arXiv:1207.0202v1 [math.PR] 1 Jul 2012

# Branching diffusion in the super-critical regime

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**Abstract**

We investigate the long-time evolution of branching diffusion processes (starting with a single particle) in inhomogeneous media. The qualitative behavior of the processes depends on the intensity of the branching. We analyze the super-critical case, when the total number of particles growing exponentially with positive probability. We study the asymptotics of the number of particles in different regions of space and describe the growth of the region occupied by the particles.

*2000 Mathematics Subject Classification Numbers:* 60J80, 35K10.

## 1 Introduction

The mathematical study of branching processes goes back to the work of Galton and Watson [15] who were interested in the probabilities of long-term survival of family names. Later it was realized that similar mathematical models could be used to describe the evolution of a variety of biological populations, in genetics [7, 8, 9, 10], and in the study of certain chemical and nuclear reactions [13, 11]. The branching processes (in particular, branching diffusions) are central in the study of the evolution of various populations such as bacteria, cancer cells, carriers of a particular gene, etc., where each member of the population may die or produce offspring independently of the rest.

In this paper we describe the long-time behavior of the population in different regions of space when, in addition to branching, the members of the population move diffusively in space and the branching mechanism depends on the location. In particular, we'll consider regions $U_{x(t)}$ centered at a point $x(t) = vt$ which is at a linear (in $t$) distance from the origin at time $t$. We will be interested in the super-critical case (when the total population grows exponentially with positive probability).

Consider a collection of particles in $\mathbb{R}^d$ that move diffusively and independently starting with one particle. Besides the diffusive motion, the particles can duplicate with the rate of duplication $v(x)$, $x \in \mathbb{R}^d$, where $x$ is the position of a given particle and $v$ is a continuous non-negative compactly supported function. Both copies start moving independently

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immediately after the duplication (annihilation of particles and creation of more than two particles from one could also be considered, but here we discuss only duplication for clarity of exposition).

Let $n_t^x(U)$ be the number of particles in a domain $U \subseteq \mathbb{R}^d$ at time $t$, assuming that at time zero there was a single particle located at $x$. The large time behavior of $n_t^x(U)$ depends crucially on the magnitude of $v$, that is on whether the operator

$$\mathcal{L}u(x) = \frac{1}{2}\Delta u(x) + v(x)u(x) \tag{1}$$

has a positive eigenvalue. This is the operator in the right hand side of the equations on the particle density and higher order correlation functions, given below. In the super-critical case (i.e., if there exists a positive eigenvalue), if $U$ is fixed, the results of [6] on the asymptotics of $n_t^x(U)$ cover, in particular, the case of compactly supported $v$. Namely, $n_t^x(U)$ grows exponentially with a random coefficient in front of the exponent. The random coefficients corresponding to different domains differ only by a multiplicative constant. (See also [14], [3]). The distribution of the random coefficient can be described in terms of its moments (see [12]). (For the asymptotic properties of branching random walks see also [1], [4], [2], [16].) If $\mathcal{L}$ has no positive eigenvalues, the total number of particles tends to a finite random limit whose distribution can be described in terms of its moments.

In the current paper we consider the super-critical case. We use the spectral techniques developed in [5], [12] to get the asymptotic formulas for the density and higher order correlation functions of the branching process. These allow us to get the asymptotics of $n_t^x(U_{tv})$, where $U_{tv}$ is a region of fixed size centered at a point whose distance from the origin grows linearly with $t$. This asymptotics is the main result of this paper.

After recalling the equations on the correlation functions (Section 2) and the asymptotic behavior of the correlation functions (Section 3), we show in Section 4 that the total number of particles, after division by an exponential factor, tends to a random limit in $L^2$. In Sections 5 and 6 we prove a similar result for fixed domains and for domains located at a linear (in $t$) distance from the origin and show that the convergence takes place not only in $L^2$ but also almost surely. In Section 7 we show that on the event that the number of particles grows exponentially, the region occupied by the particles grows linearly in $t$. In Section 8 we give the distribution of the limiting number of particles in the event that the limiting number of particles is finite.

## 2 Equations on correlation functions

Let $B_\delta$ be a ball of radius $\delta$ in $\mathbb{R}^d$. For $t > 0$ and $x, y_1, y_2, ... \in \mathbb{R}^d$ with all $y_i$ distinct, define the particle density $\rho_1(t, x, y_1)$ and the higher order correlation functions $\rho_n(t, x, y_1, ..., y_n)$ as the limits of probabilities of finding $n$ distinct particles in $B_\delta(y_1)$,...,$B_\delta(y_n)$, respectively, divided by $\mathrm{Vol}^n(B_\delta)$, under the condition that there is a unique particle at $t = 0$ located at $x$. We extend $\rho_n(t,x,y_1,...,y_n)$ by continuity to allow for $y_i$ which are not necessarily distinct. For fixed $y_1$, the density satisfies the equation

$$\partial_t \rho_1(t,x,y_1) = \frac{1}{2}\Delta \rho_1(t,x,y_1) + v(x)\rho_1(t,x,y_1), \tag{2}$$
$$\rho_1(0,x,y_1) = \delta_{y_1}(x).$$

Indeed, let $s,t>0$. Then we can write

$$\rho_1(s+t,x,y_1) = (2\pi s)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\frac{|x-z|^2}{2s}} \rho_1(t,z,y_1)dz + v(x)s\rho_1(t,x,y_1) + \alpha(s,t,x,y_1), \tag{3}$$

where the term with the integral on the right hand side is due to the effect of the diffusion on the interval $[0,s]$, the second term is due to the probability of branching on $[0,s]$, and $\alpha$ is the correction term. The correction term is present since (a) more than one instance of branching may occur before time $s$, and (b) even if a single branching occurs between the times 0 and $s$, then the original particle will be located not at $x$ but at a nearby point and the intensity of branching there is slightly different from $v(x)$. It is clear that $\lim_{s\downarrow 0}\sup_{x,y\in\mathbb{R}^d} \alpha(s,t,x,y)/s = 0$. After subtracting $\rho_1(t,x,y_1)$ from both sides of (3), dividing by $s$ and taking the limit as $s \downarrow 0$, we obtain (2).

The equations on $\rho_n$, $n>1$, are somewhat more complicated:

$$\partial_t \rho_n(t,x,y_1,...,y_n) = \frac{1}{2}\Delta \rho_n(t,x,y_1,...,y_n) + v(x)\left(\rho_n(t,x,y_1,...,y_n) + H_n(t,x,y_1,...,y_n)\right), \tag{4}$$
$$\rho_n(0,x,y_1,...,y_n) \equiv 0.$$

Here

$$H_n(t,x,y_1,...,y_n) = \sum_{U\subset Y, U\neq\emptyset} \rho_{|U|}(t,x,U)\rho_{n-|U|}(t,x,Y\setminus U),$$

where $Y=(y_1,...,y_n)$, $U$ is a proper non-empty subsequence of $Y$, and $|U|$ is the number of elements in this subsequence. Equation (4) is derived similarly to (2). The combinatorial term $H_n$ appears after taking into account the event that there is a single branching on the time interval $[0,s]$, the descendants of the first particle are found at the points in $U$ at time $s+t$, while the descendants of the second particle are found at the points of $Y\setminus U$, with the summation over all possible choices of $U$.

