Semispectral Measures and Feller Markov Kernels

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Abstract

It is well known [7, 29] that a real positive semispectral measure F is commutative if and only if there exist a self-adjoint operator A and a Markov kernel $\mu_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}(\mathbb{R}) \to [0,1]$ such that $F(\Delta) = \mu_\Delta(A)$. In quantum mechanics, it is usual to meet commutative semispectral measures for which the functions $\mu_\Delta : \sigma(A) \to [0,1]$, $\Delta \in \mathcal{B}(\mathbb{R})$, are continuous (in which case $\mu_{(\cdot)}(\cdot)$ is a strong Feller Markov kernel). An important example is the semispectral measure used in quantum mechanics to represent the unsharp position observable. In the present work we give a stronger characterization of commutative semispectral measures and study general conditions for the continuity of $\mu_\Delta : \sigma(A) \to [0,1]$. In particular,

- we show that F is commutative if and only if there exist a self-adjoint operator A and a Markov kernel $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$, $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$, such that

$$F(\Delta) = \int_\Gamma \mu_\Delta(\lambda)\, dE_\lambda,$$

  and $\mu_{(\Delta)}$ is continuous for each $\Delta \in R$ where, $R \subset \mathcal{B}(\mathbb{R})$ is a ring which generates the Borel σ-algebra of the reals $\mathcal{B}(\mathbb{R})$. Moreover, $\mu_{(\cdot)}(\cdot)$ is a Feller Markov kernel and separates the points of Γ.

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- we prove that F admits a strong Feller Markov kernel $\mu_{(\cdot)}(\cdot)$, if and only if F is uniformly continuous. Finally, we prove that if F is absolutely continuous with respect to a regular finite measure ν then, it admits a strong Feller Markov kernel.

Keywords: Semispectral Measure, Markov Kernel, C^*-algebras, Quantum Mechanics.

1 Introduction

A real semispectral measure (or Positive operator Valued measure) is a map $F : \mathcal{B}(\mathbb{R}) \to \mathcal{L}_s^+(\mathcal{H})$ from the Borel σ-algebra of the reals to the space of positive self-adjoint operators on a Hilbert space $\mathcal{H}$. If, $F(\Delta)$ is a projection operator for each $\Delta \in \mathcal{B}(\mathbb{R})$, F is called spectral measure (or Projection Valued measure). Therefore, the set of spectral measures is a subset of the set of semispectral measures. Moreover, spectral measures are in one-to-one correspondence with self-adjoint operators (spectral theorem) [38] and are used in standard quantum mechanics to represent quantum observables. It was pointed out [1, 17, 18, 28, 37, 40] that semispectral measures are more suitable than spectral measures in representing quantum observables. The quantum observables described by semispectral measures are called generalized observables or unsharp observables and play a key role in quantum information theory, quantum optics, quantum estimation theory [17, 25, 28, 41]. It is then natural to ask what are the relationships between semispectral and spectral measures. A clear answer can be given in the commutative case [1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 27, 29]. Indeed [7, 29], a real positive semispectral measure F is commutative if and only if there exist a bounded self-adjoint operator A and a Markov kernel $\mu_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that

$$F(\Delta) = \int_{\sigma(A)} \mu_\Delta(\lambda)\, dE_\lambda$$

where, E is the spectral measure corresponding to A. In other words, F is a smearing of the spectral measure E corresponding to A.

As an example we can consider the following unsharp position observable

$$Q^f(\Delta) := \int_{[0,1]} \mu_\Delta(x)\, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{1}$$

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

where, f is a positive, bounded, Borel function such that $f(y) = 0$, $y \notin [0,1]$, $\int_{[0,1]} f(y)dy = 1$, and Q_x is the spectral measure corresponding to the position operator

$$Q : L^2([0,1]) \to L^2([0,1])$$
$$\psi(x) \mapsto Q\psi := x\psi(x)$$

A possible interpretation [7] of equation (1) is that the outcomes of the measurement of Q^f are a randomization of the outcomes of the measurement of Q. It is worth noticing that (see example 5 in section 5.1) the Markov kernel

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

in equation (1) above is such that the function $x \mapsto \mu_\Delta(x)$ is continuous for each $\Delta \in \mathcal{B}(\mathbb{R})$. That is quite common in important physical applications so that it is natural to look for general conditions which assure the continuity of $\lambda \mapsto \mu_\Delta$.

The present work is devoted to the analysis of this problem. First, we give a stronger characterization of commutative semispectral measures. In particular, we show (see theorems 6 and 7) that a semispectral measure is commutative if and only if there exist a spectral resolution E and a Markov kernel $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$, $\Gamma \subset \sigma(A)$, $E(\Gamma) = \mathbf{1}$, such that

$$F(\Delta) = \int_\Gamma \mu_\Delta(\lambda)\, dE_\lambda \tag{2}$$

and $\mu_\Delta(\cdot)$ is continuous for each $\Delta \in R$ where, $R \subset \mathcal{B}(\mathbb{R})$ is a ring which generates the Borel σ-algebra of the reals $\mathcal{B}(\mathbb{R})$. It turns out that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is a Feller Markov kernel [35, 39]. Therefore, F is commutative if and only if there exists a Feller Markov kernel μ such that equation (2) is satisfied.
We also prove that the family of functions $\{\mu_\Delta\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ separates the points of $\sigma(A)$ up to a null set (see theorems 5, 6 and 7).

Then, we characterize the semispectral measures which admit a strong Feller Markov kernel, i.e., a Markov kernel μ such that the function $\lambda \mapsto \mu_\Delta(\lambda)$ is continuous for each $\Delta \in \mathcal{B}(\mathbb{R})$. In particular, we prove (see theorems 8 and 9) that a semispectral measure F admits a strong Feller Markov kernel if and only if it is uniformly continuous. As an example, we develop the details for the unsharp position observable defined in equation (1) above. Finally, we prove (see section 5) that a semispectral measure F which is absolutely continuous with respect to a regular finite measure ν is uniformly continuous (corollary 11). We give some examples of absolutely continuous semispectral measures (see example 4) and analyze the unsharp position observable which is obtained as the marginal of a phase space observable (see section 5.1).

2 Some preliminaries about Semispectral measures

In what follows, we denote by $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}([0,1])$ the Borel σ-algebra of $\mathbb{R}$ and [0,1] respectively, by $\mathbf{0}$ and $\mathbf{1}$ the null and the identity operators, by $\mathcal{L}_s(\mathcal{H})$ the space of all bounded self-adjoint linear operators acting in a Hilbert space $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle$, by $\mathcal{F}(\mathcal{H}) = \mathcal{L}_s^+(\mathcal{H})$ the subspace of all positive, bounded self-adjoint operators on $\mathcal{H}$, by $\mathcal{E}(\mathcal{H}) \subset \mathcal{F}(\mathcal{H})$ the subspace of all projection operators on $\mathcal{H}$. We use the symbols POVM and PVM to denote semispectral measures and spectral measures respectively.

Definition 1. *A Semispectral measure or Positive Operator Valued measure (for short, POVM) is a map $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ such that:*

$$F\Big(\bigcup_{n=1}^{\infty} \Delta_n\Big) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(\mathbb{R})$ and the series converges in the weak operator topology. It is said to be normalized if

$$F(\mathbb{R}) = \mathbf{1}$$

Definition 2. *A POVM is said to be commutative if*

$$\big[F(\Delta_1), F(\Delta_2)\big] = \mathbf{0}, \quad \forall\, \Delta_1\,, \Delta_2 \in \mathcal{B}(\mathbb{R}). \tag{3}$$

Definition 3. *A POVM is said to be orthogonal if*

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \;\; \textit{if} \; \Delta_1 \cap \Delta_2 = \emptyset. \tag{4}$$

Definition 4. *A Spectral measure or Projection Valued measure (for short, PVM) is an orthogonal, normalized POVM.*

