sigma_p & \text{for } \sigma_p > \dfrac{2}{m_0 + 1}, \\[2mm] \dfrac{2}{m_0 + 1} + \delta & \text{for } \sigma_p \leq \dfrac{2}{m_0 + 1}, \end{cases} \tag{2.24}$$

$$\text{with } \sigma_q < \min\{1, \frac{N}{2}(m_1 - 1) - 1\}.$$

Note that $\sigma_q \in [\sigma_p, 1)$, corresponding to some $q \in [p, p_{\max})$. Such a choice is only possible for $N \geq 2$, since

$$(m_0 + 1)\sigma_q > 2, \tag{2.25}$$

and, in particular, $\sigma_q > \frac{1}{2}$. Constant $\delta > 0$ is included to ensure strict inequality for (2.25), and is otherwise neglected by taking it sufficiently small. Let us emphasize that

$$m_1 > 1 + \frac{2}{N}(1 + \sigma_q). \tag{2.26}$$

Consider $m_j \theta_j \sigma_p \equiv \frac{N}{2}(m_j - 1) - \sigma_p$, an expression that will arise during norm interpolation. Then

$$m_j \theta_j \sigma_p > 1 \quad \text{and} \quad m_j \theta_j > 1, \tag{2.27}$$

which are due to (1.4) and $\sigma_p < 1$, respectively. Due to $q < p_{\max}$,

$$2\left(1 - \frac{1}{q}\right) < 1 + \frac{2}{N} < m_j. \tag{2.28}$$

Consider $m_j \widetilde{\theta}_j \sigma_p \equiv \frac{N}{2}(m_j - 1) - \sigma_q$, another expression that will arise during norm interpolation. Due to (2.26), and since $\sigma_p < 1 < m_0$,

$$m_j \widetilde{\theta}_j \sigma_p > 1 \quad \text{and} \quad m_j \widetilde{\theta}_j > \frac{1}{m_0}. \tag{2.29}$$

Finally, one can verify that $\frac{1}{m_0}\left(1 - \frac{m_0}{2}\right) > 1 - m_j\left(\frac{m_0}{2}\right)$ is a consequence of $m_j \geq m_1 > 1$, and since $m_1^3 - 3m_1 + 2 > 0$ when $m_0 = m_1 < 2$. The second expression of (2.29) implies:

$$(1 - \widetilde{\theta}_j)m_j\left(\frac{m_0}{2}\right) + m_j\widetilde{\theta}_j > 1. \tag{2.30}$$

3   Convergence Case

In this section, we will consider the case of $T_{crit} = T_{dec}$. We will prove that $T_{crit} = +\infty$ and convergence to a soliton by means of a bootstrap argument. The estimates shown below will be reused in Section 4.

3.1 Hypotheses and First Estimates

For $t \in [0, T_{dec})$, we decompose $u(t)$ as in (2.12). Assume that $T_{hyp} \in (0, T_{crit}]$ is the last time for which the following hypotheses hold for all $t \in [0, T_{hyp})$:

$$
\begin{aligned}
|b_-(t)| &\leq 2C_1\alpha\langle t\rangle^{-1}, \\
\|\eta(t)\|_{L^p} &\leq 2C_2\alpha\langle t\rangle^{-\sigma_p}, \\
\|\eta(t)\|_{L^q} &\leq 2C_3\alpha\langle t\rangle^{-\sigma_q}.
\end{aligned}
\tag{3.1}
$$

Universal constants C_j will be determined later. Recall that from (2.13), the definition of T_{crit}, we also have

$$
|b_+(t)| \leq \alpha\langle t\rangle^{-1}.
\tag{3.2}
$$

Under these hypotheses, the goal is now to obtain the same inequalities as (3.1) without the factor 2, by choosing $\overline{\alpha_0}$ small enough. We now fix $T = T_{hyp}$. First, we prove estimates on the parameters θ and ω.

