Efficient Detection of Symmetries of Polynomially Parametrized Curves

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Abstract

We present efficient algorithms for detecting central and mirror symmetry for the case of algebraic curves defined by means of polynomial parametrizations. The algorithms are based on an algebraic relationship between proper parametrizations of a same curve, which leads to a triangular polynomial system that can be solved in a very fast way; in particular, curves parametrized by polynomials of serious degrees/coefficients can be analyzed in a few seconds. In our analysis we provide a good number of theoretical results on symmetries of polynomial curves, algorithms for detecting rotation and mirror symmetry, and closed formulae to determine the symmetry center and the symmetry axis, when they exist. Some observations and empiric results for the case of polynomial parametrizations with floating point coefficients are also reported.

1 Introduction

This paper deals with the problem of detecting the symmetries, if any, of a curve defined by means of a polynomial parametrization, i.e. a pair $(x(t), y(t))$ where both $x(t)$ and $y(t)$ are polynomials. The question of finding the symmetries of an algebraic curve has been previously investigated mainly because of its applications in pose estimation and patter recognition. Essentially, the problem is the following: a situation that is often studied in pattern recognition is how to choose, from a database of algebraic curves representing different objects, the one that best fits a given object, also represented by an algebraic equation; this question is addressed, among many others, in [4] (where the database simulates different aircraft prototypes), [7], [10], [13] (where the

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database corresponds to silhouettes of sea animals), [14] or [15]. In order to do this one has to detect if the curve to be recognized is in fact the result of applying an affine transformation (typically translations, rotations, etc.) to some curve in the database; in turn, this implies to set the curve in a "canonical position" that makes recognition possible. The problem of detecting symmetries comes along then as a means for setting properly the curve. In particular, this question has been addressed, for example, in [4], using splines, in [1], [2], [17], by means of differential invariants, in [5], [6], [13], using a complex representation of the implicit equation of the curve, or in [4], [11], [12], [16], using moments.

It is also worth observing that in most of these papers it is assumed that the curve is given by means of its implicit equation. Exceptions to this are [4], where the curve is assumed to be a spline (i.e. a union of pieces of polynomial parametrizations), and [1], [2], and other several papers on differential invariants, where the input is considered to be a parametrization without any restriction on its functional form. Furthermore, in almost all the papers on the question, the algorithm that is provided to detect symmetries is basically numerical, and therefore the output is approximate, and not exact. This is not a problem when the form to be recognized is, up to a certain extent, fuzzy, or when there are missing data, common situations in pattern recognition. But it is undesirable if the input is exact. Up to our knowledge, the exceptions to this are the papers on differential invariants ([1], [2], etc.), and [5], [6]. However, when applied to produce an exact output, the former can only deal with parametrizations of low degree (in fact, the ultimate idea in the papers on differential invariants is to adapt the underlying theory to a numerical framework). On the other hand, the latter, i.e. [5], [6], do provide deterministic algorithms for implicit algebraic curves $f(x, y) = 0$, and yield exact answers in an efficient and elegant manner (even for serious curves). The only limitation is that these algorithms require the absolute values of the leading coefficients with respect to the variables x, y of $f(x, y)$ to be different, which is a limitation in certain cases.

In this paper, we address the problem from a different perspective. On one hand, we focus on curves defined by means of polynomial parametrizations. Even if this seems too restrictive, polynomial parametrizations are widely used, for example, in Solid Modeling and Computer Aided Geometric Design, and stay at the core of the notion of spline curve, commonly used in many applications. Furthermore, the algorithms that we provide can be directly conducted from the parametrization, and therefore do not require to compute the implicit equation of the curve (costly or even impossible for high degrees); also, no condition on the leading coefficients of the curve is needed. Finally, we assume that our input is exact (the coefficients of the parametrization are required to be real numbers, not necessarily rational) and we provide also exact, i.e. deterministic, algorithms for checking whether the curve is symmetric or not, and for determining the elements of the symmetry, in the affirmative case.

The algorithms that we provide are very efficient and are able of analyzing curves with serious coefficients and/or high degrees in just a few seconds. The idea behind these algorithms comes from Computational Real Algebraic Geometry, and exploits an algebraic property linking two "good" (in a sense that is introduced in Section 2) parametrizations of a curve. By applying this property, the analysis of symmetries is reduced to solving polynomial systems which are triangular, and that can therefore be analyzed in a fast and efficient way. Additionally, the development of our results produce a number of theoretical results on the symmetries of polynomially parametrized curves, and closed formulae for the symmetry center and the symmetry axis, in the cases when they exist.

So, in the sequel we will analyze rotation and central symmetry (i.e. symmetry with respect to a point) in Section 2, and mirror symmetry (i.e. symmetry with respect to a line) in Section 3. In both cases we report examples and timings showing the efficiency of our algorithms. In Section 4 we report some observations and empiric results on polynomials with floating point coefficients, i.e. coefficients known up to a limited precision. We finish the paper with a section on conclusions and further work.

2 Rotation and Central Symmetry

Along the paper, we let $\mathcal{C}$ be an algebraic curve admitting a real polynomial parametrization $\varphi(t) = (x(t), y(t))$, i.e. $x(t), y(t) \in \mathbb{R}[t]$ (we will summarize this by saying that $\mathcal{C}$ is a *real polynomial curve*). Furthermore, we will also require $\varphi(t)$ to be *proper*, i.e. injective for almost all values of t. For example, (t, t^2) is a proper parametrization of a parabola, while (t^2, t^4) is not (notice that whenever t moves over the complex numbers, the latter parametrizes the parabola as well, and almost all the points of the parabola are generated by two different values of t). The properness of a parametrization is considered to be a good property, since non-proper parametrizations provide a "redundant" representation of the curve, that might cause problems when plotting the curve, intersecting it with other curves, or simply determining notable points of it. From Theorem 6.11 in [9], it is guaranteed that a polynomial curve can always be properly and polynomially reparametrized; more than that, we can ensure that in fact one can always find a proper polynomial parametrization over the reals. Indeed, if $\varphi(t) = (x(t), y(t))$ is polynomial but it is not proper, from the algorithm in page 193 of [9] we have that it can be properly reparametrized without extending the ground field; furthermore, if this reparametrization is not polynomial, then the algorithm in page 199 of [9] leads to a polynomial reparametrization, again without extending the ground field. So, in the rest of the paper, without loss of generality we will assume that $\varphi(t)$ is proper.

2.1 Rotation Symmetry and Prohibitions.

We say that $\mathcal{C}$ has rotation symmetry iff there exists a point $P_0 \in \mathbb{R}^2$ and an angle $\phi \in (0, 2\pi)$ such that the rotation of center P_0 and angle ϕ leaves $\mathcal{C}$ invariant. In the special case when $\phi = \pi$, we say that $\mathcal{C}$ has central symmetry, i.e. that $\mathcal{C}$ is symmetric with respect to P_0 (the center of symmetry). The following theorem proves that in our case, the only form of rotation symmetry that a polynomial curve can exhibit is central symmetry. Here we use the notion of *infinite branch* of a parametric curve: by this notion, we mean intuitively a part of $\mathcal{C}$ that goes to infinity; for example, the parabola (t, t^2) has, according to this terminology, two infinite branches, one for $t = \infty$ and another one for $t = -\infty$. In fact, every polynomial curve has only two infinite branches; however, rational curves may have more than two (because there may be t-values where the denominator of either $x(t)$ or $y(t)$ vanishes).

Theorem 1 *Real polynomial curves cannot have any other form of rotation symmetry other than central symmetry.*

Proof. If $\mathcal{C}$ would present rotation symmetry other than central symmetry then it would consist of the union of m different copies, with $m \geq 3$. Since each copy would provide at least one infinite branch, we would get at least m infinite branches. But since $m \geq 3$, and the number of infinite branches of $\mathcal{C}$ is 2, we reach a contradiction. □

Furthermore, the following result provides a necessary condition for $\mathcal{C}$ to have central symmetry.

