Seifert fibered surgery on Montesinos knots

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Abstract

Exceptional Dehn surgeries on arborescent knots have been classified except for Seifert fibered surgeries on Montesinos knots of length 3. There are infinitely many of them as it is known that $4n+6$ and $4n+7$ surgeries on a $(-2,3,2n+1)$ pretzel knot are Seifert fibered. It will be shown that there are only finitely many others. A list of 20 surgeries will be given and proved to be Seifert fibered. We conjecture that this is a complete list.

1 Introduction

A Dehn surgery on a hyperbolic knot is *exceptional* if it is reducible, toroidal, or Seifert fibered. By Perelman's work, all other surgeries are hyperbolic. For knots in S^3, by exceptional surgery we shall always mean *nontrivial* exceptional surgery.

Given an arborescent knot, we would like to know exactly which surgeries are exceptional. We divide arborescent knots into three types. An arborescent knot is of type I if it has no Conway sphere, so it is either a 2-bridge knot or a Montesinos knot of length 3. A type II knot has a Conway sphere cutting it into two tangles, each of which is the sum of two nontrivial rational tangles, with one of them of slope 1/2. All others are of type III. In [Wu1] it was shown that all nontrivial surgeries on type III arborescent knots are Haken and hyperbolic, and all nontrivial surgeries on type II knots are laminar. In [Wu2] it was further shown that there are exactly three type II knots admitting exceptional surgery, each of which admits exactly one exceptional surgery, producing a toroidal manifold. For type I knots, Brittenham and the author determined exceptional surgeries on 2-bridge knots [BW], toroidal surgeries on Montesinos knots have been classified in [Wu3],

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and it is known that there is no reducible surgery on hyperbolic arborescent knots [Wu1].

It remains to determine small Seifert fibered surgeries on hyperbolic Montesinos knots of length 3, which is also the set of all Seifert fibered surgeries because by Ichihara and Jong [IJ1] the only *toroidal* Seifert fibered surgery on Montesinos knots is the 0 surgery on the trefoil knot and hence there is no large Seifert fibered surgery on hyperbolic Montesinos knots. For the special case of finite surgeries on Montesinos knots, the classification has been done by Ichihara and Jong [IJ2]. See also [FIKMS].

In general, let $K = K(p_1/q_1,\, p_2/q_2,\, p_3/q_3)$ be a hyperbolic Montesinos knot of length 3, and assume that it admits a nontrivial Seifert fibered surgery $K(r)$. Using immersed surfaces, it was shown in [Wu4] that we must have $\frac{1}{q_1-1} + \frac{1}{q_2-1} + \frac{1}{q_3-1} \leq 1$, hence up to relabeling we have $|q_1| = 2$, or $|q_1| = |q_2| = 3$, or $(|q_1|, |q_2|, |q_3|) = (3, 4, 5)$. In [Wu5] we studies persistently laminar branched surfaces in knot complements, and obtained restrictions on the $|p_i|$. More explicitly, if K above is a pretzel knot $K(1/q_1,\, 1/q_2,\, 1/q_3,\, n)$ then either (i) $n = 0$, or (ii) $n = -1$ and $q_i > 0$, and if K is not a pretzel knot then it is either (iii) $K(2/3,\, 1/3,\, 2/5)$, or (iv) $K(1/2,\, 1/3,\, 2/(2a+1))$ with $a \in \{3, 4, 5, 6\}$, or (v) $K(1/2,\, 2/5,\, 1/q)$ for some odd $q \geq 3$.

There are still infinitely many knots among the above, for example it includes all $(2, q_2, q_3)$ pretzel knots. In [Wu6] we studies exceptional surgeries on tubed Montesinos knots. These are the knots in solid tori obtained by tubing Montesinos tangles in some specific ways. By embedding the solid tori into S^3, we see that these knots are closely related to Montesinos knots in S^3. With this method it was shown that there are indeed infinitely many Seifert fibered surgeries on Montesinos knot of length 3, that is, each $(-2, 3, 2n+1)$ pretzel knot in S^3 admits at least two Seifert fibered surgeries, of slopes $4n + 6$ and $4n + 7$, respectively. On the other hand, using the classification theorem in [Wu6], we will show that there are only finitely many other Seifert fibered surgeries on these knots. See Theorem 2.3 below.

A few other surgeries on Montesinos knots are known to be Seifert fibered. There is that well known $(-2, 3, 7)$ pretzel knot, on which 17, 18 and 19 surgeries are Seifert fibered [FS]. Hyun-Jong Song showed that surgery on $(-3, 3, 3)$ with slope 1 is Seifert fibered, and Mattman, Miyazaki and Motegi [MMM] showed that surgery on $(-3, 3, 5)$ of slope 1 is also Seifert fibered. More examples will be given in Table 3.1 below. We conjecture that this is a complete list.

2 A finiteness theorem

Consider the knots $K_n = K(-1/2,\ 1/3,\ 1/2n+1)$. By [Wu6, Corollary 2.3], $K_n(r_n)$ is small Seifert fibered for $r_n = 6+4n$ and $7+4n$, except that $K_2(15)$ is reducible. If K' is the mirror image of K, then an orientation reversing homeomorphism of S^3 induces an orientation reversing homeomorphism from $K(r)$ to $K'(-r)$. We consider (K, r) and $(K', -r)$ as equivalent. Theorem 2.3 below shows that there are only finitely many other small Seifert fibered surgeries on length 3 Montesinos knots.

More generally, consider a two component link $L = K' \cup K''$ with K'' a trivial component. Let V be the solid torus $S^3 - \text{Int}N(K'')$, and let (V, K', r) be the manifold obtained by r surgery on K' in V. Denote by K_m the knot obtained from K' by m right-hand full twists on K''.

Lemma 2.1 *Suppose $L = K' \cup K''$ is a two component hyperbolic link in S^3 with K'' a trivial loop. Then there is a finite collection C of (m, r_n), where r_n is a slope on $\partial N(K')$, such that if $K_m(r_n)$ is non-hyperbolic, then either (i) K_m is non-hyperbolic, or (ii) (V, K', r_n) is nonhyperbolic, or (iii) $(m, r_n) \in C$.*

Proof. This is well known and follows immediately from the 2π-theorem of Gromov and Thurston. By the 2π-theorem there is a finite set C_i of slopes on each cusp T_i of $S^3 - \text{Int}N(L)$, such that if $r_i \notin C_i$ for $i = 1, 2$ then $L(r_1, r_2)$ is hyperbolic. Let $\hat{C}$ be the collection of all slopes (r_1, r_2) such that $L(r_1, r_2)$ is non-hyperbolic, and let C'_i be the set of slopes r on T_i such that r filling on T_i is non-hyperbolic. If for some r_1 there are infinitely many r_2 such that $(r_1, r_2) \in \hat{C}$ then $r_1 \in C'_1$. Similarly if there are infinitely many r_1 with $(r_1, r_2) \in C$ then $r_2 \in \hat{C}'_2$. Thus if we denote by $\hat{C}_i = \{(r_1, r_2) \mid r_i \in C'_i\}$ then $C = \hat{C} - \hat{C}_1 \cup \hat{C}_2$ is finite. Restricting the above to the set of slopes with r_1 of type $1/m$ gives the required result. $\Box$

The following result was conjectured by Gordon and proved by Lackenby and Meyerhoff [LM]. It will be referred to as the 8-Theorem below.