Lemma 3.1. *For all $t \in [0, T_{hyp})$, we have $\omega(t) \in \mathcal{I}$,*

$$
\begin{aligned}
|\dot\omega(t)| + |\dot\theta(t) - \omega(t)| &\leq C\alpha^{m_0}\langle t\rangle^{-m_0\sigma_q}, \qquad \text{which implies} \\
|\omega(t) - \omega(T)| &\leq C\alpha^{m_0}\langle t\rangle^{-m_0\sigma_q+1}, \quad \text{and} \\
|e_{\pm,\omega(t)} - e_{\pm,\omega(T)}| &\leq C\alpha^{m_0}\langle t\rangle^{-m_0\sigma_q+1}.
\end{aligned}
\tag{3.3}
$$

By exactly the same proof, we have

$$
\left|\dot{b}_\pm - e_\pm b_\pm\right| \leq C\alpha^{m_0}\langle t\rangle^{-m_0\sigma_q}.
$$

Proof. We will only prove the first inequality. To do this, we add (2.15) and (2.16), to obtain $|\dot\omega| + |\dot\theta - \omega| \leq (|\dot\omega| + |\dot\theta - \omega|)\mathcal{O}(\varepsilon) + |\langle N(\varepsilon), \text{potential}\rangle|$. With $\overline{\alpha_0}$ small enough, we get $|\dot\omega| + |\dot\theta - \omega| \leq \langle N(\varepsilon), \text{potential}\rangle$, and (3.3) follows from (2.2) and the bootstrap inequalities (3.1) and (3.2). We use the L^q control of η since it may be stronger by (2.24). $\qquad\square$

3.2 Improved Estimates

Estimate on b_-. We can now improve the estimate on b_-. Fix $e_1 = e_{-,\omega(T)} < 0$. From Lemma 3.1 we obtain:

$$
\left|\dot{b}_- - e_1 b_-\right| \lesssim \alpha^{m_0}\langle t\rangle^{-m_0\sigma_q}\left(\langle t\rangle\,|b_-(t)| + 1\right) \leq C_0\alpha^{m_0}\langle t\rangle^{-m_0\sigma_q},
$$

where $C_0 > 0$ is an explicit constant depending on $\mathcal{I}$. Moreover, since $\|\varepsilon(0)\|_{H^1\cap L^1} = \alpha$, we may assume $|b_-(0)| \leq C_0\alpha$. Integrating between 0 and t,

$$
\begin{aligned}
|b_-(t)| &\leq e^{+e_1 t}C_0\alpha + \int_0^t e^{+e_1(t-s)}C_0\alpha^{m_0}\langle s\rangle^{-m_0\sigma_q}\,ds \\
&\lesssim \alpha e^{+e_1 t} + \alpha^{m_0}\left(\int_0^{t-\frac{m_0\sigma_q}{|e_1|}\ln t} e^{+e_-(t-s)}\,ds + \int_{t-\frac{m_0\sigma_q}{|e_1|}\ln t}^t \langle s\rangle^{-m_0\sigma_q}\,ds\right),
\end{aligned}
$$

with the obvious corrections to integral bounds when $t \approx 0$. For $\overline{\alpha_0}$ sufficient small, we have shown $|b_-(t)| \lesssim \alpha\left(1 + \ln\langle t\rangle\right)\langle t\rangle^{-m_0\sigma_q} \leq C_1\alpha\langle t\rangle^{-1}$ where $C_1 > C_0$ is the appropriate universal constant.

Decomposition of η. Recall from (2.10) that $\eta = \mathcal{P}_c\eta$, implying

$$\eta = \widetilde{\eta} + (\mathcal{P}_c - \mathcal{P}_{c,T})\,\eta,$$

so that norms of η, $\widetilde{\eta}$, and $\widetilde{\eta}_{\pm}$ are all comparable. Due to (2.6), we have

$$\|u(t)\|_{L^2}^2 \approx \langle \phi_\omega + \eta, \phi_\omega + \eta \rangle = \|\phi_\omega\|_{L^2}^2 + \|\eta\|_{L^2}^2,$$

where we have ignored terms of order $b_\pm(t)$. From Lemma 3.1, we conclude

$$\|\eta(t)\|_{L^2} \le C_4 \alpha^{\frac{m_0}{2}}, \tag{3.4}$$

where the constant C_4 depends on $\mathcal{I}$. Now we estimate terms of (2.23), in some norm $L^{r'}$, with $r' \in (1,2)$. From Proposition 2.3, (2.25), and Lemma 3.1:

$$\left\| \left(\dot\theta - \omega_T\right) \mathbf{K}\,[\![\widetilde{\eta}]\!] \right\|_{L^{r'}} \le \left(\dot\theta - \omega_T\right) \|\widetilde{\eta}\|_{L^q} \le C\alpha^{2+}\langle t\rangle^{-(m_0+1)\sigma_q+1} \lesssim \alpha^{2+}\langle t\rangle^{-(1+)}. \tag{3.5}$$

