THERE EXIST MULTILINEAR BOHNENBLUST–HILLE CONSTANTS $(C_n)_{n=1}^{\infty}$ WITH $\lim_{n\to\infty}(C_{n+1}-C_n)=0$.

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ABSTRACT. After almost 80 decades of dormancy, the Bohnenblust–Hille inequalities have experienced an effervescence of new results and sightly applications in the last years. The multilinear version of the Bohnenblust–Hille inequality asserts that for every positive integer $m \geq 1$ there exists a sequence of positive constants $C_m \geq 1$ such that

$$\left(\sum_{i_1,\dots,i_m=1}^{N}\left|U(e_{i_1},\dots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C_m \sup_{z_1,\dots,z_m\in\mathbb{D}^N}\left|U(z_1,\dots,z_m)\right|$$

for all m-linear forms $U:\mathbb{C}^N\times\cdots\times\mathbb{C}^N\to\mathbb{C}$ and positive integers N (the same holds with slightly different constants for real scalars). The first estimates obtained for C_m showed exponential growth but, only very recently, a striking new panorama emerged: the polynomial Bohnenblust–Hille inequality is hypercontractive and the multilinear Bohnenblust–Hille inequality is subexponential. Despite all recent advances, the existence of a family of constants $(C_m)_{m=1}^{\infty}$ so that

$$\lim_{n\to\infty}(C_{n+1}-C_n)=0$$

has not been proved yet. The main result of this paper proves that such constants do exist. As a consequence of this, we obtain new information on the optimal constants $(K_n)_{n=1}^{\infty}$ satisfying the multilinear Bohnenblust–Hille inequality. Let γ be Euler's famous constant; for any $\varepsilon>0$, we show that

$$K_{n+1}-K_n \leq \left(2\sqrt{2}-4e^{\frac{1}{2}\gamma-1}\right)n^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)+\varepsilon},$$

for infinitely many n's. Numerically, choosing a sufficiently small value of ε,

$$K_{n+1}-K_n \leq \frac{0.8646}{n^{0.4737}}$$

for infinitely many values of $n\in\mathbb{N}$. The above results and estimates hold for both complex and real scalars.

1. Introduction and background

The polynomial and multilinear Bohnenblust–Hille inequalities have important applications in different fields, as Operator Theory, Fourier and Harmonic Analysis, Complex Analysis, Analytic Number Theory and Quantum Information Theory (see [10, 13] and

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references therein). Since its proof, in the *Annals of Mathematics* in 1931, the (multilinear and polynomial) Bohnenblust–Hille inequalities were overlooked for decades (see [3]) and only returned to the spotlights in the last few years with works of A. Defant, L. Frerick, J. Ortega-Cerdá, M. Ounaïes, D. Popa, U. Schwaring, K. Seip, among others. The polynomial Bohnenblust–Hille inequality proves the existence of a positive function $C:\mathbb{N}\to[1,\infty)$ such that for every m-homogeneous polynomial P on $\mathbb{C}^N$, the $\ell_{\frac{2m}{m+1}}$-norm of the set of coefficients of P is bounded above by $C(m)$ times the supremum norm of P on the unit polydisc. The original estimates for $C(m)$ had a growth with terms of the order $m^{m/2}$ and only in 2011 [7] rediscovered the importance of this inequality and substantially improved the estimates for $C(m)$; in the aforementioned paper it is proved that $C(m)$ is hypercontractive and more precisely

$$C(m)\leq\left(1+\frac{1}{m}\right)^{m-1}\sqrt{m}\left(\sqrt{2}\right)^{m-1}.$$

This result, besides its mathematical importance, has striking applications in different contexts. The multilinear version of the Bohnenblust–Hille inequality has a similar, *mutatis mutandis,* formulation:

Multilinear Bohnenblust–Hille inequality. For every positive integer $m\geq 1$ there exists a sequence of positive scalars $(C_m)_{m=1}^{\infty}$ in $[1,\infty)$ such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\left|U(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq C_m\sup_{z_1,\ldots,z_m\in\mathbb{D}^N}\left|U(z_1,\ldots,z_m)\right|$$

for all m-linear forms $U:\mathbb{C}^N\times\cdots\times\mathbb{C}^N\to\mathbb{C}$ and every positive integer N, where $(e_i)_{i=1}^{N}$ denotes the canonical basis of $\mathbb{C}^N$ and $\mathbb{D}^N$ represents the open unit polydisk in $\mathbb{C}^N$.

The case $m=2$ is the well-known Littlewood's 4/3 theorem (see [14, 18]). The original purpose of Littlewood's 4/3 theorem was to solve a problem of P.J. Daniell on functions of bounded variation (see [18]); on the other hand, the Bohnenblust–Hille inequality was invented to solve Bohr's famous absolute convergence problem within the theory of Dirichlet series (this subject is being recently explored by several authors; see [1, 4, 6, 8, 9] and references therein). Some independent results were proven in the 1970's where better upper bounds for C_m were obtained, but it seems that the authors were not aware of the existence of the original results by Bohnenblust and Hille.

The oblivion of the work of Bohnenblust and Hille in the past was so noticeable that Blei's book [2] published in 2001 states the Bohnenblust–Hille inequality as "the Littlewood's $2n/(n+1)$-inequality" and absolutely no mention to the paper of Bohnenblust and Hille is made. According to Blei's book the "Littlewood's $2n/(n+1)$-inequality" is originally due to A.M. Davie ([5], 1973) and (independently) to G. Johnson and W. Woodward ([16], 1974) but as a matter of fact Bohnenblust and Hille's paper preceded the aforementioned works in more than 40 years.