It is simple to see that for a PVM E, we have $E(\Delta) = E(\Delta)^2$, for any $\Delta \in \mathcal{B}(\mathbb{R})$. Then, $E(\Delta)$ is a projection operator for every $\Delta \in \mathcal{B}(\mathbb{R})$, and the PVM is a map $E : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$.
In quantum mechanics, non-orthogonal normalized POVM are also called **generalised** or **unsharp** observables and PVM **standard** or **sharp** observables. In what follows, we shall always refer to real normalized POVM and we shall use the term "measurable" for the Borel measurable functions. For any vector $x \in \mathcal{H}$ the map

$$\langle F(\cdot)x, x\rangle \,:\, \mathcal{B}(\mathbb{R}) \to \mathbb{R}, \qquad \Delta \mapsto \langle F(\Delta)x, x\rangle,$$

is a Lebesgue-Stieltjes measure. There exists a one-to-one correspondence [5] between POV measures F and POV functions $F_\lambda := F((-\infty, \lambda])$. In the following we will use the symbol $d\langle F_\lambda x, x\rangle$ to mean integration with respect to the measure $\langle F(\cdot)x, x\rangle$. We shall say that a measurable function $f : N \subset \mathbb{R} \to f(N) \subset \mathbb{R}$ is almost everywhere (a.e.) one-to-one with respect to a POVM F if it is one-to-one on a subset $N' \subset N$ such that $N - N'$ is a null set with respect to F. We shall say that a function $f : \mathbb{R} \to \mathbb{R}$ is bounded with respect to a POVM F, if it is equal to a bounded function g a.e. with respect to F, that is, if $f = g$ a.e. with respect to the measure $\langle F(\cdot)x, x\rangle$, $\forall x \in \mathcal{H}$. For any real, bounded and measurable function f and for any POVM F, there is a unique [15] bounded self-adjoint operator $B \in \mathcal{L}_s(\mathcal{H})$ such that

$$\langle Bx, x\rangle = \int f(\lambda) d\langle F_\lambda x, x\rangle, \quad \text{for each} \quad x \in \mathcal{H}. \tag{5}$$

If equation (5) is satisfied, we write $B = \int f(\lambda) dF_\lambda$ or $B = \int f(\lambda) F(d\lambda)$ equivalently.

Definition 5. *The spectrum $\sigma(F)$ of a POVM F is the closed set*

$$\big\{\lambda \in \mathbb{R} : \; F\big((\lambda - \delta, \lambda + \delta)\big) \neq 0, \, \forall \delta > 0, \; \big\}.$$

By the spectral theorem [21, 38], there is a one-to-one correspondence between PV measures E and self-adjoint operators B, the correspondence being given by

$$B = \int \lambda dE^B_\lambda .$$

Notice that the spectrum of E^B coincides with the spectrum of the corresponding self-adjoint operator B. Moreover, in this case a functional calculus can be developed. Indeed, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable real-valued function, we can define the self-adjoint operator [38]

$$f(B) = \int f(\lambda) dE^B_\lambda$$

where, E^B is the PVM corresponding to B. If f is bounded, then $f(B)$ is bounded [38].

In the following we do not distinguish between PVM and the corresponding self-adjoint operators.

Let Λ be a subset of $\mathbb{R}$ and $\mathcal{B}(\Lambda)$ the corresponding Borel σ-algebra.

Definition 6. *A real Markov kernel is a map $\mu : \Lambda \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that,*

1. *$\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(\mathbb{R})$,*
2. *$\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Lambda$.*

Definition 7. *Let ν be a measure on Λ. A map $\mu : \Lambda \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is a weak Markov kernel with respect to ν if:*

1. *$\mu_\Delta(\cdot)$ is a measurable function for each $\Delta \in \mathcal{B}(\mathbb{R})$,*
2. *$0 \leq \mu_{\mathbb{R}}(\lambda) \leq 1, \quad \nu - a.e.,$*
3. *$\mu_{\mathbb{R}}(\lambda) = 1,\ \mu_{\emptyset}(\lambda) = 0, \quad \nu - a.e.,$*
4. *for any sequence $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i \cap \Delta_j = \emptyset$,*

$$\sum_i \mu_{(\Delta_i)}(\lambda) = \mu_{(\cup_i \Delta_i)}(\lambda), \quad \nu - a.e.$$

Definition 8. *The map $\mu : \Lambda \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is a weak Markov kernel with respect to a PVM $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ if it is a weak Markov kernel with respect to each measure $\nu_x(\cdot) := \langle E(\cdot)\, x, x\rangle$, $x \in \mathcal{H}$.*

In the following, by a weak Markov kernel μ we mean a weak Markov kernel with respect to a PVM E. Moreover the function $\lambda \mapsto \mu_\Delta(\lambda)$ will be denoted indifferently by μ_Δ or $\mu_\Delta(\cdot)$.

Definition 9. *A POV measure $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ is said to be a smearing of a POV measure $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ if there exists a weak Markov kernel $\mu : \Lambda \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that,*

$$F(\Delta) = \int_\Lambda \mu_\Delta(\lambda) dE_\lambda, \quad \Delta \in \mathcal{B}(\mathbb{R}).$$

Example 1. *In the standard formulation of quantum mechanics, the operator*

$$Q : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$
$$\psi(x) \in L^2(\mathbb{R}) \mapsto Q\psi := x\psi(x)$$

is used to represent the position observable. A more realistic description of the position observable of a quantum particle is given by a smearing of Q as, for example, the optimal position semispectral measure

$$F^Q(\Delta) = \frac{1}{l\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(\int_\Delta e^{-\frac{(x-y)^2}{2l^2}} dy \Big) dE_x^Q = \int_{-\infty}^{\infty} \mu_\Delta(x)\, dE_x^Q$$

where,

$$\mu_\Delta(x) = \frac{1}{l\sqrt{2\pi}} \int_\Delta e^{-\frac{(x-y)^2}{2l^2}} dy$$

defines a Markov kernel and E^Q is the spectral measure corresponding to the position operator Q.

In the following, the symbol μ is used to denote both Markov kernels and weak Markov kernels. The symbols A and B are used to denote self-adjoint operators.

Definition 10. *Whenever F, A, and μ are such that $F(\Delta) = \mu_\Delta(A)$, $\Delta \in \mathcal{B}(\mathbb{R})$, we say that (F, A, μ) is a von Neumann triplet.*

The following theorem establishes a relationship between commutative semispectral measures and spectral measures and gives a characterization of the former. Other characterizations and an analysis of the relationships between them can be found in Ref.s [1, 27, 4, 30].

Theorem 1 ([7, 29])**.** *A semispectral measure F is commutative if and only if there exist a bounded self-adjoint operator A and a Markov kernel (weak Markov kernel) μ such that (F, A, μ) is a von Neumann triplet.*

Corollary 1. *A semispectral measure F is commutative if and only if it is a smearing of a PV measure E with bounded spectrum.*

Definition 11. *The von Neumann algebra generated by the semispectral measure F is the von Neumann algebra generated by the set $\{F(\Delta),\, \Delta \in \mathcal{B}(\mathbb{R})\}$.*

Definition 12. *If A and F in theorem 1 generate the same von Neumann algebra then A is named the sharp reconstruction of F.*

Theorem 2. *[7] The sharp reconstruction A is unique up to almost everywhere bijections.*

3 Characterization of Commutative Semispectral Measures by means of Strong Markov kernels

As we have seen in the last section, theorem 1 asserts that a semispectral measure F is commutative if and only if there exist a bounded self-adjoint operator A and a weak Markov kernel (Markov kernel) μ such that $F(\Delta) = \mu_\Delta(A)$. In the present section we study the continuity of the functions μ_Δ. First, we restrict ourselves to semispectral measures on $[0, 1]$ and then (see the appendix A) we extend the results to semispectral measures on $\mathbb{R}$. In particular, we prove (see theorem 3 below) that, if F is commutative, there exists a weak Markov kernel μ such that: a) (F, A, μ) is a von Neumann triplet, b) $\mu_{(\cdot)}(\lambda)$, $\lambda \in \sigma(A)$ is additive on a ring $\mathcal{R}(\mathcal{S})$ which generates the Borel σ-algebra of $[0, 1]$ and c) μ_Δ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$.

Then, we introduce the concept of strong Markov kernel, i.e., a weak Markov kernel $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ with respect to a PVM $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ such that $\mu_{(\cdot)}(\lambda)$ is a probability measure for each $\lambda \in \Gamma \subset \Lambda$, $E(\Gamma) = \mathbf{1}$. We prove (theorems 4 and 7) that in order to realize the smearing in corollary 1, one can use a strong Markov kernel μ such that μ_Δ is continuous for each $\Delta \in R$, where R is a ring which generates the Borel σ-algebra of the reals. It is worth remarking that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is a Feller Markov kernel. Therefore, F is commutative if and only if there exists a bounded self-adjoint operator A and a Feller Markov kernel μ such that

$$F(\Delta) = \int_{\Gamma} \mu_{\Delta}(\lambda)\, dE_{\lambda}.$$

Moreover, the family of functions $\{\mu_{\Delta}\}_{\Delta \in R}$ separates the points in Γ (see theorems 5 and 7).