Note that we use here standard notation 1+ to designate a number slightly bigger than 1. The same estimate holds for both $\|\mathbf{J}(\mathbf{V} - \mathbf{V}_T)\,[\![\widetilde{\eta}]\!]\|_{L^{r'}}$ and $\|B\|_{L^{r'}}$. Indeed, from (2.20), we have

$$\|B\|_{L^{r'}} \le C\alpha^{2+}\left(\langle t\rangle^{-2m_0\sigma_q+1} + \langle t\rangle^{-m_0\sigma_q-1}\right), \tag{3.6}$$

which is lower order provided $\sigma_q < 1$.

L^p estimate of η. To start, we estimate $\|N(\varepsilon)\|_{L^{p'}}$ using equation (2.2):

$$\|\varepsilon^{m_j}\|_{L^{p'}} \lesssim (|b_+| + |b_-|)^{m_j} + \|\eta\|_{L^{m_j p'}}^{m_j},$$

which may be interpolated as

$$\|\eta\|_{L^{m_j p'}}^{m_j} \le \|\eta\|_{L^2}^{(1-\theta_j)m_j}\|\eta\|_{L^p}^{m_j\theta_j}, \tag{3.7}$$

with $\dfrac{1}{m_j p'} = \dfrac{\theta_j}{p} + \dfrac{1-\theta_j}{2}$, or, equivalently, $\theta_j = \dfrac{\frac{1}{2} - \frac{1}{m_j p'}}{\frac{1}{2} - \frac{1}{p}}$. We note that $\theta_j \in (0,1]$ provided $2\left(1 - \dfrac{1}{p}\right) < m_j \le p - 1$, which is implied by $1 + \frac{2}{N} < m_j \le m_2$. With (3.1), (3.2) and (2.27), we have

$$\|\varepsilon^{m_j}\|_{L^{p'}} \lesssim \alpha^{1+}\langle t\rangle^{-(1+)}. \tag{3.8}$$

If $m_j > 2$,

$$\|A_j\phi^{m_j-2}\varepsilon^2\|_{L^{p'}} \lesssim (|b_+| + |b_-|)^2 + \|\eta\|_{L^q}^2 \le \alpha^2\langle t\rangle^{-(1+)},$$

which we view as a correction to (3.8).

Apply (1.6) to all terms of (2.23), with $r = p$. Since L^p is energy subcritical, we may use that $\|\widetilde{\eta}_\pm(0)\|_{H^1}$ is small to improve the bound on the linear term,

$$\|e^{t\mathbf{J}\mathbf{L}_T}\widetilde{\eta}_\pm(0)\|_{L^p} \le \frac{C_2}{4}\alpha\langle t\rangle^{-\sigma_p}.$$

The universal constant C_2 is determined by this relation. For the other terms, apply (3.5), (3.6) and (3.8) to get

$$\|\widetilde{\eta}_\pm(t)\|_{L^p} \le \frac{C_2}{4}\alpha\langle t\rangle^{-\sigma_p} + C\alpha^{1+}\int_0^t |t-s|^{-\sigma_p}\langle s\rangle^{-(1+)}\,ds. \tag{3.9}$$

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This proves $\|\eta(t)\|_{L^p} \le C_2\alpha\langle t\rangle^{-\sigma_p}$, as desired, by assuming $\overline{\alpha_0} > 0$ is sufficiently small.