Theorem 2 *Let $\mathcal{C}$ be a real polynomial curve, and let $\varphi(t) = (x(t), y(t))$ be a proper, real, polynomial parametrization of $\mathcal{C}$. If $\mathcal{C}$ has central symmetry, then $lim_{t\to\infty}x(t)$ and $lim_{t\to-\infty}x(t)$, $lim_{t\to\infty}y(t)$ and $lim_{t\to-\infty}y(t)$, exhibit different signs. In particular, $\deg_t(x(t))$ and $\deg_t(y(t))$ must be both odd.*

Proof. If $\mathcal{C}$ has central symmetry then it must exhibit two different infinite branches, that go to infinity in opposite quadrants. The rest follows easily. □

2.2 Detecting Central Symmetry

According to the preceding subsection, central symmetry is the only form of rotation symmetry that $\mathcal{C}$ can exhibit. So, let us address here the problem of detecting this kind of symmetry in an efficient way. For this purpose, previously we need the following result, that is crucial for us.

Lemma 3 *Let $\varphi_1(t)$ and $\varphi_2(t)$ be two polynomial and proper parametrizations of the same curve. Then there exists a linear function $L(t) = \alpha t + \beta$, with $\alpha, \beta \in \mathbb{R}$, such that $\varphi_1(t) = \varphi_2(\alpha t + \beta)$ and $\alpha \neq 0$.*

Proof. By Lemma 4.17 in [9], $\varphi_1(t)$ and $\varphi_2(t)$ are both proper parametrizations of $\mathcal{C}$ if and only if each one can be obtained from the other by applying a rational reparametrization $\xi(t) = \dfrac{\alpha t + \beta}{\gamma t + \delta}$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $\alpha\delta - \beta\gamma \neq 0$. However, since both $\varphi_1(t)$ and $\varphi_2(t)$ are polynomial, from Lemma 6.8 in [9] we deduce that the reparametrization $\xi(t)$ is in fact *linear*. Let us see that $\alpha, \beta \in \mathbb{R}$. Indeed, if $\alpha, \beta \in \mathbb{C}$ then since $\varphi_1(t) = \varphi_2(\alpha t + \beta)$ and $\varphi_1(t)$ is a real polynomial parametrization, we have that $\varphi_2(\overline{\alpha} + t\overline{\beta}) = \varphi_2(\alpha t + \beta)$. But then $\varphi_2(t)$ cannot be proper, because for every (real or complex) value of t, it holds that $\alpha t + \beta$ and $\overline{\alpha} t + \overline{\beta}$ generate the same point. Since $\varphi_2(t)$ is proper by hypothesis, then $\alpha, \beta \in \mathbb{R}$. Finally, since in this case $\delta = 1$ and $\gamma = 0$, then $\alpha\delta - \beta\gamma = \alpha \neq 0$. $\square$

Along the paper, it will be more convenient to write $\varphi(t)$ in complex form as

$$z(t) = x(t) + \mathbf{i} \cdot y(t)$$

Now $\mathcal{C}$ has central symmetry, with center of symmetry z_0 (z_0 corresponds to a point $P_0 = (x_0, y_0)$, written in complex form), if and only if for every t-value, $z^\star(t) = -z(t) + 2z_0$ is also a point of $\mathcal{C}$ (see Figure 1), i.e. if and only if $z^\star(t)$ is another parametrization of $\mathcal{C}$ (in complex form). Now one may check that $z^\star(t)$ also corresponds to a real polynomial parametrization; furthermore, since $z^\star(t)$ is the result of composing $\varphi(t)$ with a bijective planar transformation (the symmetry with respect to z_0), then it is also a proper parametrization. So, $\mathcal{C}$ has central symmetry if and only if $z(t)$ and $z^\star(t)$ are two *proper* parametrizations of the same curve. These observations, together with Lemma 3, give rise to the following theorem.

Theorem 4 *The curve $\mathcal{C}$ has central symmetry if and only if there exist $z_0 \in \mathbb{C}$ and a linear transformation $L(t) = \alpha t + \beta$, with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that $z^\star(t) = z(L(t))$.*

Now writing $z(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_0$, where $c_i \in \mathbb{C}$ for $i = 0, 1, \ldots, n$, and $L(t) = \alpha t + \beta$, with $\alpha, \beta \in \mathbb{R}$, the statement in Theorem 4 leads to the

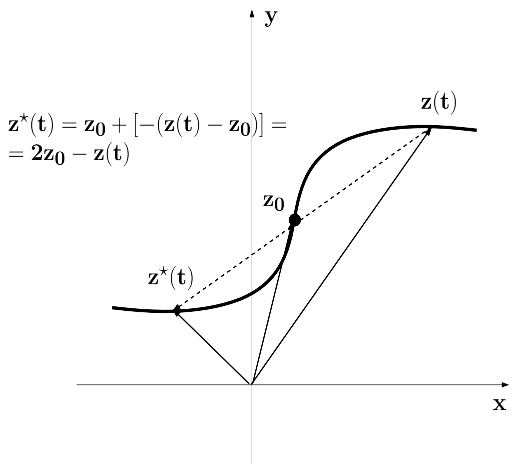

Fig. 1. Central Symmetry

following $(n+1)\times(n+1)$ polynomial system, that we denote as $\mathcal{S}$:

$$
\begin{array}{rcl}
-(c_0-z_0)+z_0 &=& c_0 + c_1\beta + c_2\beta^2 + \cdots + c_{n-1}\beta^{n-1} + c_n\beta^n \\
-c_1 &=& c_1\alpha + c_2\cdot 2\alpha\beta + \cdots + c_{n-1}(n-1)\alpha\beta^{n-2} + c_n n\alpha\beta^{n-1} \\
-c_2 &=& c_2\alpha^2 + \cdots + c_{n-1}\frac{(n-1)(n-2)}{2}\alpha^2\beta^{n-3} + c_n\frac{n(n-1)}{2}\alpha^2\beta^{n-2} \\
\vdots & & \qquad\qquad\vdots \qquad\qquad \vdots \\
-c_{n-1} &=& c_{n-1}\alpha^{n-1} + c_n n\alpha^{n-1}\beta \\
-c_n &=& c_n\alpha^n
\end{array}
$$

Since by Theorem 2 n must be odd and the case $n=1$ is trivial ($\mathcal{C}$ is a line), in the rest of the section we will assume that $n \geq 3$. Now we will refer to the above equations as $(0),(1),\ldots,(n)$, respectively (i.e. the equation (k) is the one containing α^k). Observe that Theorem 4 is equivalent to the existence of a solution (z_0,α,β), with $\alpha,\beta\in\mathbb{R}$, of the system $\mathcal{S}$. This system has a triangular structure that can be exploited for solving the problem in an efficient way: first of all, notice that from the last equation we have $\alpha^n=-1$, with n odd. So, the only possible real value of α is $\alpha=-1$. Additionally, from the equation $(n-1)$ we deduce that $\beta=\frac{-2c_{n-1}}{n\cdot c_n}$. Hence, if $\mathcal{C}$ is symmetric then this expression must yield a real number. So, the only possible value for (α,β) is $(-1,\frac{-2c_{n-1}}{n\cdot c_n})$. Finally, since the coefficient of z_0 in the equation (0) is -2, whenever $(-1,\frac{-2c_{n-1}}{n\cdot c_n})$ fulfill the equations $(1),\ldots,(n)$ we can solve for z_0 in the equation (0), and obtain the symmetry center. We summarize these ideas in the following theorem, where a closed expression for the symmetry center (in the case when it exists) is provided.

Theorem 5 *The curve $\mathcal{C}$ has central symmetry if and only if $\beta=\frac{-2c_{n-1}}{n\cdot c_n}\in\mathbb{R}$, and $\alpha=-1,\beta=\frac{-2c_{n-1}}{n\cdot c_n}$ satisfy the equations $(1),\ldots,(n)$ of the system $\mathcal{S}$.*

Moreover, in that case the center of symmetry is the point z_0 (in complex form) given by

$$z_0 = c_0 + \frac{1}{2}(c_1\beta + c_2\beta^2 + \cdots + c_n\beta^n)$$

Remark 1 *From Theorem 5.3 in [6], it follows that $\mathcal{C}$ can have at most just one center of symmetry. In our case, the existence of at most one symmetry center follows also from the above theorem.*

This theorem provides in a natural way an algorithm for checking central symmetry, and for determining the center of symmetry in the affirmative case. This is illustrated by the following example.

