A DUALITY BETWEEN NON-ARCHIMEDEAN UNIFORM SPACES AND SUBDIRECT POWERS OF FULL CLONES

JOSEPH VAN NAME

Abstract. A uniform space is said to be non-Archimedean if it is generated by equivalence relations. If λ is a cardinal, then a non-Archimedean uniform space $(X,\mathcal{U})$ is λ-totally bounded if each equivalence relation in $\mathcal{U}$ partitions X into less than λ blocks. If A is an infinite set, then let $\Omega(A)$ be the algebra with universe A and where each $a \in A$ is a fundamental constant and every finitary function is a fundamental operation. We shall give a duality between complete non-Archimedean $|A|^{+}$-totally bounded uniform spaces and subdirect powers of $\Omega(A)$. We shall apply this duality to characterize the algebras dual to supercomplete non-Archimedean uniform spaces.

1. Non-Archimedean Uniform Space Duality

In this paper, we shall assume basic facts about uniform spaces and universal algebra. The reader is referred to [2] or [3] for information about uniform spaces and to [1] for universal algebra. We shall use the entourage definition of uniform spaces, and we shall assume all complete uniform spaces are separated. If $\mathcal{A}$ is an algebra, then we shall write $V(\mathcal{A})$ for the variety generated by $\mathcal{A}$.

In [4], Marshall Stone constructed a duality between compact totally disconnected spaces and Boolean algebras. This result revolutionized the theory of Boolean algebras since it gives a way to represent Boolean algebras as topological spaces. We shall give an analogous result for uniform spaces.

A uniform space $(X,\mathcal{U})$ is said to be *non-Archimedean* if $\mathcal{U}$ is generated by equivalence relations. We say that a non-Archimedean uniform space $(X,\mathcal{U})$ is λ-totally bounded if whenever $E \in \mathcal{U}$ is an equivalence relation, then E partitions X into less than λ blocks. Clearly, if $(X,\mathcal{U})$ is λ-totally bounded, then each subspace of X is λ-totally bounded as well.

For this paper, let A be a fixed infinite set. For each $a \in A$, let $\hat{a}$ be a constant symbol. For each $f\colon A^n \to A$, let $\hat{f}$ be an n-ary function symbol. Let $\mathcal{F} = \{\hat{a}|a \in A\} \cup \bigcup_n \{\hat{f}|f\colon A^n \to A\}$. Let $\Omega(A)$ be the algebra of type $\mathcal{F}$ and with universe A where $\hat{a}^{\Omega(A)} = a$ for $a \in A$ and where $\hat{f}^{\Omega(A)} = f$ for $f\colon A^n \to A$. Therefore every n-ary function on A is given by a function symbol, so we can regard $\Omega(A)$ as the full clone of A. We shall now give a duality between subdirect powers of $\Omega(A)$ and complete non-Archimedean $|A|^{+}$-totally bounded uniform spaces. With this duality, every complete non-Archimedean uniform space can be represented algebraically simply by letting $|A|$ be at least as large as every uniform partition.

The algebra $\Omega(A)$ and the variety $V(\Omega(A))$ generated by $\Omega(A)$ have applications to mathematics besides uniform space duality. For instance, the variety $V(\Omega(A))$

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is related to the ultrapower construction and reduced power construction. In fact, one can construct ultrapowers and reduced powers from elements of the variety $V(\Omega(A))$. Also, the first order theory $\mathrm{Th}(\Omega(A))$ of $\Omega(A)$ is appealing since it is generated by the identities in $\Omega(A)$ and a single sentence. In other words, there is a $\phi \in \mathrm{Th}(\Omega(A))$ such that for each $\theta \in \mathrm{Th}(\Omega(A))$, there are identities $I_1, \dots, I_n \in \mathrm{Th}(\Omega(A))$ such that $(\phi \wedge I_1 \wedge \cdots \wedge I_n) \to \theta$.

The algebra $\Omega(A)$ serves as an infinite analogue of the two element Boolean algebra B since in B every function can be represented as a combination of the Boolean operations $\wedge, \vee, '$. Therefore the variety $V(\Omega(A))$ is analogous to the variety of Boolean algebras. The category of compact totally disconnected spaces is isomorphic to the category of complete non-Archimedean $\aleph_0$-totally bounded uniform spaces. Therefore it should be possible to reconstruct a duality between compact totally disconnected spaces and the variety of Boolean algebras, but for simplicity we shall only consider the variety $V(\Omega(A))$ when A is infinite.

An algebra $\mathcal{L} \in V(\Omega(A))$ shall be called *partitionable* if there is an injective homomorphism $\phi\colon \mathcal{L} \to \Omega(A)^I$ for some set I. Clearly, the products and subspaces of partitionable algebras are partitionable. Furthermore, each partitionable algebra is isomorphic to a subdirect product of $\Omega(A)$ since each $a \in A$ is a constant in $\Omega(A)$.

Let $Z(\mathcal{L})$ be the collection of all homomorphisms $\phi\colon \mathcal{L} \to \Omega(A)$. In this paper, the set A will always have the discrete uniformity. Now give $A^{\mathcal{L}}$ the product uniformity. Then the topology on A is the discrete topology and the topology on $A^{\mathcal{L}}$ is the product topology. Give $Z(\mathcal{L}) \subseteq A^{\mathcal{L}}$ the subspace uniformity. Then $Z(\mathcal{L})$ is a closed subspace of $A^{\mathcal{L}}$ since every convergent net $(\phi_d)_{d\in D}$ in $Z(\mathcal{L})$ converges to some $\phi \in Z(\mathcal{L})$. Thus, since $Z(\mathcal{L})$ is a closed subspace of a complete uniform space, $Z(\mathcal{L})$ is complete.