# 3 Asymptotics of the correlation functions

First we recall some basic facts about the operator $\mathcal{L}: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ (see (1)) and its resolvent $R_\lambda = (\mathcal{L}-\lambda)^{-1}$. We will assume that $v\geq 0$ is continuous, compactly supported and not identically equal to zero. It is well-known that the spectrum of $\mathcal{L}$ consists of the absolutely continuous part $(-\infty,0]$ and at most a finite number of non-negative eigenvalues:

$$\sigma(\mathcal{L}) = (-\infty,0]\cup\{\lambda_j\}, \quad 0\leq j\leq N, \quad \lambda_j\geq 0.$$

We enumerate the eigenvalues in the decreasing order. Thus, if $\{\lambda_j\} \neq \emptyset$, then $\lambda_0 = \max \lambda_j$. We assume that there is at least one positive eigenvalue. The resolvent $R_\lambda : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a meromorphic operator valued function on $\mathbb{C}' = \mathbb{C} \backslash (-\infty, 0]$.

Denote the kernel of $R_\lambda$ by $R_\lambda(x, y)$. If $v \equiv 0$ (in which case of course there are no eigenvalues), the kernel depends on the difference $x - y$ and will intermittently use the notations $R^0_\lambda(x, y)$ and $R^0_\lambda(x - y)$. The kernel $R^0_\lambda(x)$ can be expressed through the Hankel function $H^{(1)}_\nu$:

$$R^0_\lambda(x) = c_d k^{d-2}(k|x|)^{1-\frac{d}{2}} H^{(1)}_{\frac{d}{2}-1}(i\sqrt{2}k|x|), \quad k = \sqrt{\lambda}, \quad \mathrm{Re}k > 0. \tag{5}$$

We shall say that $f \in C_{\exp}(\mathbb{R}^d)$ (or simply $C_{\exp}$) if $f$ is continuous and

$$||f||_{C_{\exp}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} (|f(x)| e^{|x|^2}) < \infty.$$

The space of bounded continuous functions on $\mathbb{R}^d$ will be denoted by $C(\mathbb{R}^d)$ or simply $C$. The following simple lemma can be found in [5].

**Lemma 3.1.** *The operator $R_\lambda : C_{\exp}(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is meromorphic in $\lambda \in \mathbb{C}'$. Its poles are of the first order and are located at eigenvalues of the operator $\mathcal{L}$. For each $\varepsilon > 0$ and some $\Lambda$, the operator is uniformly bounded in $\lambda \in \mathbb{C}'$, $|\mathrm{arg}\lambda| \leq \pi - \varepsilon$, $|\lambda| \geq \Lambda$. It is of order $O(1/|\lambda|)$ as $\lambda \to \infty$, $|\mathrm{arg}\lambda| \leq \pi - \varepsilon$. The eigenvalue $\lambda_0$ of the operator $\mathcal{L}$ is simple and the corresponding eigenfunction does not change sign.*

From Lemma 3.1 it follows that the residue of $R_\lambda$ at $\lambda_0$ is the integral operator with the kernel $\psi(x)\psi(y)$, where $\psi$ is the positive eigenfunction normalized by the condition $||\psi||_{L^2(\mathbb{R}^d)} = 1$. The function $\psi$ decays exponentially at infinity. More precisely, it follows from (5) that if we write $x$ as $(\theta, |x|)$ in polar coordinates, then there is a positive continuous function $F$ such that

$$\psi(x) \sim F(\theta)|x|^{\frac{1}{2}-\frac{d}{2}} \exp(-\sqrt{2\lambda_0}|x|) \quad \text{as} \quad |x| \to \infty. \tag{6}$$

For a positive number $x$, we define the curve $\Gamma(x)$ in the complex plane as follows:

$$\Gamma(x) = \{\lambda : |\mathrm{Im}\lambda| = \sqrt{4x(x - \mathrm{Re}\lambda)},\ \mathrm{Re}\lambda \geq 0\} \bigcup \{\lambda : |\mathrm{Im}\lambda| = 2x(1 - \mathrm{Re}\lambda),\ \mathrm{Re}\lambda \leq 0\}.$$

Thus $\Gamma(x)$ is a union of a piece of the parabola with the vertex in $x$ that points in the direction of the negative real axis and two rays tangent to the parabola at the points it intersects the imaginary axis. The choice of the curve is somewhat arbitrary, yet the following properties of $\Gamma(x)$ will be important:

First, $\mathrm{Re}\lambda \leq x$ for $\lambda \in \Gamma(x)$. Second, since the rays form a positive angle with the negative real semi-axis, we have $|\mathrm{arg}\lambda| \leq \pi - \varepsilon(x)$ for all $\lambda \in \Gamma(x)$ for some $\varepsilon(x) > 0$. Third, since the rays are tangent to the parabola, and the parabola is mapped into the line $\{\lambda : \mathrm{Re}\lambda = \sqrt{x}\}$ by the mapping $\lambda \to \sqrt{\lambda}$, the image of the curve $\Gamma(x)$ under the same mapping lies in the half-plane $\{\lambda : \mathrm{Re}\lambda \geq \sqrt{x}\}$.

The integration along the vertical lines in the complex plane and along contours $\Gamma(x)$, below, is performed in the direction of the increasing complex part.

We'll need estimates on the solutions of the following parabolic equation. Let

$$\partial_t \rho(t,x) = \frac{1}{2}\Delta\rho(t,x) + v(x)\rho(t,x), \qquad \rho(0,x) = g(x) \in C_{\exp}. \tag{7}$$

We'll denote the Laplace transform of a function $f$ by $\widetilde{f}$,

$$\widetilde{f}(\lambda) = \int_0^\infty \exp(-\lambda t) f(t)dt.$$

Let $r$ be the distance between $\lambda_0$ and the rest of the spectrum of the operator $\mathcal{L}$. In the arguments that follow we'll use the symbol $A$ to denote constants that may differ from line to line.

**Lemma 3.2.** *For each $\varepsilon \in (0,r)$, the solution of (7) has the form*

$$\rho(t,x) = \exp(\lambda_0 t)\langle\psi, g\rangle\psi(x) + q(t,x), \tag{8}$$

*where*

$$||q(t,\cdot)||_C \le A(\varepsilon)\exp((\lambda_0-\varepsilon)t)||g||_{C_{\exp}}.$$

*Proof.* After the Laplace transform, the equation becomes

$$(\frac{1}{2}\Delta + v)\widetilde{\rho} - \lambda\widetilde{\rho} = -g.$$

Thus, the solution $\rho$ can be represented as

$$\rho(t,\cdot) = -\frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=\lambda_0+1} e^{\lambda t}R_\lambda g d\lambda. \tag{9}$$

The resolvent is meromorphic in the complex plane outside of the interval $(-\infty, \lambda_0 - r]$, with the only (simple) pole at $\lambda_0$ with the principal part of the Laurent expansion being the integral operator with the kernel $\psi(x)\psi(y)/(\lambda_0-\lambda)$.