Example 1. *Consider the curve $\mathcal{C}$ parametrized by $(x(t), y(t))$, where*

$$\begin{cases} x(t) = 2 + 2(2t+1)^{23} - (2t+1)^{13} + 2(2t+1)^{11} + 2(2t+1)^5 - (2t+1)^3 + 2t \\ y(t) = -2(2t+1)^{23} + (2t+1)^{13} - 2(2t+1)^{11} + 2(2t+1)^5 - (2t+1)^3 + 2t \end{cases}$$

One can check that the curve is proper by applying Theorem 4.30 in [9]. In fact, it is perhaps instructive to mention that we have constructed this curve by considering first $\psi(t) = (2t^{23} - t^{13} + 2t^{11}, 2t^5 - t^3 + t)$, which is obviously symmetric with respect to the origin, then substituting $t := 2t+1$, and finally applying the change of coordinates $\{X = 1+x+y, Y = -1-x+y\}$, which is the composition of a rotation and a translation of vector $(-1, 1)$; as a consequence, we obtain a curve which is symmetric with respect to the point $(-1, 1)$. Now one can check that the implicit equation $f(x, y)$ of $\mathcal{C}$ has degree 23, that the infinity norm of this implicit equation is close to 2^{600}, and that the leading coefficients of f w.r.t. x and y are equal. This last circumstance makes it not possible to apply the method in [5]; nevertheless, the size of the implicit equation suggests that it would be difficult to work in implicit form, anyway. Using our method and Maple 14, running in a computer with 8 Gb of RAM and a CPU revving up to 2 GHz., it takes 0.374 seconds to construct the system $\mathcal{S}$ and check that $\alpha = -1$, $\beta = -1$, $z_0 = 1 - i$ is a solution. So, we recover $(1, -1)$ as the symmetry center of the curve.

In the case when $c_{n-1} = 0$ the analysis of the system $\mathcal{S}$ is easier. Indeed, in this case from the equation $(n-1)$ we deduce that the only possible value for β is $\beta = 0$. Hence, we have the following result.

Theorem 6 *Let $\mathcal{C}$ be a real polynomial curve and let $z(t) = c_nt^n + c_{n-1}t^{n-1} + \cdots + c_1t + c_0$ be a polynomial, proper, real parametrization of $\mathcal{C}$ in complex form. If $c_{n-1} = 0$, then the following statements hold:*

(i) If $n = 3$ (i.e. $\mathcal{C}$ is a cubic) then $\mathcal{C}$ is symmetric with respect to the point c_0.

(ii) If $n \geq 4$ then $\mathcal{C}$ has central symmetry if and only if $c_k = 0$ when $k \in \{1, \ldots, n-2\}$ is even. Moreover, in that case the symmetry center is c_0.

Proof. In the case of (i), one can check that $\alpha = -1, \beta = 0$ fulfill the equations (1), (2), (3) of the system $\mathcal{S}$. Then plugging $\alpha = -1, \beta = 0$ into (0) we get $z_0 = c_0$. In the case of (ii), when we substitute $\beta = 0$ in the equations $(1), \ldots, (n-2)$ we get $c_k \cdot (-1)^k + c_k = 0$ for $k = 1, \ldots, n-2$. When k is odd this equality is clearly fulfilled, and when k is even the equality holds iff $c_k = 0$. □

To end this section, we provide information, in the following table, on some of the examples that we have tried. One may observe from the timings given that we can check curves of high degrees, algebraic coefficients, serious norms, etc. in seconds. The degrees and norms that we spell below correspond to $\varphi(t)$ (not to the implicit equation, costly or even impossible to compute in some of these cases); furthermore, for each example we make explicit whether the curve shows central symmetry or not. The column of "norm" corresponds to the maximum of the infinity norms of the components of $\varphi(t)$. In all the cases we show the timing (in seconds) corresponding to the method suggested by Theorem 5, to provide evidence of its efficiency; nevertheless, in some examples, one can get a quicker response by detecting that β is not real (in which case this type of symmetry cannot exist) or by applying Theorem 6. The first column in the table below corresponds to the number of the example in our database.

Ex.	Deg.	Norm	Cent. Sym.	Timing	Comments
2	83	2^{81}	Yes	0.718	
3	3	15	Yes	0.343	Cubic with $c_2 = 0$ (see Th. 6)
4	7	15	Yes	0.437	Curve of degree ≥ 4 with $c_{n-1} = 0$ (see Th. 6)
37	21	2^{30}	No	0.468	$\beta \notin \mathbb{R}$: with this, 0.172 secs.
38	45	194	No	0.421	$\beta \notin \mathbb{R}$: with this, 0.109 secs.
39	35	$8.06 \cdot 10^9$	Yes	1.248	Algebraic coefficients.
40	77	83	No	0.500	$\beta = 0$ but $(-1, 0)$ is not a solution of the system.
41	95	$5.89 \cdot 10^{27}$	Yes	5.850	Algebraic coefficients.

3 Mirror Symmetry

3.1 Detecting Mirror Symmetry

Along this section, as in the preceding one, we consider a real polynomial curve $\mathcal{C}$ furnished with a proper parametrization written in complex form as $z(t) = x(t) + \mathbf{i} \cdot y(t)$. Also, we write the *conjugate* of $z(t)$ as $\overline{z(t)} = x(t) - \mathbf{i} \cdot y(t)$. Now we say that $\mathcal{C}$ has mirror symmetry if there exists an axis $\mathcal{L}$ (called the *symmetry axis*) such that $\mathcal{C}$ is symmetric with respect to $\mathcal{L}$. One can see that $\mathcal{C}$ has this type of symmetry if and only if there exists a movement M, composition of a translation and a rotation, such that $M(\mathcal{C})$ is symmetric with respect to the x-axis. Hence, $\mathcal{C}$ has mirror symmetry if and only there exist $z_0 \in \mathbb{C}$, and $\phi \in [0, 2\pi)$, such that $\tilde{z}(t) = (z(t) - z_0) \cdot e^{\mathbf{i} \cdot \phi}$ is, in complex form, the parametrization of a curve $\tilde{\mathcal{C}}$, symmetric with respect to the x-axis. Here, z_0 defines the translation, and ϕ defines the angle of rotation (see Figure 2); so, $\tilde{z}(t)$ is a parametrization of $M(\mathcal{C})$. Notice that if $\mathcal{L}$ is a symmetry axis of $\mathcal{C}$, then *any* point of $\mathcal{L}$ can be chosen to be z_0. Now since the symmetric of a complex number $a \in \mathbb{C}$ with respect to the x-axis is the conjugate $\overline{a}$, it follows that $\mathcal{C}$ exhibits mirror symmetry if and only if $w(t) = (z(t) - z_0) \cdot e^{\mathbf{i} \cdot \phi}$ and $\overline{w}(t) = \overline{(z(t) - z_0) \cdot e^{\mathbf{i} \cdot \phi}}$ both parametrize the same curve $\tilde{\mathcal{C}}$. Furthermore, if $z(t)$ corresponds to a proper parametrization then $w(t)$ and $\overline{w}(t)$ define also proper parametrizations of the corresponding curves (because these parametrizations are obtained by composing a bijective planar transformation with a proper parametrization). Taking also into account Lemma 3, the following theorem, analogous to Theorem 4 in Section 2, holds.

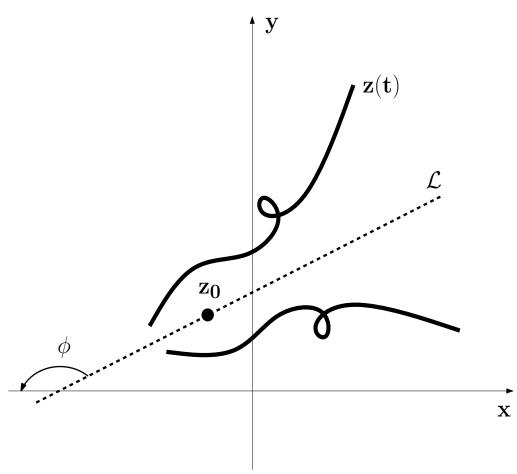

Fig. 2. Mirror Symmetry

Theorem 7 *The curve $\mathcal{C}$ exhibits mirror symmetry if and only if there exist a line $\mathcal{L}$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that for any $z_0 \in \mathcal{L}$, it holds that $\overline{w}(t) = w(\alpha t + \beta)$.*

Let us see how to take advantage of this theorem for efficiently checking, and compute in the affirmative case, the existence of mirror symmetry. We need first the following lemma.