In the following, the symbol $\mathcal{S}$ denotes the family of open intervals in $[0,1]$ with rational end-points. The symbol $\mathcal{R}(\mathcal{S})$ denotes the ring generated by $\mathcal{S}$. Notice that $\mathcal{S}$ is countable and generates the Borel σ-algebra $\mathcal{B}([0,1])$.

Theorem 3. *For any real commutative POVM $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ with spectrum in $[0,1]$, there exists a bounded self-adjoint operator A with spectrum $\sigma(A) \subset [0,1]$ and a weak Markov Kernel $\mu : \sigma(A) \times \mathcal{B}([0,1]) \to [0,1]$, such that:*

1) $\mu_{\Delta}(\cdot)$ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$,

2) $\mu_{(\cdot)}(\lambda)$ is additive on $\mathcal{R}(\mathcal{S})$,

3) $F(\Delta) = \mu_{\Delta}(A), \quad \Delta \in \mathcal{B}([0,1])$.

Proof. Let $\mathcal{A}^W(F)$ be the von Neumann algebra generated by F and $M := \{F(\Delta),\ \Delta \in \mathcal{B}([0,1])\}$. First, we prove that $\mathcal{A}^W(F)$ coincides with the von Neumann algebra generated by $\{F(\Delta)\}_{\Delta \in \mathcal{R}(\mathcal{S})}$ where, $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}([0,1])$ is the ring generated by the family $\mathcal{S}$ of open intervals with rational end-points. Notice that $\mathcal{S}$ is countable. Then, by theorem c, page 24, in Ref. [24], $\mathcal{R}(\mathcal{S})$ is countable too.
Let G denote the family of open subsets of $[0,1]$. Let us consider the set $O := \{F(\Delta),\ \Delta \in G\}$. Since the POV measure F is regular, for each Borel set Δ, there exists a decreasing family of open sets G_i such that $F(G_i) \to F(\Delta)$ strongly. Then, O is dense in M and the von Neumann algebra generated by M coincides with the von Neumann algebra generated by O. Hence,

$$\mathcal{A}^W(F) = \mathcal{A}^W(M) = \mathcal{A}^W(O). \tag{6}$$

Now, let G_1 denote the family of open intervals in $[0,1]$. Let us consider the set $O_1 = \{F(\Delta),\ \Delta \in G_1\}$. Each open set Δ is the disjoint union of a countable family of open intervals Δ_i, i.e. $\Delta = \cup_{i=1}^{\infty}\Delta_i$. Therefore,

$$\begin{aligned} F(\Delta) = F(\cup_{i=1}^{\infty}\Delta_i) &= \sum_{i=1}^{\infty} F(\Delta_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} F(\Delta_i) = \lim_{n\to\infty} F(\cup_{i=1}^{n}\Delta_i). \end{aligned}$$

Since the von Neumann algebra generated by O_1 contains $F(\cup_{i=1}^{n}\Delta_i)$, it must contain $F(\Delta) = \lim_{n\to\infty} F(\cup_{i=1}^{n}\Delta_i)$. Therefore,

$$\mathcal{A}^W(O_1) = \mathcal{A}^W(O). \tag{7}$$

Now, we prove that the von Neumann algebra $\mathcal{A}^W(O_2)$ generated by $O_2 = \{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$ coincides with $\mathcal{A}^W(O_1)$.
For each open interval (a,b) there exists a disjoint family of sets $\{\Delta_i\}_{i\in\mathbb{N}} \subset \mathcal{R}(\mathcal{S})$, $\Delta_i \subset (a,b)$, $i \in \mathbb{N}$, such that $(a,b) = \cup_{i=1}^{\infty}\Delta_i$. Then,

$$\begin{aligned} F(a,b) = F(\cup_{i=1}^{\infty}\Delta_i) &= \sum_{i=1}^{\infty} F(\Delta_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} F(\Delta_i) = \lim_{n\to\infty} F(\cup_{i=1}^{n}\Delta_i). \end{aligned}$$

Since the von Neumann algebra generated by O_2 contains $F(\cup_{i=1}^{n}\Delta_i)$ for each $n \in \mathbb{N}$, it must contain $F(\Delta) = \lim_{n\to\infty} F(\cup_{i=1}^{n}\Delta_i)$. Therefore, $\mathcal{A}^W(O_1) = \mathcal{A}^W(O_2)$ and, by equations (6) and (7),

$$\mathcal{A}^W(O_2) = \mathcal{A}^W(O_1) = \mathcal{A}^W(O) = \mathcal{A}^W(F) \tag{8}$$

which proves that $\mathcal{A}^W(F)$ coincides with the von Neumann algebra generated by the set $\{F(\Delta)\}_{\Delta\in\mathcal{R}(\mathcal{S})}$.

Now, we proceed to the proof of the existence of A. Let us consider the set O_2. Let $\{\Delta_i\}_{i\in\mathbb{N}}$ be an enumeration of the set $\mathcal{R}(\mathcal{S})$. Let $E^{(i)}$ denote the spectral measure corresponding to $F(\Delta_i) \in O_2$. We have $F(\Delta_i) = \int x\, dE_x^{(i)}$. Therefore, for each $i,k \in \mathbb{N}$ there exists a division $\{\Delta_j^{(i,k)}\}_{j=1,\ldots,m_{i,k}}$ of $[0,1]$ such that

$$\Big\| \sum_{j=1}^{m_{i,k}} x_j^{(i,k)}\, E^{(i)}(\Delta_j^{(i,k)}) - F(\Delta_i) \Big\| \le \frac{1}{k}. \tag{9}$$

By the spectral theorem [21] the von Neumann algebra $\mathcal{A}^W(F)$ contains all the projection operators in the spectral resolution of $F(\Delta)$, $\Delta \in \mathcal{B}([0,1])$.

Therefore, the von Neumann algebra $\mathcal{A}^W(D)$ generated by the set $D := \{E^{(i)}(\Delta_j^{i,k}),\, j \leq m_{i,k},\, i,k \in \mathbb{N}\}$ is contained in $\mathcal{A}^W(F)$ and then

$$\mathcal{A}^W(D) \subset \mathcal{A}^W(F) = \mathcal{A}^W(O_2). \tag{10}$$

Moreover, the C^*-algebra $\mathcal{A}^C(D)$ generated by D contains the C^*-algebra $\mathcal{A}^C(O_2)$ generated by O_2 (see equation (9)). Summing up the preceding observations, we have

$$\mathcal{A}^C(O_2) \subset \mathcal{A}^C(D) \subset \mathcal{A}^W(F).$$

By the double commutant theorem [31],

$$\mathcal{A}^W(F) = [\mathcal{A}^C(O_2)]'' \subset [\mathcal{A}^C(D)]'' = \mathcal{A}^W(D)$$

so that (see equation 10),

$$\mathcal{A}^W(D) = \mathcal{A}^W(F). \tag{11}$$

By theorem 11, page 871 in Ref. [21], the spectrum Λ of $\mathcal{A}^C(D)$ is homeomorphic to a closed subset of $\prod_{i=1}^{\infty}\{0,1\}$. Let $\pi : \Lambda \to \prod_{i=1}^{\infty}\{0,1\}$ denote the homeomorphism between the two spaces.

Now, if we identify Λ with a closed subset of $\prod_{i=1}^{\infty}\{0,1\}$, we can prove the existence of a continuous function distinguishing the points of Λ. Indeed, let $\pi(\lambda) = \bar{x} := (x_1, \dots, x_n, \dots) \in \prod_{i=1}^{\infty}\{0,1\}$. The function

$$f(\lambda) = \sum_{i=1}^{\infty} \frac{x_i}{3^i}$$

is continuous and injective and then it distinguishes the points of Λ. Moreover, since Λ and $[0,1]$ are Hausdorff, the map $f : \Lambda \to f(\Lambda)$ is a homeomorphism.

By theorem 1, page 895, in Ref. [21], there exists a spectral measure $\widetilde{E} : \mathcal{B}(\Lambda) \to \mathcal{F}(\mathcal{H})$ such that the map

$$\begin{aligned} T : \mathcal{C}(\Lambda) &\to B(\mathcal{H}) \\ g \mapsto T(g) &= \int_{\Lambda} g(\lambda) d\widetilde{E}_\lambda \end{aligned} \tag{12}$$

defines an isometric *-isomorphism between $\mathcal{A}^C(D)$ and $\mathcal{C}(\Lambda)$.