L^q *estimate of* η. We consider now the L^q norm of η. From (2.23), we have

$$
\begin{aligned}
\|\widetilde{\eta}_\pm(t)\|_{L^q} &\lesssim \|e^{t\mathbf{JL}_T}\widetilde{\eta}_\pm(0)\|_{L^q} \\
&\quad + \int_0^t \|e^{(t-s)\mathbf{JL}_T}\mathcal{P}_\pm\left(\left(\dot\theta - \omega\right)\mathbf{K}\,[\![\widetilde{\eta}]\!] + \mathbf{J}(\mathbf{V} - \mathbf{V}_T)\,[\![\widetilde{\eta}]\!] + B\right)\|_{L^q}ds \\
&\quad + \int_0^t \|e^{(t-s)\mathbf{JL}_T}\mathcal{P}_\pm N(\varepsilon)\|_{L^q}\,ds \\
&\equiv \mathbf{I} + \mathbf{II} + \mathbf{III}.
\end{aligned}
\tag{3.10}
$$

To estimate these terms, we use dispersive estimate (1.6). We first get

$$
\mathbf{I} \lesssim \langle t\rangle^{-\sigma_q}\|\widetilde{\eta}_\pm(0)\|_{L^1\cap H^1} \le \frac{C_3}{2}\alpha\langle t\rangle^{-\sigma_q}.
\tag{3.11}
$$

The universal constant C_3 is determined by this relation.

Term $\mathbf{III}$ is treated in a similar way as for the L^p estimate. We have

$$
\|\varepsilon^{m_j}\|_{L^{q'}} \lesssim (|b_+| + |b_-|)^{m_j} + \|\eta\|_{L^{m_jq'}}^{m_j},
$$

where now we interpolate according to

$$
\|\eta\|_{L^{m_jq'}}^{m_j} \le \|\eta\|_{L^2}^{(1-\widetilde{\theta}_j)m_j}\|\eta\|_{L^p}^{m_j\widetilde{\theta}_j},
\tag{3.12}
$$

with $\dfrac{1}{m_jq'} = \dfrac{1-\widetilde{\theta}_j}{2} + \dfrac{\widetilde{\theta}_j}{p}$, or, equivalently, $\widetilde{\theta}_j = \dfrac{\frac{1}{2} - \frac{1}{m_jq'}}{\frac{1}{2} - \frac{1}{p}}$. We note that $\widetilde{\theta}_j \in (0,1]$ provided $2\left(1 - \dfrac{1}{q}\right) < m_j \le p\left(1 - \dfrac{1}{q}\right)$. The former inequality is (2.28), while the latter inequality is true for any $q \ge p \ge m_j + 1$. From (3.4), the terms of (3.12) give α with the exponent $(1 - \widetilde{\theta}_j)m_j\frac{m_0}{2} + m_j\widetilde{\theta}_j$, which is greater than 1 by (2.30). We have

$$
\|\varepsilon^{m_j}\|_{L^{q'}} \lesssim \alpha^{1+}\langle t\rangle^{-m_j\widetilde{\theta}_j\sigma_p},
\tag{3.13}
$$

and if $m_j > 2$,

$$
\|A_j\phi^{m_j-2}\varepsilon^2\|_{L^{q'}} \lesssim (|b_+| + |b_-|)^2 + \|\eta\|_{L^q}^2 \le \alpha^2\langle t\rangle^{-2\sigma_q},
\tag{3.14}
$$

and both decay faster than $\langle t\rangle^{-1}$, due to (2.29) and (2.25), respectively. With (1.6),

$$
\mathbf{III} \lesssim \alpha^{1+}\int_0^t |t-s|^{-\sigma_q}\langle s\rangle^{-(1+)}\,ds \lesssim \alpha^{1+}\langle t\rangle^{-\sigma_q}.
\tag{3.15}
$$

The integral is a simple calculation since $\sigma_q < 1$. Term $\mathbf{II}$ may be included with $\mathbf{III}$. Its contribution is controlled by (3.5) and (3.6) applied with $r = q$.

Assuming $\overline{\alpha_0} > 0$ is sufficiently small, we have shown that $\|\eta(t)\|_{L^q} \le C_3\alpha\langle t\rangle^{-\sigma_q}$.