Let $\ell_1, \dots, \ell_n \in \mathcal{L}$. Then let $\mathcal{E}^{\sharp}_{\ell_1,\dots,\ell_n}$ be the equivalence relation $A^{\mathcal{L}}$ where for $r, s \in A^{\mathcal{L}}$ we have $(r, s) \in \mathcal{E}^{\sharp}_{\ell_1,\dots,\ell_n}$ if and only if $r(\ell_1) = s(\ell_1), \dots, r(\ell_n) = s(\ell_n)$. Then the equivalence relations $\mathcal{E}^{\sharp}_{\ell_1,\dots,\ell_n}$ generate the uniformity on $A^{\mathcal{L}}$. Take note that each $\mathcal{E}^{\sharp}_{\ell_1,\dots,\ell_n}$ partitions $A^{\mathcal{L}}$ into $|A|^n = |A|$ blocks, so the uniform space $A^{\mathcal{L}}$ is $|A|^+$-totally bounded. Let $\mathcal{E}_{\ell_1,\dots,\ell_n}$ be the restriction of $\mathcal{E}^{\sharp}_{\ell_1,\dots,\ell_n}$ to $Z(\mathcal{L})$. Then the equivalence relations $\mathcal{E}_{\ell_1,\dots,\ell_n}$ generate the uniformity on $Z(\mathcal{L})$. In particular, $Z(\mathcal{L})$ is a $|A|^+$-totally bounded non-Archimedean uniform space.

Let $(X, \mathcal{U})$ be a uniform space. Then let $\mathfrak{B}_A(X, \mathcal{U})$ be the collection of all uniformly continuous mappings from X to A. Clearly $\mathfrak{B}_A(X, \mathcal{U})$ is a subdirect product of $\Omega(A)$, so $\mathfrak{B}_A(X, \mathcal{U})$ is a partitionable algebra.

If $(X, \mathcal{U})$ is a uniform space, then for each $x \in X$, we have $\pi_x\colon \mathfrak{B}_A(X, \mathcal{U}) \to \Omega(A)$ be a homomorphism where π_x is the projection mapping defined by $\pi_x(f) = f(x)$. Therefore define a mapping $\mathcal{C}\colon (X, \mathcal{U}) \to Z(\mathfrak{B}_A(X, \mathcal{U}))$ by $\mathcal{C}(x) = \pi_x$. In other words, if $x \in X$, and $f\colon (X, \mathcal{U}) \to A$ is uniformly continuous, then $\mathcal{C}(x)f = f(x)$. If there is any confusion about the space $(X, \mathcal{U})$, then we shall write $\mathcal{C}_{(X,\mathcal{U})}$ for the mapping $\mathcal{C}$.

Now let $\mathcal{L} \in V(\Omega(A))$. If $\ell \in \mathcal{L}$, then let $\ell^\star\colon Z(\mathcal{L}) \to A$ be the mapping defined by $\ell^\star(\phi) = \phi(\ell)$. We claim that $\ell^\star$ is uniformly continous. Assume that $(\phi, \theta) \in \mathcal{E}_\ell$. Then $\phi(\ell) = \theta(\ell)$, so $\ell^\star(\phi) = \ell^\star(\theta)$, and hence $(\ell^\star(\phi), \ell^\star(\theta)) \in E$ for each equivalence relation E on A. Therefore $\ell^\star$ is uniformly continuous, so $\ell^\star \in \mathfrak{B}_A(Z(\mathcal{L}))$. In light of the above discussion, we define a function $\rho\colon \mathcal{L} \to \mathfrak{B}_A(Z(\mathcal{L}))$

by $\rho(\ell)=\ell^\star$. Therefore $\rho(\ell)(\phi)=\phi(\ell)$ for $\phi\in Z(\mathcal{L}),\ell\in\mathcal{L}$. We will write $\rho_{\mathcal{L}}$ for the mapping ρ to specify the domain of ρ in case there may be confusion.

Exercise 1.1. *If $f\colon A^n\to A$ is injective (surjective), then $\hat{f}^{\mathcal{L}}\colon\mathcal{L}^n\to\mathcal{L}$ is injective (surjective) for each $\mathcal{L}\in V(\Omega(A))$.*

Theorem 1.2. *The equivalence relations $\mathcal{E}_\ell$ generate the uniformity on $Z(\mathcal{L})$.*

Proof. Assume that $\ell_1,\dots,\ell_n\in\mathcal{L}$. Let $i\colon A^n\to A$ be injective. Then $\hat{i}^{\mathcal{L}}$ is also injective. Now let $\ell=\hat{i}^{\mathcal{L}}(\ell_1,\dots,\ell_n)$. Assume $\phi,\theta\in Z(\mathcal{L})$ and $(\phi,\theta)\in\mathcal{E}_\ell$. Then $\phi(\ell)=\theta(\ell)$, so $\phi(\hat{i}^{\mathcal{L}}(\ell_1,\dots,\ell_n))=\theta(\hat{i}^{\mathcal{L}}(\ell_1,\dots,\ell_n))$. Therefore, $i(\phi(\ell_1),\dots,\phi(\ell_n))=i(\theta(\ell_1),\dots,\theta(\ell_n))$, so since i is injective, we have $\phi(\ell_1)=\theta(\ell_1),\dots,\phi(\ell_n)=\theta(\ell_n)$, thus $(\phi,\theta)\in\mathcal{E}_{\ell_1,\dots,\ell_n}$. In other words, we have $\mathcal{E}_\ell\subseteq\mathcal{E}_{\ell_1,\dots,\ell_n}$. Therefore the equivalence relations $\mathcal{E}_\ell$ generate the uniformity on $Z(\mathcal{L})$. □

