MAXIMAL FUNCTIONS FOR MULTIPLIERS ON COMPACT MANIFOLDS

Abstract. Let P be a self-adjoint positive elliptic (-pseudo) differential operator on a compact manifold M without boundary. For a function $m \in L^\infty[0,\infty)$ satisfying a Hörmander-Mikhlin type condition, Seeger and Sogge [11] proved that the multiplier theorem $\|m(P)f\|_{L^p(M)} \le C_p\|f\|_{L^p(M)}$ holds. In this paper, we prove that $\| \sup_{1\le i\le N} |m_i(P)f|\|_{L^p} \le C_p(\log(N+1))^{1/2}\|f\|_{L^p}$ holds when $\{m_i\}_{i=1}^N$ uniformly satisfy the condition. This result is sharp when M is n dimensional torus.

1. INTRODUCTION

Suppose a function $m \in L^\infty(\mathbb{R}^n)$ satisfies the Hörmander-Mikhlin condition

$$\sup_{\lambda\in\mathbb{R}^+} \|\phi(\xi)m(\lambda\xi)\|_{L_2^\alpha} \le A, \qquad \alpha > \frac{n}{2} \tag{1.1}$$

with a nonzero function $\phi \in C_c^\infty$ supported on $[\frac{1}{2}, 2]$. Then the muliplier operator $m(D)f(x) := \mathcal{F}^{-1}(m(\xi)\hat{f}(\xi))(x)$ is well-known. That is,

$$\|m(D)f\|_{L^p} \le C_k\|f\|_{L^p}, \qquad 1 < p < \infty.$$

[A more history] There are many variations related to this result. Let us introduce one of them here. We consider N mulipliers $m_1, \dots, m_N$ satisyfing uniformly condition (1.1) and we seek to find the minimal growth of a function $A(N)$ as N goes to infinity which gives the bound

$$\| \sup_{1\le i\le N} |m_i(D)f|\|_p \le A(N)\|f\|_p \tag{1.2}$$

for all $f \in S$ and $N \in \mathbb{N}$. In [5], Christ et al. found an example which shows that $A(N) \ge c\sqrt{\log(N+1)}$ and they proved that $A(N) = O(\log(N+1))$ using an extrapolation argument. In a subsequent paper [8] Grafakos et al. obtained the sharp result $A(N) = O(\log(N+1)^{1/2})$.

In this paper, we consider the same problem on compact manifolds. On the setting of compact manifolds, Seeger and Sogge [11] obatined a multiplier theorem which is the analogue of the Hormander-Mikhlin multiplier theorem on Euclidean space.

Let M be a compact boundaryless manifold of dimension $n \ge 2$. We consider a first order elliptic pseudo-differential operator P. We assume that P is positive and

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2000 *Mathematics Subject Classification.* Primary.

self-adjoint with respect to a C^∞ density dx on M. It imply that $L^2(M) = L^2(M, dx)$ can be decomposed as

$$L^2(M) = \sum_{j=1}^{\infty} E_j$$

with the eigenspaces E_j corresponding to eigenvalues λ_j. Here we assume that $\{\lambda_j\}$ is arranged as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Let e_j be the projection onto the eigenspace E_j. Then,

$$f = \sum_{j=1}^{\infty} e_j(f), \qquad \forall f \in L^2(M)$$

where the summation converges in L^2.

For a bounded function m defined on $[0, \infty)$, the operator $m(P)$ is defined by

$$m(P)f = \sum_{j=1}^{\infty} m(\lambda_j) e_j(f). \tag{1.3}$$

The operator $m(P)$ is always bounded on $L^2(M)$ from the identity $\|f\|^2_{L^2(M)} = \sum \|e_j(f)\|^2_{L^2(M)}$.

But, we need some smoothness condition on the function m to have that $m(P)$ is bounded on $L^p(M)$ with $p \neq 2$.

Suppose that $\beta \in C_0^\infty((1/2, 2))$ satisfies $\sum_{-\infty}^{\infty} \beta(2^j s) = 1, s > 0$. Under the assumption

$$\sup_{\lambda > 0} \lambda^{-1} \int_{-\infty}^{\infty} |\lambda^\alpha D_s^\alpha (\beta(s/\lambda) m(s))|^2 ds < \infty, \ 0 \leq \alpha \leq s \tag{1.4}$$

with $s > \frac{n}{2}$, the main theorem in [11] is that

$$\|m(P)f\|_{L^p(M)} \leq C_p \|f\|_p, \qquad 1 < p < \infty.$$

The main theorem in this paper is the following:

Theorem 1.1. *Suppose* $1 < r \leq \frac{2(n+1)}{n+3}$ *and* $m_1, \cdots, m_N$ *uniformly satisfy the condition* (1.4) *with* $s > \frac{n}{r}$. *Then,*

$$\| \sup_{1 \geq i \geq N} |m_i(P) f| \|_{L^p(M)} \leq C_p (\log(N+1))^{1/2} \|f\|_p$$

holds with $r < p < \infty$.