3.3 Bootstrap Conclusions

It was shown in the previous section that, for all $t \in [0, T_{hyp})$,

$$
\begin{aligned}
|b_-(t)| &\leq C_1 \alpha \langle t \rangle^{-1}, \\
\|\eta(t)\|_{L^p} &\leq C_2 \alpha \langle t \rangle^{-\sigma_p}, \\
\|\eta(t)\|_{L^q} &\leq C_3 \alpha \langle t \rangle^{-\sigma_q}.
\end{aligned}
\tag{3.16}
$$

As these are continuously evolving quantities, by (3.1) it must be that $T_{hyp} = T_{crit}$. Together with the estimate for b_+, this proves proximity to the soliton family as $t \to T_{crit}$, and so by Lemma 2.2 the decomposition can be extended past $t = T_{crit}$. Therefore, if $T_{crit} = T_{dec}$, then $T_{dec} = +\infty$.

By integration of the first equation of (3.3), there exists $\omega_+ \in \mathcal{I}$, $|\omega(0) - \omega_+| \lesssim \alpha^{m_0}$ such that the remaining estimates of (3.3) hold. It is essential here that $m_0 \sigma_q > 1$. From (3.16), we have that $\|u(t) - \phi_{\omega(t)} e^{i\theta(t)}\|_{L^p} \leq C\alpha \langle t \rangle^{-\sigma_p}$. Together we have, for all $t > 0$,

$$
\|u(t) - \phi_{\omega_+} e^{i\theta(t)}\|_{L^p} \lesssim \alpha \langle t \rangle^{-\sigma_p} + \alpha^{m_0} \langle t \rangle^{-m_0 \sigma_q + 1}.
$$

For Theorem 1.1, we use $\mu = \min\{\sigma_p, m_0 \sigma_q - 1\} > 0$.

4 Escape Case

In this section, we consider the case of $T_{crit} < T_{dec}$. The arguments of Section 3 apply for $t \in [0, T_{crit})$. We extend these arguments to prove that parameter $|b_{+,\omega(t)}|$ grows exponentially for an interval of time after T_{crit}.

4.1 Hypotheses and First Estimates

Assume that $T_{exit} \in (T_{crit}, T_{dec}]$ is the last time for which the following hypotheses (4.1)-(4.3) hold for all $t \in [0, T_{exit})$:

$$
\begin{aligned}
|b_-(t)| &\leq 2C_5 |b_+(t)|, \\
\|\eta(t)\|_{L^p} &\leq 2C_6 \left(|b_+(t)|^{m_0} + \alpha \langle t \rangle^{-\sigma_p}\right), \\
\|\eta(t)\|_{L^q} &\leq 2C_7 \left(|b_+(t)| + \alpha \langle t \rangle^{-\sigma_q}\right).
\end{aligned}
\tag{4.1}
$$

Universal constants C_j will be determined later. In place of (2.13), we make two further hypotheses: for all $s < t \in [T_{crit}, T_{exit})$,

$$
e^{\frac{2}{3} e_2 (t-s)} \leq \frac{b_+(t)}{b_+(s)} \leq e^{\frac{3}{2} e_2 (t-s)},
\tag{4.2}
$$

where $e_2 = e_{+,\omega(T_{crit})}$, the positive eigenvalue of $\mathbf{L}_T$ for $T = T_{crit}$. Recall that $e_1 = e_{-,\omega(T_{crit})} = -e_2 < 0$. Our final hypothesis is, for all $t \in [0, T_{exit})$,

$$
|b_+(t)| < 3\alpha_0.
\tag{4.3}
$$

As a particular consequence of (4.2) and (2.13), note that for $t \in (T_{crit}, T_{exit})$,

$$
|b_+(t)| > |b_+(T_{crit})| \geq \alpha \langle T_{crit} \rangle^{-1} > \alpha \langle t \rangle^{-1}.
\tag{4.4}
$$

Under these hypotheses, our goal is to obtain hypotheses (4.1) without the factor 2, and hypothesis (4.2) with tighter exponents. We will argue that hypothesis (4.3) **cannot** be improved, and conclude the expected exit behavior.