Theorem 1.3. *1. Let $\mathcal{L}\in V(\Omega(A))$. Then $\rho\colon\mathcal{L}\to\mathfrak{B}_A(Z(\mathcal{L}))$ is a surjective homomorphism, and ρ is an isomorphism if and only if $\mathcal{L}$ is partitionable.*

2. If $(X,\mathcal{U})$ is a uniform space, then the mapping $\mathcal{C}\colon(X,\mathcal{U})\to Z(\mathfrak{B}_A(X,\mathcal{U}))$ is uniformly continuous and $\mathcal{C}''(X)$ is dense in $Z(\mathfrak{B}_A(X,\mathcal{U}))$. If $(X,\mathcal{U})$ is separated and non-Archimedean, then $\mathcal{C}$ is injective. If $(X,\mathcal{U})$ is separated non-Archimedean and $|A|^+$-totally bounded, then $\mathcal{C}$ is an embedding. If $(X,\mathcal{U})$ is complete non-Archimedean and $|A|^+$-totally bounded, then $\mathcal{C}$ is an isomorphism.

Proof. 1. If $\ell\in\mathcal{L}$, then we have $\rho(\ell)=(\rho(\ell)(\phi))_{\phi\in Z(\mathcal{L})}=(\phi(\ell))_{\phi\in Z(\mathcal{L})}$. Therefore ρ is a homomorphism since ρ is a homomorphism in each coordinate.

To prove surjectivity, assume that $f\colon Z(\mathcal{L})\to A$ is uniformly continuous. Then there is an $\ell\in\mathcal{L}$ where if $(\phi,\theta)\in\mathcal{E}_\ell$, then $f(\phi)=f(\theta)$. In other words, if $\phi(\ell)=\theta(\ell)$, then $f(\phi)=f(\theta)$. Therefore there is a function $g\colon A\to A$ where $f(\phi)=g(\phi(\ell))$ whenever $\phi\in Z(\mathcal{L})$. Furthermore, we have $f(\phi)=g(\phi(\ell))=\phi(\hat{g}^{\mathcal{L}}(\ell))=\rho(\hat{g}^{\mathcal{L}}(\ell))(\phi)$ for each $\phi\in Z(\mathcal{L})$. Therefore $\rho(\hat{g}^{\mathcal{L}}(\ell))=f$. Thus the mapping ρ is surjective.

Now assume $\mathcal{L}$ is partitionable. Then for each pair of distinct $\ell_1,\ell_2\in\mathcal{L}$ there is a homomorphism $\phi\colon\mathcal{L}\to A$ with $\rho(\ell_1)(\phi)=\phi(\ell_1)\neq\phi(\ell_2)=\rho(\ell_2)(\phi)$. Therefore $\rho(\ell_1)\neq\rho(\ell_2)$. We conclude that ρ is injective. Likewise, if we assume ρ is an isomorphism, then since $\mathfrak{B}_A(Z(\mathcal{L}))$ is partitionable, we have $\mathcal{L}$ be partitionable as well.

2. Since $\mathcal{C}\colon(X,\mathcal{U})\to Z(\mathfrak{B}_A(X,\mathcal{U}))\subseteq A^{\mathfrak{B}_A(X,\mathcal{U})}$, we have $\mathcal{C}$ be uniformly continuous if and only if $\mathcal{C}$ is uniformly continuous in every coordinate $f\in\mathfrak{B}_A(X,\mathcal{U})$. However, we have $\mathcal{C}(x)=(\mathcal{C}(x)(f))_{f\in\mathfrak{B}_A(X,\mathcal{U})}=(f(x))_{f\in\mathfrak{B}_A(X,\mathcal{U})}$, so $\mathcal{C}$ is uniformly continuous.

We shall now show that $\mathcal{C}''(X)$ is dense in $Z(\mathfrak{B}_A(X,\mathcal{U}))$. The uniformity on $Z(\mathfrak{B}_A(X,\mathcal{U}))$ is generated by the equivalence relations $\mathcal{E}_f$ where $f\in\mathfrak{B}_A(X,\mathcal{U})$. The blocks in the equivalence relation $\mathcal{E}_f$ are the nonempty sets of the form $U_{f,a}=\{\phi\in Z(\mathfrak{B}_A(X,\mathcal{U}))|\phi(f)=a\}$. Therefore it suffices to show that $\mathcal{C}''(X)$ intersects each non-empty block $U_{f,a}$.

Now assume that $U_{f,a}$ is non-empty. Then there is a $\phi\in Z(\mathfrak{B}_A(X,\mathcal{U}))$ with $\phi(f)=a$. We claim that $f(x)=a$ for some $x\in X$. Therefore, assume that $f(x)\neq a$ for all $x\in X$. Let $i\colon A\to A$ be a mapping where $i(a)\neq a$ and $i(b)=b$ for $b\neq a$. Then we have $f=i\circ f=\hat{i}^{\mathfrak{B}_A(X,\mathcal{U})}(f)$, so $\phi(f)=\phi(\hat{i}^{\mathfrak{B}_A(X,\mathcal{U})}(f))=i(\phi(f))\neq a$. Thus, by contrapositive, if $\phi(f)=a$, then $f(x)=a$ for some $x\in X$.