By Lemma 3.1, the norm of $R_\lambda$ does not exceed $A/|\lambda|$ near infinity to the right of $\Gamma(\lambda_0-\varepsilon)$. Therefore, the same integral as in (9) but along the segment parallel to the real axis connecting a point $\lambda_0+1+ib$ with the contour $\Gamma(\lambda_0-\varepsilon)$ tends to zero when $b\to\infty$. Therefore, we can replace the contour of integration in (9) by $\Gamma(\lambda_0-\varepsilon)$. The residue gives the main term, while the integral over $\Gamma(\lambda_0-\varepsilon)$ gives the remainder term. □

**Lemma 3.3.** *Let $K \subset \mathbb{R}^d$ be a compact set. For each $\varepsilon \in (0,r)$, the function $\rho_1(t,x,y)$ satisfies*

$$\rho_1(t,x,y) = \exp(\lambda_0 t)\psi(x)\psi(y) + q_1(t,x,y),$$

*where*

$$\sup_{x\in K}|q_1(t,x,y)| \le A(\varepsilon)\exp((\lambda_0-\varepsilon)t - |y|\sqrt{2(\lambda_0-\varepsilon)}) \tag{10}$$

*for* $t \geq 1/2$. *Moreover,*

$$\int_{\mathbb{R}^d} q_1(t,x,y)\psi(y)dy = 0 \tag{11}$$

*for each* $t > 0$, $x \in \mathbb{R}^d$, *and*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_1(t,x,y)dy \leq A\exp(\lambda_0 t). \tag{12}$$

*Proof.* First, let us show that

$$\langle \psi, \rho_1(t,\cdot,y)\rangle = \exp(\lambda_0 t)\psi(y) \tag{13}$$

for each $t > 0$, $y \in \mathbb{R}^d$. Indeed,

$$0 = \int_0^t \langle (\frac{\partial}{\partial s} + \mathcal{L})(\exp(-\lambda_0 s)\psi), \rho_1\rangle ds =$$

$$\langle \exp(-\lambda_0 s)\psi, \rho_1\rangle|_{s=0}^t + \int_0^t \langle (\exp(-\lambda_0 s)\psi), (-\frac{\partial}{\partial s} + \mathcal{L})\rho_1\rangle ds =$$

$$\langle \exp(-\lambda_0 t)\psi, \rho_1(t,\cdot,y)\rangle - \langle \psi, \rho_1(0,\cdot,y)\rangle = \exp(-\lambda_0 t)\langle \psi, \rho_1(t,\cdot,y)\rangle - \psi(y),$$

which proves (13). The relationship (13) immediately implies (11).

Next, let $K'$ be a compact set that contains $\mathrm{supp}(v) \cup K$ in its interior. In order to prove (10), consider first the case when $y \in K'$. Apply (8) with $t$ replaced by $t' = t - 1/2$ and $g = \rho_1(1/2,\cdot,y)$. In order to calculate the main term of the asymptotics, we note that $||g||_{C_{\exp}}$ is bounded uniformly in $y \in K'$ and

$$\langle \psi, g\rangle = \exp(\frac{1}{2}\lambda_0)\psi(y),$$

as follows from (13). Therefore, (8) implies that

$$\rho_1(t,x,y) = \exp(\lambda_0 t)\psi(y)\psi(x) + \exp((\lambda_0 - \varepsilon)t)q(t,x,y),$$

where $||q(t,\cdot,y)||_C \leq A(K')$ for all $y \in K'$. We still need to consider the case when $y \notin K'$.

Let $u(t,x,y) = \rho_1(t,x,y) - p_0(t,x,y)$, where $p_0$ is the fundamental solution of the heat equation. Then $u$ satisfies the non-homogeneous version of (7) with the right hand side $f = -v(x)p_0(t,x,y)$ and $g \equiv 0$. Note that $f$ is a smooth function since $y \notin K'$. Solving this equation for $u$ using the Laplace transform, as in the proof of Lemma 3.2, we obtain

$$u(t,\cdot,y) = -\frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=\lambda_0+1} e^{\lambda t}R_\lambda(-v\widetilde{p}_0(\lambda,\cdot,y))d\lambda$$

$$= -\frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=\lambda_0+1} e^{\lambda t}R_\lambda(vR^0_\lambda(\cdot,y))d\lambda \tag{14}$$

$$= \exp(\lambda_0 t)\langle \psi, vR^0_{\lambda_0}(\cdot,y)\rangle\psi - \frac{1}{2\pi i}\int_{\Gamma(\lambda_0-\varepsilon)} e^{\lambda t} R_\lambda(vR^0_\lambda(\cdot,y))d\lambda,$$

where the first term on the right hand side is due to the residue at $\lambda = \lambda_0$. The first term can be re-written as

$$\exp(\lambda_0 t)\langle \psi, vR^0_{\lambda_0}(\cdot,y)\rangle\psi(x) =$$
$$\exp(\lambda_0 t)(R^0_{\lambda_0}(v\psi))(y)\psi(x) = -\exp(\lambda_0 t)\psi(y)\psi(x).$$

The last equality here follows from the fact that $\psi$ is an eigenfunction with eigenvalue $\lambda_0$, that is

$$(\frac{1}{2}\Delta - \lambda_0)\psi = -v\psi.$$

In order to estimate the second term on the right hand side of (14), we note that from (5) it follows that

$$|R^0_\lambda(x,y)| \le A(l)|\sqrt{\lambda}|^{\frac{d}{2}-\frac{3}{2}}|x-y|^{\frac{1}{2}-\frac{d}{2}}|\exp(-\sqrt{2\lambda}|y-x|)|$$

if $|\lambda|, |y-x| \ge l$. Thus

$$||vR^0_\lambda(\cdot,y)||_{C_{\exp}} \le A(\varepsilon)|y|^{\frac{1}{2}-\frac{d}{2}}|\sqrt{\lambda}|^{\frac{d}{2}-\frac{3}{2}}\exp(-\sqrt{2(\lambda_0-\varepsilon)}|y|)$$

for $y \notin K'$, $\lambda \in \Gamma(\lambda_0-\varepsilon)$ due to the fact that $\mathrm{Re}\sqrt{\lambda} \ge \sqrt{\lambda_0-\varepsilon}$ for $\lambda \in \Gamma(\lambda_0-\varepsilon)$ and $|y-x| \ge l$ for $x \in \mathrm{supp}(v)$, $y \notin K'$.