This paper is organized as follows. In section 2, we shall study the properties of kernels of mulitpliers on compact manifolds. Aslo, the remainder terms will be estimated. Then, in section 3, we shall estimate the interaction between homogeneous martingales and the main kernels. Then, we shall prove the main theorem in section 4.

2. KERNELS OF MULTIPLIERS ON COMPACT MANIFOLDS

In this section, we shall study the properties of kernels corresponding to the multipliers on compact manifolds. Firstly, we shall recall fundamental materials in [12] and we shall exploit more porperties which will be useful later.

Let M be a compact manifold and P be a first-order selfadjoint positive elliptic operators on M. Let $E_j : L^2 \to L^2$ be the projection maps onto the one-dimensional eigenspace ε_j with eigenvalu λ_j. Then, by the spectral theorem, we have

$$P = \sum_{j=1}^{\infty} \lambda_j E_j.$$

If we let $\{e_j(x)\}$ the orthonormal basis adapted to the spectral decomposition, we have

$$E_j f(x) = e_j(x) \int_M f(y)\overline{e_j(y)}dy$$

From (1.3), the kernel of $m(P)$ is equal to

$$\sum_j m(\lambda_j)e_j(x)e_j(y).$$

On the other hand, we mainly study the kernels from the following formula

$$m(P) = \int_{-\infty}^{\infty} e^{itP}\hat{m}(t)f dt$$

and the following theorem:

Theorem 2.1 ([12, Theorem]). *Let M be a compact C^∞ manifold and let $P \in \psi^1_{cl}(M)$ be elliptic and self-adjoint with respect to a positive C^∞ density dx. Then there is an $\epsilon > 0$ such that when $|t| < \epsilon$,*

$$e^{itP} = Q(t) + R(t)$$

where the remainder has kernel $R(t, x, y) \in C^\infty([-\epsilon, \epsilon] \times M \times M)$ and the kernel $Q(t, x, y)$ is supported in a small neighborhood of the diagonal in $M \times M$. Furthermore, suppose that local coordinate are chosen in a patch $\Omega \subset M$ so that dx agrees with Lebesque measure in the corresponding open subset $\tilde{\Omega} \subset \mathbb{R}^n$; then, if $\omega \subset \Omega$ is relatively compact, $Q(t, x, y)$ takes the following form when $(t, x, y) \in [-\epsilon, \epsilon] \times M \times \omega$.

$$Q(t, x, y) = (2\pi)^{-n} \int e^{i[\phi(x,y,\xi)+tp(y,\xi)]} q(t, x, y, \xi)d\xi.$$

We find the eigenfunction v corresponding to the first eigenvalue λ_1, that is $Pv_1 = \lambda_1 v_1$. We may assume that v is positive. Then,

$$\int_M e_j(x)v(x)dx = 0 \qquad , j = 2, 3, \cdots . \tag{2.1}$$

From (2.1), it will be useful to use $v(x)dx$ for our choice of the density of the manifold M.

Now we decompose our mutipliers dyadically. For this we find two function $\phi_0 \in C_0^\infty[0, 1)$ and $\phi \in C_0^\infty(1/4, 1)$ such that $\sum_{j=0}^\infty \phi_j^3(s) = 1$ for all $s \geq 0$ where we let $\phi_j(s) = \phi(s/2^j)$ for $j \geq 1$. Then

$$m(P)f = \sum_{j=1}^\infty \phi_j(P)(m(P)\phi_j(P))\phi_j(P)f.$$

We let $m_j(s) = m(s)\phi_j(s)$. Then

$$m_j(P) = \int e^{itP}\widehat{m_j}(t)dt. \tag{2.2}$$

We fix a function $\rho \in S(\mathbb{R})$ satisfying $\rho(t) = 1, |t| \leq \frac{\epsilon}{2}$ and $\rho(t) = 0, |t| > \epsilon$. Then we split the integral (2.2) as

$$\begin{aligned} m_j(P) &= \int e^{itP}\widehat{m_j}(t)\rho(t)dt + \int e^{itP}\widehat{m_j}(t)(1-\rho(t))dt \\ &=: \tilde{m}_j(P) + r_j(P). \end{aligned}$$

Proposition 2.2. *For* $1 \leq p \leq \frac{2(n+1)}{n+3}$,

$$\|\sum_{j=1}^\infty \phi_j(P)r_j(P)\phi_j(P)f\|_{L^\infty} \lesssim \|f\|_p \quad \text{if } s > \frac{n}{p}.$$