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Lemma 4.1. *For all $s < t \in [T_{crit}, T_{exit})$, $\omega(t) \in \mathcal{I}$ and:*

$$|\dot{\omega}(t)| + |\dot{\theta}(t) - \omega(t)| \leq C \left(|b_+(t)|^{m_0} + \alpha^{m_0} \langle t \rangle^{-m_0 \sigma_q} \right), \qquad and$$

$$|\omega(t) - \omega(s)| \leq C \left(|b_+(t)|^{m_0} + \alpha^{m_0} \langle s \rangle^{-m_0 \sigma_q + 1} \right), \quad which \ implies$$

$$|e_{\pm, \omega(t)} - e_{\pm, \omega(s)}| \leq C \left(|b_+(t)|^{m_0} + \alpha^{m_0} \langle s \rangle^{-m_0 \sigma_q + 1} \right).$$

Proof. The argument is the same as for Lemma 3.1, based on the following estimate: $|\langle N(\varepsilon), \text{potential} \rangle| \lesssim |b_+(t)|^{m_0} + \alpha^{m_0} \langle t \rangle^{-m_0 \sigma_q}$. When integrating quantities in $|b_+(t)|^r$, for any exponent $r > 0$, we use (4.2):

$$\int_s^t |b_+(\tau)|^r \, d\tau \leq |b_+(t)|^r \int_s^t e^{-\frac{2}{3} r \, e_2(t-\tau)} \, d\tau. \tag{4.5}$$

$\square$

4.2 Improved Estimates

Growth estimate for b_+. By Lemma 4.1, the dominant forcing term of (2.18) is $\langle N(\varepsilon), [\![Y_-]\!] \rangle$, and it reads: $\dot{b}_+ - e_2 b_+ = \mathcal{O}\left(|b_+|^{m_0} + \alpha^{m_0} \langle t \rangle^{-m_0 \sigma_q} \right)$. Taking $\overline{\alpha_0}$ to be sufficiently small, we may assume $|\dot{b}_+ - e_2 b_+| < \frac{1}{5} e_2 |b_+|$. After integration, this is a stronger statement than (4.2):

$$e^{\frac{4}{5} e_2(t-s)} \leq \frac{b_+(t)}{b_+(s)} \leq e^{\frac{6}{5} e_2(t-s)}. \tag{4.6}$$

Estimate on b_-. The same argument applied to (2.17) gives

$$|b_-(t)| \leq e^{+e_1(t - T_{crit})} |b_-(T_{crit})| + \int_{T_{crit}}^t e^{+e_1(t-\tau)} |b_+(\tau)| \, d\tau.$$

Compare the estimate for b_- from the previous chapter (3.1) with (2.13) to see that $|b_-(T_{crit})| \leq C_1 |b_+(T_{crit})|$. We conclude that $|b_-(t)| \leq C_5 |b_+(t)|$ for some C_5.

Decomposition of η. Analogous to (3.4), and using (2.4)-(2.9), we have

$$\|\eta(t)\|_{L^2} \leq C_8 \left(|b_+(t)|^{\frac{m_0}{2}} + \alpha^{\frac{m_0}{2}} \right). \tag{4.7}$$

Under our new hypotheses, let us revisit (3.5) and (3.6). As before, we use (2.20), Lemma 4.1 and (2.25) to conclude:

$$\left\| \left(\dot{\theta} - \omega_T \right) \mathbf{K} [\![\tilde{\eta}]\!] \right\|_{L^{r'}} + \left\| \mathbf{J}(\mathbf{V} - \mathbf{V}_T) [\![\tilde{\eta}]\!] \right\|_{L^{r'}} + \|B\|_{L^{r'}} \ll \left(|b_+(t)|^{m_0} + \alpha^{m_0} \langle t \rangle^{-(1+)} \right). \tag{4.8}$$

L^p estimate of η. For $t \in [T_{crit}, T_{exit})$, we estimate $\|N(\varepsilon)\|_{L^{p'}}$ using (3.7) and (4.1):

$$\|N(\varepsilon)\|_{L^{p'}} \lesssim |b_+|^{m_0} + \sum_j \|\eta\|_{L^2}^{(1-\theta_j)m_j} C_6^{m_j \theta_j} \left(|b_+(t)|^{m_0} + \alpha \langle t \rangle^{-\sigma_p} \right)^{m_j \theta_j}. \tag{4.9}$$

Due to (2.27) and (4.3), we may bound (4.9) by $\frac{C_6}{C_9} \left(|b_+|^{m_0} + \alpha^{1+} \langle t \rangle^{-(1+)} \right)$, for any universal constant $C_9 > 0$, by taking the universal constant C_6 sufficiently large and $\overline{\alpha_0}$ sufficiently