The 8-Theorem (Lackenby-Meyerhoff) *If M is a hyperbolic manifold and r_1, r_2 are two exceptional slopes on a torus component of ∂M, then $\Delta(r_1, r_2) \leq 8$.*

Consider the links $L = K' \cup K''$ in Figure 2.1 below, where K'' is the trivial knot, and K' is a in the solid torus $V = S^3 - \text{Int}N(K'')$ obtained by adding two strings to a Montesinos tangle $T(p_1/q_1,\ p_2/q_2)$, as shown in

Figure 2.1(a)-(b). These are called *tubed Montesinos knots* in [Wu6]. Denote by $K^0(p_1/q_1,\ p_2/q_2)$ the knot in Figure 2.1(a), and by $K^1(p_1/q_1,\ p_2/q_2)$ the one in Figure 2.1(b). We always assume that K' is a knot in V, so q_1, q_2 are not both even. Denote by (V, K', r) the manifold obtained by r surgery on K' in V. A knot in V is considered to be equivalent to its mirror image in the lemma below.

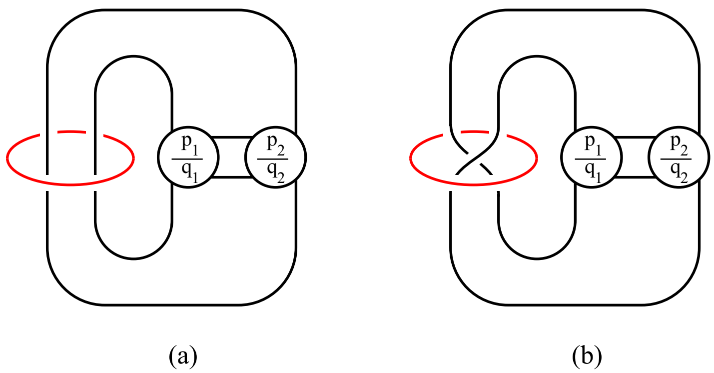


Figure 2.1

Lemma 2.2 *Suppose $q_i \geq 2$, r is a nontrivial slope, and (V, K', r) is non-hyperbolic. Then (K', r) is equivalent to one of the following pairs. The surgery is small Seifert fibered for $r = 7$ in (2), and toroidal otherwise.*

(1) $K = K^a(1/q_1,\ 1/q_2)$, $|q_i| \geq 2$, $a = 0, 1$, and r is the pretzel slope.

(2) $K = K^1(-1/2,\ 1/3)$, $r = 6, 7, 8$.

Proof. Exceptional surgeries for all tubed Montesinos knots in solid torus have been classified in [Wu6, Theorem 5.5]. This lemma is the above theorem applied to the case that K' is a tubed Montesinos knot of length 2 in V. □

Theorem 2.3 *Besides the $4n+6$ and $4n+7$ surgeries on the $(-2, 3, 2n+1)$ pretzel knots, there are only finitely many small Seifert fibered surgeries on hyperbolic Montesinos knots K of length 3. Moreover, K is equivalent to one of the following.*

(1) (q_1, q_2, q_3) pretzel knot, $|q_1| \leq |q_2| \leq |q_3| \leq 17$, and either $|q_1| = 2$ or $|q_1| = |q_2| = 3$.

(2) $(3,\ 3,\ 2n,\ -1)$ pretzel knot, $2 \leq n \leq 8$.

(3) $K(-1/2,\ 2/5,\ 1/(2n+1))$ for some $n > 0$.

(4) Ten individual knots: $(3, \pm 4, \pm 5)$ *pretzel knots,* $(3, 4, 5, -1)$ *pretzel knot,* $K(-2/3, 1/3, 2/5)$*, and* $K(-1/2, 1/3, 2/(2a+1))$ *for* $a = 3, 4, 5, 6$*.*

Proof. By [Wu5], if K is a hyperbolic Montesinos knot of length 3 and $K(r)$ is atoroidal and Seifert fibered, then K is one of the following knots.

(a) $K = (q_1, q_2, q_3)$ pretzel knot, and either $|q_1| = 2$, or $|q_1| = |q_2| = 3$, or $(|q_1|, |q_2|, |q_3|) = (3, 4, 5)$;

(b) $K = (q_1, q_2, q_3, -1)$ pretzel knot with $q_i \geq 3$, and either $q_1 = q_2 = 3$ or $(q_1, q_2, q_3) = (3, 4, 5)$;

(c) $K = K(-2/3, 1/3, 2/5)$;

(d) $K = K(1/2, 1/3, 2/(2a+1))$ and $a \in \{3, 4, 5, 6\}$;

(e) $K = K(-1/2, 2/5, 1/q)$ for some $q \geq 3$ odd.

Besides the 10 individual knots listed in (5), there are several infinite families of knots among the above. We divide these into four cases as follows.

Case 1. *K is a (q_1, q_2, q_3) pretzel knot with $2 = |q_1| \leq |q_2| \leq |q_3|$.*

Up to equivalence we may assume $q_1 = -2$. Let $L = K_1 \cup K_2$ be the link in Figure 2.1(b), where $p_1/q_1 = -1/2$, $p_2/q_2 = 1/q_2$, and K_1 denotes the trivial circle in the figure. Denote by $L(r_1, r_2)$ the manifold obtained from L by r_i surgery on K_i. By Kirby Calculus, we see that $L(1/n, r_2) = K(r)$, where $K = K(-1/2, 1/q_2, 1/(1-2n))$, and $r = r_2 - 4n$. On the other hand, if we denote by V the exterior of K_1, and put $M = (V, K_2, r_2)$, then $L(r_1, r_2) = M(r_1)$, the manifold obtained by Dehn filling along slope r_1 on ∂M.

By Lemma 2.2, if M is non-hyperbolic then $q_2 = 3$ and $r_2 = 6, 7, 8$, in which case $K(r)$ is a $6-4n$, $7-4n$ or $8-4n$ surgery on $K(-1/2, 1/3, 1/(1-2n))$. The first two are Seifert fibered and have been excluded in the statement, while the last one is the pretzel slope, in which case $K(r)$ is toroidal [Wu3] and hence cannot be Seifert fibered [IJ1]. Thus we may assume that $M = (V, K_2, r_2)$ is hyperbolic. It is easy to see that $M(1/0)$ is the r_2 surgery on the $(2, 5)$ torus knot and hence is nonhyperbolic. Therefore by the 8-Theorem we see that if $M(1/n)$ is non-hyperbolic then $|n| \leq 8$, hence $|q_3| = |1 - 2n| \leq 17$.