Proof. It suffices to show that

$$\|\phi_j(P)r_j(P)\phi_j(P)f\|_{L^\infty} \lesssim 2^{j(\frac{n}{p}-s)}\|f\|_p.$$

We have

$$\begin{aligned} \|\phi_j(P)r_j(P)\phi_j(P)f\|_{L^\infty}^2 &\lesssim \lambda^{\frac{n-1}{2}}\|\phi_j(P)r_j(P)\phi_j(P)f\|_{L^2}^2 \\ &\lesssim \lambda^{n-1}\lambda^{2n(\frac{1}{p}-\frac{1}{2})-1}\sum_{k=0}^\infty \sup_{\tau\in[k,k+1)} |\tau_\lambda(\tau)|^2\|f\|_{L^p}^2 \\ &\lesssim \lambda^{\frac{2n}{p}-1}\lambda^{-2s+1}\|f\|_{L^p}^2 = \lambda^{\frac{2n}{p}-2s}\|f\|_{L^p}^2. \end{aligned}$$

The second inequality comes form Theorem 5.1.1 in [12]. The last ineqaulity is due to (5.3.4) in [12].

□

Lemma 2.3. *For* $1 < p \leq \frac{2(n+1)}{n+3}$ *and* m *satisfying the condition* (1.4), *we have*

$$\|r_j(P)f\|_{L^\infty} \leq C\lambda^{\frac{n}{p}-s}\|f\|_p$$

Proof. It follows from the same argument as above with observing that $(\hat{m}_j(\cdot)(1-\rho(\cdot)))^\vee(\tau) = O(((|\tau| + 2^j)^{-N})$ for any $N \in \mathbb{N}$ if $\tau \notin [2^{j-2}, 2^{j+2}]$. $\square$

And we have

$$\int e^{itP}\widehat{m_j}(t)\rho(t)dt = \int (Q(t) + R(t))\widehat{m_j}(t)\rho(t)dt.$$

Observe that

$$\int R(t)\rho(t)\widehat{m_j}(t)dt = \int \widehat{R(\cdot)\rho(\cdot)}(t)m(t)\phi(\frac{t}{2^j})dt.$$

From the support of $\phi(\frac{\cdot}{2^j})$ and the fact that $m \in L^\infty(\mathbb{R})$ we induce that

$$\int R(t,x,y)\rho(t)\widehat{m_j}(t)dt = O_N(2^{-jN}) \quad \text{for all } N \in \mathbb{N}. \tag{2.3}$$

So we may only consider $\int Q(t)\widehat{m_j}(t)\rho(t)dt$. And it was proved in [12] that

Lemma 2.4. *Let* $\tilde{K}_j(x,y)$ *be the kernel of* $\int Q(t)\widehat{m_j}(t)\rho(t)dt$. *Then we have* $\tilde{K}_j(x,y) = 2^{nj}K_j(2^jx, 2^jy)$ *where* K_j *satisfying*

$$\int |D_y^\alpha K_j(x,y)|^2(1+|x-y|)^{2s}dx \leq C, \quad 0 \leq |\alpha| \leq 1.$$

However, we need to bound higher-order integral of K_j for later use. To obtain this, we find a C^∞ function ζ supported on $(\frac{1}{8}, 2)$ such that $\zeta = 1$ on $(\frac{1}{4}, 1)$. Then we have $\zeta\phi = \phi$ and also $\zeta_j\phi_j = \phi_j$. From this, we have

$$\begin{aligned} m_j(P) &= m_j(P)\zeta_j(P) \\ &= \tilde{m}_j(P)\circ\zeta_j(P) + r_j(P)\circ\zeta_j(P). \end{aligned}$$

We can treat $r_j(P)\zeta_j(P$ by the same way for $r_j(P)$. for the first term,

$$\tilde{m}_j(P)\circ\zeta_j(P) = \int Q(t)\widehat{m_j}(t)\rho(t)dt \circ \zeta_j(P) + O(2^{-jN})\zeta_j(P).$$

Here, we only need to concern the first term. From Lemma (2.3), we have

$$\zeta_j(P) = \int Q(s)\cdot\widehat{\zeta_j}(s)\rho(s)ds + O(2^{-jN}).$$

So the remainder term causes no problem and we may only concern the operator

$$\int Q(t)\tilde{m}_j(t)\rho(t)dt \circ \int Q(s)\widehat{\zeta_j}(s)\rho(s)ds \tag{2.4}$$

We notice that above two operators are both local operators, that is, their kernels have their supports on near the digonal set in $M \times M$. Therefore, the kernel of the operator (2.4) has also support near the diagonal. We let $\tilde{L}_j(x,y)$ be the kernel of