Lemma 8 *A polynomial curve cannot have more than one symmetry axis.*

Proof. If $\mathcal{C}$ has m symmetry axes then by symmetry, each infinite branch gives us m infinite branches more. However, since $\mathcal{C}$ is polynomial then it has only two infinite branches, and therefore $m = 1$. $\square$

We will also need the following result, where we use the notation and terminology of Theorem 7.

Lemma 9 *If $\overline{w}(t) = w(\alpha t + \beta)$ for $\alpha, \beta \in \mathbb{R}$, then $\alpha = -1$.*

Proof. If $\overline{w}(t) = w(\alpha t + \beta)$ for $\alpha, \beta \in \mathbb{R}$, then by Theorem 7 the curve $\mathcal{C}$ is symmetric with respect to a certain axis, and $w(t)$ parametrizes a curve $\tilde{\mathcal{C}}$ which is symmetric with respect to the x-axis. Let us write $w(t) = (a_p t^p + a_{p-1}t^{p-1} + \cdots, b_q t^q + b_{q-1}t^{q-1} + \cdots)$. Because of the symmetry of $\tilde{\mathcal{C}}$ with respect to the x-axis, we have that q must be odd (so that the sign of $y(t)$ when t goes to ∞ or $-\infty$ is different). Now since $w(\alpha t + \beta) = \overline{w}(t)$, then

$$b_q \alpha^q t^q + \cdots = -b_q t^q + \cdots$$

Hence, $\alpha^q = -1$. Since q is odd, $b_q \neq 0$ and α is real, then we deduce that $\alpha = -1$. $\square$

Now taking into account Theorem 7 and the expression of $w(t)$, after some calculations we obtain the following system $\mathcal{W}$, similar to the system $\mathcal{S}$ in Section 2:

$$\begin{array}{rcl}
(\overline{c}_0 - \overline{z}_0)\cdot e^{-\mathbf{i}\cdot 2\phi} & = & (c_0 - z_0) + c_1\beta + c_2\beta^2 + \cdots + c_{n-1}\beta^{n-1} + c_n\beta^n \\
\overline{c}_1 \cdot e^{-\mathbf{i}\cdot 2\phi} & = & c_1\alpha + c_2\cdot 2\alpha\beta + \cdots + c_{n-1}(n-1)\alpha\beta^{n-2} + c_n n\alpha\beta^{n-1} \\
\overline{c}_2 \cdot e^{-\mathbf{i}\cdot 2\phi} & = & c_2\alpha^2 + \cdots + c_{n-1}\frac{(n-1)(n-2)}{2}\alpha^2\beta^{n-3} + c_n\frac{n(n-1)}{2}\alpha^2\beta^{n-2} \\
\vdots & & \vdots \\
\overline{c}_{n-1} \cdot e^{-\mathbf{i}\cdot 2\phi} & = & c_{n-1}\alpha^{n-1} + c_n n\alpha^{n-1}\beta \\
\overline{c}_n \cdot e^{-\mathbf{i}\cdot 2\phi} & = & c_n\alpha^n
\end{array}$$

We denote the above equations as $(0), (1), \ldots, (n)$, respectively (i.e. the equation (k) is the one containing the power α^k). Notice that the equation (0) is the only one containing z_0, and is linear in z_0. Furthermore, from Theorem 7 one can see that the existence of mirror symmetry is equivalent to the existence of $\phi \in [0, 2\pi)$, $\alpha, \beta \in \mathbb{R}$ such that: (i) these values fulfill the equations $(1), \ldots, (n)$; (ii) for these values, the equation (0) corresponds to a real line (which would be the symmetry axis $\mathcal{L}$). Now by Lemma 9, if $\mathcal{W}$ is consistent then $\alpha = -1$; also, by combining the last two equations we get that

$$\beta = \xi(\alpha) = \frac{c_n \bar{c}_{n-1} \alpha - \bar{c}_n c_{n-1}}{n|c_n|^2}$$

And since $\alpha = -1$, we get

$$\beta = \xi(-1) = -\frac{c_n \bar{c}_{n-1} + \bar{c}_n c_{n-1}}{n|c_n|^2}$$

In the numerator of this expression we recognize $c_n \bar{c}_{n-1} + \overline{c_n \bar{c}_{n-1}}$ (i.e. the sum of a complex number and its conjugate), which is a real number; so, $\beta \in \mathbb{R}$. Now in order to verify whether the equation (0) corresponds to a real line, the following results are useful. The first one can be proven in a straightforward way.

Lemma 10 *Any real line $Ax + By + C = 0$ is transformed by means of the complex change $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$ into $\overline{\gamma} z + \gamma \bar{z} + C = 0$, where $\gamma = \frac{A}{2} + i\frac{B}{2}$. Conversely, any equality $\overline{\gamma} z + \gamma \bar{z} + C = 0$ with C real corresponds to a real line.*

The above lemma is used for proving the following result. Here, we denote $Q(\beta) = c_1 \beta + c_2 \beta^2 + \cdots + c_n \beta^n$, and

$$Q^\star(\beta) = Q(\beta) \cdot \frac{\bar{c}_n}{|c_n|} \cdot \sqrt{(-1)^n}$$

Proposition 11 *The equation* (0) *of the system $\mathcal{W}$ corresponds to a real line if and only if $Q^\star(\beta) \in \mathbb{R}$.*

Proof. After substituting $-e^{-\mathbf{i}2\phi} = e^{\mathbf{i}(\pi - 2\phi)} = e^{\mathbf{i}2(\frac{\pi}{2} - \phi)}$ in the equation (0), and dividing the whole equation by $e^{\mathbf{i}(\frac{\pi}{2} - \phi)}$, we get

$$e^{-\mathbf{i}(\pi/2 - \phi)} z_0 + e^{\mathbf{i}(\frac{\pi}{2} - \phi)} \bar{z}_0 = \bar{c}_0 \cdot e^{\mathbf{i}(\frac{\pi}{2} - \phi)} + c_0 \cdot e^{-\mathbf{i}(\frac{\pi}{2} - \phi)} + Q(\beta) \cdot e^{-\mathbf{i}(\frac{\pi}{2} - \phi)}$$

Now from Lemma 10, and taking into account that the conjugate of $e^{-\mathbf{i}(\pi/2 - \phi)}$ is $e^{\mathbf{i}(\pi/2 - \phi)}$, it holds that the above expression corresponds to a real line iff the right hand-side of the equation is a real number. However, one may see that

$$\bar{c}_0 \cdot e^{\mathbf{i}(\frac{\pi}{2} - \phi)} + c_0 \cdot e^{-\mathbf{i}(\frac{\pi}{2} - \phi)}$$

is already real, because it is the sum of a complex number, and its conjugate. So, we get a real line iff $Q(\beta)\cdot e^{-\mathbf{i}(\frac{\pi}{2}-\phi)}$ is a real number. Finally, using that $-e^{-\mathbf{i}2\phi}=e^{\mathbf{i}2(\frac{\pi}{2}-\phi)}$ and the equation (n), one has that

$$e^{-\mathbf{i}(\frac{\pi}{2}-\phi)}=\frac{1}{e^{\mathbf{i}(\frac{\pi}{2}-\phi)}}=\frac{1}{\sqrt{-e^{-\mathbf{i}2\phi}}}=\frac{1}{\sqrt{\frac{-c_n(-1)^n}{\bar{c}_n}}}=\sqrt{\frac{-\bar{c}_n}{c_n(-1)^n}}$$

Multiplying and dividing by $\bar{c}_n$ into the square root, the result follows easily.
□