Case 2. *K is a (q_1, q_2, q_3) pretzel knot with $3 = |q_1| = |q_2| \leq |q_3|$.*

Similar to the above, we have $L = K_1 \cup K_2$ as in Figure 2.1(a) or 2.1(b), according to whether $q_3 = 2n$ or $2n + 1$. In this case $|q_1| = |q_2| = 3$, so $K_2 = K^0(\pm 3, 3)$ or $K^1(\pm 3, 3)$. By Lemma 2.2, the only exceptional surgery on K_2 in V is the surgery along the pretzel slope, corresponding to the toroidal surgery along the pretzel slope of K [Wu3]. For all other nontrivial slopes r, $M = (V, K_2, r)$ is hyperbolic. Note that when $q_3 = 2n$, $M(1/0)$ is the r-surgery on the connected sum of two trefoil knot and hence is nonhyperbolic, therefore as above, we see that if $|n| > 8$ then r surgery on K is hyperbolic. When $q_3 = 2n+1$ is old and $q_1 = q_2 = 3$, the knot with $n = -1$ is the trefoil knot, so the above argument applies and we conclude that $|n - (-1)| \leq 8$ if K admits a nontrivial small Seifert fibered surgery. Thus $|q_3| = |2n+1| \leq 17$.

Now consider the case that K is the $(-3, 3, 2n+1)$ pretzel knot. In this case we need to use a recent result of Boyer, Gordon and Zhang. Since K has a Seifert surface of genus 1, by [BGZ, Theorem 1.5] $K(p/q)$ is hyperbolic unless $|p| \leq 3$. By [Wu5, Theorem 6.6] we know that the knot complement has a persistently laminar branched surface with two meridional cusps, hence if $q \neq 1$ then the lamination is genuine in $K(p/q)$, so by [Br] $K(p/q)$ cannot be Seifert fibered. K is the twist knot 6_1 when $n = 0$, and its mirror image when $n = -1$, one of which has small Seifert surgery slopes $1, 2, 3$ and the other $-1, -2, -3$. It now follows by the same argument as above that for $p/q = \pm 1, \pm 2, \pm 3$, the surgery $K(p/q)$ is hyperbolic unless $|q_3| \leq 17$.

Case 3. *K is a $(3, 3, -1, q_3)$ pretzel knot with $q_3 \geq 3$.*

Since K is a knot, q_3 must be even, say $q_3 = 2n$, so we have $n \geq 2$. As above, let $L = K_1 \cup K_2$, where K_1 is trivial and K_2 is a tubed knot $K^0(1/3, -2/3)$ in the solid torus $V = S^3 - \text{Int}N(K_1)$. Now r surgery on K is equivalent to $r - 4n$ surgery on K_2 followed by $1/n$ Dehn filling on ∂V, with respect to the standard meridian-longitude coordinate of K_1. By Lemma 2.2 there is no exceptional surgery on K_2 in V, hence $(V, K_2, r - 4n)$ is hyperbolic. Also note that when $n = 0$ the knot K is the connected sum of two trefoils, hence all surgeries are non-hyperbolic. It follows that $K(r)$ is hyperbolic unless $n \leq 8$.

Case 4. *$K = K(-1/2,\ 2/5,\ 1/(2n+1)$ for some $n \geq 1$.*

Let $L = K_1 \cup K_2$, where K_1 is trivial and $K_2 = K^1(-1/2, 2/5)$ in $V = S^3 - \text{Int}N(K_1)$. $K(r)$ is the same as $r - 4n$ surgery on K_2 followed by $1/n$ filling on ∂V. By Lemma 2.2 we see that (V, K_2, r) is always hyperbolic for any nontrivial r, hence by Lemma 2.1 there are only finitely many exceptional surgeries on the set of hyperbolic knots K as above. $\square$

We note that the argument above does not provide a bound for n for the knots of type $K(-1/2, 2/5, 1/(2n+1))$, although by the theorem such bound does exist. However, using computer assistant proof it seems likely that $n \leq 9$. See the discussion about *Snappex* after Conjecture 4.1.

3 Seifert fibered surgeries

Gordon conjectured that a Seifert fibered surgery on a hyperbolic knot is an integral surgery. The following lemma shown that this is true for most Montesinos knots of length 3.

Lemma 3.1 *Suppose K is a hyperbolic Montesinos knot of length 3 such that $K(r)$ is small Seifert fibered and r is a nontrivial non-integral slope. Then K is equivalent to either (i) a $(-2, p_2, p_3)$ pretzel knot with $3 \leq p_2 \leq p_3 \leq 17$, or (ii) a $(3, 3, -1, 2n)$ knot with $2 \leq n \leq 8$, or (iii) the $(3, 4, 5, -1)$ pretzel knot.*

Proof. By the proof of [Wu6, Theorem 6.6] K has a persistently laminar branched surface with two meridional cusps unless it is a pretzel knot of type $(p_1, p_2, p_3, -1)$ with $p_i > 1$. Such branched surface becomes genuinely laminar after nonintegral surgery because the component containing the Dehn filling solid torus is a solid torus with cusps intersecting a meridian disk at least 4 times, hence by [Br] the surgered manifold cannot be a small Seifert fibered manifold. The result follows by comparing this with the list of knots in Theorem 2.3. □

The lemma can be used in searching for Seifert fibered surgeries. We may now use Snappy [CDW] to check surgeries on the list of knots in Theorem 2.3. $K(r)$ is likely to be Seifert fibered if the program gives nearly zero volume. Since most of those knots in the list are strongly invertible, one can then try to use the Montesinos trick to show that the manifold is indeed Seifert fibered. The following table gives the list of Seifert fibered surgeries on hyperbolic Montesinos knots of length 3, where $M(r_1, r_2, r_3)$ denotes the closed 3-manifold which is the double branched cover of S^3 with branch set a Montesinos link $K(r_1, r_2, r_3)$. It is well known that all such $M(r_1, r_2, r_3)$ are small Seifert fibered.

	K	r	$K(r)$
(1)	$K(-1/2,\ 1/3,\ 1/2n+1)$	$r = 4n+6$	$M(1/2,\ -1/4,\ 2/2n-5)$
		$r = 4n+7$	$M(-1/3,\ 3/5,\ 1/(n-2))$
(2)	$K(-1/2,\ 1/3,\ 1/7)$	$r = 17$	$M(-1/2,\ 1/3,\ -2/5)$
(3)	$K(-1/2,\ 1/3,\ 2/5)$	$r = 3$	$M(-2/15,\ 1/2,\ -1/3)$
		$r = 4$	$M(-2/7,\ 1/2,\ -1/6)$
		$r = 5$	$M(3/5,\ -1/3,\ -1/5)$
(4)	$K(-1/2,\ 1/5,\ 2/5)$	$r = 7$	$M(3/4,\ -2/5,\ -1/4)$
		$r = 8$	$M(-1/5,\ 1/2,\ -2/9)$
(5)	$K(-1/2,\ 1/7,\ 2/5)$	$r = 11$	$M(-1/4,\ -2/7,\ 2/3)$
(6)	$K(-1/2,\ 1/3,\ 2/7)$	$r = -1$	$M(-3/4,\ 1/3,\ 3/8)$
		$r = 0$	$M(1/5,\ 3/10,\ -1/2)$
		$r = 1$	$M(1/2,\ -2/3,\ 3/19)$
(7)	$K(-1/2,\ 1/3,\ 2/9)$	$r = 2$	$M(-3/8,\ -3/2,\ -1/4)$
		$r = 3$	$M(8/11, -1/2, -1/5)$
		$r = 4$	$M(-3/20, -1/2, 2/3)$
(8)	$K(-1/2,\ 1/3,\ 2/11)$	$r = -2$	$M(2/7,\ 2/5,\ -2/3)$
		$r = -1$	$M(2/9, 2/7, -1/2)$
(9)	$K(-1/3,\ 1/3,\ 1/4)$	$r = 1$	$M(-1/2,\ 1/5,\ 2/7)$
(10)	$K(-1/3,\ 1/3,\ 1/6)$	$r = 1$	$M(-1/2,\ 1/3,\ 2/13)$
(11)	$K(-1/3,\ 1/3,\ 1/3)$	$r = 1$	$M(1/2, -1/5, -2/7)$
(12)	$K(-1/3,\ 1/3,\ 1/5)$	$r = 1$	$M(-1/3, -1/4, 3/5)$
(13)	$K(-2/3,\ 1/3,\ 2/5)$	$r = -5$	$M(2/5,\ 2/5,\ -3/4)$