$\int Q(s)\hat{\zeta}_j(s)\rho(s)ds$. Then, by Lemma (2.4), we may let $\tilde{L}_j(x,y) = 2^{jn}L_j(2^jx, 2^jy)$ with L_j satisfying

$$\int |D_y^\alpha K_j(x,y)|^2(1+|x-y|)^{2N}dx \le C_N, \quad 0 \le |\alpha \le 1$$

for any $N \in \mathbb{N}$. We let $\tilde{H}_j$ be the kernel of the operator (2.4). Then we have

$$\tilde{H}_j(x,z) = \int \tilde{K}_j(x,y)\tilde{L}_j(y,z)dy. \tag{2.5}$$

Lemma 2.5. *We have* $\tilde{H}_j(x,z) = 2^{jn}H_j(2^jx, 2^jz)$ *with* H_j *satisfying*

$$\int |H_j(x,z)|^q(1+|x-z|)^{sq}dz \le C$$

for each $q \ge 2$. *And*

$$\int H_j(x,z)dz = O_N(2^{-jN})$$

for any $N \in \mathbb{N}$.

Proof. We write (2.5) as

$$\begin{aligned} 2^{jn}H_j(2^jx,2^jz) &= \int 2^{jn}K_j(2^jx,2^jy)2^{jn}L_j(2^jy,2^jz)dy \\ &= \int 2^{jn}K_j(2^jx,y)L_j(y,2^jz)dy. \end{aligned}$$

So,

$$H_j(x,z) = \int K_j(x,y)L_j(y,z)dy.$$

Using this, we have

$$\begin{aligned} (1+|x-z|)^s|H_j(x,y)| &= (1+|x-z|)^s\int K_j(x,y)L_j(y,z)dy \\ &\le \int K_j(x,y)(1+|x-y|)^s \cdot L(y,z)(1+|y-z|)^s dy \\ &\le (\int |K_j(x,y)|^2(1+|x-y|)^{2s}dy)^{1/2} \cdot (\int |L_j(y,z)|^2(1+|y-z|)^{2s}dy)^{1/2} \end{aligned}$$

On the one hand,

$$\begin{aligned} |\int K_j(x,z)dz| &= |\int[\int K_j^1(x,y)dx]K_j^2(y,z)dy| \\ &\le \int O(2^{-Nj})|K_j^2(y,z)|dy \\ &= O(2^{-Nj}). \end{aligned}$$

□

Therefore, we have

$$\tilde{m}_j(P) \circ \zeta_j(P) = \tilde{H}_j(x,z) + O_N(2^{-jN})$$

and

$$m_j(P) = \tilde{H}_j(x,z) + O_N(2^{-jN}) + r_j(P) \circ \zeta_j(P).$$

As a corollay, we obtain the following.

Corollary 2.6. *We just let $\phi_j(x,y)$ be the kernel of $\phi_j(P)$ and we split $\phi_j(x,y) = \tilde{\phi}_j(x,y) + R_j(x,y)$ as above. Then,*

$$\int |\tilde{\phi}_j(x,y)| 2^{jn}(1+2^j|x-y|)^N dy \leq C,$$

$$\int \tilde{\phi}_j(x,y)dy = O(2^{-Nj})$$

and $|R_j(x,y)| = O(2^{-Nj})$.

We now state another corollary comes from the same proof of Lemma 2.5 in [2].

Corollary 2.7.

$$|\tilde{H}_j * f(x)| \lesssim M_r f(x) \cdot \|m_j\|_{L_2^\alpha},$$

We have

$$m(P)f = \sum_{j=1}^{\infty} m_j(P)\phi_j(P)f = \sum_{j=1}^{\infty}(\tilde{\phi}_j(P) + O(2^{-jN}))m_j(P)\phi_j(P)f$$

and

$$\begin{aligned}\sum_{j=1}^{\infty} \tilde{\phi}_j(P)m_j(P)\phi_j(P)f &= \sum_{j=1}^{\infty} \tilde{\phi}_j(P)(\tilde{H}_j(x,z) + O_N(2^{-jN}) + r_j(P) \circ \zeta_j(P)) \circ \phi_j(P)f \\ &= \sum_{j=1}^{\infty} \tilde{\phi}_j(P)\tilde{H}_j(x,z)\phi_j(P)f + \sum_{j=1}^{\infty} \tilde{\phi}_j(P)(O_N(2^{-jN}) + r_j(P) \circ \zeta_j(P))\phi_j(P)f\end{aligned}$$

The second term can be treated by Proposition 2.2. And we split $\phi_j(P)$ as $\phi_j(P)f = (\tilde{\phi}_j(P) + \tilde{R}_j(P))f$. Then

$$\sum_{j=1}^{\infty} \tilde{H}_j(x,z)\phi_j(P)f = \sum_{j=1}^{\infty} \tilde{H}_j(x,z)\tilde{\phi}_j(P)f + \sum_{j=1}^{\infty} \tilde{H}_j(x,z)O(2^{-Nj})f.$$