Recently, the Bohnenblust–Hille inequality has attracted the attention of many authors (see [10, 11] and references therein). A series of very recent works (see [21, 22, 7, 10, 12, 13, 19, 20, 23, 26]) have been investigating estimates for C_m. The first estimates for the constants C_m indicate that one should expect an exponential growth for the optimal constants $(K_n)_{n=1}^{\infty}$ satisfying the Bohnenblust–Hille inequality:

- $K_n \leq n^{\frac{n+1}{2n}} 2^{\frac{n-1}{2}}$ ([3], 1931),
- $K_n \leq 2^{\frac{n-1}{2}}$ ([5, 17], 1970's),
- $K_n \leq \left(\frac{2}{\sqrt{\pi}}\right)^{n-1}$ ([25], 1995).

It is worth mentioning that the Bohnenblust–Hille inequality also holds for the case of real scalars. In this paper, for the sake of simplicity, we shall work with real scalars. As a matter of fact, since the upper estimates (5) also hold for the complex case (because these estimates are clearly bigger than the best known estimates for the complex case (see [23])) our whole procedure encompasses both the real and complex cases. We just stress that, if we deal with the complex setting separately we will probably be able to produce slightly better estimates, since the best known constants for complex scalars are smaller than the constants for the real case and present an even better asymptotic behavior (see [23, 22]).

Up to now the best values of these constants are unknown (for details see [2, Remark i, page 178] or [12, 23] and references therein). Only very recently (see [19]) it was proved that the multilinear Bohnenblust–Hille inequality is subexponential and that there exists a sequence $(C_n)_{n=1}^{\infty}$ satisfying the multilinear Bohnenblust–Hille inequality such that

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 1. \tag{1}$$

Notwithstanding the recent advances a lot of mystery remains on the estimates of the optimal constants satisfying the multilinear (and polynomial) Bohnenblust–Hille inequality.

In [21], which can be considered as a continuation of [19], a *dichotomy theorem* for the candidates of constants satisfying the multilinear Bohnenblust–Hille inequality is proved and, as a consequence, provides some new information on the optimal constants. In [21] a sequence of positive real numbers $(R_n)_{n=1}^{\infty}$ is said to be *well-behaved* if there are $L_1, L_2 \in [0, \infty]$ such that

$$\lim_{n\to\infty} \frac{R_{2n}}{R_n} = L_1 \text{ and } \lim_{n\to\infty} (R_{n+1} - R_n) = L_2.$$

As a consequence of the main result in [21] the following one is proved:

Theorem 1.1. *([21]) The optimal constants $(K_n)_{n=1}^{\infty}$ satisfying the Bohnenblust–Hille inequality is*

(i) subexponential and not well-behaved

or

(ii) well-behaved with

$$\lim_{n\to\infty} \frac{K_{2n}}{K_n} \in \left[1, \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right]$$

and

$$\lim_{n\to\infty}\left(K_{n+1}-K_n\right)=0,$$

where γ denotes the Euler's famous constant $\gamma := \lim_{m\to\infty}\left((-\log m)+\sum_{k=1}^{m}k^{-1}\right)$.

Up to now there is no solution to the problem below:

Problem 1.2. *Is there a sequence $(C_n)_{n=1}^{\infty}$ satisfying the multilinear Bohnenblust–Hille inequality such that*

$$\lim_{n\to\infty} C_{n+1}-C_n=0? \tag{2}$$

In this paper, among other several results, we solve this problem positively (details are given in Subsection 1.2 and Section 3). The main results of this paper are the solution to the above problem (Theorem 3.1) and some surprising consequences of this result (see Subsubsection 1.2.1 and Theorems 4.1, 5.1 and Corollary 5.2).

1.1. **A chronological overview of recent results.** In view of the large amount of recent papers and preprints related to the subject, we shall dedicate some space to locate the contribution of the present paper in the actual panorama of the subject.

- In ([11], 2009), the bilinear version of Bohnenblust–Hille inequality (known as Littlewood's 4/3 theorem) is explored in a new direction and this paper rediscovers the importance of the Bohnenblust–Hille inequality.
- The paper ([10], 2011) is a remarkable work of A. Defant, D. Popa and U. Schwarting providing a new proof of the Bohnenblust–Hille inequality which also led to interesting vector-valued generalizations.
- In ([7], 2011) it is proved that the polynomial Bohnenblust–Hille inequality is hypercontractive. Several striking applications are presented.
- In ([23], 2012) new constants satisfying the multilinear Bohnenblust–Hille inequality are presented, based on the arguments of the new proof of the Bohnenblust–Hille theorem from [10]. An improvement of this approach (for the case of complex scalars) is presented in ([22], 2012).
- In ([19], 2012) some numerical investigations on the asymptotic growth of the constants satisfying the multilinear Bohnenblust–Hille inequality are presented; in this direction, in ([12], 2012) some quite surprising results are obtained:

  Theorem ([19]). There exists a sequence $(C_n)_{n=1}^{\infty}$ satisfying the multilinear Bohnenblust–Hille inequality such that

  $$\lim_{n\to\infty}\frac{C_{n+1}}{C_n}=1.$$

  Theorem ([19, Appendix]). The best constants $(K_n)_{n=1}^{\infty}$ satisfying the multilinear Bohnenblust–Hille inequality have a subexponential growth. In particular, if there is a constant $L>0$ so that

  $$\lim_{n\to\infty}\frac{K_{n+1}}{K_n}=L,$$

then $L = 1$.

- In ([21], 2012) a Dichotomy Theorem is proved and, as a consequence, for example, it is shown that the optimal constants satisfying the multilinear Bohnenblust–Hille inequality do not have a polynomial growth.
- In ([13, 20], 2012), in a completely different line of attack, the authors obtain lower bounds for the constants of the multilinear and polynomial Bohnenblust–Hille inequalities.
- In ([26], 2012) an explicit formula for some recursive formulae for constants satisfying the multilinear Bohnenblust–Hille inequality (from [12, 23]) is obtained (the original formulae on [12, 23] were obtained via a complicated recursive formula).

1.2. **Remarks and summary of the contributions of this paper.** In this subsection we relate the main results of this paper to the recent advances of this subject. We need to recall some notation. We shall work with the case of real scalars but, as mentioned before, the same results hold in the case of complex scalars.