Table 3.1 Seifert fibered surgeries

Theorem 3.2 *For each knot K and slope r in the table, r surgery on K produces a Seifert fibered manifold $K(r)$ as shown in the table.*

Proof. (1) is given in [Wu6, Theorem 5.5]. (2) is well known, see for example [CGLS]. Most of the others can be proved using the Montesinos trick.

Consider a strongly invertible knot K in S^3 with axis X intersecting K twice. π-rotation along X gives a quotient map $\rho : (S^3, X, K) \to (\bar{S}, \bar{X}, \bar{K})$, where $\bar{S}$ is a 3-sphere, $\bar{X}$ a trivial circle, and $\bar{K}$ an arc with its two endpoints on $\bar{X}$. For example, when K is the knot $K(-1/2,\ 1/3,\ 2/5)$ in Figure 3.1(1), the pair $\bar{X}$ and $\bar{K}$ are shown in Figure 3.1(2). The quotient of $N(K)$ is a 3-ball $\bar{N}$ in $\bar{S}$, drawn as a thick arc in Figure 3.1(3). Shrinking $\bar{N}$ to a round ball gives Figure 3.1(4). Put $\alpha = \bar{X} \cap \bar{N}$. We may consider $(\bar{N}, \alpha)$ as a rational tangle of slope ∞, and set up coordinates so that a longitude on $\partial N(K)$ projects to a curve of slope 0 on $\partial \bar{N}$, which is considered as a pillow case with the four points $X \cap \partial \bar{N}$ as cone point, so every essential simple closed curve on $\partial \bar{N}$ has a slope; see [HT]. The Montesinos trick [Mon] says that $K(r)$ can be obtained by replacing $(\bar{N}, \alpha)$ with a rational tangle of slope $-r$ to obtain a link $L[-r]$ in $\bar{S}$, then taking the double branched cover of $\bar{S}$ along $L[-r]$. More generally, if $\bar{N}$ is deformed so that α is the ∞ tangle and the longitude projects to a curve of slope r_0 on $\partial \bar{N}$ then a curve of slope r on $\partial N(K)$ projects to a curve of slope $r_0 - r$ on $\partial \bar{N}$, hence $K(r)$ is the double branched cover of $\bar{S}$ along $L[r_0 - r]$. Thus if $L[r_0 - r]$ is a Montesinos link $K(a, b, c)$ then $K(r)$ is the Seifert fibered manifold $M(a, b, c)$.

Continue with the example above. We can simplify Figure 3.1(4) to that of Figure 3.1(5). To determine the framing, consider the bounded checkboard surface F for the diagram in Figure 3.1(1). It is easy to check that ∂F has slope 6 on $\partial N(K)$. The quotient of F is a disk $\bar{F}$ which deforms to a disk in Figure 3.1(3) whose intersection with $\partial \bar{N}$ is an arc of slope 0. In other words, the longitude of $N(K)$ projects to a curve of slope r_0 on $\partial N(K)$ such that a curve of slope $r = 6$ projects to a curve of slope $r_0 - r = 0$, hence $r_0 = 6$. Thus $K(3)$ is the double branched cover of $L[r_0 - 3] = L[3]$, shown in Figure 3.1(6). It can be deformed to that in Figure 3.1(7) and then further to Figure 3.1(8), which is a Montesinos knot $K(-2/15,\ 1/2,\ -1/3)$. Hence $K(3) = M(-2/15,\ 1/2,\ -1/3)$. Similarly, $K(4)$ is the double branched cover of $L[r_0 - 4] = L[2]$ shown in Figure 3.1(9), which is isotopic to $K(-2/7,\ 1/2,\ -1/6)$ in Figure 3.1(10), and $K(5)$ is the double branched cover of $L[1]$ in Figure 1(11), isotopic to $K(3/5,\ -1/3,\ -1/5)$ in Figure 3.1(12). This completes the proof for the three Seifert fibered surgeries on $K(-1/2,\ 1/3,\ 2/5)$.

The proofs for cases (4)–(10) are similar. Surgeries on $K(-1/2,\ 1/5,\ 2/5)$ and $K(-1/2,\ 1/5,\ 2/7)$ are given in Figure 3.2, $K(-1/2,\ 1/3,\ 2/7)$ in Figure 3.3, $K(-1/2,\ 1/3,\ 2/9)$ in Figure 3.4, $K(-1/2,\ 1/3,\ 2/11)$ in Figure 3.5, and $K(-1/3,\ 1/3,\ 1/4)$ and $K(-1/3,\ 1/3,\ 1/6)$ in Figure 3.6.

For case (13), the knot can be written as $K = K(1/3, -3/5, 1/3)$ and can be drawn as in Figure 3.7(1). This is also a strongly invertible knot, only that the axis is not the one passing through all 3 tangles as in examples above. The quotients $\bar{X}$ and $\bar{N}$ are shown in Figure 3.7(2), which is isotopic to that in Figure 3.7(3). Using a symmetric spanning surface as above, one can check that the longitude projects to a curve of slope -6 on $\partial \bar{N}$, hence $K(-5)$ is the double branched cover of $L[-1]$ in Figure 3.7(4), which is isotopic to the Montesinos knot $K(2/5, 2/5, -3/4)$ in Figure 3.7(5). The proof for the case (11) is similar and is shown in Figures 3.7(6)–(9).