Hence, using the estimate on the norm of $R_\lambda : C_{\exp} \to C$ from Lemma 3.1, we obtain

$$||R_\lambda(vR^0_\lambda(\cdot,y))||_C \le A(\varepsilon)|\sqrt{\lambda}|^{\frac{d}{2}-\frac{5}{2}}\exp(-\sqrt{2(\lambda_0-\varepsilon)}|y|), \quad \lambda \in \Gamma(\lambda_0-\varepsilon).$$

Therefore, since $\mathrm{Re}\lambda \le \lambda_0 - \varepsilon$ for $\lambda \in \Gamma(\lambda_0-\varepsilon)$ and the factor $e^{\lambda t}$ decays exponentially along $\Gamma(\lambda_0-\varepsilon)$, the $C$-norm of the second term on the right hand side of (14) does not exceed $A(\varepsilon)\exp((\lambda_0-\varepsilon)t - |y|\sqrt{2(\lambda_0-\varepsilon)})$. The term $p_0(t,x,y)$ with $x \in K$, $y \notin K'$, $t \ge 1/2$, is estimated by the same expression, possibly with a different constant $A(\varepsilon)$. Indeed, if $t \ge 1/2$, then

$$p_0(t,x,y) \le A\exp(-|y-x|^2/2t) \le A\exp((\lambda_0-\varepsilon)t - |y-x|\sqrt{2(\lambda_0-\varepsilon)})$$

since

$$|y-x|^2/2t + (\lambda_0-\varepsilon)t - |y-x|\sqrt{2(\lambda_0-\varepsilon)} = (|y-x|/\sqrt{2t} - \sqrt{(\lambda_0-\varepsilon)t})^2 \ge 0.$$

This completes the proof of (10). In order to prove (12), we again write $u(t,x,y) = \rho_1(t,x,y) - p_0(t,x,y)$. For fixed $x$, apply the Duhamel formula to the equation

$$\partial_t u(t,x,y) = \frac{1}{2}\Delta u(t,x,y) + v(y)\rho_1(t,x,y)$$

with the initial data $u(0,x,\cdot)\equiv 0$. Now (12) follows since the $L^1$-norm of $v(y)\rho_1(t,x,y)$ is bounded by $A\exp(\lambda_0 t)$ uniformly in $x$, as follows from (10). $\square$

We'll need additional notations in order to describe the asymptotics of $\rho_n$ with $n>1$. Let $\alpha^1_\varepsilon(t,y) = \psi(y)$ and $\alpha^2_\varepsilon(t,y) = \exp(-\varepsilon t - |y|\sqrt{2(\lambda_0-\varepsilon)})$. Consider all possible sequences $\sigma=(\sigma_1,...,\sigma_n)$ with $\sigma_i\in\{1,2\}$. By $\Pi^n_\varepsilon(t,y_1,...,y_n)$ we denote the quantity

$$\Pi^n_\varepsilon(t,y_1,...,y_n) = \sup_{\sigma\neq(1,...,1)} \alpha^{\sigma_1}_\varepsilon(t,y_1)\cdot...\cdot\alpha^{\sigma_n}_\varepsilon(t,y_n).$$

Let $P_t : C_{\exp}\to C$ be the operator that maps the initial function $g$ to the solution $\rho(t,\cdot)$ of equation (7). Let $P^0_t g(x) = \exp(\lambda_0 t)\langle\psi,g\rangle\psi(x)$ and $P^1_t = P_t - P^0_t$. Lemma 3.2 states that

$$||P^1_t|| \leq A(\varepsilon)\exp((\lambda_0-\varepsilon)t).$$

The particular form of $P^0_t$ then implies that

$$||P_t|| \leq ||P^0_t|| + ||P^1_t|| \leq A'\exp(\lambda_0 t). \tag{15}$$

For $g\in C_{\exp}$ and $n\geq 2$, we denote

$$I_n(g) := R_{n\lambda_0}g = \int_0^\infty \exp(-n\lambda_0 s)P_s g ds \in C.$$

Note that

$$\int_0^t \exp(n\lambda_0 s)P_{t-s}g ds = \exp(n\lambda_0 t)\int_0^t \exp(-n\lambda_0 s)P_s g ds$$
$$= \exp(n\lambda_0 t)(I_n(g) + O(\exp(-(n-1)\lambda_0 t))) \quad \text{as} \quad t\to\infty. \tag{16}$$

The functions $f_1, f_2, ...$ are defined inductively: $f_1=\psi$ and

$$f_n = \sum_{k=1}^{n-1}\frac{n!}{k!(n-k)!}I_n(vf_kf_{n-k}), \quad n\geq 2.$$

**Lemma 3.4.** *Let $K\subset\mathbb{R}^d$ be a compact set. For each $\varepsilon\in(0,r)$, the function $\rho_n$ satisfies*

$$\rho_n(t,x,y_1,...,y_n) = \exp(n\lambda_0 t)f_n(x)\psi(y_1)\cdot...\cdot\psi(y_n) + q_n(t,x,y_1,...,y_n), \tag{17}$$

*where*

$$\sup_{x\in K}|q_n(t,x,y_1,...,y_n)| \leq A_n(\varepsilon)\exp(n\lambda_0 t)\Pi^n_\varepsilon(t,y_1,...,y_n) \tag{18}$$

*for $t\geq 1/2$.*

*Proof.* For $n=1$, the relation (17) coincides with the statement of Lemma 3.3. Let us assume that (17) holds for all natural numbers up to and including $n-1$. A generic subsequence $U \subset Y = (y_1, ..., y_n)$ will be written as $U = (z_1, ..., z_{|U|})$ and its complement as $Y \setminus U = (\overline{z}_1, ..., \overline{z}_{n-|U|})$. By the Duhamel principle applied to the equation for $\rho_n$, we obtain

$$\rho_n(t, \cdot, y_1, ..., y_n) = \int_0^t P_{t-s}(v \sum_{U \subset Y, U \neq \emptyset} \rho_{|U|}(s, \cdot, z_1, ..., z_{|U|}) \rho_{n-|U|}(s, \cdot, \overline{z}_1, ..., \overline{z}_{n-|U|}))ds$$

$$= \int_0^t P_{t-s}(v \sum_{U \subset Y, U \neq \emptyset} \exp(|U|\lambda_0 s) f_{|U|}(\cdot) \psi(z_1) \cdot ... \cdot \psi(z_{|U|})$$
$$\times \exp((n-|U|)\lambda_0 s) f_{n-|U|}(\cdot) \psi(\overline{z}_1) \cdot ... \cdot \psi(\overline{z}_{n-|U|}))ds \tag{19}$$
$$+2 \int_0^t P_{t-s}(v \sum_{U \subset Y, U \neq \emptyset} \exp(|U|\lambda_0 s) f_{|U|}(\cdot) \psi(z_1) \cdot ... \cdot \psi(z_{|U|}) q_{n-|U|}(s, \cdot, \overline{z}_1, ..., \overline{z}_{n-|U|}))ds$$
$$+ \int_0^t P_{t-s}(v \sum_{U \subset Y, U \neq \emptyset} q_{|U|}(s, \cdot, z_1, ..., z_{|U|}) q_{n-|U|}(s, \cdot, \overline{z}_1, ..., \overline{z}_{n-|U|}))ds.$$