The fact that f distinguishes the points of Λ, implies that the self-adjoint operator

$$A = \int_{\Lambda} f(\lambda)\, d\widetilde{E}_\lambda$$

is a generator of the von Neumann algebra $\mathcal{A}^W(D) = \mathcal{A}^W(F)$. Indeed, by the Stone-Weierstrass theorem, $\mathcal{C}(\Lambda)$ is singly generated, in particular f is a generator. Then, the isomorphism between $\mathcal{A}^C(D)$ and $\mathcal{C}(\Lambda)$ assures that $\mathcal{A}^C(D)$ is singly generated and that A is a generator. Hence, $\mathcal{A}^W(F) = \mathcal{A}^W(D) = [\mathcal{A}^C(D)]''$ is singly generated. In particular, A generates $\mathcal{A}^W(F)$, i.e., $\mathcal{A}^W(F) = \mathcal{A}^W(A)$.

Now, we proceed to the proof of the existence of the weak Markov kernel μ.

By (12), for each $\Delta \in \mathcal{R}(\mathcal{S})$, there exists a continuous function $\gamma_\Delta \in \mathcal{C}(\Lambda)$ such that

$$F(\Delta) = \int_\Lambda \gamma_\Delta(\lambda)\, d\widetilde{E}_\lambda.$$

Now, we show that, for each $\Delta \in \mathcal{R}(\mathcal{S})$, there is a continuous function $\nu_\Delta : \sigma(A) \to [0,1]$ from the spectrum of A to the interval $[0,1]$ such that $\nu_\Delta(f(\lambda)) = \gamma_\Delta(\lambda)$, $\lambda \in \Lambda$, and $F(\Delta) = \nu_\Delta(A)$.
To prove this, let us consider the function

$$\nu_\Delta(t) := (\gamma_\Delta \circ f^{-1})(t), \quad \Delta \in \mathcal{R}(\mathcal{S}).$$

It is continuous since it is the composition of continuous functions and,

$$\nu_\Delta(f(\lambda)) = \gamma_\Delta(f^{-1}(f(\lambda))) = \gamma_\Delta(\lambda).$$

Moreover,

$$\nu_\Delta(A) = F(\Delta), \quad \forall \Delta \in \mathcal{R}(\mathcal{S}).$$

Indeed, by the change of measure principle (page 894, ref. [21]),

$$\begin{aligned} F(\Delta) &= \int_\Lambda \gamma_\Delta(\lambda)\, d\widetilde{E}_\lambda = \int_\Lambda \gamma_\Delta(f^{-1}(f(\lambda)))\, d\widetilde{E}_\lambda \\ &= \int_{\sigma(A)} \gamma_\Delta(f^{-1}(t))\, dE_t = \int_{\sigma(A)} \nu_\Delta(t)\, dE_t = \nu_\Delta(A) \end{aligned}$$

where $\sigma(A) = f(\Lambda)$ is the spectrum of A and E is the spectral measure corresponding to A defined by the relation $E(\Delta) = \widetilde{E}(f^{-1}(\Delta))$, $\Delta \in \mathcal{B}([0,1])$ (see corollary 10, page 902, in Ref. [21]).
For each $\lambda \in \sigma(A)$, the map $\nu_{(\cdot)}(\lambda) : \mathcal{R}(\mathcal{S}) \to [0,1]$ defines an additive set function. Indeed, let $\Delta \in \mathcal{R}(\mathcal{S})$ be the disjoint union of the sets $\Delta_1, \Delta_2 \in$

$\mathcal{R}(\mathcal{S})$. Then,

$$\begin{aligned}\int \nu_{(\Delta_1\cup\Delta_2)}(\lambda)\,dE_\lambda &= F(\Delta_1\cup\Delta_2) = F(\Delta_1)+F(\Delta_1)\\ &= \int \nu_{\Delta_1}(\lambda)\,dE_\lambda + \int \nu_{\Delta_2}(\lambda)\,dE_\lambda\\ &= \int \big[\nu_{\Delta_1}(\lambda)+\nu_{\Delta_2}(\lambda)\big]\,dE_\lambda\end{aligned}$$

so that, by the continuity of the functions $\nu_{(\Delta_1)}(\lambda)$ and $\nu_{(\Delta_2)}(\lambda)$, we get (see theorem 1, page 895, in Ref. [21])

$$\nu_{(\Delta_1)}(\lambda)+\nu_{(\Delta_2)}(\lambda) = \nu_{(\Delta_1\cup\Delta_2)}(\lambda), \quad \forall\lambda\in\sigma(A).$$

Now, we extend ν to all the Borel σ-algebra of $[0,1]$.
Since A is the generator of $\mathcal{A}^W(F)$, for each $\Delta\in\mathcal{B}([0,1])$, there exists a Borel function ω_Δ such that.

$$F(\Delta) = \int_{\sigma(A)} \omega_\Delta(t)\,dE_t = \int_\Lambda (\omega_\Delta\circ f)(\lambda)\,d\widetilde{E}_\lambda$$

Then, we can consider the map $\mu:\sigma(A)\times\mathcal{B}([0,1])\to[0,1]$ defined as follows

$$\mu_\Delta(\lambda) = \begin{cases}\nu_\Delta(\lambda) & if \quad \Delta\in\mathcal{R}(\mathcal{S})\\ \omega_\Delta(\lambda) & if \quad \Delta\notin\mathcal{R}(\mathcal{S}).\end{cases}$$

Since μ coincides with ν on $\mathcal{R}(\mathcal{S})$ it is additive on $\mathcal{R}(\mathcal{S})$.
In order to prove that μ is a weak Markov kernel, let us consider a set $\Delta\in\mathcal{B}([0,1])$ which is the disjoint union of the sets $\{\Delta_i\}_{i\in\mathbb{N}}$, $\Delta_i\in\mathcal{B}([0,1])$. Then,

$$\begin{aligned}\int \mu_{(\cup_{i=1}^\infty\Delta_i)}(x)\,dE_x &= \int \mu_\Delta(x)dE_x = F(\Delta) = \sum_{i=1}^\infty F(\Delta_i)\\ &= \sum_{i=1}^\infty \int \mu_{\Delta_i}(x)\,dE_x = \int \sum_{i=1}^\infty \mu_{\Delta_i}(x)\,dE_x\end{aligned}$$

so that, by Corollary 9, page 900, in Ref. [21],

$$\sum_{i=1}^\infty \mu_{\Delta_i}(x) = \mu_\Delta(x), \quad E-a.e,$$

which implies that $\mu:[0,1]\times\mathcal{B}([0,1])\to[0,1]$ is a weak Markov kernel. In particular (F,A,μ) is a von Neumann triplet. □

Notice that also the converse of theorem 3 is true. Indeed, $F(\Delta) = \mu_\Delta(A)$, $\Delta \in \mathcal{B}[0,1]$, implies that $\{F(\Delta)\}_{\Delta\in\mathcal{B}[0,1]}$ is commutative.

Now, we show that theorem 3 can be strengthened. In order to do that we need the following definition.

Definition 13. *Let $E : \mathcal{B}(\Lambda) \to \mathcal{E}(\mathcal{H})$ be a PVM. The map $\mu_{(\cdot)}(\cdot) : \Lambda\times\mathcal{B}(\mathbb{R}) \to [0,1]$ is a strong Markov kernel with respect to E if it is a weak Markov kernel and there exists a set $\Gamma \subset \Lambda$, $E(\Gamma) = \mathbf{1}$, such that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is a Markov kernel with respect to E.*

Theorem 4. *A POV measure $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if there exists a bounded self-adjoint operator A with spectrum $\sigma(A) \subset [0,1]$ and a strong Markov kernel $\mu : \sigma(A) \times \mathcal{B}([0,1]) \to [0,1]$ such that*

1. *(F, A, μ) is a von Neumann triplet,*
2. *$\mu_\Delta \in \mathcal{C}(\sigma(A))$, $\quad \Delta \in \mathcal{R}(\mathcal{S})$,*
3. *μ is additive on $\mathcal{R}(\mathcal{S})$.*

Proof. By theorem 3, F is commutative if and only if there is a self-adjoint operator A and a weak Markov kernel $\nu : \sigma(A) \times \mathcal{B}([0,1]) \to [0,1]$ such that

i. (F, A, ν) is a von Neumann triplet,

ii. $\nu_\Delta \in \mathcal{C}(\sigma(A))$, $\quad \Delta \in \mathcal{R}(\mathcal{S})$,

iii. ν is additive on $\mathcal{R}(\mathcal{S})$.