The $(-3, 3, 5)$ pretzel knot in case (12) does not seem to be strongly invertible and hence cannot be proved using the method above. Fortunately this has been done by Mattman, Miyazaki and Motegi. Figure 3.7 in [MMM] shows that 1 surgery on the $(-3, 3, 5)$ pretzel knot yields the manifold $M(-1/3, -1/4, 3/5)$. $\square$

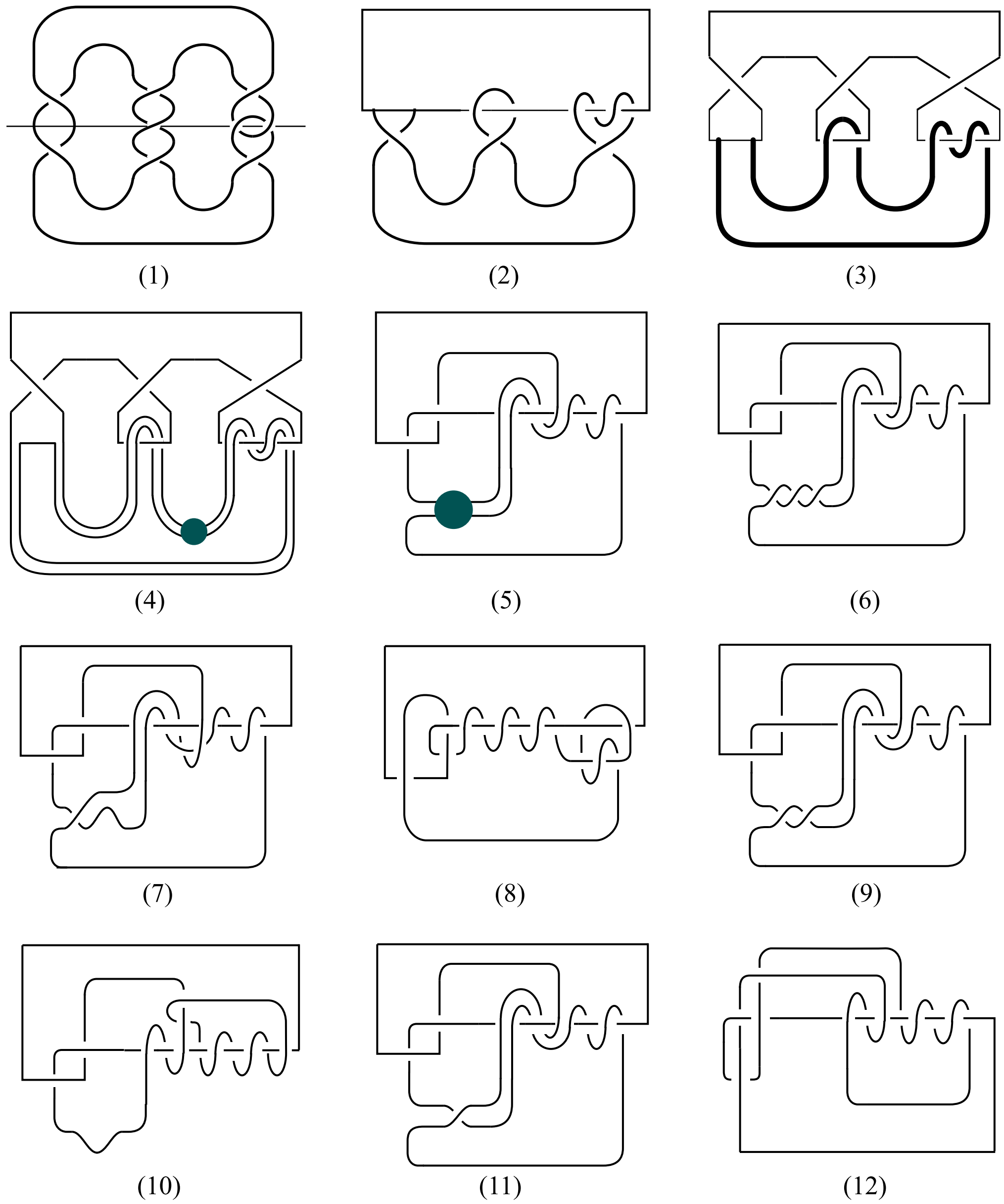


Figure 3.1

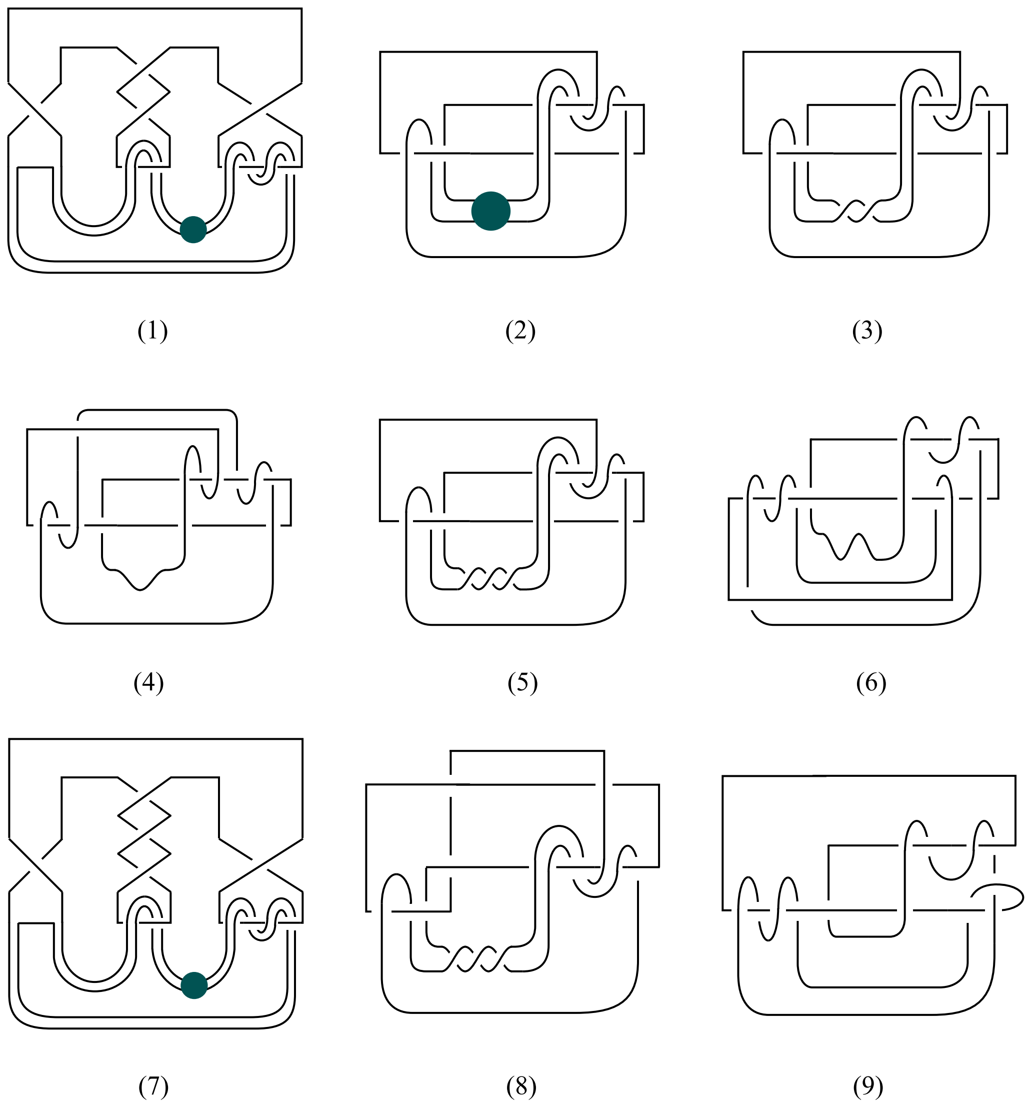
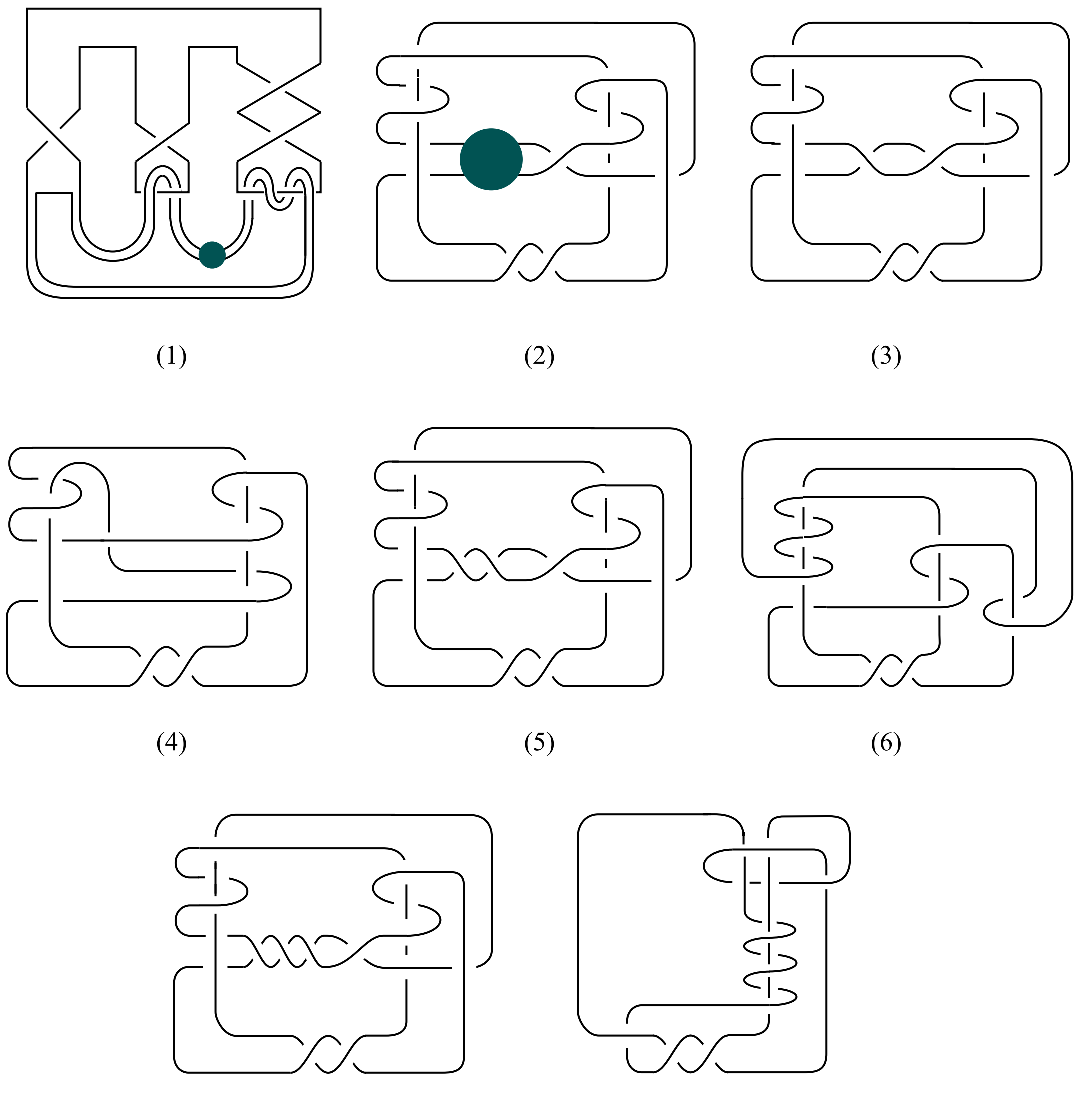


Figure 3.2



(1) (2) (3)

(4) (5) (6)

(7) (8)