So, we can put together all these observations to characterize the existence of mirror symmetry. For this purpose, it is preferable to consider a new system $\mathcal{W}'$, which is obtained from $\mathcal{W}$ by substituting

$$e^{-\mathbf{i}\cdot 2\phi}=\frac{c_n}{\bar{c}_n}\alpha^n$$

(from the last equation) into the n first equations, and dividing out each equation (k) by α^k:

$$\begin{array}{lllllll}
(\bar{c}_0-\bar{z}_0)c_n\cdot\alpha^n = & \bar{c}_n\cdot[(c_0-z_0)+ & c_1\beta & + c_2\beta^2 & +\cdots+ & c_{n-1}\beta^{n-1} & + \; c_n\beta^n] \\
\bar{c}_1c_n\cdot\alpha^{n-1} = & & \bar{c}_n\cdot[c_1+ & 2c_2\beta & +\cdots+ & c_{n-1}(n-1)\beta^{n-2} & + \; c_n n\beta^{n-1}] \\
\bar{c}_2c_n\cdot\alpha^{n-2} = & & & \bar{c}_n\cdot[c_2 & +\cdots+ & c_{n-1}\frac{(n-1)(n-2)}{2}\beta^{n-3} & + \; c_n\frac{n(n-1)}{2}\beta^{n-2}] \\
\quad\vdots & & & & & \vdots & \vdots \\
\bar{c}_{n-1}c_n\cdot\alpha = & & & & & \bar{c}_n\cdot[c_{n-1} & + \; c_n n\beta]
\end{array}$$

We denote these equations as $[0],[1],\ldots,[n-1]$, respectively. Furthermore, we write the equation $[0]$ as $r(z_0,\bar{z}_0,\beta)=0$. Then the following result, which can be deduced from the results in this section, holds. Observe that this result provides a characterization for the existence of mirror symmetry, and a closed expression for the symmetry axis, when it exists.

Theorem 12 *The curve $\mathcal{C}$ has mirror symmetry if and only if the following two conditions hold: (1) $\alpha=-1$, $\beta=\xi(-1)$ fulfill the equations $[1],\ldots,[n-1]$; (2) $Q^\star(\beta)\in\mathbb{R}$. Moreover, in that case the symmetry axis of $\mathcal{C}$ (in complex form) is $r(z,\bar{z},\beta)$, i.e.*

$$\bar{c}_nz-c_n(-1)^n\bar{z}+\bar{c}_0c_n(-1)^n-\bar{c}_nc_0-(c_1\beta+c_2\beta^2+\cdots+c_n\beta^n)=0$$

Remark 2 *One can check that the conditions (1) and (2) are independent, in general. Take for instance a cubic parametrized by $z(t)=t^3+c_2t^2+c_1t+c_0$, i.e. $c_3=1$. In this case, one can check that $\alpha=-1$ and $\beta=\xi(-1)=$*

$-\dfrac{1}{3}(c_2+\bar{c}_2)$ *fulfill the equations [1], [2] iff* $\bar{c}_1 = c_1 - \dfrac{1}{3}c_2^2 + \dfrac{1}{3}\bar{c}_2^2$*. However,* $Q^\star(\beta) = -\dfrac{i}{27}(c_2+\bar{c}_2)\cdot(9c_1 - 2c_2^2 - c_2\bar{c}_2 + \bar{c}_2^2)$*. So, for example* $c_1 = 1+2/3i$*,* $c_2 = 1+i$*,* $c_3 = 1$ *satisfy [1], [2] but the value of* $Q^\star(\beta)$ *in that case is* $-14i/27$*, which is not real.*

Remark 3 *One may see that if $\mathcal{C}$ has a symmetry axis $\mathcal{L}$, then both must intersect. Indeed, since $\mathcal{C}$ is symmetric with respect to $\mathcal{L}$, if it has a point in the half-plane over $\mathcal{L}$ then it must have also another point in the half-plane below $\mathcal{L}$. But since $\mathcal{C}$ is a continuous curve (i.e. there is no t-value where it becomes infinite), both points must be connected, and therefore the symmetry axis must be crossed.*

Theorem 12 provides also the following result on the existence of symmetries of $\mathcal{C}$.

Proposition 13 *The following statements are true:*

(1) If $\bar{c}_n - (-1)^n c_n = 0$, and $\mathcal{C}$ has a symmetry axis, then it is parallel to the x-axis.
(2) If $(-1)^n c_n + \bar{c}_n = 0$, and $\mathcal{C}$ has a symmetry axis, then it is parallel to the y-axis.

Proof. Let us see (1). For this purpose, assume that $\mathcal{L}$ is a symmetry axis for $\mathcal{C}$. The intersection of $\mathcal{L}$ with the x-axis can be found by substituting $\bar{z}_0 = z_0$ in the closed expression for z_0 provided in the statement of Theorem 12. This substitution yields $(\bar{c}_n - (-1)^n c_n)z_0 + \cdots = 0$ (where $\cdots$ comprises terms where z_0 is not present), and hence when $\bar{c}_n - (-1)^n c_n = 0$ no value for z_0 can be found. So, (1) follows. The statement (2) is proven in a similar way but imposing $\bar{z}_0 = -z_0$, instead of $\bar{z}_0 = z_0$. □

The above proposition can be used for proving the following, clearer, result on the existence of symmetries. Here we denote $\deg_t(x(t)) = r$, $\deg_t(y(t)) = s$.

Theorem 14 *The following statements hold:*

(i) If $r > s$ and r is even, and $\mathcal{C}$ has a symmetry axis, then it is parallel to the x-axis.
(ii) If $r > s$ and r is odd, and $\mathcal{C}$ has a symmetry axis, then it is parallel to the y-axis.
(iii) If $r < s$ and s is odd, and $\mathcal{C}$ has a symmetry axis, then it is parallel to the x-axis.
(iv) If $r < s$ and s is even, and $\mathcal{C}$ has a symmetry axis, then it is parallel to the y-axis.

Proof. If $r > s$ then $n = r$, and c_n is real, i.e. $\bar{c}_n = c_n$. Now if n, i.e. r, is even we are in the hypotheses of statement (i), and the result follows from the statement (1) of Proposition 13; if n, i.e. r, is odd we we are in the hypotheses of statement (ii), which follows from the statement (2) of Proposition 13. Finally if $r < s$ then $n = s$, and c_n is a purely imaginary number, i.e. $\bar{c}_n = -c_n$; then we argue as before. □

Theorem 12 leads to an algorithm for computing the symmetry axis of $\mathcal{C}$ (if any). This algorithm is illustrated in the following example.

Example 2. *Consider the curve $\mathcal{C}$ parametrized by $(x(t), y(t))$, where*

$$\begin{cases} x(t) = (2t+1)^{20} + (2t+1)^{18} + (2t+1)^{10} + 1 + (2t+1)^{21} - 3(2t+1)^5 + (2t+1)^3 \\ y(t) = -(2t+1)^{20} - (2t+1)^{18} - (2t+1)^{10} - 1 + (2t+1)^{21} - 3(2t+1)^5 + (2t+1)^3 \end{cases}$$

This curve is proper, and has been constructed starting from a simpler curve, then applying a change of parameters and finally rotating the curve $\frac{\pi}{4}$ radians. One can check that the implicit equation $f(x, y)$ of $\mathcal{C}$ has degree 21, that the infinity norm of this implicit equation is close to 2^{500}, and that the absolute values of the leading coefficients of f w.r.t. x and y are equal and of size 2^{441}. Using our method, it takes 1.388 seconds to construct the system $\mathcal{W}'$ and check that:

- $\alpha = -1$, $\beta = \xi(-1)$, *fulfill the equations* $[1], \ldots, [n-1]$, *and* $Q^\star(\beta) = 2\sqrt{2} \in \mathbb{R}$. *So, from Theorem 12, $\mathcal{C}$ has one symmetry axis.*
- *The equation of the symmetry axis is obtained by substituting the above values for α, β in the equation [0], which yields $iz - \bar{z} = 0$, i.e. $y = x$.*

As in the preceding section, in the case $c_{n-1} = 0$ we can provide a sharper result. In this case $\beta = 0$ and therefore $Q^\star(\beta) = 0$; so, the second condition in Theorem 12 always holds. In addition to this, the equations of $\mathcal{W}'$ have the form $m_k(\alpha) = c_n \bar{c}_k \alpha^k - \bar{c}_n c_k = 0$ for $k = 1, \ldots, n-2$. So, when $\alpha = -1$ we have

$$m_k(\alpha) = \begin{cases} c_n\bar{c}_k - \overline{c_n\bar{c}_k} & \text{if k is even} \\ -(c_n\bar{c}_k + \overline{c_n\bar{c}_k}) & \text{if k is odd} \end{cases}$$

Hence, the next result, which essentially follows from the above expression, follows.