However, we have $\mathcal{C}(x)(f) = f(x) = a$, so $\mathcal{C}(x) \in U_{f,a}$. Therefore $\mathcal{C}''(X)$ is dense in $Z(\mathfrak{B}_A(X,\mathcal{U}))$.

Now assume that $(X,\mathcal{U})$ is separated and non-Archimedean. Then we shall show that $\mathcal{C}$ is injective. Assume that $x, y \in X, x \neq y$. Then since $(X,\mathcal{U})$ is separated and non-Archimedean, there is a uniformly continuous function $f\colon X \to A$ such that $f(x) \neq f(y)$. Therefore $\mathcal{C}(x)(f) = f(x) \neq f(y) = \mathcal{C}(y)(f)$, and hence $\mathcal{C}(x) \neq \mathcal{C}(y)$. We conclude that $\mathcal{C}$ is injective.

Now assume that $(X,\mathcal{U})$ is separated, non-Archimedean, and $|A|^+$-totally bounded. Then we shall show that $\mathcal{C}$ is an embedding. Assume that $E \in \mathcal{U}$ is an equivalence relation. Then since $(X,\mathcal{U})$ is $|A|^+$-totally bounded, there is a function $f\colon X \to A$ where $f(x) = f(y)$ if and only if $(x, y) \in E$. Clearly f is uniformly continuous, so $f \in \mathfrak{B}_A(X,\mathcal{U})$ and $\mathcal{E}_f$ is an equivalence relation on $Z(\mathfrak{B}_A(X,\mathcal{U}))$. Now assume that $x, y \in X$. Then $(x, y) \in E$ if and only if $f(x) = f(y)$ if and only if $\mathcal{C}(x)(f) = \mathcal{C}(y)(f)$ if and only if $(\mathcal{C}(x), \mathcal{C}(y)) \in \mathcal{E}_f$. Therefore $\mathcal{C}$ is an embedding.

If $(X,\mathcal{U})$ is complete, non-Archimedean, and $|A|^+$-totally bounded, then we have $\mathcal{C}$ be an embedding, and $Z(\mathfrak{B}_A(X,\mathcal{U}))$ is the completion of $\mathcal{C}''(X)$. However, if X is complete, we have $\mathcal{C}''(X) = Z(\mathfrak{B}_A(X,\mathcal{U}))$. Therefore, in this case, $\mathcal{C}$ is a uniform homeomorphism. □

Let $\mathcal{L}, \mathcal{M} \in V(\Omega(A))$ and assume that $\phi\colon \mathcal{L} \to \mathcal{M}$ is a homomorphism. Then let $Z(\phi)\colon Z(\mathcal{M}) \to Z(\mathcal{L})$ be the function defined by $Z(\phi)(\theta) = \theta \circ \phi$ for homomorphisms $\theta\colon \mathcal{M} \to A$. One can easily show that the mappings $Z(\phi)$ are uniformly continuous and Z is a functor from the variety $V(\Omega(A))$ to the category of uniform spaces. Now assume that $(X,\mathcal{U}), (Y,\mathcal{V})$ are uniform spaces and $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ is uniformly continuous. Then define a mapping $\mathfrak{B}_A(f)\colon \mathfrak{B}_A(Y,\mathcal{V}) \to \mathfrak{B}_A(X,\mathcal{U})$ by $\mathfrak{B}_A(f)(g) = g \circ f$. Then each $\mathfrak{B}_A(f)$ is a homomorphism. Furthermore, $\mathfrak{B}_A$ gives a functor from the category of uniform spaces to the variety $V(\Omega(A))$.

Theorem 1.4. (1) *Let $f\colon (X,\mathcal{U}) \to (Y,\mathcal{V})$ be uniformly continuous. Then $Z(\mathfrak{B}_A(f)) \circ \mathcal{C}_{(X,\mathcal{U})} = \mathcal{C}_{(Y,\mathcal{V})} \circ f$.*

$$\begin{array}{ccc} (X,\mathcal{U}) & \xrightarrow{f} & (Y,\mathcal{V}) \\ \downarrow{\scriptstyle \mathcal{C}} & & \downarrow{\scriptstyle \mathcal{C}} \\ Z(\mathfrak{B}_A(X,\mathcal{U})) & \xrightarrow{Z(\mathfrak{B}_A(f))} & Z(\mathfrak{B}_A(Y,\mathcal{V})) \end{array}$$

(2) *Let $\phi\colon \mathcal{L} \to \mathcal{M}$ be a homomorphism. Then we have $\mathfrak{B}_A(Z(\phi)) \circ \rho_{\mathcal{L}} = \rho_{\mathcal{M}} \circ \phi$.*

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\phi} & \mathcal{M} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ \mathfrak{B}_A(Z(\mathcal{L})) & \xrightarrow{\mathfrak{B}_A(Z(\phi))} & \mathfrak{B}_A(Z(\mathcal{M})) \end{array}$$

(3) *The pair of functions $Z(\rho_{\mathcal{L}})\colon Z(\mathfrak{B}_A(Z(\mathcal{L}))) \to Z(\mathcal{L})$ and $\mathcal{C}_{Z(\mathcal{L})}\colon Z(\mathcal{L}) \to Z(\mathfrak{B}_A(Z(\mathcal{L})))$ are inverses.*