Figure 3.3

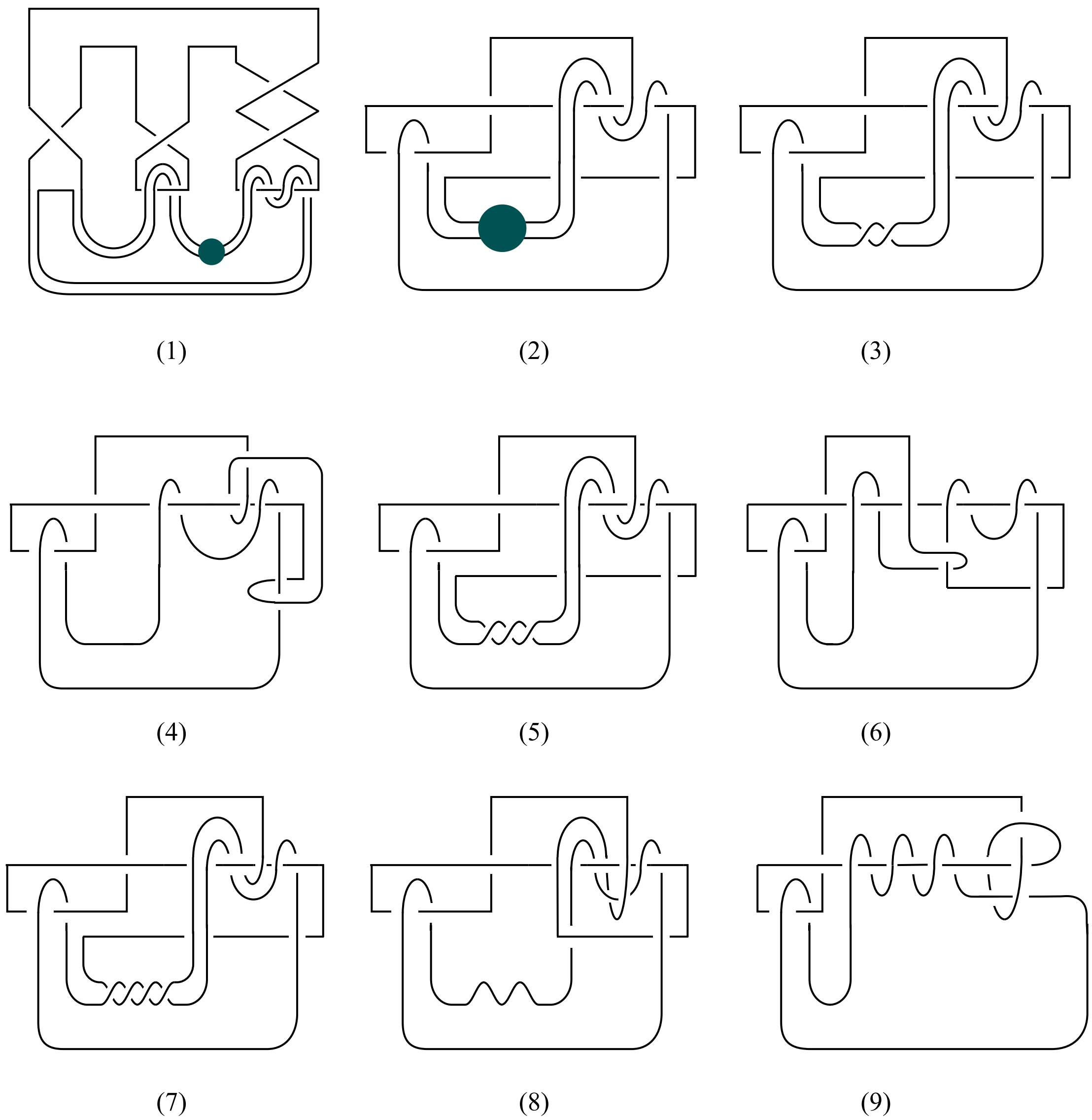


Figure 3.4

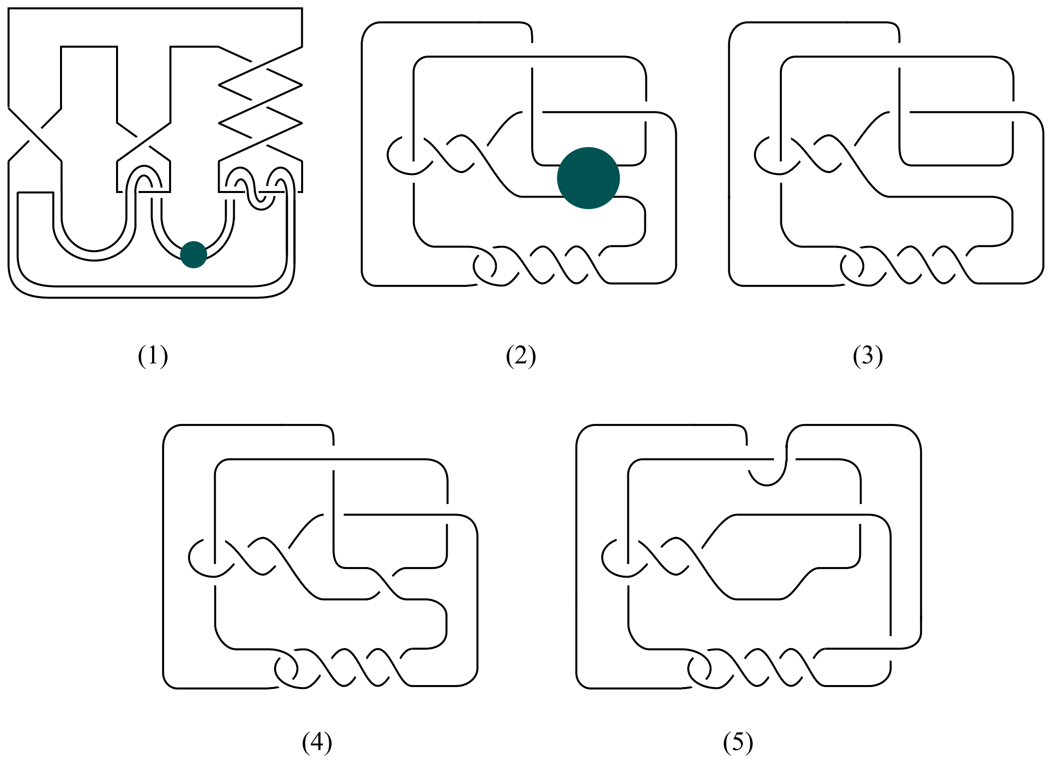


Figure 3.5

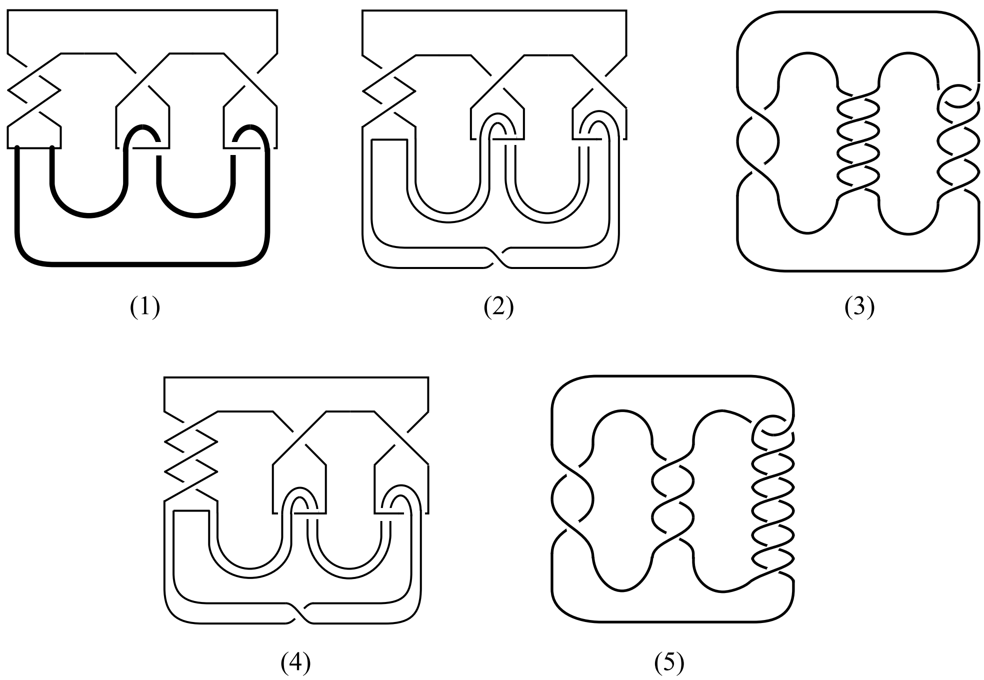


Figure 3.6

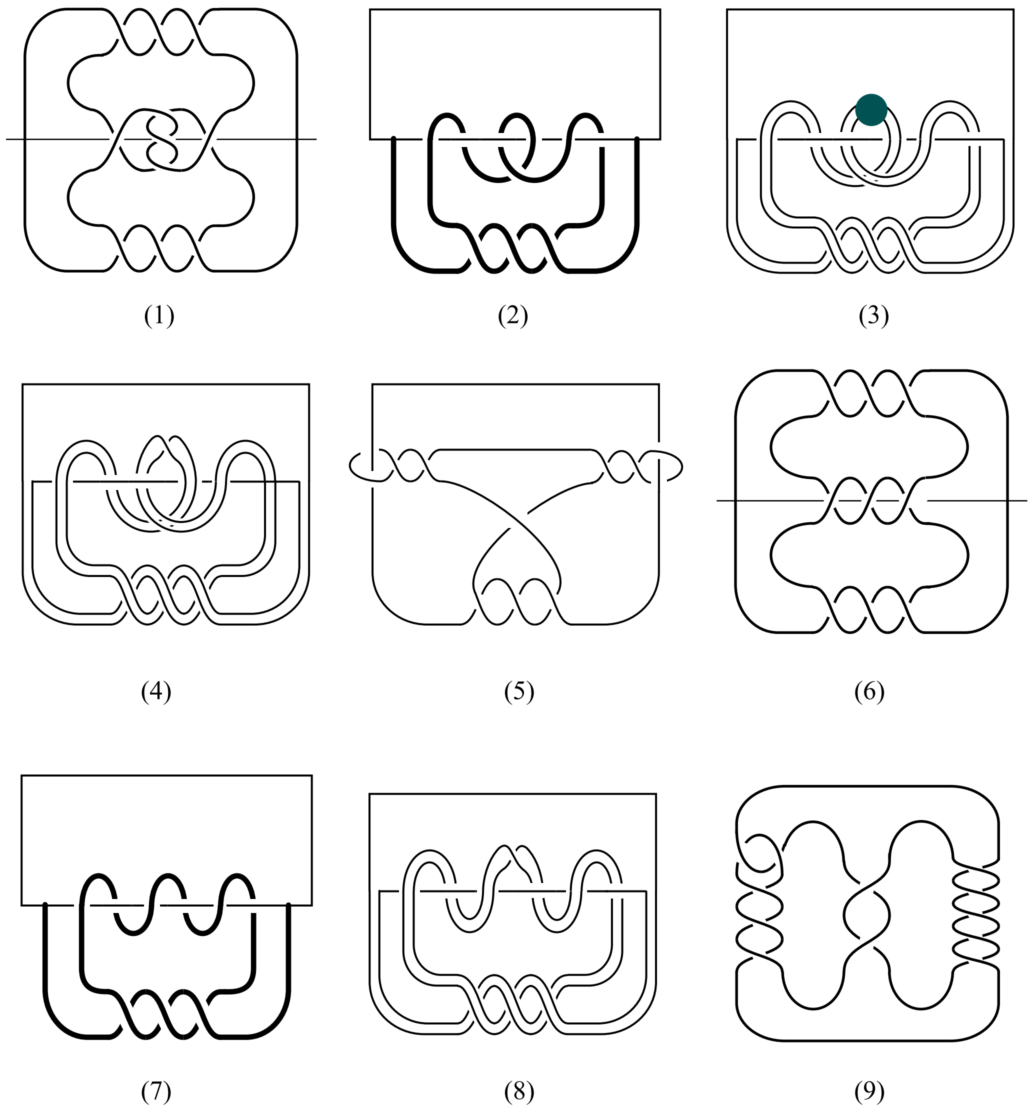


Figure 3.7

4 A conjecture and some computer assistant approach

Using Snappea or Snappy, one can test the knots in Theorem 2.3 to find the set of slopes along which Dehn surgeries produce manifolds with near zero volume. Snappy volume represents the Gromov norm of the manifold, hence if the surgery is Seifert fibered then the volume should be zero. We have following conjecture.

Conjecture 4.1 *A nontrivial Dehn surgery on a hyperbolic Montesinos knot of length 3 is Seifert fibered if and only if it is equivalent to one of those in table 3.1.*

A computer program *Snappex* has been written, which combines *Snap* of Oliver Goodman [Gm] with a template written by Harriet Moser [Mos2]. See [Wu7]. *Snap* uses the *Snappea* core of Jeff Weeks [We] and the high precision package *Para* to calculate hyperbolic structure for 3-manifolds, while the Moser Script uses the *Snap* output as its input and then attempt to verify the hyperbolicity of the manifold rigorously. This is based on [Mos1], in which Moser showed that there is a genuine hyperbolic structure for the manifold in a neighborhood of the *Snap* solution if the latter satisfies certain conditions. Given a knot K, one can use *Snap* to find a hyperbolic structure, use Moser Script to verify it, then use *Snap* output to find all slopes of length at most 2π, and then use Moser Script to check whether each of these is a hyperbolic surgery. *Snappex* makes this procedure automatic. Thus given a knot K, *Snappex* will give a list of slopes which contains all possible exceptional slopes. Assuming the accuracy of the programs involved and the correctness of compilers, *Snappex* rigorously proves that Dehn surgery on K along any slope not in the above list must be hyperbolic.