The second and third integrals on the right hand side of (19) contribute only to the remainder term. Indeed, consider the contribution to the second integral from the term with a given $U$:

$$\int_0^t P_{t-s}(v \exp(|U|\lambda_0 s) f_{|U|}(\cdot) \psi(z_1) \cdot ... \cdot \psi(z_{|U|}) q_{n-|U|}(s, \cdot, \overline{z}_1, ..., \overline{z}_{n-|U|}))ds$$

$$\leq A\psi(z_1)...\psi(z_{|U|}) \int_0^t P_{t-s}(v \exp(|U|\lambda_0 s) f_{|U|}(\cdot)$$
$$\times \exp((n-|U|)\lambda_0 s) \Pi_\varepsilon^{n-|U|}(s, \overline{z}_1, ..., \overline{z}_{n-|U|}))ds$$
$$\leq A\psi(z_1) \cdot ... \cdot \psi(z_{|U|}) \int_0^t \exp(\lambda_0(t-s)) \exp(n\lambda_0 s) \Pi_\varepsilon^{n-|U|}(s, \overline{z}_1, ..., \overline{z}_{n-|U|}))ds$$
$$\leq A \exp(n\lambda_0 t) \psi(z_1)...\psi(z_{|U|}) \Pi_\varepsilon^{n-|U|}(t, \overline{z}_1, ..., \overline{z}_{n-|U|}) \leq A \exp(n\lambda_0 t) \Pi_\varepsilon^n(t, y_1, ..., y_n),$$

where the first inequality follows from the inductive assumption and the second one from (15). The third integral on the right hand side of (19) is estimated similarly. It remains to consider the first integral. It is equal to

$$\psi(y_1) \cdot ... \cdot \psi(y_n) \int_0^t \exp(n\lambda_0 s) P_{t-s}(v \sum_{U \subset Y, U \neq \emptyset} f_{|U|} f_{n-|U|})ds$$

$$= \psi(y_1) \cdot ... \cdot \psi(y_n) \exp(n\lambda_0 t) \left(f_n(\cdot) + O(\exp(-(n-1)\lambda_0 t))\right),$$

where the last equality follows from (16). Thus we obtain the main term from the right hand side of (17) plus the correction

$$\psi(y_1)\cdot ... \cdot\psi(y_n)\exp(n\lambda_0 t)O(\exp(-(n-1)\lambda_0 t))$$

for which the estimate (18) holds since $\psi(y_1)\exp(-\lambda_0 t)\le \alpha^2_\varepsilon(t,y_1)$ due to (6). □

# 4 Growth of the total number of particles

We denote the probability space on which the branching process is defined by $(\Omega,\mathcal{F},\mathrm{P})$. Let $\mathcal{F}_t$, $t\ge 0$, be the filtration generated by the process. We'll write $L^2$ for $L^2(\Omega,\mathcal{F},\mathrm{P})$. Let $N^x_t$ be the number of particles in $\mathbb{R}^d$ at time $t$, assuming that at $t=0$ there was a single particle located at $x$.

In this section we prove the basic result on the convergence of $N^x_t/e^{\lambda_0 t}$ in $L^2$. The almost sure convergence and the asymptotics of the number of particles in a (possibly time-dependent) region of space will be considered in the following sections.

**Theorem 4.1.** *There is a random variable $\xi^x$ such that*

$$\frac{N^x_t}{e^{\lambda_0 t}}\to\xi^x \quad as \quad t\to\infty, \tag{20}$$

*where the convergence takes place in $L^2$.*

*Proof.* Observe that for $0<s\le t$,

$$\mathrm{E}(N^x_s N^x_t)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_2(s,x,y_1,z)\rho_1(t-s,z,y_2)dzdy_1dy_2+\int_{\mathbb{R}^d}\rho_1(t,x,y_2)dy_2. \tag{21}$$

Indeed, fix $y_1,y_2\in\mathbb{R}^d$. Then the probability that there is a particle in an infinitesimal neighborhood of $y_1$ at time $s$, while a different particle present at time $s$ gives rise to a particle in an infinitesimal neighborhood of $y_2$ at time $t$ is equal to

$$\left(\int_{\mathbb{R}^d}\rho_2(s,x,y_1,z)\rho_1(t-s,z,y_2)dz\right)dy_1dy_2.$$

The probability that a particle in an infinitesimal neighborhood of $y_1$ at time $s$ gives rise to a particle in an infinitesimal neighborhood of $y_2$ at time $t$ is equal to

$$\rho_1(s,x,y_1)\rho_1(t-s,y_1,y_2)dy_1dy_2.$$

After adding the contributions from the two events and integrating in $y_1$ and $y_2$, we obtain (21).

Combining (21) with (17) and using that $f_1 = \psi$, we see that

$$\mathrm{E}(N_s^x N_t^x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[e^{2\lambda_0 s} f_2(x)\psi(y_1)\psi(z)e^{\lambda_0(t-s)}\psi(z)\psi(y_2)+$$

$$e^{2\lambda_0 s} f_2(x)\psi(y_1)\psi(z)q_1(t-s,z,y_2) + q_2(s,x,y_1,z)\rho_1(t-s,z,y_2)]dzdy_1dy_2+$$

$$\int_{\mathbb{R}^d}\rho_1(t,x,y_2)dy_2 =: I^1_{s,t}(x) + I^2_{s,t}(x) + I^3_{s,t}(x) + I^4_{s,t}(x).$$

Note that

$$I^1_{s,t} = e^{\lambda_0(s+t)} f_2(x)(\int_{\mathbb{R}^d}\psi)^2$$

since $\int_{\mathbb{R}^d}\psi^2(z)dz = 1$. Also observe that $I^2_{s,t}(x) = 0$ since $\int_{\mathbb{R}^d}\psi(z)q_1(t-s,z,y_2)dz = 0$ by (11). Finally,

$$\sup_{x\in K}|I^3_{s,t}(x) + I^4_{s,t}(x)| \leq Ae^{\lambda_0(t+s)-\varepsilon s}$$

by Lemma 3.4 and (12). Therefore,

$$\sup_{x\in K}|\mathrm{E}(N_s^x N_t^x) - e^{\lambda_0(s+t)} f_2(x)(\int_{\mathbb{R}^d}\psi)^2| \leq Ae^{\lambda_0(t+s)-\varepsilon s}.$$

Thus we have

$$\mathrm{E}\left(\frac{N_s^x}{e^{\lambda_0 s}} - \frac{N_t^x}{e^{\lambda_0 t}}\right)^2 = \frac{\mathrm{E}(N_s^x)^2}{e^{2\lambda_0 s}} + \frac{\mathrm{E}(N_t^x)^2}{e^{2\lambda_0 t}} - 2\frac{\mathrm{E}(N_s^x N_t^x)}{e^{\lambda_0(s+t)}} \leq Ae^{-\varepsilon s}$$

for $x \in K$. This shows that $N_t^x/e^{\lambda_0 t}$ is a Cauchy family of random variables as $t \to \infty$, and we have convergence in $L^2$. □

**Remark.** It is possible to show (see [12]) that all the moments of the variables $N_t^x/e^{\lambda_0 t}$ converge to those of $\xi^x$. The moments of the limiting distribution are

$$\mathrm{E}(\xi^x)^n = \left(\int_{\mathbb{R}^d}\psi(y)dy\right)^n f_n(x).$$

They were shown to determine the distribution of $\xi^x$ uniquely.

**Remark.** In dimensions $d \geq 3$ the limiting random variable $\xi^x$ is equal to zero with positive probability. Indeed, since the diffusion is transient, there is a positive probability that the original particle wanders off to infinity without branching. Let $B^x$ be the event that the number of particles stays bounded (and therefore tends to a finite limit as $t \to \infty$) and $E^x = \{\xi^x > 0\}$ be the event that the number of particles grows exponentially. It is possible to show (see [12], for example) that $\mathrm{P}(B^x \cup E^x) = 1$ for each $x$.

# 5 Growth of the number of particles in a domain

Let $n_t^x(U)$ denote the number of particles in a domain $U \subseteq \mathbb{R}^d$, assuming that at $t = 0$ there was a single particle located at $x$. Let

$$\alpha(U) = \frac{\int_U \psi(y)dy}{\int_{\mathbb{R}^d} \psi(y)dy} .$$

The asymptotics of the number of particles in $U$ is given by the following theorem.