(4) *The pair of functions $\mathfrak{B}_A(\mathcal{C}_{(X,\mathcal{U})})\colon \mathfrak{B}_A(Z(\mathfrak{B}_A(X,\mathcal{U}))) \to \mathfrak{B}_A(X,\mathcal{U})$ and $\rho_{\mathfrak{B}_A(X,\mathcal{U})}\colon \mathfrak{B}_A(X,\mathcal{U}) \to \mathfrak{B}_A(Z(\mathfrak{B}_A(X,\mathcal{U})))$ are inverses.*

Proof. (1) Let $x \in X$ and let $g \in \mathfrak{B}_A(Y,\mathcal{V})$. Then we have

$$[(Z(\mathfrak{B}_A(f)) \circ \mathcal{C})(x)](g) = [Z(\mathfrak{B}_A(f))(\mathcal{C}(x))](g)$$

$$= [\mathcal{C}(x) \circ \mathfrak{B}_A(f)]g = \mathcal{C}(x)[\mathfrak{B}_A(f)(g)]$$
$$= \mathcal{C}(x)(g \circ f) = g(f(x)) = \mathcal{C}(f(x))(g).$$

Therefore $\mathcal{C} \circ f = Z(\mathfrak{B}_A(f)) \circ \mathcal{C}$.

(2) This proof is analogous to part 1. Let $\ell \in \mathcal{L}$ and let $\theta \in Z(\mathcal{M})$. Then we have

$$[(\mathfrak{B}_A(Z(\phi)) \circ \rho)(\ell)](\theta) = [\mathfrak{B}_A(Z(\phi))(\rho(\ell))](\theta)$$
$$= [\rho(\ell) \circ Z(\phi)]\theta = \rho(\ell)(Z(\phi)(\theta))$$
$$= \rho(\ell)(\theta \circ \phi) = \theta \circ \phi(\ell) = \theta(\phi(\ell)) = \rho(\phi(\ell))(\theta).$$

Therefore $\rho \circ \phi = \mathfrak{B}_A(Z(\phi)) \circ \rho$.

(3) The uniform space $Z(\mathcal{L})$ is complete, so $\mathcal{C}_{Z(\mathcal{L})}$ is a uniform homeomorphism. It therefore suffices to show that $Z(\rho_{\mathcal{L}}) \circ \mathcal{C}_{Z(\mathcal{L})} \colon Z(\mathcal{L}) \to Z(\mathcal{L})$ is the identity map. Therefore let $\phi \colon \mathcal{L} \to \Omega(A)$ is a homomorphism and $\ell \in \mathcal{L}$. Then we have

$$[Z(\rho_{\mathcal{L}}) \circ \mathcal{C}_{Z(\mathcal{L})}(\phi)](\ell) = [Z(\rho_{\mathcal{L}})(\mathcal{C}_{Z(\mathcal{L})}(\phi))](\ell)$$
$$= [\mathcal{C}_{Z(\mathcal{L})}(\phi) \circ \rho_{\mathcal{L}}](\ell) = \mathcal{C}_{Z(\mathcal{L})}(\phi)(\rho_{\mathcal{L}}(\ell)) = \rho_{\mathcal{L}}(\ell)(\phi) = \phi(\ell).$$

We therefore conclude that $Z(\rho_{\mathcal{L}}) \circ \mathcal{C}_{Z(\mathcal{L})}$ is the identity map.

(4) This proof this analogous to 3. Since $\mathfrak{B}_A(X,\mathcal{U})$ is partitionable, we have $\rho_{\mathfrak{B}_A(X,\mathcal{U})}$ be an isomorphism. We therefore need to show that $\mathfrak{B}_A(\mathcal{C}_{(X,\mathcal{U})}) \circ \rho_{\mathfrak{B}_A(X,\mathcal{U})} \colon \mathfrak{B}_A(X,\mathcal{U}) \to \mathfrak{B}_A(X,\mathcal{U})$ is the identity map. Thus, assume that $f \in \mathfrak{B}_A(X,\mathcal{U})$ and $x \in X$. Then

$$[\mathfrak{B}_A(\mathcal{C}_{(X,\mathcal{U})}) \circ \rho_{\mathfrak{B}_A(X,\mathcal{U})}(f)](x) = [\mathfrak{B}_A(\mathcal{C}_{(X,\mathcal{U})})(\rho_{\mathfrak{B}_A(X,\mathcal{U})}(f))](x)$$
$$= (\rho_{\mathfrak{B}_A(X,\mathcal{U})}(f) \circ \mathcal{C}_{(X,\mathcal{U})})(x) = \rho_{\mathfrak{B}_A(X,\mathcal{U})}(f)(\mathcal{C}_{(X,\mathcal{U})}(x))$$
$$= \mathcal{C}_{(X,\mathcal{U})}(x)(f) = f(x).$$

Therefore $\mathfrak{B}_A(\mathcal{C}_{(X,\mathcal{U})}) \circ \phi_{\mathfrak{B}_A(X,\mathcal{U})}$ is the identity map.

□

2. A Characterization of non-Archimedean Supercomplete Spaces

A congruence θ on $\mathcal{L}$ is said to be *partitionable* if $\mathcal{L}/\theta$ is partitionable. Let $PC(\mathcal{L})$ denote the collection of all partitional congruences of $\mathcal{L}$. One can easily see that $PC(\mathcal{L})$ consists of all congruences of the form $\bigcap_{\theta \in R} \ker(\theta)$ where $R \subseteq Z(\mathcal{L})$.