The Greek letter γ shall denote Euler's famous constant,

$$\gamma := \lim_{m\to\infty}\left(-\log m + \sum_{k=1}^{m}\frac{1}{k}\right) \approx 0.5772.$$

Also, henceforth, we use the notation

$$A_p := \sqrt{2}\left(\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}\right)^{1/p}, \tag{3}$$

for $p > p_0 \approx 1.847$ and

$$A_p := 2^{\frac{1}{2}-\frac{1}{p}} \tag{4}$$

for $p \le p_0 \approx 1.847$. The precise definition of p_0 is the following: $p_0 \in (1,2)$ is the unique real number so that

$$\Gamma\left(\frac{p_0+1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

The constants A_p are precisely the best constants satisfying Khinchine's inequality (these constants are due to U. Haagerup [15]). In [23] it was proved that the following constants satisfy the multilinear Bohnenblust–Hille inequality:

$$C_m = \begin{cases} 1 & \text{if } m = 1, \\ \left(A_{\frac{2m}{m+2}}^{m/2}\right)^{-1} C_{\frac{m}{2}} & \text{if } m \text{ is even, and} \\ \left(A_{\frac{2m-2}{m+1}}^{\frac{-1-m}{2}} C_{\frac{m-1}{2}}\right)^{\frac{m-1}{2m}} \left(A_{\frac{2m+2}{m+3}}^{\frac{1-m}{2}} C_{\frac{m+1}{2}}\right)^{\frac{m+1}{2m}} & \text{if } m \text{ is odd.} \end{cases} \tag{5}$$

From now on the notation C_m shall represent the constants in (5). Up to now these are the best (smallest) constants satisfying the (real, and consequently also the complex) multilinear Bohnenblust–Hille inequality (it was not known if the sequence $(C_m)_{m=1}^{\infty}$ is increasing; in [12] it was proved that if the above sequence $(C_m)_{m=1}^{\infty}$ is increasing, then

$$\lim_{n\to\infty} \frac{C_m}{C_{m-1}} = 1. \tag{6}$$

If $(C_m)_{m=1}^{\infty}$ is not increasing, then the sequence

$$C'_n = \begin{cases} 1 & \text{if } n = 1, \\ DC'_{n/2} & \text{if } n \text{ is even, and} \\ D\left(C'_{\frac{n-1}{2}}\right)^{\frac{n-1}{2n}} \left(C'_{\frac{n+1}{2}}\right)^{\frac{n+1}{2n}} & \text{for } n \text{ odd,} \end{cases} \tag{7}$$

is such that

$$\lim_{n\to\infty} \frac{C'_{n+1}}{C'_n} = 1.$$

Above, D (whose precise value was not known) is any common upper bound for the sequences

$$\left(A_{\frac{2m}{m+2}}^{-m/2}\right)_{m=1}^{\infty} \tag{8}$$

and

$$\left(\left(A_{\frac{2m-2}{m+1}}^{\frac{-1-m}{2}}\right)^{\frac{m-1}{2m}} \left(A_{\frac{2m+2}{m+3}}^{\frac{1-m}{2}}\right)^{\frac{m+1}{2m}}\right)_{m=1}^{\infty}. \tag{9}$$

In [12] it is proved that both sequences tend to $\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \approx 1.4403$ but no information about their eventual monotonicity is provided. To summarize, in [19] is shown that there exists a sequence of constants $(Z_m)_{m=1}^{\infty}$ satisfying the multilinear Bohnenblust–Hille inequality and so that

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = 1,$$

but the precise formula of the constants Z_m depends on the (unknown) value of D or, of course, on the (unknown) monotonicity of the constants (5).

In the the present paper, among other results, we solve both problems by proving that:

- The sequence (5) is increasing.
- $D = \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \approx 1.4403$ (and, of course, this value is sharp).

This information has useful consequences. The fact that $D < 2$ shall be crucial for the proof of one of the main results of this paper:

Theorem 3.1. There exist multilinear Bohnenblust–Hille constants $(R_n)_{n=1}^{\infty}$ with $R_{n+1} - R_n \to 0$.

For the proof of the above result it is crucial that $\left(\frac{D}{2}\right)^n \to 0$ and so we do need the information that $D < 2$. The concrete estimate for D allows us to deal with a simple presentation of good (small) estimates for the constants of multilinear Bohnenblust–Hille inequality. More precisely (using the value of D) now we know that the sequence

$$S_n = \begin{cases} \left(\sqrt{2}\right)^{n-1} & \text{if } n = 1, 2 \\ \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right) S_{n/2} & \text{for } n \text{ even,} \\ \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right) \left(S_{\frac{n-1}{2}}\right)^{\frac{n-1}{2n}} \left(S_{\frac{n+1}{2}}\right)^{\frac{n+1}{2n}} & \text{for } n \text{ odd} \end{cases} \tag{10}$$

satisfies the multilinear Bohnenblust–Hille inequality. This estimate for D can also be used in the explicit formula for the constants (10) presented in [26].