Since the second term is trivially bounded,we only need to consider

$$\tilde{m}(P)f = \sum_{j=1}^{\infty} \tilde{\phi}_j(P)\tilde{H}_j(x,z)\tilde{\phi}_j(P)f.$$

Now we modify the kernel $\tilde{\phi}_j(x,y)$ to $\bar{\phi}_j$ so that we have

$$\int \bar{\phi}_j(x,t)dx = 0 \quad \text{for all } y.$$

This fact will be used in the proof of Lemma 3.3. Since $\int \tilde{\phi}_j(x,y)dx = O(2^{-Nj})$ we can modify it to $\bar{\phi}_j$ so that $\int \bar{\phi}_j(x,y)dx = 0$ and $\phi_j^m = \tilde{\phi}_j - \bar{\phi}_j$ satisfy

$$\int |\phi_j^m(x,y)| 2^{jn}(1+2^j|x-y|)^N dy = O(2^{-N_0 j})$$

with sufficiently large $N_0 >> 1$. Then, the summation with the operators with the kernels ϕ_j^m causes no problem. So we may only deal with

$$\bar{m}(P)f = \sum_{j=1}^{\infty} \bar{\phi}_j(P)\tilde{H}_j(x,z)\tilde{\phi}_j(P)f.$$

We let $\bar{\phi}_j(x,y) = 2^{jn}\phi_j'(2^j x, 2^j y)$.

3. MARTINGALES ON HOMOGENEOUS SPACE

We introduce the following things on homogeneous space in [4] which may be regarded as dyadic cubes on Euclidean space. Open set Q_α^k will role as dyadic cubes of sidelengths 2^{-k} (or more precisely, δ^k) with the two conventions : 1. For each k, the index α will run over some unspecified index set dependent on k. 2. For two sets with $Q_\alpha^{k+1} \subset Q_\beta^k$, we say that Q_β^k is a parent of Q_α^{k+1}, and Q_α^{k+1} a child of Q_β^k.

Theorem 3.1 (Theorem 14, [4]). *Let X be a space of homogenous type. Then there exists a family of subset $Q_\alpha^k \subset X$, defined for all integers k, and constants $\delta, \epsilon > 0, C < \infty$ such that*

- $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0 \ \forall k$
- *for any α, β, k, l with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$*
- *each Q_α^k has exactly one parent for all $k \geq 1$*
- *each Q_α^k has at least one child*
- *if $Q_\alpha^{k+1} \subset Q_\beta^k$ then $\mu(Q_\alpha^{k+1}) \geq \epsilon\mu(Q_\beta^k)$*
- *for each (α,k) there exists $x_{\alpha,k} \in X$ such that $B(x_{\alpha,k}, \delta^k) \subset Q_\alpha^k \subset B(x_{\alpha,k}, C\delta^k)$.*

Moreover,

$$\mu\{y \in Q_\alpha^k : \rho(y, X \setminus Q_\alpha^k) \leq t\delta^k\} \leq Ct^\epsilon \mu(Q_\alpha^k) \ for\ 0 < t \leq 1,\ for\ all\ \alpha, k. \quad (3.1)$$

Expectation operators are defined by

$$\mathbb{E}_k f(x) = \mu(Q_\alpha^k)^{-1} \int_{Q_\alpha^k} f d\mu \quad \text{for } x \in Q_\alpha^k$$

and by $\mathbb{D}_k f(x) = \mathbb{E}_{k+1} f(x) - \mathbb{E}_k f(x)$. Then, many results on the dyadic martingales on Euclidean space still hold in the setting of homogeneous space with the above expectations operators. We define the square function for the martingale as

$$S(f) = (\sum_{k\geq 0} |\mathbb{D}_k f(x)|^2)^{1/2}$$

We state a homogeneous space version of a lemma in [1] which was used in [GH]. There is a constant $c_d > 0$ so that for all $\lambda > 0,\ 0 < \epsilon < \frac{1}{2}$, the following inequality holds.