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small. Now apply (1.6) and use both (4.9) and, for $t \in [0, T_{crit})$, the estimates from the previous chapter that led to (3.9):

$$\|\tilde{\eta}_\pm(t)\|_{L^p} \leq \frac{C_2}{2}\alpha\langle t\rangle^{-\sigma_p} + C\alpha^{1+} \int_0^t |t-s|^{-\sigma_p} \langle s\rangle^{-(1+)} \, ds$$
$$+ \frac{C_6}{C_9} \int_{T_{crit}}^t |t-s|^{-\sigma_p} |b_+(s)|^{m_0} \, ds.$$

From (3.9), the first terms are bounded by $C_2\alpha\langle t\rangle^{-\sigma_p}$. Assume that $C_6 \geq C_2$. Integrate the final term with (4.5), and take C_9 to be the constant factor. This completely determines C_6. We have shown $\|\tilde{\eta}_\pm(t)\|_{L^p} < C_6 \left(\alpha\langle t\rangle^{-\sigma_p} + |b_+(t)|^{m_0}\right)$.

L^q *estimate of η.* Recall the terms **I**, **II** and **III** of (3.10). The estimate (3.11) of term **I** still applies. For term **III**, we first consider (3.12) for $t > T_{crit}$, using (4.1), (4.4) and (4.7),

$$\|\eta\|_{L^{m_j q'}}^{m_j} \lesssim o(1) \left(|b_+|^{m_0} + \alpha\langle t\rangle^{-\sigma_p}\right)^{m_j\tilde{\theta}_j} \lesssim |b_+(t)|^{1+},$$

where the second inequality relied on (2.29) and (2.30) for the exponents of b_+ and α, respectively. If $m_j > 2$,

$$\|A_j\phi^{m_j-2}\varepsilon^2\|_{L^{q'}} \lesssim |b_+|^2 + \alpha^2\langle t\rangle^{-2\sigma_q},$$

as in (3.14). As before, all terms decay faster than $\langle t\rangle^{-1}$ and we have:

$$\mathbf{III} \lesssim \int_{T_{crit}}^t |b_+(s)|^{1+} \, ds + \alpha^{1+} \int_0^t |t-s|^{-\sigma_q} \langle s\rangle^{-(1+)} \, ds,$$

which are the same integrals as (4.5) and (3.15), respectively. Term **II** may be included for $s \in [0, t)$, controlled by a combination of (3.5), (3.6) and (4.8):

$$\mathbf{II} + \mathbf{III} \lesssim |b_+(t)|^{1+} + \alpha^{1+}\langle t\rangle^{-\sigma_q}.$$

This concludes our estimate of $\|\tilde{\eta}_\pm\|_{L^q}$.

4.3 Bootstrap Conclusions

In the previous section, we proved (4.6) and that, for all $t \in [T_{crit}, T_{exit})$,

$$\begin{aligned}
|b_-(t)| &\leq C_5 |b_+(t)|, \\
\|\eta(t)\|_{L^q} &\leq C_7 \left(|b_+(t)| + \alpha\langle t\rangle^{-\sigma_q}\right), \\
\|\eta(t)\|_{L^p} &\leq C_6 \left(|b_+(t)|^{m_0} + \alpha\langle t\rangle^{-\sigma_p}\right).
\end{aligned} \qquad (4.10)$$

These are continuously evolving quantities. From (4.3), (4.4) and (4.6), we conclude that

$$T_{exit} \leq T_{crit} + \frac{5}{4e_2} \ln\left(3\frac{\alpha_0}{\alpha}\langle T_{crit}\rangle\right) < +\infty.$$

Assuming $\overline{\alpha_0}$ is sufficiently small, Lemma 2.2, (4.3) and (4.10) imply that the decomposition can be extended past time T_{exit}, and hence $T_{exit} < T_{dec}$. The only possible failure at time $t = T_{exit}$ is (4.3), and so we conclude that $b_+(T_{exit}) = 3\alpha_0$. The conclusion of Theorem 1.1 then follows for some $\omega_+ \approx \omega(T_{exit})$.

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Acknowledgments

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