The sequence $(R_n)_{n=1}^{\infty}$ in our main result is a slight modification of the sequence (10). A natural question is why not to work directly with the sequence (7) or (10)? The answer is that the recursive formulations of the sequences $(C_n)_{n=1}^{\infty}$ and $(S_n)_{n=1}^{\infty}$ turn the estimates of $C_{n+1} - C_n$ and $S_{n+1} - S_n$ quite complicated. As a matter of fact, we do not even know if the limit $\lim_{n\to\infty} C_{n+1} - C_n$ exists, although we have some vestige that it seems to be zero. Just to illustrate the puzzling behavior of $C_{n+1} - C_n$ we show the following list of numerical calculations:

$$\begin{aligned} C_{351} - C_{350} &\approx 0.03053 \\ C_{510} - C_{509} &\approx 0.01778 \\ C_{516} - C_{515} &\approx 0.03356 \\ C_{1000} - C_{999} &\approx 0.01320 \\ C_{1330} - C_{1329} &\approx 0.01712. \end{aligned}$$

It might occur that this sequence fluctuates to zero, but a formal proof seems quite an unpleasant task. We recall that in [12] it is proved that

$$\lim_{n\to\infty} \frac{C_{2n}}{C_n} = \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \approx 1.4403$$

and from [21, Proposition] we know that the limit of $C_{n+1} - C_n$ is different from zero if and only if the limit does not exist (so we just need to prove that the limit exists). Since our main goal is to prove the existence of a sequence $(R_n)_{n=1}^{\infty}$ satisfying the Bohnenblust–Hille inequality with $R_{n+1} - R_n \to 0$, and its consequences to the nature of the growth of the optimal satisfying the multilinear Bohnenblust–Hille inequality, there is no damage in making slight changes on the sequence $(C_n)_{n=1}^{\infty}$.

Remark 1.3. *The values of the sequence (10) could be constructed with values much closer to the those of the sequence (5). We just need to define the first values of (10) as exactly the same as those from (5) (for example the first* 100 *values) and afterwards we make the corresponding changes. In this case the values of the two sequences would be extremely close.*

1.2.1. *Consequences of the main theorem.* The proof of the Main Theorem furnishes some information on the optimal constants satisfying the multilinear Bohnenblust–Hille inequality (see Theorem 4.1).

The solution to Problem 1.2 is constructive; we show an explicit sequence of constants with the desired property. The previous estimates obtained in [12, 22, 23] may eventually be also solutions to the Problem 1.2, but due their forbidding recursive formulae, it seems rather difficult to verify if these constants in fact share the property (2). Even the closed formula for the multilinear Bohnenblust–Hille constants presented in [26] lodge some technical difficulties when estimating the difference $R_{n+1} - R_n$.

We also stress that in all previous related papers there was not available information on the monotonicity of the limits involving the Gamma function and this lack of information was a peremptory barrier for the estimate of $C_{n+1} - C_n$.

The constants $(R_n)_{n=1}^{\infty}$ that we obtain here with the property (2) are slightly bigger than the constants from [12, 22, 23] but, on the other hand, they are constructed in a more simple fashion so that with a careful control of the monotonicity of the expressions involving the Gamma Function, we are finally able to show that $R_{n+1} - R_n \to 0$.

We also estimate how the difference $R_{n+1} - R_n$ tends (monotonely) to 0^+. In fact we have

$$R_{n+1} - R_n \leq (0.8646)\, n^{-0.47368}$$

for every positive integer n. More precisely our constants are so that

$$R_{n+1} - R_n \leq \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma - 1}\right) n^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)}$$

and thus

$$R_{n+1} - R_n = o\left(n^{\log_2\left(2^{-3/2}e^{1-(\gamma/2)}\right)+\varepsilon}\right)$$

for all $\varepsilon > 0$. As a consequence of the fact that $R_{n+1} - R_n \to 0$ we prove the following results:

- (Theorem 4.1) Let $(K_n)_{n=1}^{\infty}$ be the sequence of best constants satisfying the multilinear Bohnenblust–Hille inequality. If there is an $L \in [0, \infty]$ so that

  $$\lim_{n\to\infty} K_{n+1} - K_n = L,$$

  then

  $$L = 0.$$

- (Theorem 5.1) Let $(K_n)_{n=1}^{\infty}$ be the optimal constants of the multilinear Bohnenblust–Hille inequality. For any $\varepsilon > 0$, we have

  $$K_{n+1} - K_n \leq \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma - 1}\right) n^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)+\varepsilon}$$

  for infinitely many n. Numerically, choosing a small epsilon,

  $$K_{n+1} - K_n \leq 0.8646 \left(\frac{1}{n}\right)^{0.4737}$$

- (Corollary 5.2) The optimal multilinear Bohnenblust–Hille constants $(K_n)_{n=1}^{\infty}$ satisfy
$$\liminf\,(K_{n+1}-K_n)\leq 0.$$

These results complements recent information given in [21].

2. First results: technical lemmata

Our first result, and crucial for our goals, is the proof that the sequence $\left(A_{\frac{2m}{m+2}}^{-m/2}\right)_{m=1}^{\infty}$ is increasing. We stress that this is not an obvious result. In fact, since the sequence $(A_p)_{p\geq 1}$ is composed by the best constants satisfying the Khinchine inequality, using the monotonicity of the L_p-norms we can conclude that

$$\left(A_{\frac{2m}{m+2}}\right)_{m=1}^{\infty}\subset(0,1)$$

is increasing. Hence

$$\left(A_{\frac{2m}{m+2}}^{-1}\right)_{m=1}^{\infty}\subset(1,\infty)$$

is decreasing; thus, since $(m/2)_{m=1}^{\infty}$ is increasing, no straightforward conclusion on the monotonicity of $\left(A_{\frac{2m}{m+2}}^{-m/2}\right)_{m=1}^{\infty}$ can be inferred. The key result used in the proof of the following lemmata is an useful theorem due to F. Qi [24] asserting that

$$\left(\frac{\Gamma(s)}{\Gamma(r)}\right)^{\frac{1}{s-r}}$$

increases with $r,s>0$.

Lemma 2.1. *The sequence* $\left(A_{\frac{2m}{m+2}}^{-m/2}\right)_{m=1}^{\infty}$ *is increasing. In particular*

$$C_{2m}\leq\left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right)C_m$$

for all m.

Proof. Since

$$\frac{2m}{m+2}>p_0\approx 1.847$$

for all $m\geq 25$, the formula (3) holds only for $m\geq 25$; but a direct inspection (using (4)) shows that the sequence is increasing for $m<25$.