Theorem 15 *Assume that $c_{n-1} = 0$. Then $\mathcal{C}$ presents mirror symmetry if and only if one of the following two conditions hold: (i) for every $k = 1, \ldots, n-2$, $c_n \cdot \bar{c}_k \in \mathbb{R}$; (ii) for every $k \in \{1, \ldots, n-2\}$, $c_n \cdot \bar{c}_k \in \mathbb{R}$ when k is even, and*

$c_n \cdot \bar{c}_k$ is a pure imaginary number (i.e. with null real part) when k is odd.

3.2 Some More Prohibitions and Observations

If $\mathcal{C}$ exhibits mirror symmetry and $\mathcal{L}$ is a symmetry axis, then almost all elements in the family of lines $Ax + By + C = 0$ normal to $\mathcal{L}$ must intersect $\mathcal{C}$ at an even number of points (the finitely many exceptions will be those lines that contain some point of $\mathcal{C} \cap \mathcal{L}$). We can exploit this for deducing the non-existence of mirror symmetry in some cases. In order to do this, we need the following previous result.

Lemma 16 *Let $p(t), q(t)$ be two real polynomials, and let $A, B, C \in \mathbb{R}$. Also, let $R_{A,B,C}(t) = Ap(t) + Bq(t) + C$, and let $R'_{A,B,C}(t)$ denote its derivative with respect to t. Then there does not exist $(A_0, B_0) \in \mathbb{R}^2$ such that $R_{A_0,B_0,C}(t)$ has multiple roots for almost all $C \in \mathbb{R}$.*

Proof. For a particular value of $C \in \mathbb{R}$, $R_{A_0,B_0,C}(t)$ has multiple roots iff $R_{A_0,B_0,C}(t) = A_0p(t) + B_0q(t) + C$ and $R'_{A_0,B_0,C}(t) = A_0p'(t) + B_0q'(t)$ have a common factor. Moreover, $R_{A_0,B_0,C}(t)$ has multiple roots for almost all $C \in \mathbb{R}$ iff the former polynomials have a common factor regardless of the value of C. But this cannot happen because the second polynomial does not depend on C. □

Then we have the following result.

Theorem 17 *Let $r = \deg_t(x(t))$, $s = \deg_t(y(t))$. The following statements hold:*

(1) If r (resp. s) is odd, then $\mathcal{C}$ cannot have any symmetry axis parallel to the x-axis (resp. y-axis).
(2) If $r = s$ and r, s are odd, then $\mathcal{C}$ has at most one symmetry axis $Ax + By + C = 0$ parallel to the vector (a_r, a_s), where a_r, a_s are the leading coefficients of $x(t), y(t)$, respectively.

Proof. Let us see (1). If $\mathcal{C}$ has a symmetry axis parallel to the x-axis (resp. the y-axis), then the family of lines $x = a$ (resp. $y = b$) must intersect $\mathcal{C}$ at an even number of points, for almost all $a \in \mathbb{R}$ (resp. $b \in \mathbb{R}$); however, if r (resp. s) is odd this cannot happen. Now in order to see (2), one observes that if $\mathcal{C}$ has a symmetry axis $\mathcal{L}$, then the family of normal lines to $\mathcal{L}$ has the form $A_0x + B_0y + C = 0$ for a certain $(A_0, B_0) \in \mathbb{R}^2$, where C is a real parameter. The intersection of this family with $\mathcal{C}$ amounts to $Ax(t) + By(t) + C = 0$, and this equation must have an even number of solutions for almost all $C \in \mathbb{R}$. Now assume that the degree of this equation is odd, and let δ stand for the number of real solutions of the equation; in that case, either $A_0a_r + B_0a_s = 0$, or the equation has multiple solutions. However, by Lemma 16 the latter cannot happen for almost all values of C, and (2) follows. $\square$

Example 2, in the subsection 2, provides an example for the statement (2) of this theorem. Now by putting together Theorem 14 and Theorem 17, we get the following corollary of Theorem 17.

Corollary 18 *If both $r = \deg_t(x(t))$ and $s = \deg_t(y(t))$ are odd, and $r \neq s$, then $\mathcal{C}$ does not exhibit mirror symmetry.*

The results in Theorem 14, Theorem 17 and Corollary 18 are displayed in the table below, where the notation of Theorem 17 is maintained. Observe that the results in the preceding sections do not say anything for the case when both $x(t)$ and $y(t)$ have equal and even degrees.

r odd, **s** odd	$\mathbf{r} < \mathbf{s}$:	No mirror symmetry
	$\mathbf{r} = \mathbf{s}$:	Axis parallel to (a_r, a_s), if any.
	$\mathbf{r} > \mathbf{s}$	No mirror symmetry
r odd, **s** even	Axis parallel to y-axis, if any.	
r even, **s** odd	Axis parallel to x-axis, if any.	
r even, **s** even	$\mathbf{r} < \mathbf{s}$:	Axis parallel to y-axis, if any.
	$\mathbf{r} = \mathbf{s}$:	(nothing to say)
	$\mathbf{r} > \mathbf{s}$:	Axis parallel to x-axis, if any.

Finally, one might wonder whether central symmetry and mirror symmetry can coincide at the same time. Let us see that the answer, in our case, is negative. For this purpose, we first need the following result.

Proposition 19 *Let $\mathcal{C}$ be an algebraic curve, and assume that it presents rotation symmetry, with center P_0 and angle θ. If $\mathcal{L}$ is an axis of symmetry of $\mathcal{C}$, then $P_0 \in \mathcal{L}$.*

Proof. Assume by contradiction that $P_0 \notin \mathcal{L}$, and let $P_0' \neq P_0$ be the symmetric of P_0 w.r.t. $\mathcal{L}$. Furthermore, let $\tilde{\mathcal{C}}$ be the symmetric of $\mathcal{C}$ w.r.t. $\mathcal{L}$. Since $\mathcal{C}$ is by hypothesis symmetric w.r.t. $\mathcal{L}$, we have $\tilde{\mathcal{C}} = \mathcal{C}$. Hence, a rotation of $\mathcal{C}$ around P_0 with angle θ amounts to a rotation of $\tilde{\mathcal{C}}$ around P_0' with angle $-\theta$. But since $\tilde{\mathcal{C}} = \mathcal{C}$, it holds that $\mathcal{C}$ has two symmetry centers, namely P_0 and P_0'. However this cannot happen because $\mathcal{C}$ is algebraic, and therefore it can have just one symmetry center (see Theorem 5.3 in [6]). So, $P_0 \in \mathcal{L}$. $\square$

Now we can state the result.

Theorem 20 *Polynomial curves cannot exhibit central symmetry and mirror symmetry at the same time.*

Proof. If $\mathcal{C}$ has a center of symmetry P_0 and a symmetry axis $\mathcal{L}$, then by Proposition 19 it holds that $P_0 \in \mathcal{L}$. In that case, it is easy to see that $\mathcal{C}$ must also have another symmetry axis $\mathcal{L}'$, namely the normal line to $\mathcal{L}$ at P_0. But by Lemma 8, this is not possible. $\square$

3.3 More Examples

We finish this section with a table showing data on some of the examples that we have tried. From this table, and as it also happens in the case of central symmetry, we can see that the method provided here can deal efficiently with serious inputs. In all the cases we provide the timing (in secs.) of the algorithm deriving from Theorem 12; however, under certain circumstances (i.e. when $Q^\star(\beta) \notin \mathbb{R}$, when $c_{n-1} = 0$, or in some cases corresponding to the table in Subsection 3.2) we can use some of the results in the preceding subsections to derive an answer faster.