$$\begin{aligned} &\text{meas}(\{x : \sup_{k\geq 0} |\mathbb{E}_k g(x) - \mathbb{E}_0 g(x)| > 2\lambda, S(g) < \epsilon\lambda\}) \\ &\quad \leq C\text{exp}(-\frac{C_d}{\epsilon^2})\text{meas}(\{x : \sup_{k\geq 0} |\mathbb{E}_k g(x)| > \lambda\}); \end{aligned}$$

see [[1]. Corollary 3.1]. Now we choose a bump function $\psi \in C_0^\infty$ which is supported on $[\frac{1}{4}, 4]$ and equal to 1 on $[\frac{1}{2}, 2]$. Let $\psi_j(\xi) = \psi(2^{-j}\xi)$. Then $m_j(\xi) = \psi_j^2(\xi) m_j(\xi)$ holds and we have

$$\bar{m}_j(P) = \tilde{\phi}_j(P) m_j(P) \tilde{\phi}_j(P).$$

So $\bar{m}_j(P) f = \bar{\phi}_j * (m_j(P)(\tilde{\phi}_j(P)f))$ and

$$\begin{aligned} \mathbb{D}_k(\bar{m}(P)f) &= \mathbb{D}_k(\sum_{j\in\mathbb{Z}} \bar{m}_j(P)f) && (3.2) \\ &= \sum_{j\in\mathbb{Z}} \mathbb{D}_k(\bar{\phi}_j * (\bar{m}_j(P)\tilde{\phi}_j(P)f)). && (3.3) \end{aligned}$$

We introduce

$$G_r(f) = (\sum_{k\in\mathbb{Z}} (\mathcal{M}(|\tilde{\phi}_k f|^r))^{2/r})^{1/2}$$

Fefferman-Stein [6] inequality is

$$\|G_r(f)\|_p \leq C_{p,r} \|f\|_p, \qquad 1 < r < 2, r < p < \infty.$$

In (3.2), some cancellation between the martingale operators $\mathbb{D}_k$ and the convolution operators with the kernel k_{ψ_j} exists when the difference between their scales δ^k and 2^{-j} is larger than some constant. This leads to

Proposition 3.2. *$Tf = \bar{m}(P)f$ and $1 < r \leq \infty$. Then, for $x \in G$, $S(Tf)(x) \leq A_r \|m\|_{L_2^\alpha} G_r(f)(x)$.*

We need the following lemma which explains the cancellation property.

Lemma 3.3. $|\mathbb{E}_k(\bar{\phi}_j f)(x) \leq 2^{(-(\log\delta)k - j)/q'} M_q f(x)$ *if* $j > (-\log\delta)k + 10$.
$|\mathbb{B}_k(\bar{\phi}_j f)(x)| \leq 2^{((\log\delta)k + j)/q'} M_q f(x)$ *if* $j < (-\log\delta)k - 10$.

Proof. Find Q^k_α such that $x \in Q^k_\alpha$.

$$\begin{aligned}
\mathbb{E}_k(\bar{\phi}_j f)(x) &= \frac{1}{\mu(Q^k_\alpha)} \int_{Q^k_\alpha} (\bar{\phi}_j f)(y) dy \\
&= \frac{1}{\mu(Q^k_\alpha)} \int_{Q^k_\alpha} [\int_G 2^{nj} \phi'_j(2^j y, 2^j z) f(z) dz] dy \\
&= \frac{1}{\mu(Q^k_\alpha)} \int_G [\int_{Q^k_\alpha} 2^{nj} \phi'_j(2^j y, 2^j z) dy] f(z) dz.
\end{aligned}$$

We now divide f according to its domain. We let

- $B = \{y : \text{dist } (y, \partial Q^k_\alpha) \le 2^{-[(-\log \delta)k + m/2]}\}$
- $A_1 = Q^k_\alpha \cap B^c$
- $A_2 = (Q^k_\alpha)^c \cap B^c$.

We divide f as $f = f_{A_1} + f_{A_2} + f_B = f\chi_{A_1} + f\chi_{A_2} + f\chi_B$. We note that f_B can be treated in the same way given in [8]. So we may only consider for f_A and f_B. Firstly, for f_{A_1},

$$\bar{\phi}_j f_{A_1} f(y) = \int 2^{nj} \phi'_j(2^j y, 2^j z) \chi_{A_1}(z) f(z) dz$$

and

$$\mathbb{E}_k(\bar{\phi}_j f_{A_1}(x)) = \frac{1}{\mu(Q^k_\alpha)} \int_G [\int_{Q^k_\alpha} 2^{nj} \phi'_j(2^j y, 2^j z) dy] \chi_{A_1}(z) f(z) dz$$

$$\begin{aligned}
|\int_{Q^k_\alpha} 2^{nj} \phi'_j(2^j y, 2^j z) dy| &= |\int_{(Q^k_\alpha)^c} 2^{nj} \phi'_j(2^j y, 2^j z) dy| \\
&\le \int_{(Q^k_\alpha)^c} 2^{nj} |\phi'_j(2^j y, 2^j z)| dy \\
&\le \int_{|z| \le 2^{-[(-\log \delta)k + m/2]}} 2^{nj} |\phi'_j(2^n z)| dz \\
&= \int_{|z| \ge 2^m} |\phi'_j(z)| dz \le 2^{-mc}
\end{aligned}$$