For $m\geq 25$, note that

$$A_{\frac{2m}{m+2}}^{m/2}=\frac{1}{\sqrt{2}}\left(\frac{\Gamma\left(\frac{3m+2}{2m+4}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^{\frac{m+2}{4}}.$$

But from [24, Theorem 2] we know that

$$\left(\left(\frac{\Gamma\left(\frac{3m+2}{2m+4}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^{\frac{m+2}{-2}}\right)_{m=1}^{\infty}$$

is increasing. Thus

$$\left(\left(\frac{\Gamma\left(\frac{3m+2}{2m+4}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^{\frac{m+2}{2}}\right)_{m=1}^{\infty}$$

is decreasing and

$$\left(\left(\frac{\Gamma\left(\frac{3m+2}{2m+4}\right)}{\Gamma\left(\frac{3}{2}\right)}\right)^{\frac{m+2}{4}}\right)_{m=1}^{\infty}$$

is also decreasing and the proof is done. □

A first consequence of this lemma solves a question left open in [12].

Proposition 2.2. *The sequence*

$$C_n = \begin{cases} 1 & \text{if } n = 1 \\ \left(A_{\frac{2n}{n+2}}^{n/2}\right)^{-1} C_{\frac{n}{2}} & \text{if } n \text{ is even} \\ \left(A_{\frac{2n-2}{n+1}}^{\frac{-1-n}{2}} C_{\frac{n-1}{2}}\right)^{\frac{n-1}{2n}} \left(A_{\frac{2n+2}{n+3}}^{\frac{1-n}{2}} C_{\frac{n+1}{2}}\right)^{\frac{n+1}{2n}} & \text{if } n \text{ is odd.} \end{cases}$$

is increasing.

Proof. We proceed by induction. The first values are checked directly. Let us suppose that the result is valid for all positive integers smaller than $n-1$ and use induction.

First case. n is even.

Note that

$$C_n \leq C_{n+1}$$

if and only if

$$\frac{C_{n/2}}{A_{\frac{2n}{n+2}}^{n/2}} \leq \left(\frac{C_{n/2}}{A_{\frac{2n}{n+2}}^{(n+2)/2}}\right)^{\frac{n}{2(n+1)}} \cdot \left(\frac{C_{\frac{n+2}{2}}}{A_{\frac{2n+4}{n+4}}^{n/2}}\right)^{\frac{n+2}{2(n+1)}}$$

and this is equivalent to

$$\left(C_{n/2}\right)^{\frac{n+2}{2(n+1)}} \left(\left(A_{\frac{2n}{n+2}}^{n/2}\right)^{-1}\right)^{\frac{n}{2(n+1)}} \leq \left(C_{\frac{n+2}{2}}\right)^{\frac{n+2}{2(n+1)}} \left(\left(A_{\frac{2n+4}{n+4}}^{(n+2)/2}\right)^{-1}\right)^{\frac{n}{2(n+1)}}.$$

But the last inequality is true. In fact, from the induction hypothesis we have

$$C_{n/2} \leq C_{\frac{n+2}{2}}$$

and from Lemma 2.1 we know that

$$\left(A_{\frac{2n}{n+2}}^{n/2}\right)^{-1} \leq \left(A_{\frac{2n+4}{n+4}}^{(n+2)/2}\right)^{-1}$$

holds.

Second case. n is odd.

A similar argument shows that

$$C_n \leq C_{n+1}$$

if and only if

$$\frac{\left(C_{(n-1)/2}\right)^{\frac{n-1}{2n}}}{\left(A_{\frac{2n-2}{n+1}}^{(n-1)/2}\right)^{\frac{n+1}{2n}}} \leq \frac{\left(C_{(n+1)/2}\right)^{\frac{n-1}{2n}}}{\left(A_{\frac{2n+2}{n+3}}^{(n+1)/2}\right)^{\frac{n+1}{2n}}}$$

and this inequality is true using the induction hypothesis and Lemma 2.1. □

Lemma 2.3. *The sequence*

$$\left(\left(\left(A_{\frac{2m+2}{m+3}}^{\frac{m-1}{2}}\right)^{-1}\right)^{\frac{m+1}{2m}} \cdot \left(\left(A_{\frac{2m-2}{m+1}}^{\frac{m+1}{2}}\right)^{-1}\right)^{\frac{m-1}{2m}}\right)_{m=1}^{\infty}$$

is bounded by

$$D := \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right).$$

Proof. Let

$$X_m := A_{\frac{2m}{m+2}}^{-m/2}$$

for all m. From Lemma 2.1 we know that $(X_m)_{m=1}^{\infty}$ is increasing and bounded by D. Note that

$$\left(\left(A_{\frac{2m-2}{m+1}}^{\frac{m-1}{2}}\right)^{-1}\right) = X_{m-1} \leq X_{m+1} = \left(\left(A_{\frac{2m+2}{m+3}}^{\frac{m+1}{2}}\right)^{-1}\right).$$

Thus we have

$$\begin{aligned}
&\left(\left(A_{\frac{2m+2}{m+3}}^{\frac{m-1}{2}}\right)^{-1}\right)^{\frac{m+1}{2m}} \cdot \left(\left(A_{\frac{2m-2}{m+1}}^{\frac{m+1}{2}}\right)^{-1}\right)^{\frac{m-1}{2m}} \\
&= \left(\left(A_{\frac{2m+2}{m+3}}^{\frac{m+1}{2}}\right)^{-1}\right)^{\frac{m-1}{2m}} \cdot \left(\left(A_{\frac{2m-2}{m+1}}^{\frac{m-1}{2}}\right)^{-1}\right)^{\frac{m+1}{2m}} \\
&= \left(X_{m+1}\right)^{\frac{m-1}{2m}} \left(X_{m-1}\right)^{\frac{m+1}{2m}} \\
&\leq X_{m+1}.
\end{aligned}$$