**Theorem 5.1.** *For each measurable $U \subseteq \mathbb{R}^d$, we have*

$$\frac{n_t^x(U)}{e^{\lambda_0 t}} \to \alpha(U)\xi^x \quad as \quad t \to \infty, \tag{22}$$

*where the convergence takes place in $L^2$.*

*Proof.* By Theorem 4.1 it is sufficient to prove that

$$\frac{n_t^x(U) - \alpha(U)N_t^x}{e^{\lambda_0 t}} \to 0 \ \text{ as } \ t \to \infty.$$

Observe that

$$\mathrm{E}(N_t^x)^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho_2(t,x,y_1,y_2)dy_1dy_2 + \int_{\mathbb{R}^d} \rho_1(t,x,y_1)dy_1, \tag{23}$$

$$\mathrm{E}(n_t^x(U)N_t^x) = \int_{\mathbb{R}^d}\int_U \rho_2(t,x,y_1,y_2)dy_1dy_2 + \int_U \rho_1(t,x,y_1)dy_1, \tag{24}$$

$$\mathrm{E}(n_t^x(U))^2 = \int_U\int_U \rho_2(t,x,y_1,y_2)dy_1dy_2 + \int_U \rho_1(t,x,y_1)dy_1. \tag{25}$$

Upon expanding $(n_t^x(U) - \alpha(U)N_t^x)^2 = (n_t^x(U))^2 - 2\alpha(U)n_t^x(U)N_t^x + (\alpha(U)N_t^x)^2$, using Lemma 3.4 for the asymptotics of $\rho_1$ and $\rho_2$ and collecting all the lower order terms in the remainder $R(t,x)$, we obtain

$$\mathrm{E}\left(\frac{n_t^x(U) - \alpha(U)N_t^x}{e^{\lambda_0 t}}\right)^2 =$$

$$f_2(x)(\int_U\int_U \psi(y_1)\psi(y_2)dy_1dy_2 - 2\alpha(U)\int_{\mathbb{R}^d}\int_U \psi(y_1)\psi(y_2)dy_1dy_2 +$$

$$(\alpha(U))^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi(y_1)\psi(y_2)dy_1dy_2) + R(t,x) =$$

$$f_2(x)\left(\int_U \psi - \alpha(U)\int_{\mathbb{R}^d}\psi\right)^2 + R(t,x) = R(t,x),$$

where $\sup_{x\in K} R(t,x) \le Ae^{-\varepsilon t}$. This shows that $(n_t^x(U) - \alpha(U)N_t^x)/e^{\lambda_0 t}$ is a Cauchy family, thus completing the proof. $\square$

Now let us examine the number of particles in the vicinity of a point that is at a linear in $t$ distance from the origin. Let $b = \sqrt{\lambda_0/2}$ ($b$ will be seen to be the rate of growth of the region where the particles can be found with probability that tends to one). Let $v \in \mathbb{R}^d$ be a vector with $0 < |v| < b$ and $U_{tv} = U + tv$ the domain obtained from $U$ by translation by the vector $tv$. Let

$$g(t) = g(U,t,v) = \frac{e^{\lambda_0 t}\int_{U_{tv}} \psi(y)dy}{\int_{\mathbb{R}^d} \psi(y)dy}.$$

Note that the asymptotics of $g(t)$ can be obtained from (6). In particular,

$$A_1 t^{\frac{1-d}{2}} e^{(\lambda_0 - \sqrt{2\lambda_0}|v|)t} \le g(t) \le A_2 t^{\frac{1-d}{2}} e^{(\lambda_0 - \sqrt{2\lambda_0}|v|)t} \tag{26}$$

if $U$ is bounded.

**Theorem 5.2.** *Let $U$ be a bounded domain. For each $v \in \mathbb{R}^d$ such that $|v| < b$, we have*

$$\frac{n_t^x(U_{tv})}{g(t)} \to \xi^x \quad as \quad t \to \infty, \tag{27}$$

*where the convergence takes place in $L^2$.*

*Proof.* From (23)-(25) we obtain

$$\mathrm{E}\left(\frac{n_t^x(U_{tv})}{g(t)} - \frac{N_t^x}{e^{\lambda_0 t}}\right)^2 = f_2(x)\left(\frac{e^{\lambda_0 t}}{g(t)}\int_{U_{tv}}\psi - \int_{\mathbb{R}^d}\psi\right)^2 + R(t,x), \tag{28}$$

where

$$R(t,x) = (\frac{1}{g^2(t)}\int_{U_{tv}}\int_{U_{tv}} q_2(t,x,y_1,y_2)dy_1dy_2 - \frac{2}{g(t)e^{\lambda_0 t}}\int_{U_{tv}}\int_{\mathbb{R}^d} q_2(t,x,y_1,y_2)dy_1dy_2 +$$
$$\frac{1}{e^{2\lambda_0 t}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_2(t,x,y_1,y_2)dy_1dy_2 + \frac{1}{g^2(t)}\int_{U_{tv}} \rho_1(t,x,y_1)dy_1 -$$
$$\frac{2}{g(t)e^{\lambda_0 t}}\int_{U_{tv}} \rho_1(t,x,y_1)dy_1 + \frac{1}{e^{2\lambda_0 t}}\int_{\mathbb{R}^d} \rho_1(t,x,y_1)dy_1).$$

The first term on the right hand side of (28) is equal to zero as follows from the definition of $g(t)$. Applying Lemma 3.4 to estimate the integrand in each of the terms in $R(t,x)$, it is easy to see that $R(t,x)$ tends to zero exponentially fast as $t \to \infty$. Indeed, let us consider the first term. By (26) and (18), with an arbitrary $\varepsilon \in (0,r)$, we have the estimate

$$|\frac{1}{g^2(t)}\int_{U_{tv}}\int_{U_{tv}} q_2(t,x,y_1,y_2)dy_1dy_2| \le$$

$$A(\varepsilon)\frac{e^{2\lambda_0 t-\varepsilon t-t|v|\sqrt{2(\lambda_0-\varepsilon)}}(e^{-\varepsilon t-t|v|\sqrt{2(\lambda_0-\varepsilon)}}+t^{\frac{1-d}{2}}e^{-t|v|\sqrt{2\lambda_0}})}{t^{1-d}e^{2(\lambda_0-\sqrt{2\lambda_0}|v|)t}} \leq \tag{29}$$

$$A(\varepsilon)(t^{d-1}e^{2t\left(|v|\sqrt{2\lambda_0}-\varepsilon-|v|\sqrt{2(\lambda_0-\varepsilon)}\right)}+t^{\frac{d-1}{2}}e^{t\left(|v|\sqrt{2\lambda_0}-\varepsilon-|v|\sqrt{2(\lambda_0-\varepsilon)}\right)}).$$

Note that $|v|\sqrt{2\lambda_0}-\varepsilon-|v|\sqrt{2(\lambda_0-\varepsilon)}<0$ since $|v|\in(0,\sqrt{\lambda_0/2})$. Therefore, the right hand side of (29) tends to zero exponentially fast as $t\to\infty$. The other terms in the expression for $R(t,x)$ can be dealt with in the same fashion. □

**Remark.** With the help of Lemma 3.4 it is possible to show that we have the convergence of all the moments in (22) and (27).