So

$$\begin{aligned}
|\mathbb{E}_k(\bar{\phi}_j f(x))| &\le \frac{1}{\mu(Q^k_\alpha)} \int_G 2^{-mc} 1_{A_1}(z) f(z) dz \\
&\le 2^{-mc} M f(x)
\end{aligned}$$

We now consider f_{A_2}.

$$\mathbb{E}_k(\bar{\phi}_j f(x)) = \frac{1}{\mu(Q^k_\alpha)} \int_G [\int_{Q^k_\alpha} 2^{nj} \phi'_j(2^j y, 2^j z) dy] 1_{A_2}(z) f(z) dz$$

We have

$$\int_{Q_\alpha^k} 2^{nj}\phi_j'(2^j y, 2^j z)dy = \frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k}[\int_G 2^{nj}\phi_j'(2^j y, 2^j z)1_{A_2}(z)f(z)dz]dy$$

Observe $|y-z| \leq 2^{-[(-\log\delta)k+m/2]}$ and from Lemma 6.36 in [7],

$$|\int_G 2^{nj}\phi_j'(2^j y, 2^j z)1_{A_2}f(z)dz| \leq Mf(y)\cdot 2^{-m\alpha/2}.$$

So

$$\begin{aligned}\mathbb{E}_k(\bar{\phi}_j f_{K_2}(x)) &\leq \frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k} Mf(y)dy 2^{-m\alpha/2} \\ &\lesssim Mf(x)\cdot 2^{-m\alpha/2}.\end{aligned}$$

We now prove the second statement.

$$\begin{aligned}\mathbb{E}_k(\bar{\phi}_j f)(x) - \mathbb{E}_{k+1}(\bar{\phi}_j f)(x) &= \frac{1}{\mu(Q_\alpha^{k+1})}\int_{Q_\alpha^{k+1}}(\bar{\phi}_j f)(y)dy - \frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k}(\bar{\phi}_j f)(y)dy \\ &= \int_G f(z)[\frac{1}{\mu(Q_\alpha^{k+1})}\int_{Q_\alpha^{k+1}} 2^{nj}\phi_j'(2^j y, 2^j z)dy - \frac{1}{\mu(Q_\alpha^k)}\int_{Q_\alpha^k} 2^{nj}\phi_j'(2^j y, 2^j z)\end{aligned}$$

$$\begin{aligned} & \tfrac{1}{\mu(Q_\alpha^{k+1})}\textstyle\int_{Q_\alpha^{k+1}} 2^{nj}\phi_j'(2^j y, 2^j z) - 2^{nj}\phi_j'(2^j y, 2^j z)dy \\ = & \tfrac{1}{\mu(Q_\alpha)}\textstyle\int_{Q_\alpha^{k+1}} 2^{nj}\phi_j'(2^j(x-y), 2^j(x-z)) - 2^{nj}\phi_j'(2^j x, 2^j z)dy \\ = & \tfrac{1}{\mu(Q_\alpha^{k+1})}\textstyle\int_{Q_\alpha^{k+1}}[\int_0^1 2^{nj}\frac{d}{dt}\phi_j'(2^j t\cdot(y-x), 2^j t\cdot(x-z))dy \\ = & \tfrac{1}{\mu(Q_\alpha^{k+1})}\textstyle\int_{Q_\alpha^{k+1}}[\int_0^1 2^{nj}2^j(y-x)\cdot\nabla\phi_j'(2^j t\cdot(y-x).2^j t\cdot(x-z))dy \\ \leq & \tfrac{1}{\mu(Q_\alpha^{k+1})}\textstyle\int_{Q_\alpha^{k+1}}[\int_0^1 2^{nj}2^j\cdot\delta^k|\nabla\phi_j'(2^n x, 2^n z)|dy \\ \leq & 2^{nj}2^j\delta^k(1+2^j|x-z|)^{-N}, \quad x \in Q_\alpha^k.\end{aligned}$$

So

$$|\mathbb{D}_k(\tilde{\phi}_j f)(x)| \lesssim \int_G f(z)2^{nj}2^j\delta^k(1+2^j|y-z|)^{-N}dz$$

Observe that $|x-z| - c\delta^k \leq |x-z| - c|x-y| \leq c|y-z|$. Thus

$$2^j(|x-z|) - c2^j\delta^k \leq 2^j|y-z|$$

Thus

$$\frac{1}{2} + 2^j|x-z| \leq 2^j|y-z|$$

□

proof of Proposition 3.2.

$$\begin{aligned} |\mathbb{B}_k(Tf)| &= |\sum_{j\in\mathbb{Z}} \mathbb{B}_k(\bar{\phi}_j \tilde{H}_j(x,z)\tilde{\phi}_j(P)f)| \\ &\leq \sum_{j\in\mathbb{Z}} 2^{-|k|\log\delta|-j|} M^r(\tilde{\phi}_j f) \end{aligned}$$

$$|\mathbb{B}_k(Tf)|^2 \leq (\sum_{j\in\mathbb{Z}} 2^{-|k|\log\delta|-j|}) \sum_{j\in\mathbb{Z}} 2^{-||k\log\delta|-j|}(M_r(\tilde{\phi}_j f))^2$$