Since $(X_m)_{m=1}^{\infty}$ is increasing and bounded by D we conclude that

$$\left(\left(\left(A_{\frac{2m+2}{m+3}}^{\frac{m-1}{2}}\right)^{-1}\right)^{\frac{m+1}{2m}} \cdot \left(\left(A_{\frac{2m-2}{m+1}}^{\frac{m+1}{2}}\right)^{-1}\right)^{\frac{m-1}{2m}}\right)_{m=1}^{\infty}$$

is also bounded by D. □

3. There exist multilinear Bohnenblust–Hille constants $(R_n)_{n=1}^{\infty}$ with $R_{n+1} - R_n \to 0$

In this section we prove one of the main results of this paper (Theorem 3.1). We note that $(S_n)_{n=1}^{\infty}$ (defined in (10)) is increasing and satisfies the multilinear Bohnenblust–Hille inequality. The proof of the first assertion is straightforward; for the proof of the second assertion we just need to observe that the sequence $(C_n')_{n=1}^{\infty}$ in (7) is so that $C_n' \leq S_n$ for all n. We recall that a closed formula for the constants $(S_n)_{n=1}^{\infty}$ with a generic D in the place of $\left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right)$ appears in [26]. Since $(S_n)_{n=1}^{\infty}$ is increasing, the new sequence defined by

$$M_n = \begin{cases} \left(\sqrt{2}\right)^{n-1} & \text{if } n = 1,2 \\ \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right) M_{\frac{n}{2}} & \text{if } n \text{ is even, and} \\ \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right) M_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

is so that

$$C_n' \leq S_n \leq M_n$$

and a “uniform perturbation” of this sequence $(M_n)_{n=1}^{\infty}$ shall be the desired sequence.

Let

$$D := \left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right) \approx 1.4403.$$

and, for all $k \geq 1$, consider

$$B_k := \{2^{k-1}+1, \ldots, 2^k\}.$$

It is simple to note that for all $n \geq 2$ we have

$$M_n = \sqrt{2} D^{k-1} \text{ whenever } n \in B_k$$

and for this reason $\lim_{n\to\infty} M_n - M_{n-1}$ does not exist. Now consider the sequence $(R_n)_{n=1}^{\infty}$, which is a slight uniform perturbation of the sequence $(M_n)_{n=1}^{\infty}$:

$$R_n := \sqrt{2}\left(D^{k-1} + (j_n - 1)\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right)\right), \text{ whenever } n \in B_k \tag{11}$$

where j_n is the position of n in the order of the elements of B_k.

It is plain that

$$M_n \leq R_n$$

for all $n \geq 3$ and, as we shall see,

$$(R_{n+1} - R_n)_{n=1}^{\infty}$$

is decreasing. Using the definition of $(R_n)_{n=1}^{\infty}$ with a careful handling of the expressions involved it is not difficult to estimate how $R_{n+1} - R_n$ decreases to zero:

Theorem 3.1. *The sequence (11) satisfies the multilinear Bohnenblust–Hille inequality and* $(R_{n+1} - R_n)_{n=1}^{\infty}$ *is decreasing and converges to zero. Moreover*

$$R_{n+1} - R_n \leq \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma - 1}\right) n^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)} \tag{12}$$

for all n. *Numerically,*

$$R_{n+1} - R_n \leq (0.8646)\, n^{-0.47368}$$

Proof. Of course $(R_n)_{n=1}^{\infty}$ satisfies the multilinear Bohnenblust–Hille inequality. Let us show that $(R_{n+1} - R_n)_{n=1}^{\infty}$ is decreasing. In fact, if $n \in B_k$, we have two possibilities:

First case: $n + 1 \in B_k$.

In this case

$$\begin{aligned} &R_{n+1} - R_n \\ &= \sqrt{2}D^{k-1} + \sqrt{2}\,(j_{n+1} - 1)\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right) - \left(\sqrt{2}D^{k-1} + \sqrt{2}\,(j_n - 1)\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right)\right) \\ &= \sqrt{2}\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right) \end{aligned}$$

i.e., $(R_{n+1} - R_n)_{n=1}^{\infty}$ is constant and equal to $\sqrt{2}\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right)$.

Second case: $n + 1 \in B_{k+1}$.

In this case $n = 2^k$ and $n + 1 = 2^k + 1$, and thus

$$\begin{aligned} &R_{n+1} - R_n \\ &= \sqrt{2}D^k + \sqrt{2}\,(1 - 1)\left(\frac{D^{k+1} - D^k}{2^k}\right) - \left(\sqrt{2}D^{k-1} + \sqrt{2}\left(2^{k-1} - 1\right)\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right)\right) \\ &= \sqrt{2}\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right), \end{aligned}$$

obtaining the same value. But, since $D < 2$ we have

$$\frac{D^k - D^{k-1}}{2^{k-1}} > \frac{D^{k+1} - D^k}{2^k}$$

and we conclude that $(R_{n+1} - R_n)_{n=1}^{\infty}$ is decreasing.

If we consider the subsequence

$$\left(R_{2^k+1} - R_{2^k}\right)_{k=1}^{\infty}$$

we have

$$\begin{aligned}\lim_{k\to\infty}\left(R_{2^k+1}-R_{2^k}\right) &= \sqrt{2}\lim_{k\to\infty}\left(\frac{D^k-D^{k-1}}{2^{k-1}}\right)\\ &= \sqrt{2}\,(D-1)\lim_{k\to\infty}\left(\frac{D}{2}\right)^{k-1}\\ &= 0,\end{aligned} \tag{13}$$

since $D<2$. Hence

$$\lim_{n\to\infty} R_{n+1}-R_n = 0.$$

Now we estimate the difference $R_{n+1}-R_n$.