Theorem 2.1. *Let $\mathcal{L} \in V(\Omega(A))$. Let $R \subseteq Z(\mathcal{L})$. Then let $\phi \in Z(\mathcal{L})$. Then $\phi \in \overline{R}$ if and only if $\bigcap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi)$.*

Proof. $\rightarrow$ Assume $\phi \in \overline{R}$. Also assume $\ell, \mathfrak{m} \in \mathcal{L}$ and $(\ell, \mathfrak{m}) \in \bigcap_{\theta \in R} \ker(\theta)$. Then $\theta(\ell) = \theta(\mathfrak{m})$ for $\theta \in R$. Since $\phi \in \overline{R}$, there is a $\theta \in R$ with $(\phi, \theta) \in \mathcal{E}_{\ell,\mathfrak{m}}$, so $\phi(\ell) = \theta(\ell) = \theta(\mathfrak{m}) = \phi(\mathfrak{m})$. Therefore $(\ell, \mathfrak{m}) \in \ker(\phi)$. We conclude that $\bigcap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi)$.

$\leftarrow$ Assume $\bigcap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi)$. Then let $\ell \in \mathcal{L}$ and assume $\phi(\ell) = a$. Let $b \in A$ be an element with $b \neq a$. Let $i \colon A \to A$ be the map where $i(a) = a$ and $i(c) = b$ for $c \neq a$. Then $\phi(\hat{i}^{\mathcal{L}}(\ell)) = i(\phi(\ell)) = i(a) = a \neq b = \phi(\hat{b}^{\mathcal{L}})$, so $(\hat{i}^{\mathcal{L}}(\ell), \hat{b}^{\mathcal{L}}) \notin \ker(\phi)$, hence $(\hat{i}^{\mathcal{L}}(\ell), \hat{b}^{\mathcal{L}}) \notin \ker(\theta)$ for some $\theta \in R$. Therefore $b = \theta(\hat{b}^{\mathcal{L}}) \neq \theta(\hat{i}^{\mathcal{L}}(\ell)) = i(\theta(\ell))$. Thus $\theta(\ell) = a = \phi(\ell)$. Therefore $(\phi, \theta) \in \mathcal{E}_{\ell}$. Since $\ell \in \mathcal{L}$ is arbitrary, we have $\phi \in \overline{R}$. □

We shall now give a Galois correspondence between closed sets in $Z(\mathcal{L})$ and partitionable congruences in $\mathcal{L}$. Let $f\colon P(\mathcal{L}^2) \to P(Z(\mathcal{L})), g\colon P(Z(\mathcal{L})) \to P(\mathcal{L}^2)$ be the mappings where

$$f(R) = \{\phi \in Z(\mathcal{L}) | (a,b) \in \ker(\phi) \text{ for all } (a,b) \in R\} = \{\phi \in Z(\mathcal{L}) | R \subseteq \ker(\phi)\}$$

and where

$$g(S) = \{(a,b) \in \mathcal{L}^2 | (a,b) \in \ker(\phi) \text{ for all } \phi \in S\} = \bigcap_{\phi \in S} \ker(\phi).$$

Let $C = g \circ f, D = f \circ g$. Then C and D are closure operators. In other words, we have $C(R) \subseteq C(C(R))$, and if $R \subseteq S$, then $C(R) \subseteq C(S)$ for $R, S \subseteq \mathcal{L}^2$. Let $C^* = \{R \subseteq \mathcal{L}^2 | C(R) = R\} = \{C(R) | R \subseteq \mathcal{L}^2\}$ and let $D^* = \{S \subseteq Z(\mathcal{L}) | D(S) = D\} = \{D(S) | S \subseteq Z(\mathcal{L})\}$. Let $f^*\colon C^* \to D^*, g^*\colon D^* \to C^*$ be the restriction of the functions f and g. Then the functions f^* and g^* are inverse functions.

Theorem 2.2. *The mapping D is the topological closure operator induced by the uniformity on $Z(\mathcal{L})$.*

Proof. Let $R \subseteq Z(\mathcal{L})$. Then

$$D(R) = f \circ g(R) = f(\bigcap_{\theta \in R} \ker(\theta)) = \{\phi \in Z(\mathcal{L}) | \bigcap_{\theta \in R} \ker(\theta) \subseteq \ker(\phi)\} = \overline{R}.$$

□

If $(X, \mathcal{U})$ is a uniform space, then let $H(X)$ be the collection of all closed subsets of X. Clearly $D^* = H(Z(\mathcal{L}))$ and $C^* = PC(\mathcal{L})$. Therefore we have $f^*\colon PC(\mathcal{L}) \to H(Z(\mathcal{L}))$ and $g^*\colon H(Z(\mathcal{L})) \to PC(\mathcal{L})$.

We shall now characterize the partitionable algebras $\mathcal{L}$ where $S(\mathcal{L})$ is supercomplete. For each $E \in \mathcal{U}$, let $\overline{E}$ be the binary relation on $H(X)$ where $(C, D) \in \overline{E}$ if and only if $C \subseteq E[D] = \{x \in X | (z, x) \in E \text{ for some } z \in D\}$ and $D \subseteq E[C]$. Then the relations $\overline{E}$ generate a uniformity on $H(X)$. Therefore $H(X)$ is a uniform space. With this uniformity, we shall call $H(X)$ the hyperspace of X. A separated uniform space X is said to be *supercomplete* if $H(X)$ is complete.