By theorem 2 in Ref. [7], starting from $\nu : \sigma(A) \times \mathcal{R}(\mathcal{S}) \to [0,1]$ it is possible to define a Markov kernel $\omega : \sigma(A) \times \mathcal{B}([0,1]) \to [0,1]$ such that (F, A, ω) is a von Neumann triplet. Then, by item i. above, for each $\Delta \in \mathcal{B}([0,1])$,

$$\int \nu_\Delta(\lambda)\, dE_\lambda = F(\Delta) = \int \omega_\Delta(\lambda)\, dE_\lambda$$

hence,

$$\omega_\Delta(\lambda) = \nu_\Delta(\lambda), \quad E - a.e. \tag{13}$$

Now, let $\{\Delta_i\}_{i\in\mathbb{N}}$ be an enumeration of $\mathcal{R}(\mathcal{S})$. By equation (13), for each $i \in \mathbb{N}$, there is a set N_i, $E(N_i) = \mathbf{0}$, such that

$$\omega_{\Delta_i}(\lambda) = \nu_{\Delta_i}(\lambda), \quad \lambda \in [0,1] - N_i. \tag{14}$$

Then, for each $i \in \mathbb{N}$,

$$\omega_{\Delta_i}(\lambda) = \nu_{\Delta_i}(\lambda), \quad \lambda \in [0,1] - N \tag{15}$$

where,

$$N := \cup_{i=1}^{\infty} N_i, \quad E(N) = \mathbf{0}.$$

Therefore, for almost all $\lambda \in \sigma(A)$, $\nu_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$.
Now, we can define the map

$$\mu_{(\cdot)}(\lambda) = \begin{cases} \nu_{(\cdot)}(\lambda) & \lambda \in N \\ \omega_{(\cdot)}(\lambda) & \lambda \in [0,1] - N \end{cases}$$

If we put $\Gamma = [0,1] - N$, we have that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is a Markov kernel. Therefore, $\mu_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is a strong Markov kernel.
Notice that, for each $\Delta \in \mathcal{R}(\mathcal{S})$ and $\lambda \in \sigma(A)$,

$$\mu_\Delta(\lambda) = \nu_\Delta(\lambda)$$

so that, μ_Δ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$ and additive on $\mathcal{R}(\mathcal{S})$. We also have,

$$\mu_\Delta(A) = \omega_\Delta(A) = F(\Delta), \quad \Delta \in \mathcal{R}(\mathcal{S}).$$

□

It is worth analyzing whether the set functions $\{\mu_{(\cdot)}(\lambda)\}_{\lambda \in \sigma(A)}$ are distinct or, in other words, if the family of functions $\{\mu_\Delta\}_{\Delta \in \mathcal{B}(\mathbb{R})}$ separates the points of $\sigma(A)$. The following theorem answers in the positive.

Theorem 5. *Let (F, A, μ) be the von Neumann triplet whose existence was proved in theorem 3. Then, there exists a set $\Gamma \subseteq \sigma(A)$, $E^A(\Gamma) = \mathbf{1}$, such that the family of functions $\{\mu_\Delta(\cdot)\}_{\Delta \in \mathcal{B}([0,1])}$ separates the points of Γ.*

Proof. In the following we use the same notation that we used in the proof of theorem 3. In particular, $\mathcal{A}^W(F)$ denotes the von Neumann algebra generated by $\{F(\Delta)\}_{\Delta \in \mathcal{B}([0,1])}$, $O_2 := \{F(\Delta)\}_{\Delta \in \mathcal{R}(\mathcal{S})}$ and $\mathcal{A}^C(O_2)$ is the C^*-algebra generated by O_2. We recall that the von Neumann algebra generated by $\mathcal{A}^C(O_2)$ coincides with $\mathcal{A}^W(F) = \mathcal{A}^W(A)$ where, A is the generator of $\mathcal{A}^W(F)$ whose existence was proved in theorem 3. By the Gelfand-Naimark theorem [21, 36], there is a * isomorphism ϕ between $\mathcal{A}^C(O_2)$ and the algebra of continuous functions $\mathcal{C}(\Lambda_2)$ where Λ_2 is the spectrum of $\mathcal{A}^C(O_2)$. Moreover,

$$f \in \mathcal{C}(\Lambda_2) \mapsto \phi(f) = \int_{\Lambda_2} f(\lambda)\, d\widetilde{E}_\lambda$$

where, $\widetilde{E}$ is the spectral measure from the Borel σ algebra $\mathcal{B}(\Lambda_2)$ to $\mathcal{E}(\mathcal{H})$ whose existence is assured by theorem 1, page 895, in Ref. [21]. The Gelfand-Naimark isomorphism ϕ can be extended to a homomorphism between the algebra of the Borel functions on Λ_2 and the von Neumann algebra $\mathcal{A}^W(F) = \mathcal{A}^W(A)$ generated by $\mathcal{A}^C(O_2)$ (see Ref. [20], page 360, section 3). Therefore, there is a Borel function h such that

$$A = \int_{\Lambda_2} h(\lambda)\, d\widetilde{E}_\lambda \tag{16}$$

Let $\{\Delta_i\}_{i\in\mathbb{N}}$ denote an enumeration of the set $\mathcal{R}(\mathcal{S})$. Since $\mathcal{A}^C(O_2)$ is the smallest uniform closed algebra containing $\{F(\Delta_i)\}_{i\in\mathbb{N}}$, $\mathcal{C}(\Lambda_2)$ is the smallest uniform closed algebra of functions containing $\{\nu_{\Delta_i} := \phi^{-1}(F(\Delta_i))\}_{i\in\mathbb{N}}$. In other words $\{\nu_{\Delta_i}\}_{i\in\mathbb{N}}$ generates $\mathcal{C}(\Lambda_2)$. The Stone-Weierstrass theorem [21] assures that $\{\nu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points in Λ_2.
On the other hand, the fact that (F, A, μ) is a von Neumann triplet, implies that, for each $\Delta_i \in \mathcal{R}(\mathcal{S})$, there is a Borel function μ_{Δ_i} such that

$$\int_{\Lambda_2} \nu_{\Delta_i}(\lambda)\, d\widetilde{E}_\lambda = F(\Delta_i) = \mu_{\Delta_i}(A) = \int_{\Lambda_2} \mu_{\Delta_i}(h(\lambda))\, d\widetilde{E}_\lambda.$$

Then, for each $\Delta_i \in \mathcal{R}(\mathcal{S})$, there is a set $M_i \subset \Lambda_2$, $\widetilde{E}(M_i) = \mathbf{1}$, such that

$$\mu_{\Delta_i}(h(\lambda)) = \nu_{\Delta_i}(\lambda), \quad \lambda \in M_i. \tag{17}$$

Let $M := \cap_{i=1}^{\infty} M_i$. Then,

$$\widetilde{E}(M) = \lim_{n\to\infty} \widetilde{E}(\cap_{i=1}^{n} M_i) = \lim_{n\to\infty} \prod_{i=1}^{n} \widetilde{E}(M_i) = \mathbf{1}$$

and, for each $i \in \mathbb{N}$,

$$(\mu_{\Delta_i} \circ h)(\lambda) = \nu_{\Delta_i}(\lambda), \quad \lambda \in M \subseteq \Lambda_2. \tag{18}$$

Since $\{\nu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points in Λ_2, it separates the points in M. Then, equation (18) implies that $\{\mu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points in $\Gamma := h(M)$. Moreover[1],

$$E^A(\Gamma) = E^A(h(M)) = \widetilde{E}[h^{-1}(h(M))] = \mathbf{1}$$

where, E^A is the spectral measure defined by the relation

$$E^A(\Delta) = \widetilde{E}(h^{-1}(\Delta))$$

and such that,

$$A = \int x\, dE^A_x$$

while, $h^{-1}(h(M))$ is a Borel set containing M.
We have proved that the set of functions $\{\mu_{\Delta_i}\}_{i\in\mathbb{N}}$ separates the points of Γ and that $E^A(\Gamma) = \mathbf{1}$. In other words,

$$\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda'), \quad \lambda \neq \lambda', \quad \lambda, \lambda' \in \Gamma.$$

□

As a consequence of theorem 4 and theorem 5, we have the following theorem 6.