Let k be such that $n\in B_k$; we thus have

$$2^{k-1}+1\leq n\leq 2^k$$

and

$$\log_2\left(\frac{n}{2}\right)\leq \log_2\left(2^{k-1}\right)=k-1.$$

Using that $\frac{D}{2}<1$ we conclude that

$$R_{n+1}-R_n\leq \left(\frac{D}{2}\right)^{k-1}\sqrt{2}\,(D-1)\leq \left(\frac{D}{2}\right)^{\log_2\left(\frac{n}{2}\right)}\sqrt{2}\,(D-1)$$

and a simple calculation gives us

$$\begin{aligned}R_{n+1}-R_n &\leq \left(\frac{e^{1-\frac{1}{2}\gamma}}{2\sqrt{2}}\right)^{\log_2\left(\frac{n}{2}\right)}\left(\left(e^{1-\frac{1}{2}\gamma}\right)-\sqrt{2}\right)\\ &= \left(2\sqrt{2}-4e^{\frac{1}{2}\gamma-1}\right)n^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)}\end{aligned}$$

□

Corollary 3.2. *If $\varepsilon>0$, then*

$$R_{n+1}-R_n = o\left(n^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)+\varepsilon}\right).$$

As we know, the constants defined in (11) are slightly bigger than the constants from (7), (10); but we stress that there is no damage, asymptotically speaking. More precisely, the limits of $\left(\frac{R_{2n}}{R_n}\right)_{n=1}^{\infty}$ and $\left(\frac{R_{n+1}}{R_n}\right)_{n=1}^{\infty}$ are exactly the same of $\left(\frac{C_{2n}}{C_n}\right)_{n=1}^{\infty}$ and $\left(\frac{C_{n+1}}{C_n}\right)_{n=1}^{\infty}$:

Proposition 3.3. *The sequence $\left(\frac{R_{2n}}{R_n}\right)_{n=1}^{\infty}$ is decreasing and*

$$\lim_{n\to\infty}\frac{R_{2n}}{R_n}=\left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right). \tag{14}$$

Also

$$\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = 1. \tag{15}$$

Proof. The proof that $\left(\frac{R_{2n}}{R_n}\right)_{n=1}^{\infty}$ is decreasing needs some care with the details, but is essentially straightforward and we omit.

Let k be so that $2n \in B_k$; then j_{2n} is even. Also, we have $n \in B_{k-1}$ and note that $j_n = \frac{j_{2n}}{2}$. Hence

$$\frac{R_{2n}}{R_n} = \frac{\sqrt{2}\left(D^{k-1} + (j_{2n}-1)\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right)\right)}{\sqrt{2}\left(D^{k-2} + \left(\frac{j_{2n}}{2}-1\right)\left(\frac{D^{k-1}-D^{k-2}}{2^{k-2}}\right)\right)}.$$

Considering the subsequence given for $j_{2n} = 2$ we have

$$\begin{aligned}
\frac{D^{k-1} + (2-1)\left(\frac{D^k - D^{k-1}}{2^{k-1}}\right)}{D^{k-2} + (1-1)\left(\frac{D^{k-1}-D^{k-2}}{2^{k-2}}\right)} &= \frac{D^{k-1} + \left(\frac{D^k - D^{k-1}}{2^{k-1}}\right)}{D^{k-2}} \\
&= \frac{2^{k-1}D^{k-1} + D^k - D^{k-1}}{2^{k-1}D^{k-2}} \\
&= \frac{D^{k-2}\left(2^{k-1}D + D^2 - D\right)}{2^{k-1}D^{k-2}} \\
&= \frac{2^{k-1}D + D^2 - D}{2^{k-1}} \overset{k\to\infty}{\longrightarrow} D.
\end{aligned}$$

Combining this fact with the monotonicity of the sequence we obtain (14). The proof of (15) is similar. □

4. On the optimal constants: part 1

In this section $(R_n)_{n=1}^{\infty}$ denotes the sequence defined in (11). As a consequence of Theorem 3.1 we have some new information on the growth of the optimal constants satisfying the multilinear Bohnenblust–Hille inequality. This information complements (although not formally generalizes) some recent results from [21]:

Theorem 4.1. *Let $(K_n)_{n=1}^{\infty}$ be the sequence of the optimal constants satisfying the multilinear Bohnenblust–Hille inequality. If there is a constant $M \in [-\infty, \infty]$ so that*

$$\lim_{n\to\infty} (K_{n+1} - K_n) = M$$

then $M = 0$.

Proof. The case $M \in [-\infty, 0)$ is clearly not possible. Let us first suppose that $M \in (0, \infty)$. Let n_0 be a positive integer so that

$$n \geq n_0 \Rightarrow K_{n+1} - K_n > \frac{M}{2}$$

and n_1 be a positive integer so that

$$n \geq n_1 \Rightarrow R_{n+1} - R_n < \frac{M}{4}.$$

So, if $n \geq n_2 := \max\{n_1, n_0\}$, then

$$K_n - K_{n_2} > \left(\frac{M}{2}\right)(n - n_2)$$

and

$$R_n - R_{n_2} < \left(\frac{M}{4}\right)(n - n_2).$$

Let $N > n_2$ be so that

$$\left(\frac{M}{2}\right)(N - n_2) + K_{n_2} > R_{n_2} + \left(\frac{M}{4}\right)(N - n_2).$$

Note that the is possible since

$$\left(\frac{M}{2}\right)(n - n_2) - \left(\frac{M}{4}\right)(n - n_2) \to \infty.$$

For this N we have

$$K_N > \left(\frac{M}{2}\right)(N - n_2) + K_{n_2} > R_{n_2} + \left(\frac{M}{4}\right)(N - n_2) > R_N,$$

which is a contradiction. The case $M = \infty$ is a simple adaptation of the previous case. $\square$

5. On the optimal constants: part 2

From the previous results we know that

$$R_{n+1} - R_n \leq \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma - 1}\right) n^{\log_2\left(2^{-3/2} e^{1 - \frac{1}{2}\gamma}\right)}.$$

Summing the above inequalities it is simple to show that

$$R_n \leq 1 + \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma - 1}\right) \sum_{j=1}^{n-1} j^{\log_2\left(2^{-3/2} e^{1 - \frac{1}{2}\gamma}\right)}. \tag{16}$$