Take note that if $\mathcal{L}$ is an algebra and $\ell \in \mathcal{L}$, then we have $\phi \in \mathcal{E}_\ell[C]$ if and only if there is some $\theta \in C$ with $(\theta, \phi) \in \mathcal{E}_\ell$. In other words, $\phi \in \mathcal{E}_\ell[C]$ if and only if $\phi(\ell) \in \{\theta(\ell) | \theta \in C\}$. Therefore $(C, D) \in \overline{\mathcal{E}_\ell}$ if and only if $\{\theta(\ell) | \theta \in C\} = \{\phi(\ell) | \phi \in D\}$.

Exercise 2.3. *Every finitely generated algebra $\mathcal{L} \in V(\Omega(A))$ is generated by a single element.*

A *locally partitionable* congruence is a congruence θ on $\mathcal{L}$ so that whenever $\mathcal{M} \subseteq \mathcal{L}$ is a finitely generated subalgebra, we have $\theta \cap \mathcal{M}^2$ be a partitionable congruence.

Let $LPC(\mathcal{L})$ denote the set of all locally partitionable congruences on $\mathcal{L}$. $LPC(\mathcal{L})$ is closed under arbitrary intersection, so $LPC(\mathcal{L})$ is a complete lattice. Let $FS(\mathcal{L})$ be the collection of all finitely generated subalgebras of $\mathcal{L}$. We shall now give $LPC(\mathcal{L})$ a complete uniformity by representing $LPC(\mathcal{L})$ as an inverse limit.

If $\mathcal{M}, \mathcal{N}$ are finitely generated subalgebras of $\mathcal{L}$ and $\mathcal{M} \subseteq \mathcal{N}$, then define a function $E_{\mathcal{N},\mathcal{M}}\colon PC(\mathcal{N}) \to PC(\mathcal{M})$ by letting $E_{\mathcal{N},\mathcal{M}}(\theta) = \theta \cap \mathcal{M}^2$. One can easily show that $(PC(\mathcal{N}))_{\mathcal{N} \in FS(\mathcal{L})}$ is an inverse system of sets with transitional mappings $E_{\mathcal{N},\mathcal{M}}$. Let $IL(\mathcal{L})$ be the inverse limit $\overset{Lim}{\longleftarrow} PC(\mathcal{N})$. Give each $PC(\mathcal{N})$ the discrete uniformity and give $\overset{Lim}{\longleftarrow} PC(\mathcal{N})$ the inverse limit uniformity. Let $\mathcal{E}_{\mathcal{N}}$ be the equivalence relation on $IL(\mathcal{L})$ where we have $(\theta_{\mathcal{M}})_{\mathcal{M}\in FS(\mathcal{L})}, (\psi_{\mathcal{M}})_{\mathcal{M}\in FS(\mathcal{L})} \in \mathcal{E}_{\mathcal{N}}$ if and only if $\theta_{\mathcal{N}} = \psi_{\mathcal{N}}$. Then the equivalence relations $\mathcal{E}_{\mathcal{N}}$ generate the uniformity on $IL(\mathcal{L})$.

Let $\Gamma\colon LPC(\mathcal{L}) \to IL(\mathcal{L})$ be the mapping defined by letting $\Gamma(\theta) = (\theta \cap \mathcal{M}^2)_{\mathcal{M}\in FS(\mathcal{L})}$. Conversely, define a mapping $\Delta\colon IL(\mathcal{L}) \to LPC(\mathcal{L})$ be the mapping defined by $\Delta((\theta_{\mathcal{M}})_{\mathcal{M}\in FS(\mathcal{L})}) = \bigcup_{\mathcal{M}} \theta_{\mathcal{M}}$.

Exercise 2.4. *The functions Γ and Δ are inverses.*

Now give $LPC(\mathcal{L})$ the uniformity such that the maps Γ and Δ are uniform homeomorphisms. Now for each finitely generated subalgebra $\mathcal{N} \subseteq \mathcal{L}$, let $\mathcal{F}_{\mathcal{N}}$ be the equivalence relation on $LPC(\mathcal{L})$ where $(\theta, \psi) \in \mathcal{F}_{\mathcal{N}}$ if and only if $\theta \cap \mathcal{N}^2 = \psi \cap \mathcal{N}^2$. Clearly $(\theta, \psi) \in \mathcal{F}_{\mathcal{N}}$ if and only if $(\Gamma(\theta), \Gamma(\psi)) \in \mathcal{E}_{\mathcal{N}}$. Therefore the equivalence relations $\mathcal{F}_{\mathcal{N}}$ generate the uniformity on $LPC(\mathcal{L})$.

Exercise 2.5. *Let $(X,\mathcal{U})$ be a non-Archimedean uniform space. Let $\mathcal{N} \subseteq \mathfrak{B}_A(X,\mathcal{U})$ be a finitely generated subalgebra. Then there is a partition P such that if $r\colon X \to P$ is the function where $x \in r(x)$ for all $x \in X$, then $\mathcal{N} = \{f \circ r | f\colon P \to A\}$. Furthermore, if θ is a partitionable congruence on $\mathcal{N}$, then there is an $V \subseteq X$ where if $f, g \in \mathcal{N}$, then $(f, g) \in \theta$ if and only if $f(x) = g(x)$ for all $x \in V$.*

Theorem 2.6. *Let $\mathcal{L}$ be partitionable. Then $PC(\mathcal{L})$ is dense in $LPC(\mathcal{L})$.*