So

$$S(Tf)(x) = (\sum_{k=1}^{\infty} |\mathbb{B}_k(Tf)|^2)^{1/2} \lesssim (\sum_{n\in\mathbb{Z}} |M_q(\tilde{\phi}_j f)|^2)^{1/2}$$

□

4. PROOF OF MAIN THEOREM

We need to bound

$$\| \sup_{1\leq i\leq N} |T_i f| \|_p = (p4^p \int_0^\infty \lambda^{p-1} \mathrm{meas}(\{x : \sup_i |T_i f(x)| > 4\lambda\}) d\lambda)^{1/p}$$

by some constant time of $\sqrt{\log(N+1)}\|f\|_p$. By propostion 3.2 we have the pointwise bound

$$S(T_i f) \leq A_r B G_r(f). \tag{4.1}$$

We bound the level set as

$$\{x : \sup_{1\leq i\leq N} |T_i f(x)| > 4\lambda\} \subset E_{\lambda,1} \cup E_{\lambda,2} \cup E_{\lambda,3},$$

where with

$$\epsilon_N := (\frac{c_d}{10\log(N+1)})^{1/2}$$

and

$$\begin{aligned} E_{\lambda,1} &= \{x : \sup_{1\leq i\leq N} |T_i f(x) - \mathbb{E}_{-N} T_i f(x)| > 2\lambda, G_r(f)(x) \leq \frac{\varepsilon_N \lambda}{A_r B}\}, \\ E_{\lambda,2} &= \{x : G_r(f)(x) > \frac{\varepsilon_N \lambda}{A_r B}\}, \\ E_{\lambda,3} &= \{x : \sup_{1\leq i\leq N} |\mathbb{E}_0 T_i f(x) > 2\lambda\}. \end{aligned}$$

By (4.1),

$$E_{\lambda,1} \subset \bigcup_{i=1}^{N} \{x : |T_i f(x)| > 2\lambda, S(T_i f) \leq \varepsilon_N \lambda\}$$

and we have

$$\begin{aligned} \text{meas}(E_{\lambda,1}) &\leq \sum_{i=1}^{N} \text{meas}(\{x : |T_i f(x) - \mathbb{E}_{-N} T_i f(x)| > 2\lambda, S(T_i f) \leq \varepsilon_N \lambda\}) \\ &\leq \sum_{i=1}^{N} C \exp(-\frac{c_d}{\varepsilon_N^2}) \text{meas}(\{x : \sup_k |\mathbb{E}_k(T_i f)| > \lambda\}). \end{aligned}$$

Therefore

$$\begin{aligned} &(p \int_0^\infty \lambda^{p-1} \text{meas}(E_{\lambda,1}) d\lambda)^{1/p} \\ &\qquad \lesssim (\textstyle\sum_{i=1}^N \exp(-\frac{c_d}{\varepsilon_N^2}) \| \sup_k |\mathbb{E}_k(T_i f)\|_p^p)^{1/p} \\ &\qquad \lesssim (\textstyle\sum_{i=1}^N \exp(-\frac{c_d}{\varepsilon_N^2}) \|T_i f\|_p^p)^{1/p} \\ &\qquad \lesssim B(N \exp(-\tfrac{c_d}{\varepsilon_N^2}))^{1/p} \|f\|_p \lesssim B\|f\|_p \end{aligned}$$

uniformly in N. By a change of variables,

$$\begin{aligned} (p \int_0^\infty \lambda^{p-1} \text{meas}(E_{\lambda,2}) d\lambda)^{1/p} &= \frac{A_r B}{\varepsilon_N} \|G_r(f)\|_p \\ &\lesssim B\sqrt{\log(N+1)} \|f\|_p. \end{aligned}$$

Finally, from the Fefferman-Stein inequality

$$\text{meas}(E_{\lambda,3}) \leq \sum_{i=1}^{N} \text{meas}(\{x : |\mathbb{E}_{-N} T_i f(x)| > 2\lambda\})$$

and thus

$$\begin{aligned} (p \int_0^\infty \lambda^{p-1} \text{meas}(E_{\lambda,3}) d\lambda)^{1/p} &= 2\| \sup_{i=1,\ldots,N} |\mathbb{E}_{-N}(T_i f)| \|_p \\ &\lesssim \sup_{i=1,\cdots,N} \|T_i f\|_p \\ &\lesssim \|f\|_p. \end{aligned}$$

Above three inequalities conclude the proof.

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