Ex.	Degree	Norm	Mirr. Sym.	Timing	Comments
6	21	2^{31}	Yes	1.388	
7	21	3.5	Yes	0.983	Algebraic coefficients.
8	69	276	No	3.229	Case $c_{n-1} = 0$ (see Th. 15; with this, 0.203 secs.)
9	45	461	No	1.653	Case $c_{n-1} = 0$ (see Th. 15; with this, 0.203 secs.)
10	45	194	No	1.653	
11	60	844	Yes	2.465	r even, s odd (see Th. 14)
12	91	2^{89}	Yes	15.803	
42	77	2^{71}	No	2.433	$Q^\star(\beta) \notin \mathbb{R}$; with this, 2.293 secs.
43	35	2^{38}	No	2.901	r, s odd (see Th. 14; with this, 0 secs.)
44	56	2^{52}	Yes	23.385	Algebraic coefficients.

4 The Floating Point Case

In the preceding sections, we were assuming that the input was given exactly. However, it can happen that the coefficients are known only up to a certain precision ϵ, that can be understood as the number of correct decimal positions of the coefficients. We discuss this situation here. The overall idea is that while in the exact case one first tests whether there exists a certain type of symmetry, and one computes the symmetry center or the symmetry axis when the answer is affirmative, here it is better to proceed just the converse way: one uses the results in the sections before to provide the *temptative* symmetry center, or symmetry axis, and afterwards one evaluates whether the curve behaves in approximately a symmetric way with respect to the computed point/line.

4.1 Central Symmetry

One can use the closed formula provided by Theorem 5 also in this case. However, the inaccuracy of the coefficients causes the estimation of the symmetry center by this formula to be also inaccurate. In order to measure this inaccuracy, we can differentiate the formula for z_0 provided in Theorem 5, and interpret the norm of the differential as a measure for the error the value z_0 is known with. Hence, we have that

$$dz_0 = dc_0 + \frac{1}{2}(dc_1\beta + dc_2\beta^2 + \cdots + dc_n\beta^n) + \frac{1}{2}(c_1 + 2c_2\beta + \cdots + nc_n\beta^{n-1})d\beta$$

Furthermore, $d\beta = \frac{-2}{n} \cdot \frac{dc_{n-1}\cdot c_n - c_{n-1}\cdot dc_n}{c_n^2}$. Now, let $|c|$ denote the infinity norm of c; then we have that $|c_i| \leq |c|$ for all $i = 0, 1, \ldots, n$. Also, $|dc_i| \leq \epsilon$ for $i = 1, \ldots, n$. Hence, substituting the expression for $d\beta$ in the above expression, taking norms, and making use of usual properties on norms, we have that

$$|dz_0| \leq \epsilon \cdot \left[1 + \left(\frac{3}{2}n + 1\right)\max\{1, |\beta|^n\}\right]$$

Notice here that $|\beta| = \left(\frac{2|c_{n-1}/c_n|}{n}\right)^n$, and hence for a fixed value of $|c_{n-1}/c_n|$ the above bound grows linearly as n goes to ∞. This bound, however, has two drawbacks: (i) it can be small although central symmetry is not present; (ii) it can be slightly high even when central symmetry is present. So, in practice it is only useful when it is small, and one knows that the curve has this kind of symmetry. An alternative possibility for evaluating central symmetry is to take into account the test provided by Theorem 5, i.e. checking whether the equations $(1), \ldots, (k)$ of the system $\mathcal{S}$ (see Subsection 2.2) are fulfilled. These equations correspond to polynomial functions $f_k(c_0, c_1, \ldots, c_n, \alpha, \beta)$, that in presence of symmetry should vanish when evaluated at $\alpha = -1, \beta = \frac{-2\beta_{n-1}}{n\beta}$ and the coefficients of the parametrization. In our case, because of the limited precision the c_i's are known with, even in presence of approximate central symmetry these functions do not evaluate as 0. However, experimentation shows (see later) that when the curve exhibits central symmetry these values tend to be small (whenever the parametrization is not too big, in norm or degree). One can prove (by proceeding as before) an upper bound $\mathcal{O}(\epsilon \cdot 2^n \cdot n \cdot \max\{1, |\beta|^n\})$ for the inaccuracy of these evaluations, but in our experiments we have observed that this bound is generally too high. A measure like $L = \max\{|\tilde{f}_1|, \ldots, |\tilde{f}_n|\}$, where the $\tilde{f}_k$'s stand for the evaluations of the f_k's, gives a better idea. Another possibility is to evaluate the central symmetry with respect to z_0 in a statistical fashion. For this purpose, we observe that from the results of Subsection 2.2, the symmetric of $\varphi(t)$ with respect to the point z_0 (if this is really a symmetry center) is $\varphi(-t+\beta)$. So, we generate 400 t-values, giving rise to points $P_t = \varphi(t)$'s, and we compute the $\tilde{P}_t = \varphi(-t+\beta)$'s; then for each t-value we compute the difference between the distance $d(P_t, \tilde{P}_t)$, and the value of $2 \cdot d(P_t, z_0)$ (i.e. twice the distance $d(P_t, z_0)$). We represent this difference by δP_t, and its relative value (expressed as a percentage) by $\delta \hat{P}_t$, i.e. $\delta \hat{P}_t = \frac{\delta P_t}{d(P_t, z_0)} \cdot 100$. Finally, the arithmetic mean M (after taking out the outliers) of these values measures whether z_0 is really a symmetry center, or not.

In the table below we provide some information concerning 10 of the examples that we have tried. In each case, we spell: the number of the example, the degree of the parametrization (i.e. the maximum power of t appearing in it), the infinity norm of the parametrization, the precision (i.e. the accuracy of the coefficients), and the values of the three above parameters: $|dz_0|$, L, M. Furthermore, in each case we have explored the curve graphically and we have confirmed the existence or not of approximate symmetry,

Ex.	Degree	Norm	Prec.	Symm.	$\|dz_0\|$	L	M
16	13	1964.14	10^{-6}	Yes	0.033	0.001	0.0026
17	5	28.41	10^{-3}	Yes	0.273	$2.23 \cdot 10^{-7}$	$3.35 \cdot 10^{-7}$
18	9	3.0006	10^{-4}	No	0.015	42.82	9.18
19	29	3.0006	10^{-4}	No	0.0045	14.85	9.70
20	29	10.002	10^{-4}	No	0.0045	26.742	12.94
21	25	3.002	10^{-4}	Yes	0.0039	0.	$5.40 \cdot 10^{-8}$
22	39	14.0028	10^{-4}	No	0.0061	31.25	21.47
23	57	182.73	10^{-4}	Yes	0.00875	0.0076	$1.75 \cdot 10^{-4}$
24a	53	2^{50}	10^{-6}	Yes	$6.11 \cdot 10^{12}$	$1.8 \cdot 10^{11}$	$4.11 \cdot 10^{15}$
24b	53	2^{50}	10^{-16}	Yes	611.28	$7.44 \cdot 10^{-15}$	10^{-4}

It is interesting to observe that in examples like 16 or 24b, the bound for $|dz_0|$ works worse than L or M. On the other hand, one may notice several examples (18, 19, 20, 22) where $|dz_0|$ is small, although there is not central symmetry.

Also, the example 24a shows a curve with very high coefficients, that we have approximated with precision 10^{-6}; even though the curve is clearly symmetric (it is a small perturbation of a symmetric curve with algebraic coefficients), the three parameters were very big and the symmetry was not recognized by any of them. We needed to use much higher precision (10^{-16}) to recognize the symmetry (see Example 24b).