Theorem 6. *A POV measure $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if there exist a bounded self-adjoint operator A with spectrum $\sigma(A) \subset [0,1]$, a strong Markov kernel $\mu : \sigma(A) \times \mathcal{B}([0,1]) \to [0,1]$ and a ring $\mathcal{R}(\mathcal{S})$ of subset of $[0,1]$ such that*

1. *(F, A, μ) is a von Neumann triplet,*
2. *$\mu_\Delta \in \mathcal{C}(\sigma(A))$, $\quad \Delta \in \mathcal{R}(\mathcal{S})$.*
3. *there is a set $\Gamma \subset \sigma(A)$, $E^A(\Gamma) = \mathbf{1}$, such that $\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda')$, $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Gamma$ (where, E^A is the spectral measure corresponding to A).*

[1] Notice that $h(M)$ is a Borel set. In orther to prove that, we first recall that Λ_2 is a Polish space (that is, a complete, separable, space [32]). Indeed, by theorem 11, page 871, in Ref. [21], it is homeomorphic to a closed subspace of the Cartesian product $\prod_{i=1}^{\infty} \sigma(F(\Delta_i))$, where $\sigma(F(\Delta_i))$ is a complete separable metric space, and by theorem 2, page 406, and theorem 6, page 156, in Ref. [33], it is complete and separable. Moreover, h is measurable and injective on M. Therefore, Soulsin's theorem (see theorem 9 page 440 and Corollary 1 page 442 in Ref. [32]) assures that $h(M)$ is a Borel set.

4. $\mathcal{R}(\mathcal{S})$ generates $\mathcal{B}([0,1])$.

Theorem 6 can be generalized to the case of POVM with spectrum in $\mathbb{R}$. See the appendix A for the details.

Theorem 7. *A POV measure $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if there exist a bounded self-adjoint operator A with spectrum in $\sigma(A) \subset [0,1]$, a strong Markov kernel $\mu : \sigma(A) \times \mathcal{B}(\mathbb{R}) \to [0,1]$ and a ring R of subset of $\mathbb{R}$ such that*

1. *(F, A, μ) is a von Neumann triplet,*
2. *$\mu_\Delta \in \mathcal{C}(\sigma(A)), \quad \Delta \in R$.*
3. *there is a set $\Gamma \subset \sigma(A)$, $E^A(\Gamma) = \mathbf{1}$, such that $\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda')$, $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Gamma$ (where, E^A is the spectral measure corresponding to A).*
4. *R generates $\mathcal{B}(\mathbb{R})$.*

4 Characterization of Semi-spectral Measures by means of Feller Markov Kernels

The following corollary is a consequence of theorem 7 and characterizes the commutative semispectral measures by means of Feller Markov kernel.

Definition 14. *A Feller Markov kernel is a Markov kernel $\mu_{(\cdot)}(\cdot) : \Lambda \times \mathcal{B}(\mathbb{R}) \to [0,1]$ such that the function*

$$G(\lambda) = \int_\Lambda f(t)\, d\mu_t(\lambda), \quad \lambda \in \Lambda$$

is continuous and bounded whenever f is continuous and bounded.

Corollary 2. *A semispectral measure $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if there exist a bounded self-adjoint operator A with spectrum $\sigma(A) \subset [0,1]$, a set $\Gamma \subset \Lambda$, $E^A(\Gamma) = \mathbf{1}$ and a Feller Markov kernel $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$ such that*

1) $F(\Delta) = \int_\Gamma \mu_\Delta(\lambda)\, dE^A_\lambda$

2) μ *separates the points of* Γ.

Proof. Let F, A, and μ be as in theorem 7. By theorem 7, $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is a Markov kernel. By theorem 6, it separates the points of Γ, i.e., $\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda')$, $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Gamma$. Therefore, Item 2 is proved. It remains to prove that $\mu_{(\cdot)}(\cdot) : \Gamma \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is a Feller Markov kernel. By theorem 7, μ_Δ is continuous for each $\Delta \in R$. Notice that for each open set $O \in \mathcal{B}(\mathbb{R})$, there is a countable family of sets $\Delta_i \in R$ such that $O = \cup_{i=1}^\infty \Delta_i$. Therefore, by theorem 2.2 in Ref. [16],

$$\lim_{n\to\infty} \int f(t)\, \mu_t(\lambda_n) = \int f(t)\, \mu_t(\lambda), \quad f \in \mathcal{C}_b(\mathbb{R})$$

whenever $\lim_{n\to\infty} \lambda_n = \lambda$ and $\mathcal{C}_b(\mathbb{R})$ is the space of bounded, continuous functions. □

5 Characterization of Semi-spectral Measures which admit strong Feller Markov Kernels

In the last section we proved that each commutative semispectral measure admits a strong Markov kernel μ such that μ_Δ is a continuous function for each $\Delta \in \mathcal{R}(\mathcal{S})$ where, $\mathcal{R}(\mathcal{S})$ is a ring which generates the Borel σ-algebra $\mathcal{B}([0,1])$.

In the present section we characterize the commutative semispectral measures for which the Markov kernel μ, whose existence was proved in theorem 1, is such that μ_Δ is continuous for each $\Delta \in \mathcal{B}([0,1])$. Whenever such a Markov kernel exists, we say that the semispectral measure admits a strong Feller Markov kernel. In particular, we prove that a commutative semispectral measure F admits a strong Feller Markov kernel if and only if F is uniformly continuous. First, we restrict ourselves to semispectral measures with spectrum in $[0,1]$. In the appendix A, we extend the results to the case of semispectral measures with unbounded spectrum (see theorem A2).

Definition 15. *Let F be a POVM. Let $\Delta = \cup_{i=1}^\infty \Delta_i$, $\Delta_i \cap \Delta_j = \emptyset$. If $\lim_{n\to\infty} \sum_{i=1}^n F(\Delta_i) = F(\Delta)$ in the uniform operator topology then we say that F is uniformly continuous.*

Notice that the term uniformly continuous derives from the fact that the σ-additivity of F in the uniform operator topology is equivalent to the continuity in the uniform operator topology. Analogously, the σ-additivity of F in the weak operator topology is equivalent to the continuity of F in the weak operator topology [15].

Definition 16. *A Markov kernel $\mu_{(\cdot)}(\cdot) : [0,1] \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is said to be strong Feller if μ_Δ is a continuous function for each $\Delta \in \mathcal{B}(\mathbb{R})$.*

Definition 17. *We say that a commutative POVM admits a strong Feller Markov kernel if there exists a strong Feller Markov kernel μ such that $F(\Delta) = \int \mu_\Delta(\lambda)\, dE_\lambda$, where E is the sharp reconstruction of F.*

Theorem 8. *A commutative POVM $F : \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ admits a strong Feller Markov kernel if and only if it is uniformly continuous.*

In order to prove the theorem we need the following lemmata.

Lemma 1. *Suppose F uniformly continuous. Suppose μ and A as in theorem 3. Then, for each $\lambda \in \sigma(A)$, $\mu_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(\mathcal{S})$.*

Proof. Let $\Delta, \Delta_i \in \mathcal{R}(\mathcal{S})$, $\Delta_i \cap \Delta_j = \emptyset$, $\cup_{i=1}^{\infty} \Delta_i = \Delta$. Then,

$$\mathbf{0} = u - \lim_{n\to\infty} \big(F(\Delta) - F(\cup_{i=1}^{n} \Delta_i)\big) = u - \lim_{n\to\infty} \int \Big(\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\Big)\, dE_\lambda.$$

By the uniform continuity of F and theorem 1, page 895, in Ref. [21], it follows that, $\forall \epsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n > \bar{n}$ implies,

$$\begin{aligned} \|\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\|_\infty &= \|\int \Big(\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\Big)\, dE_\lambda\| \\ &= \|F(\Delta) - F(\cup_{i=1}^{n} \Delta_i)\| \le \epsilon. \end{aligned} \tag{19}$$

By equation (19),

$$|\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)| \le \epsilon, \quad \forall \lambda \in [0,1].$$

□

Lemma 1 assures that μ is σ-additive on $\mathcal{R}(\mathcal{S})$. Therefore (see proposition 2 in Ref. [7]), the map $\mu : \sigma(A) \times \mathcal{R}(\mathcal{S}) \to [0,1]$ can be extended to a Markov kernel $\widetilde{\mu} : \sigma(A) \times \mathcal{B}([0,1]) \to [0,1]$ whose restriction to $\mathcal{R}(\mathcal{S})$ coincides with μ and such that $F(\Delta) = \widetilde{\mu}_\Delta(A)$.