# 6 The almost sure convergence

**Theorem 6.1.** *The convergence in (20) takes place almost surely. If $U$ is a domain with a smooth boundary, then the convergence in (22) and (27) takes place almost surely.*

*Proof.* We will only prove the almost sure convergence in (27) since the other statements can be proved similarly. Fix an arbitrary $\delta>0$ and $K>0$. By the Borel-Cantelli lemma, it is sufficient to demonstrate that there is an increasing sequence $t_n\to\infty$ such that

$$\sum_{n=1}^{\infty}\mathrm{P}(\sup_{t\in[t_n,t_{n+1}]}|\frac{n_t^x(U_{tv})}{g(t)}-\xi^x|>\delta,\ \xi^x<K)<\infty.$$

From the proof of Theorem 5.2 it follows that $n_t^x(U_{tv})/g(t)$ converges to $\xi^x$ in $L^2$ exponentially fast. Let $\gamma>0$ be such that

$$\mathrm{E}(\frac{n_t^x(U_{tv})}{g(t)}-\xi^x)^2\leq Ae^{-\gamma t} \tag{30}$$

for some constant $A$. We take $t_n=2\ln n/\gamma$, $n\geq 1$. By the Chebyshev inequality,

$$\mathrm{P}(|\frac{n_{t_n}^x(U_{t_nv})}{g(t_n)}-\xi^x|>\frac{\delta}{5})\leq\frac{25Ae^{-\gamma t_n}}{\delta^2}=\frac{25A}{\delta^2 n^2},$$

and therefore

$$\sum_{n=1}^{\infty}\mathrm{P}(|\frac{n_{t_n}^x(U_{t_nv})}{g(t_n)}-\xi^x|>\frac{\delta}{5})<\infty. \tag{31}$$

It remains to show that

$$\sum_{n=1}^{\infty}\mathrm{P}(\sup_{t\in[t_n,t_{n+1}]}|\frac{n_t^x(U_{tv})}{g(t)}-\frac{n_{t_n}^x(U_{t_nv})}{g(t_n)}|>\frac{4\delta}{5},\ \xi^x<K)<\infty,$$

which is equivalent to the following two inequalities holding at the same time

$$\sum_{n=1}^{\infty} \mathrm{P}( \sup_{t\in[t_n,t_{n+1}]} \frac{n_t^x(U_{tv})}{g(t)} - \frac{n_{t_n}^x(U_{t_nv})}{g(t_n)} > \frac{4\delta}{5},\ \xi^x < K) < \infty, \tag{32}$$

$$\sum_{n=1}^{\infty} \mathrm{P}( \sup_{t\in[t_n,t_{n+1}]} (-\frac{n_t^x(U_{tv})}{g(t)} + \frac{n_{t_n}^x(U_{t_nv})}{g(t_n)}) > \frac{4\delta}{5},\ \xi^x < K) < \infty. \tag{33}$$

We will only prove (32) since (33) can be proved similarly. Let us show that the term $n_{t_n}^x(U_{t_nv})/g(t_n)$ in (32) can be replaced by a more convenient expression. For $r > 0$, let $U^r = \{x \in \mathbb{R}^d : \mathrm{dist}(x, U) < r\}$ be the $r$-neighborhood of $U$. Let $g^r(t) = g(U^r, t, v)$. Since $U$ is a smooth domain, from the definition of $g$ it follows that $r$ can be chosen to be sufficiently small so that

$$\sup_{t\in[t_n,t_{n+1}]} |\frac{g^r(t_n) - g(t)}{g(t)}| < \frac{\delta}{5}/(K + \frac{\delta}{5}) \tag{34}$$

for all sufficiently large $n$. Moreover, since $r > 0$ and $t_{n+1} - t_n \to \infty$, we have

$$\bigcup_{t\in[t_n,t_{n+1}]} U_{tv} \subset U_{t_nv}^{\frac{r}{2}} \tag{35}$$

for all sufficiently large $n$. As in (31), we have

$$\sum_{n=1}^{\infty} \mathrm{P}(|\frac{n_{t_n}^x(U_{t_nv}^r)}{g^r(t_n)} - \xi^x| > \frac{\delta}{5}) < \infty. \tag{36}$$

Now,

$$\sum_{n=1}^{\infty} \mathrm{P}( \sup_{t\in[t_n,t_{n+1}]} |\frac{n_{t_n}^x(U_{t_nv}^r)}{g(t)} - \frac{n_{t_n}^x(U_{t_nv})}{g(t_n)}| > \frac{3\delta}{5},\ \xi^x < K) \leq$$
$$\sum_{n=1}^{\infty} \mathrm{P}( \sup_{t\in[t_n,t_{n+1}]} |\frac{n_{t_n}^x(U_{t_nv}^r)}{g^r(t_n)} - \frac{n_{t_n}^x(U_{t_nv})}{g(t_n)}| > \frac{2\delta}{5},\ \xi^x < K) +$$
$$\sum_{n=1}^{\infty} \mathrm{P}( \sup_{t\in[t_n,t_{n+1}]} |\frac{n_{t_n}^x(U_{t_nv}^r)}{g(t)} - \frac{n_{t_n}^x(U_{t_nv}^r)}{g^r(t_n)}| > \frac{\delta}{5},\ \xi^x < K).$$

The first series on the right hand side is finite by (31) and (36). The second series is estimated from above by

$$\sharp\{n : \sup_{t\in[t_n,t_{n+1}]} (K + \frac{\delta}{5}) |\frac{g^r(t_n) - g(t)}{g(t)}| > \frac{\delta}{5}\} + \sum_{n=1}^{\infty} \mathrm{P}(|\frac{n_{t_n}^x(U_{t_nv}^r)}{g^r(t_n)} - \xi^x| > \frac{\delta}{5}).$$

The first term of this expression is finite as follows from (34), while the second one is finite by (36). Therefore, (32) will follow if we demonstrate that

$$\sum_{n=1}^{\infty} \mathrm{P}( \sup_{t \in [t_n, t_{n+1}]} \frac{n^x_t(U_{tv}) - n^x_{t_n}(U^r_{t_n v})}{g(t)} > \frac{\delta}{5},\ \xi^x < K) < \infty. \tag{37}$$

Similarly to (31), we have

$$\sum_{n=1}^{\infty} \mathrm{P}(|\frac{N^x_{t_n}}{e^{\lambda_0 t_n}} - \xi^x| > c) < \infty$$

for each constant $c > 0$. Combining this with (26) and (35), we see that (37) will follow if we show that for each $\alpha, R > 0$ we have

$$\sum_{n=1}^{\infty} \mathrm{P}( \sup_{t \in [t_n, t_{n+1}]} n^x_t(U^{\frac{r}{2}}_{t_n v}) - n^x_{t_n}(U^r_{t_n v}) > n^\alpha,\ N^x_{t_n} < Re^{\lambda_0 t_n}) < \infty. \tag{38}$$

Roughly speaking, we need to show that the number of particles that visit a smaller region $U^{\frac{r}{2}}_{t_n v}$ over a short interval of time $[t_n, t_{n+1}]$ can't significantly exceed the number of particles that are in the larger region $U^r_{t_n v}$ at the time $t_n$. Let us fix $n$ and study the $n$-th term in the series (38). After conditioning on $\mathcal{F}_{t_n}$, the questions becomes the following: Suppose we have $m \leq Re^{\lambda_0 t_n}$ particles at time zero located at $x_1, ..., x_m$. Let $y_1, ..., y_M$ be their descendants at time $t = t_{n+1} - t_n$. Each of these has a starting point in the set $\{x_1, ..., x_m\}$. We are interested in the probability that at least $n^\alpha$ descendants of the original particles were at a distance $r/2$ from their respective starting points prior to the time $t = t_{n+1} - t_n$.