Proof. Since $\mathcal{L}$ is partitionable, we may assume that $\mathcal{L} = \mathfrak{B}_A(X,\mathcal{U})$ for some complete non-Archimedean $|A|^+$-totally bounded uniform space $(X,\mathcal{U})$. Let $\theta \in LPC(\mathcal{L})$ and assume that $\mathcal{N} \subseteq \mathfrak{B}_A(X,\mathcal{U})$ is finitely generated. Then there is a $V \subseteq X$ where for $f, g \in \mathcal{N}$, we have $(f, g) \in \theta$ if and only if $f(x) = g(x)$ for all $x \in V$. Now let $V^\sharp$ be the congruence in $\mathfrak{B}_A(X,\mathcal{U})$ where $(f, g) \in V^\sharp$ if and only if $f(x) = g(x)$ for $x \in V$. Then $V^\sharp$ is a partitionable congruence with $V^\sharp \cap \mathcal{N}^2 = \theta \cap \mathcal{N}^2$. Therefore $(V^\sharp, \theta) \in \mathcal{F}_{\mathcal{N}}$. We conclude that $PC(\mathcal{L})$ is dense in $LPC(\mathcal{L})$. □

Exercise 2.7. *Assume $a_i \in A$ for $i \in I$ and $b_j \in A$ for $j \in J$. Then $\{a_i | i \in I\} = \{b_j | j \in J\}$ if and only if for each pair of functions $f, g\colon A \to A$, we have $\forall i \in I, f(a_i) = g(a_i) \Leftrightarrow \forall j \in J, f(b_j) = g(b_j)$.*

Theorem 2.8. *The mappings $f^*\colon PC(\mathcal{L}) \to H(Z(\mathcal{L}))$ and $g^*\colon H(Z(\mathcal{L})) \to PC(\mathcal{L})$ are uniform homeomorphisms.*

Proof. We only need to show that g^* is a uniform homeomorphism. Since $Z(\mathcal{L})$ is generated by equivalence relations $\mathcal{E}_\ell$, the equivalence relations $\overline{\mathcal{E}_\ell}$ generate $H(Z(\mathcal{L}))$. We have $(C, D) \in \overline{\mathcal{E}_\ell}$ if and only if

$$\{\theta(\ell) | \theta \in C\} = \{\theta(\ell) | \theta \in D\}$$

if and only if for $f, g\colon A \to A$ we have

$$\forall \phi \in C, f(\phi(\ell)) = g(\phi(\ell)) \leftrightarrow \forall \phi \in D, f(\phi(\ell)) = g(\phi(\ell))$$

if and only if for each $f, g\colon A \to A$ we have

$$\forall \phi \in C, \phi(\hat{f}^{\mathcal{L}}(\ell))) = \phi(\hat{g}^{\mathcal{L}}(\ell)) \leftrightarrow \forall \phi \in D, \phi(\hat{f}^{\mathcal{L}}(\ell))) = \phi(\hat{g}^{\mathcal{L}}(\ell))$$

if and only if whenever $f, g\colon A \to A$ we have

$$(\hat{f}^{\mathcal{L}}(\ell), \hat{g}^{\mathcal{L}}(\ell)) \in \bigcap_{\phi \in C} \ker(\phi) \leftrightarrow (\hat{f}^{\mathcal{L}}(\ell), \hat{g}^{\mathcal{L}}(\ell)) \in \bigcap_{\phi \in D} \ker(\phi)$$

if and only if

$$g^*(C) \cap \langle \ell \rangle^2 = \bigcap_{\phi \in C} \ker(\phi) \cap \langle \ell \rangle^2 = \bigcap_{\phi \in D} \ker(\phi) \cap \langle \ell \rangle^2 = g^*(D) \cap \langle \ell \rangle^2$$

if and only if $(g^*(C), g^*(D)) \in \mathcal{F}_{\langle \ell \rangle}$. Therefore g^* is a uniform homeomorphism. □

Theorem 2.9. *Let $\mathcal{L}$ be a partitionable algebra. Then $Z(\mathcal{L})$ is supercomplete if and only if every locally partitionable congruence on $\mathcal{L}$ is partitionable.*

Proof. However, since $H(Z(\mathcal{L}))$ is uniformly homeomorphic to $PC(\mathcal{L})$, we have $H(Z(\mathcal{L}))$ be complete if and only if $PC(\mathcal{L})$ is complete. Since $PC(\mathcal{L})$ is a dense subspace of the complete space $LPC(\mathcal{L})$, we have $PC(\mathcal{L})$ be complete if and only if $PC(\mathcal{L}) = LPC(\mathcal{L})$ if and only if each locally partitionable congruence on $\mathcal{L}$ is partitionable. Therefore $Z(\mathcal{L})$ is supercomplete if and only if every locally partitionable congruence on $\mathcal{L}$ is partitionable. □

Exercise 2.10. *A partitionable algebra $\mathcal{L}$ is finitely generated if and only if $Z(\mathcal{L})$ is discrete. A partitionable algebra $\mathcal{L}$ is countably generated if and only if $Z(\mathcal{L})$ is uniformizable by a metric.*

We shall now prove a purely algebraic result using hyperspaces.

Corollary 2.11. *If $\mathcal{L}$ is a countably generated partitionable algebra, then every locally partitionable congruence is partitionable.*

Proof. If $\mathcal{L}$ is a countably generated partitionable algebra, then $Z(\mathcal{L})$ is uniformizable by a metric. However, in [2][p. 30], it is shown that every complete metric space is supercomplete. Therefore since $Z(\mathcal{L})$ is supercomplete, every locally partitionable congruence is partitionable. □

References

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Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Avenue Tampa, FL 33620, USA
E-mail address: jvanname@mail.usf.edu