4.2 Mirror Symmetry

In the exact case, one computes the symmetry axis $\mathcal{L}$ by using the (0) equation of the system $\mathcal{W}'$. However, in the approximate case $\mathcal{L}$ is better determined by separately computing ϕ, and a point P_0 of $\mathcal{L}$, written as z_0 in complex form. Unless the axis is parallel to the x-axis (or $\bar{c}_n - c_n(-1)^n$ is very small in norm) we choose P_0 to be the intersection of $\mathcal{L}$ with the x-axis (in complex form, this will correspond to a real number, therefore satisfying $\bar{z}_0 = z_0$); if $\mathcal{L}$ is parallel to the x-axis (or $\bar{c}_n - c_n(-1)^n$ is small), we will compute instead the intersection with the y-axis (a purely imaginary number, hence fulfilling $\bar{z}_0 = -z_0$). In order to compute ϕ, and whenever we are in a case where the value of ϕ cannot be determined from the beginning (see the table in Subsection 3.2), we will use the equation [n] of $\mathcal{W}$; so, we can easily see that

$$\tan(2\phi) = -\frac{\mathrm{Im}(r_n)}{\mathrm{Re}(r_n)},$$

where $r_n = \frac{c_n(-1)^n}{\bar{c}_n}$. This computation is well-conditioned whenever $\mathrm{Re}(r_n)$ is not too small. Afterwards, we can compute z_0 from the equation [0]: here we will get a formula for z_0 with $\bar{c}_n + \delta(-1)^n c_n$ in the denominator, with $\delta = -1$ whenever we impose $\bar{z}_0 = z_0$, and $\delta = 1$ whenever $\bar{z}_0 = -z_0$. As a consequence, if this denominator is small in norm we have a bad-conditioned case. A general analysis of the sensitivity of ϕ, z_0 in this case is more complicated than in the case of central symmetry, and involves not only general parameters (precision, degree, norm, etc.) but also other questions, like the closeness of $|\bar{c}_n+\delta(-1)^n c_n|$ to 0. So, in this case we suggest two possible ways for assessing the existence of symmetry with respect to $\mathcal{L}$:

(i) The equations of the polynomial $\mathcal{W}'$ correspond to polynomial expressions $g_k(c_0, c_1, \ldots, c_n, \alpha, \beta) = 0$. Hence, a first indicator of the existence or not of symmetry is the maximum absolute value L of the g_k's, when evaluated at $\alpha = -1$, $\beta = \xi(-1)$, and the parametrization.

(ii) Statistical criteria. In our experiments we have used the following: we generate points of $\mathcal{C}$ (by substituting 400 values of t in $z(t)$). If $\mathcal{L}$ is really a symmetry axis, one can check that the symmetric of $z(t)$ with respect to $\mathcal{L}$

is

$$\tilde{z}(t) = e^{-2i\phi} \cdot (\bar{z}(t) - \bar{z}_0) + z_0 = \frac{c_n(-1)^n}{\bar{c}_n} \cdot (\bar{z}(t) - \bar{z}_0) + z_0$$

Thus, denoting the distance of $z(t)$ to $\mathcal{L}$ as d_t, and the distance between $z(t)$ and $\tilde{z}(t)$ as $\tilde{d}_t$, one might consider the arithmetic mean of the relative errors $\delta_t = \dfrac{|\tilde{d}_t - 2d_t|}{\tilde{d}_t}$, in percentage. In our case, we improved this a little by: (a) computing the intersection points $t_1, \ldots, t_r$ of $\mathcal{L}$ and $\mathcal{C}$; (b) giving 400 values around each of these points (i.e. 400 for the first one, another 400 for the second, etc.); (c) computing the arithmetic means $\bar{x}_1, \ldots, \bar{x}_r$ of the deviations of the points obtained around each t_i; (d) finally, taking the median of $\bar{x}_1, \ldots, \bar{x}_r$. We decided to compute the intersections in order to avoid using points with very high distances from $\mathcal{L}$, which might lead to numerical problems. Also, we took the median of $\bar{x}_1, \ldots, \bar{x}_r$ to avoid errors due to fake intersections detected by the Maple solver, which we detected specially for high degrees/norms. The statistical parameter produced this way is denoted, in the following, as M.

The problem in the case of (i) is that L can be big even when the curve is approximately symmetric, due to inaccuracies. In the case of M, we found that some polynomial curves which are non symmetric when examined "localy", show however small (and even very small) values for M. This suggests a "global" symmetry, i.e. symmetry from a statistical point of view, which is what M is really evaluating. We see an example of this in Figure 3. At the left, we have the curve

$$\varphi(t) = (-73.073 + 97.097t - 62.062t^3 - 56.056t^9 + 87.087t^{10} + 1.001t^{14} + 1.001t^{15},$$
$$-17.017 + 71.071t^5 - 44.044t^6 + 80.080t^8 - 82.082t^{11} + 1.001t^{14} + 62.062t^{15})$$

plotted for $t \in [-1, 1]$, and at the right we have the same curve for $t \in [-10, 10]$. We see that the curve is not "locally" symmetric, although "globally" it is. For this curve, we have $L = 194$ and $M = 5.973 \cdot 10^{-5}$, therefore showing global symmetry.

In the following table we provide information on some of the examples that we have tried. We see that only two of the 10 curves shown in the table show a high value of M; so, we must conclude a common tendency between polynomial curves to mirror symmetry. Also, we find several curves with big values for L, but small values for M. Finally, it is worth observing that the examples 25, 29, 30, 34, 37 correspond to symmetric curves that have been slightly perturbed, while the other examples in the table correspond to perturbations of non-symmetric curves. Hence, we can see that in general the closeness to

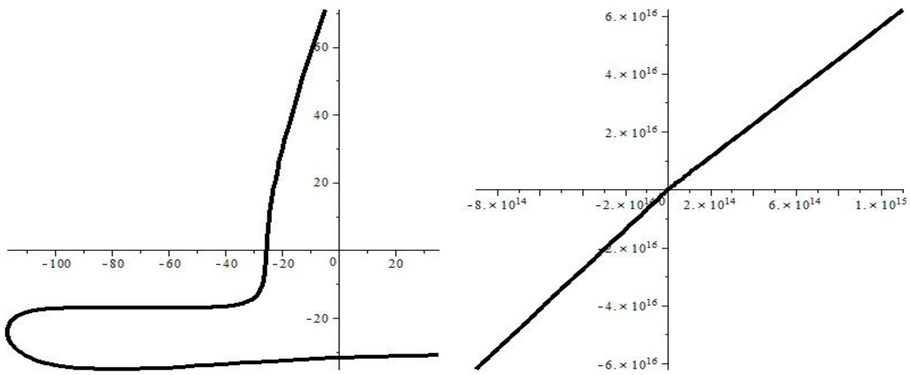

Fig. 3. Global Symmetry

"local" symmetry is well recognized; an exception is the curve 34, because of its high coefficients (check the norm column for this curve).

Example	Degree	Norm	Precision	L	M
25	21	4.5	10^{-10}	10^{-9}	0.00003
26	69	275.99	10^{-4}	663.16	52.53
27	45	461	10^{-4}	970	0.01087
28	8	97	10^{-4}	192.51	270.576
29	60	843.991	0.001	0.002	0.0
30	30	3.003	0.01	3.27	0.01159
31	51	5.005	0.01	11.55	0.25
34	21	$2.59 \cdot 10^9$	0.01	17588.46531	0.00018
35	6	120.32	0.01	266.45	4.03558
36	17	97.097	0.01	166.166	$2.64 \cdot 10^{-8}$
37	20	3.006	0.01	3.673	2.95707

5 Conclusions and Further Work

Here we have presented an efficient method for detecting the symmetries, and computing them in the affirmative case, of algebraic curves defined by means of polynomial parametrizations. The method is a combination of geometric and algebraic ideas: on the one hand, we observe that the nature of the symmetry leads to a new parametrization of the curve; on the other hand, whenever we start from a proper parametrization, this second parametrization must also be proper, and therefore is related with the first one by means of a certain algebraic relationship (see Lemma 3). This way we obtain a polynomial system with three unknowns but with a triangular structure, that can therefore be solved in a fast and efficient way. In fact, we provide closed expressions for the temptative solutions of the system, and for the elements defining the symmetries of $\mathcal{C}$, if any. Furthermore, we have also reported the experimentation done in the case of floating point inputs, showing that our results can be also useful in that case. Now it is natural to wonder if this method can be extended to other situations. In the case of rational, not simply polynomial, parametrizations of plane curves, we can apply the same strategy to build proper parametrizations, which however are related by means of a *linear rational* change of coordinates (see Lemma 4.17 in [9]), and not simply a linear one; this leads to a polynomial system with more unknowns than in the polynomial case, where the triangular structure is lost. In the case of space curves, the difficulty comes from the fact that space rotations are harder to treat; in particular complex numbers cannot be used anymore, although they can be replaced by quaternions (see for example [3]). Finally, in the case of surfaces there are also results relating proper parametrizations of a same surface (see [8]); however, here the number of parameters increases, which leads to a more difficult framework. So, in all these cases the method must be adapted. Our current efforts are focused to adapt the ideas presented in this paper to these situations, which can therefore be considered as on-going work.

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