A similar procedure is carried out by *Snappex* for links of 2 components. Using this and some theoretical arguments one can show that $n \leq 9$ for the knots in Theorem 2.3(3). Consider the link $L = K' \cup K''$ in Figure 2.1 with $p_1/q_1 = -1/2$ and $p_2/q_2 = 2/5$. Then the knot in Theorem 2.3(3), which we denote by K_n for any given n, is obtained from L by $-1/n$ surgery on K'' Using Kirby Calculus [Ro] it can be shown that s surgery on K is equivalent to $(s - 4n, -1/n)$ surgery on L. Running *Snappex* on this link gives the candidate list C, and one can check to see that if $(r_1, r_2) \in C$ then either $r_2 = 1/n$ with $n \leq 4$, or $r_1 \in \{-2, -1, 0, 1, 1/0\}$. We need to show that if $L(r, -1/n)$ is small Seifert fibered for $r = -2, -1, 0, 1$ then $n \leq 9$.

Consider the case $r = 1$. By Lemma 2.2, the manifold $L(r, \emptyset)$ is hyperbolic for all $r \neq 1/0$; in particular, $M = L(1, \emptyset)$ is hyperbolic. Note that -1 surgery on the second component of L yields the knot $K_1 = K(-1/2, 2/5, 1/3)$, which by Theorem 2.3 has Seifert fibered surgeries of slopes $s = 3, 4, 5$. By the above, we have that $L(-1, -1/1) = K_1(3)$ is non-hyperbolic, hence by the 8-Theorem $L(-1, -1/n) = K_n(3)$ is hyperbolic for $n > 9$. Similarly for $r = 0, -1$.

The above does not work for $r = -2$. Fortunately $s = r + 4n$ is the boundary of a non-orientable checkboard spanning surface F with $\chi(F) = -2$. Cutting along F produces a handlebody M of genus 3. Considering $\partial N(F) \cap M$ as horizontal surface and $\partial M \cap \partial N(K)$ as vertical surface, we obtain a cusped manifold. It can be shown that the horizontal surface is incompressible (at least for $n > 2$) and extends to an incompressible surface in the surgered manifold $K_n(s)$, and M is not an I-bundle. It follows from [Br] that $K_n(s)$ cannot be small Seifert fibered.

Back to Conjecture 4.1. We now have a list of a few hundred knots to check. *Snappex* has a command to find a candidate list of exceptional slopes for all these knots. There are several hundred surgeries that remains on this list, which need to be verified using some other methods. A few non-integral slopes can be excluded using Lemma 3.1. All but a couple of the remaining slopes are integral slopes, which are shown to be "apparently hyperbolic" by Casson's *Geo* program [Ca], which provides strong supporting evidence for the conjecture.

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