The expected number of descendants of a single particle that cover distance $r/2$ (from the initial position of the particle) in time $t$ decays faster than $e^{-\beta/t}$ as $t \to 0$ for some $\beta > 0$. Therefore, by the Chebyshev inequality, the $n$-th term in (38) is estimated from above by $Re^{\lambda_0 t_n} e^{-\beta/(t_{n+1} - t_n)} n^{-\alpha}$, which decays exponentially in $n$, as follows from the definition of $t_n$. Therefore the series (38) converges, which completes the proof. □

# 7 Limiting shape of the region occupied by particles

Let $B(r)$ denote the ball of radius $r$ centered at the origin. Recall that $b = \sqrt{\lambda_0/2}$.

**Theorem 7.1.** *For each $\delta > 0$, there exists a random variable $T = T(\delta)$ ($T < \infty$ almost surely) with the following properties:*

*(a) There are no particles outside $B((b + \delta)t)$ for $t \geq T$.*

*(b) On the event $\xi^x > 0$ the union of the unit neighborhoods of the particles cover $B((b - \delta)t)$ for all $t \geq T$.*

*Sketch of the proof.* We'll only verify the statement for a sequence of times $t_n = c\ln n$ for a certain $c$. Namely, we'll show that there is a random variable $N$ and a constant $c > 0$ such that:

$(a')$ There are no particles outside of $B((b+\delta)t_n)$ for $n \geq N$.

$(b')$ On the event $\xi^x > 0$ the union of the unit neighborhoods of the particles cover $B((b-\delta)t_n)$ for $n \geq N$.

The transition from the sequence of times to the continuous time can be then accomplished similarly to the way it was done in the proof of Theorem 6.1.

By the Chebyshev inequality,

$$\mathrm{P}(n^x_{t_n}(B((b+\delta)t_n)) \geq 1) \leq \mathrm{E}n^x_{t_n}(B((b+\delta)t_n)) = \int_{B((b+\delta)t_n)} \rho_1(t_n, x, y)dy \leq$$

$$A_1 \int_{B((b+\delta)t_n)} (\exp(\lambda_0 t_n - |y|\sqrt{2\lambda_0}) + \exp((\lambda_0 - \varepsilon)t_n - |y|\sqrt{2(\lambda_0-\varepsilon)}))dy \leq$$

$$A_2 \left(\exp(t_n(\lambda_0 - \sqrt{2\lambda_0}(b+\delta))) + \exp(t_n(\lambda_0 - \varepsilon - (b+\delta)\sqrt{2(\lambda_0-\varepsilon)}))\right),$$

where the second inequality is due to Lemma 3.3 and (6). By choosing a sufficiently small $\varepsilon > 0$, we can make the right hand side of the last formula smaller than $A_2 \exp(-t_n\delta\sqrt{\lambda_0})$. Therefore,

$$\sum_{n=1}^{\infty} \mathrm{P}(n^x_{t_n}(B((b+\delta)t_n)) \geq 1) \leq A_2 \sum_{n=1}^{\infty} e^{-t_n\delta\sqrt{\lambda_0}} \leq A_2 \sum_{n=1}^{\infty} n^{-2} < \infty$$

if we choose $t_n = c_1 \ln n$ with $c_1 \geq 2/(\delta\sqrt{\lambda_0})$. By the Borel-Cantelli lemma, there is a random variable $N_1$ such that $(a')$ holds (with $N_1$ instead of $N$).

In order to establish $(b')$, note that for each $n$, the ball $B((b-\delta)t_n)$ can be covered by the balls $B^1_n, ..., B^{m(n)}_n$ of radius $1/2$ centered at $t_n v^1, ..., t_n v^{m(n)}$ in such a way that the centers of the balls are inside $B((b-\delta)t_n)$ and $m(n) = O(t^d_n)$. Let $g^k_n = g(B(1/2), t_n, v^k)$. As in (30), there is $\gamma > 0$ such that

$$\mathrm{E}(\frac{n^x_{t_n}(B^k_n)}{g^k_n} - \xi^x)^2 \leq Ae^{-\gamma t_n}.$$

By the Chebyshev inequality, for each $k = 1, ..., m(n)$ and each $K > 0$,

$$\mathrm{P}(n^x_{t_n}(B^k_n) = 0, \quad \xi^x \geq K) \leq \mathrm{P}(n^x_{t_n}(B^k_n) \leq Kg^k_n/2, \quad \xi^x \geq K) \leq \frac{4Ae^{-\gamma t_n}}{K^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \mathrm{P}(n^x_{t_n}(B^k_n) = 0 \quad \text{for some } k, \quad \xi^x \geq K) \leq \sum_{n=1}^{\infty} A_2(K)e^{-\gamma t_n} t^d_n.$$

The series on the right hand side of this inequality converges if we choose $t_n = c_2 \ln n$ with $c_2 \geq 2/\gamma$. Since $K > 0$ was arbitrary, by the Borel-Cantelli lemma, $(b')$ holds (with $N_2$ instead of $N$). It remains to take $c = \max(c_1, c_2)$ and then $N = \max(N_1, N_2)$. □

# 8 Limiting distribution in the case of finitely many particles

In this section we make a couple of remarks concerning the distribution of the total number of particles on the event $\xi^x = 0$.

**Remark.** Let $N^x_\infty = \lim_{t\to\infty} N^x_t$. This random variable is finite almost surely on the event $\xi^x = 0$ (this can be proved similarly to the way it was done for the corresponding statement in the critical case in [12]). In other words, with probability one, the total number of particles either grows exponentially or tends to a finite limit. The latter event has nonzero probability if and only if $d \geq 3$.

**Remark.** Let $d \geq 3$ and define $M^n(x) = \mathrm{P}(N^x_\infty = n)$, $n \geq 1$. The quantities $M^n(x)$ satisfy a recursive system of partial differential equations. Namely,

$$\frac{1}{2}\Delta M^1(x) = v(x), \tag{39}$$

with the condition at infinity

$$\lim_{|x|\to\infty} M^1(x) = 1.$$

For $n \geq 2$, we have

$$\frac{1}{2}\Delta M^n(x) = v(x)\sum_{k=1}^{n-1} M^k(x)M^{n-k}(x), \tag{40}$$

$$\lim_{|x|\to\infty} M^n(x) = 0.$$

Equations (39) and (40) can be obtained by considering the behavior of the initial particle on the time interval $[0,\delta]$ such that $\delta \downarrow 0$, with the left hand side accounting for the diffusive motion and the right hand side for the branching.

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