Lemma 2. *Suppose F uniformly continuous. Suppose μ and A as in theorem 3. Let $\widetilde{\mu}$ be the extension of μ defined above. Then, for each open interval Δ, the function $\widetilde{\mu}_\Delta$ is continuous.*

Proof. For each open interval, there exists an increasing family of sets $\Delta_i \in \mathcal{S}$ such that $\Delta_i \uparrow \Delta$. Indeed, if $\Delta = (a,b)$, the family of sets $\{(a_i,b_i) \in \mathcal{S}\}_{i\in\mathbb{N}}$ such that $a_i > a_{i+1} \geq a$, $\lim_{i\to\infty} a_i = a$, $b_i < b_{i+1}$, $\lim_{i\to\infty} b_i = b$, is increasing and $\cup_{i=1}^\infty \Delta_i = \Delta$. Then,

$$\int \widetilde{\mu}_\Delta(\lambda)\, dE_\lambda = F(\Delta) = u - \lim_{i\to\infty} F(\Delta_i) = u - \lim_{i\to\infty} \int \widetilde{\mu}_{\Delta_i}(\lambda)\, dE_\lambda.$$

By the uniform continuity of F and theorem 1, page 895, in Ref. [21], it follows that, $\forall \epsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n, m > \bar{n}$ implies,

$$\begin{aligned} \|\widetilde{\mu}_{\Delta_n}(\lambda) - \widetilde{\mu}_{\Delta_m}(\lambda)\|_\infty &= \| \int [\widetilde{\mu}_{\Delta_n}(\lambda) - \widetilde{\mu}_{\Delta_m}(\lambda)]\, dE_\lambda \| \\ &= \|F(\Delta_n) - F(\Delta_m)\| \leq \epsilon. \end{aligned} \tag{20}$$

By equation (20),

$$|\widetilde{\mu}_{\Delta_n}(\lambda) - \widetilde{\mu}_{\Delta_m}(\lambda)| \leq \epsilon, \quad \forall \lambda \in \sigma(A). \tag{21}$$

Since $\widetilde{\mu}$ is a Markov kernel,

$$\lim_{i\to\infty} \widetilde{\mu}_{\Delta_i}(\lambda) = \widetilde{\mu}_\Delta(\lambda), \quad \forall \lambda \in \sigma(A).$$

Moreover, by equation (21), the convergence is uniform and this proves the continuity of $\widetilde{\mu}_\Delta$. □

Lemma 3. *Suppose F, $\widetilde{\mu}$ and A be as in lemma 2. Then, for each open set Δ, the function $\widetilde{\mu}_\Delta$ is continuous.*

Proof. Each open set Δ is the disjoint union of a countable family of open intervals, i.e., $\Delta = \cup_{i=1}^\infty \Delta_i$, $\Delta_i = (a_i, b_i)$. Let us define the set $\widetilde{\Delta}_n := \cup_{i=1}^n \Delta_i$. Therefore, $\widetilde{\Delta}_n \uparrow \Delta$. Moreover, $\mu_{\widetilde{\Delta}_n}$ is continuous for each $n \in \mathbb{N}$, and

$$u - \lim_{i\to\infty} F(\widetilde{\Delta}_n) = F(\Delta).$$

Then, the same reasoning we used in the preceding lemma allows us to conclude that the family of continuous functions $\mu_{\widetilde{\Delta}_n}$ converges uniformly to μ_Δ. □

The following lemma states the continuity of μ_{G_δ} for each G_δ set [32].

Lemma 4. *Suppose F, $\widetilde{\mu}$ and A be as in lemma 2. Then, for each G_δ set, the function $\widetilde{\mu}_{G_\delta}$ is continuous.*

Proof. For each G_δ set there exists [15] a family of open sets $\{G_i\}_{i\in\mathbb{N}}$, $G_\delta \subset G_i$, such that $\cap_{i=1}^{\infty} G_i = G_\delta$. Then, by the uniform continuity of F,

$$F(G_\delta) = F(\cap_{i=1}^{\infty} G_i) = u - \lim_{n\to\infty} F(\cap_{i=1}^{n} G_i) = u - \lim_{n\to\infty} F(\widetilde{G}_n)$$

where, $\widetilde{G}_n := \cap_{i=1}^{n} G_i$ and $\widetilde{G}_n \downarrow G_\delta$.
By theorem 1, page 895, in Ref. [21], it follows that, $\forall \epsilon > 0$, there exists a number $\bar{n} \in \mathbb{N}$, such that $n, m > \bar{n}$ implies,

$$\|\mu_{\widetilde{G}_n}(\lambda) - \mu_{\widetilde{G}_m}(\lambda)\|_\infty = \| \int (\mu_{\widetilde{G}_n}(\lambda) - \mu_{\widetilde{G}_m}(\lambda))\, dE_\lambda \| \leq \epsilon. \tag{22}$$

Since $\widetilde{\mu}$ is a Markov kernel, for each $\lambda \in \sigma(A)$,

$$\lim_{i\to\infty} \widetilde{\mu}_{\widetilde{G}_i}(\lambda) = \widetilde{\mu}_{G_\delta}(\lambda).$$

Moreover, by equation (22) the convergence is uniform and then $\widetilde{\mu}_{G_\delta}$ is continuous. □

Now, we are ready to prove theorem 8.

Proof of theorem 8. In order to prove the first part of the theorem, we use transfinite induction [32, 19]. Let G_0 be the family of open sets in $[0,1]$, ω_1 the first uncountable ordinal and G_α, $\alpha < \omega_1$ the Borel hierarchy (see page 236 in Ref. [32]). In particular, $G_1 = G_\delta$, $G_2 = G_{\delta\sigma}$, $G_3 = G_{\delta\sigma\delta}, \ldots$ and $G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$ for each limit ordinal α. By means of the same reasoning that we used in the proof of lemma 1, lemma 2, lemma 3 and lemma 4, one can prove the continuity of $\widetilde{\mu}_\Delta$ whenever Δ is of the kind $G_{\delta,\sigma}, G_{\delta\sigma\delta}\ldots$. Analogously, if $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in G_\alpha$ then, $\widetilde{\mu}_\Delta$ is continuous for each Δ in $G_{\alpha+1}$. Indeed, each set in $G_{\alpha+1}$ is either the countable union or the countable intersection of sets in G_α and the reasoning in lemma 3 or lemma 4 can be used. If α is a limit ordinal and $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in G_\beta$, $\beta < \alpha$, then, $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in G_\alpha = (\cup_{\beta<\alpha} G_\beta)_\sigma$. Indeed, each set in G_α is the countable union of sets in $\cup_{\beta<\alpha} G_\beta$ and the reasoning in lemma 2 can be used. Therefore, by transfinite induction, $\widetilde{\mu}_\Delta$ is continuous for each $\Delta \in \cup_{\alpha<\omega_1} G_\alpha = \mathcal{B}([0,1])$ [32].

In order to prove the second part of the theorem we show that the existence of a strong Feller Markov kernel implies the uniform continuity of F.
Suppose that there exists a strong Feller Markov kernel μ such that $F(\Delta) = \mu_\Delta(\lambda)$. Since μ is a Markov kernel it is σ-additive. Then,

$$\lim_{i\to\infty} \big(\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\big) = 0, \quad \lambda \in \sigma(A).$$

where, $\Delta, \Delta_i \in \mathcal{B}([0,1])$, $\cup_{i=1}^{\infty} \Delta_i = \Delta$.
By hypothesis,

$$\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda) \in \mathcal{C}(\sigma(A)), \quad \forall n \in \mathbb{N}.$$

Then, by theorem B1 in appendix B,

$$u - \lim_{i\to\infty} \big(\mu_\Delta(\lambda) - \sum_{i=1}^{n} \mu_{\Delta_i}(\lambda)\big) = 0.$$

By theorem 1, page 895, in Ref. [21], $\|F(\Delta)\| = \|\mu_\Delta\|_\infty$, hence

$$\lim_{n\to\infty} \|F(\Delta) - F(\cup_{i=1}^{n} \Delta_i)\| = \lim_{n\to\infty} \|\mu_\Delta - \sum_{i=1}^{n} \mu_\Delta\|_\infty = 0.$$

which proves that F is uniformly continuous. □

In the case of a semispectral measure with spectrum in $\mathbb{R}$, we have the following extension of theorem 8 (see theorem A2 in the appendix A).