If $\varepsilon > 0$, let us define

$$T_n = 1 + \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma - 1}\right) \sum_{j=1}^{n-1} j^{\log_2\left(2^{-3/2} e^{1 - \frac{1}{2}\gamma}\right) + \varepsilon}.$$

Then

$$T_{n+1} - T_n = \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma - 1}\right) n^{\log_2\left(2^{-3/2} e^{1 - \frac{1}{2}\gamma}\right) + \varepsilon}$$

It is simple to prove that the set

$$A_\varepsilon := \{n : K_{n+1} - K_n \leq T_{n+1} - T_n\}$$

is infinite. In fact, if A_ε was finite, let n_ε be its minimum. So, for all $n > n_\varepsilon$ we would have

$$K_{n+1} - K_n > T_{n+1} - T_n.$$

Also, for any $N > n_\varepsilon + 1$, summing both sides for $n = n_\varepsilon + 1$ to $n = N$, we have

$$K_{N+1} - K_{n_\varepsilon+1} > T_{N+1} - T_{n_\varepsilon+1}.$$

We finally obtain

$$K_{N+1} - T_{N+1} > K_{n_\varepsilon+1} - T_{n_\varepsilon+1}$$

and it is a contradiction, since

$$K_{N+1} - T_{N+1} \leq R_{N+1} - T_{N+1} \leq$$

$$\leq \left(1 + \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma-1}\right) \sum_{j=1}^{N} j^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)}\right) - \left(1 + \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma-1}\right) \sum_{j=1}^{N} j^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)+\varepsilon}\right)$$

and this last expression tends to $-\infty$. Thus, we have:

Theorem 5.1. *Let $(K_n)_{n=1}^{\infty}$ be the optimal constants satisfying the multilinear Bohnenblust–Hille constants. For any $\varepsilon > 0$, we have*

$$K_{n+1} - K_n \leq \left(2\sqrt{2} - 4e^{\frac{1}{2}\gamma-1}\right) n^{\log_2\left(2^{-3/2}e^{1-\frac{1}{2}\gamma}\right)+\varepsilon} \tag{17}$$

for infinitely many n's.

Estimating the values in (17) and choosing a sufficiently small $\varepsilon > 0$ we can assert that

$$K_{n+1} - K_n \leq \frac{0.8646}{n^{0.4737}}.$$

It seems quite likely that the optimal constants of the multilinear Bohnenblust–Hille inequality have an uniform growth. The above theorem induces us to conjecture that the estimate holds for all n.

Corollary 5.2. *The optimal multilinear Bohnenblust–Hille constants $(K_n)_{n=1}^{\infty}$ satisfy*

$$\liminf_{n} (K_{n+1} - K_n) \leq 0.$$

The following straightforward consequence of (16) seems to be of independent interest:

Theorem 5.3. *The optimal multilinear Bohnenblust–Hille constants $(K_n)_{n=1}^{\infty}$ satisfy*

$$K_n \leq 1 + (0.8646) \sum_{j=1}^{n-1} \left(\frac{1}{j}\right)^{0.4737}$$

for all $n \geq 2$.

6. Is there a strong multilinear Bohnenblust–Hille inequality?

Of course, there are still a lot of open questions related to the growth of the optimal constants satisfying the multilinear (and polynomial) Bohnenblust–Hille inequalities to be solved. For example, it is *not* clear that the optimal constants $(K_n)_{n=1}^{\infty}$ satisfying the multilinear Bohnenblust–Hille inequality grow to infinity. Is it true? It seems that the original estimates induce us to think that in fact $K_n \to \infty$, but it purports to exist no other evidence for this.

Although there still remains in a veil of mystery, combining all the information obtained thus far we believe that the possibility of boundedness of the constants of the multilinear Bohnenblust–Hille inequality should be seriously considered. We prefer not to conjecture that it is true, but we pose it as an open problem:

Problem 6.1. *Is there an universal constant* $C_{\mathbb{K}}$ *so that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\left|U(e_{i_1},\ldots,e_{i_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq C_{\mathbb{K}}\sup_{z_1,\ldots,z_m\in\mathbb{D}^N}\left|U(z_1,\ldots,z_m)\right|$$

for every positive integer $m\geq 1$*, all* m*-linear forms* $U:\mathbb{K}^N\times\cdots\times\mathbb{K}^N\to\mathbb{K}$ *and every positive integer* N*?*

Conjecture 6.2. *If the answer is positive, we conjecture that* $C_{\mathbb{R}}=2$ *and* $C_{\mathbb{C}}\leq 2$*.*

We justify our conjecture that $C_{\mathbb{R}}=2$ motivated by the lower bounds obtained in [13] for the constants of the multilinear Bohnenblust–Hille inequality (real case)

$$C_m\geq 2^{1-\frac{1}{m}}. \tag{18}$$

We stress that the case $m=2$ in (18) is sharp, i.e., $\sqrt{2}$ is the optimal constant for the 2-linear Bohnenblust–Hille inequality (real case). As a matter of fact, if we consider $m=1$, then the formula (18) also provides a sharp value. So, since in each level m, the lower estimate for C_m is obtained by the same induction argument (for details, see [13]) and since the cases $m=1,2$ provide sharp constants, we speculate that it is not impossible that the formula (18) gives the exact constants for the Bohnenblust–Hille constants. We reinforce our belief by observing the several recent works showing that the growth of the constants in the Bohnenblust–Hille inequality is it in fact quite slower than the original estimates have predicted.

It is well-known (although not formally proved) that the constants for the case of real scalars are smaller that the constants for the complex case. For $m=2$, for example, $C_2=\sqrt{2}$ in the real case and $C_2\leq\frac{2}{\sqrt{\pi}}<\sqrt{2}$ in the complex case. Besides, the growth of the constants in the complex case seems to be slower than the growth in the real case (see [21, 22]). So, if our conjecture is correct, it seems natural that $C_{\mathbb{C}}\leq C_{\mathbb{R}}$.

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