Theorem 9. *A commutative semispectral measure $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ is uniformly continuous if and only if it admits a strong Feller Markov kernel.*

Example 2. *Let us consider the following unsharp position observable*

$$Q^f(\Delta) := \int_{[0,1]} \mu_\Delta(x)\, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{23}$$

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

where, f is a bounded, continuous function such that $f(y) = 0$, $y \notin [0,1]$ and

$$\int_{[0,1]} f(y)\, dy = 1,$$

and Q_x is the spectral measure corresponding to the position operator

$$\begin{aligned} Q : L^2([0,1]) &\to L^2([0,1]) \\ \psi(x) &\mapsto (Q\psi)(x) := x\psi(x) \end{aligned}$$

Notice that, for each $\Delta \in \mathcal{B}(\mathbb{R})$, $\mu_\Delta : [0,1] \to [0,1]$ is continuous. Indeed, by the uniform continuity of f, for each $\epsilon > 0$, there is a $\delta > 0$ such that $|x - x'| \leq \delta$ implies $|f(x-y) - f(x'-y)| \leq \epsilon$, for each y. Therefore,

$$\begin{aligned} |\mu_\Delta(x) - \mu_\Delta(x')| &= \Big| \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy - \int_{\mathbb{R}} \chi_\Delta(x'-y)\, f(y)\, dy \Big| \\ &= \Big| \int_\Delta [f(x-y) - f(x'-y)]\, dy \Big| \leq \epsilon \int_{\Delta\cap[-1,1]} dy \leq 2\epsilon \end{aligned}$$

By theorem 8 and the continuity of μ_Δ, $\Delta \in \mathcal{B}(\mathbb{R})$, Q^f is uniformly continuous. That can be proved as follows. Suppose $\Delta_i \downarrow \Delta$ and $f(y) \leq M$, $y \in \mathbb{R}$. Since, for each $x \in [0,1]$,

$$\mu_{\Delta_i - \Delta}(x) = \int_{\Delta_i - \Delta} f(x-y)\, dy \leq M \int_{(\Delta_i - \Delta)\cap[-1,1]} dx$$

we have that, for each $\psi \in \mathcal{H}$, $|\psi|^2 = 1$,

$$\langle \psi, Q^f(\Delta_i - \Delta)\psi \rangle = \int_{[0,1]} \mu_{\Delta_i - \Delta}(x)\, |\psi|^2(x)\, dx \leq M \int_{(\Delta_i - \Delta)\cap[-1,1]} dx$$

which proves the uniform continuity of Q^f.

In the case of uniformly continuous POV measures, we can prove a necessary condition for the norm-1-property.

Definition 18 ([26])**.** *A semispectral measure F has the norm-1-property if $\|F(\Delta)\| = 1$, for each $\Delta \in \mathcal{B}(\mathbb{R})$ such that $F(\Delta) \neq \mathbf{0}$.*

Theorem 10. *Let F be uniformly continuous. Then, F has the norm-1-property only if $\|F(\{\lambda\})\| \neq 0$ for each $\lambda \in \sigma(F)$.*

Proof. We proceed by contradiction. Suppose that F has the norm-1 property and that there exists $\lambda \in \sigma(F)$, such that $\|F(\{\lambda\})\| = 0$. Let $(a_i, b_i) \subset \mathcal{B}([0,1])$ be a sequence of open intervals such that, $a_i < \lambda < b_i$, $(a_{i+1}, b_{i+1}) \subset (a_i, b_i)$, $\lim_{i\to\infty} a_i = \lambda$, $\lim_{i\to\infty} b_i = \lambda$. Then, $(a_i, b_i) \downarrow \{\lambda\}$. Moreover, by the uniform continuity of F and the norm-1 property,

$$1 = \lim_{i\to\infty} \|F((a_i, b_i))\| = \lim_{i\to\infty} \|F((a_i, b_i)) - F(\{\lambda\})\| = 0.$$

□

Example 3. *Let Q^f be as in example 2. Theorem 10 implies that Q^f cannot have the norm-1 property. Indeed, for each $\lambda \in \mathbb{R}$,*

$$Q^f(\{\lambda\})\psi = \lim_{i\to\infty} Q^f([\lambda, \lambda_i))\psi = \lim_{i\to\infty} \mu_{[\lambda,\lambda_i)}(x)\psi(x) = 0, \quad \forall \psi \in \mathcal{H}$$

where, $\lambda, \lambda_i \in \mathbb{R}$, $\lambda_i \to \lambda$.

6 Absolutely continuous semispectral measures

In the present section, we prove that absolutely continuous commutative POV measures admit a strong Feller Markov kernel. Then, we apply the result to the case of the unsharp position observable.

Definition 19. *[40, 41] A POV measure $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ is absolutely continuous with respect to a measure $\nu : \mathcal{B}(\mathbb{R}) \to [0,1]$ if there exists a positive number c such that $\|F(\Delta)\| \leq c\,\nu(\Delta)$, for each $\Delta \in \mathcal{B}(\mathbb{R})$.*

Theorem 11. *Let F be absolutely continuous with respect to a finite measure ν. Then, F is uniformly continuous.*

Proof. Suppose $\Delta_i \uparrow \Delta$. We have

$$\begin{aligned}\lim_{n\to\infty} \|F(\Delta) - F(\Delta_i)\| &= \lim_{n\to\infty} \|F(\Delta - \Delta_i)\| \\ &\leq c \lim_{n\to\infty} \nu(\Delta - \Delta_i) = 0.\end{aligned}$$

which proves that F is uniformly continuous. □

Corollary 3. *Let F be absolutely continuous with respect to a finite measure ν. Then, F is commutative if and only if there exist a self-adjoint operator A and a strong Feller Markov kernel $\mu : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ such that:*

$$F(\Delta) = \mu_\Delta(A), \quad \Delta \in \mathcal{B}(\mathbb{R}) \tag{24}$$

Proof. By theorem 11, F is uniformly continuous. Then, theorem 9 implies the thesis. □

Example 4. *Let us consider the unsharp position operator defined as follows.*

$$Q^f(\Delta) := \int_{[0,1]} \mu_\Delta(x)\, dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{25}$$

$$\mu_\Delta(x) := \int_{\mathbb{R}} \chi_\Delta(x-y)\, f(y)\, dy, \quad x \in [0,1]$$

where, f is a positive, bounded, Borel function such that $f(x) = 0$, $x \notin [0,1]$,

$$\int_{[0,1]} f(x)dx = 1,$$

and Q_x is the spectral measure corresponding to the position operator

$$Q : L^2([0,1]) \to L^2([0,1])$$
$$\psi(x) \mapsto Q\psi := x\psi(x)$$

Q^f is absolutely continuous with respect to the measure

$$\nu(\Delta) = M \int_{\Delta\cap[-1,1]} dx.$$

Indeed, for each $\psi \in \mathcal{H}$, $|\psi|^2 = 1$,

$$\langle \psi, Q^f(\Delta)\psi\rangle = \int_{[0,1]} \mu_\Delta(x)\, \psi^2(x)\, dx \leq M \int_{\Delta\cap[-1,1]} dx$$

where, the inequality

$$\mu_\Delta(x) = \int_\Delta f(x-y)\, dy \leq M \int_{\Delta\cap[-1,1]} dx$$

has been used.
Therefore, by theorem 11, $Q^f(\Delta)$ is uniformly continuous.

6.1 Unsharp Position Observable

In the present subsection, we study an important kind of absolutely continuous POV measures, the unsharp position observables obtained as the marginals of a covariant phase space observable.

In the following $\mathcal{H} = L^2(\mathbb{R})$, Q and P denote position and momentum observables respectively and $*$ denotes convolution, i.e. $(f*g)(x) = \int f(y)g(x-y)dy$. Let us consider the joint position-momentum POV measure [1, 17, 18, 23, 28, 37, 41, 42]

$$F(\Delta \times \Delta') = \int_{\Delta \times \Delta'} U_{q,p}\,\gamma\,U^*_{q,p}\,dq\,dp$$

where, $U_{q,p} = e^{-iqP}e^{ipQ}$ and $\gamma = |f\rangle\langle f|$, $f \in L^2(\mathbb{R})$, $\|f\|_2 = 1$. The marginal

$$Q^f(\Delta) := F(\Delta \times \mathbb{R}) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |f|^2)(x)\,dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \tag{26}$$

is an unsharp position observable. Notice that the map $\mu_\Delta(x) := \mathbf{1}_\Delta * |f(x)|^2$ defines a Markov kernel.

Moreover, Q^f is absolutely continuous with respect to the Lebesgue measure. Indeed,

$$\begin{aligned} Q^f(\Delta) = F(\Delta \times \mathbb{R}) &= \int_{\Delta \times \mathbb{R}} U_{q,p}\,\gamma\,U^*_{q,p}\,dq\,dp \\ &= \int_\Delta dq \int_{\mathbb{R}} U_{q,p}\,\gamma\,U^*_{q,p}\,dp \\ &= \int_\Delta \widehat{Q}(q)\,dq \leq \int_\Delta \mathbf{1}\,dq \end{aligned}$$

where,

$$\widehat{Q}(q) = \int_{\mathbb{R}} U_{q,p}\,\gamma\,U^*_{q,p}\,dp.$$

Although Q^f is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, it is not uniformly continuous. That does not contradict theorem 11 since the Lebesgue measure on $\mathbb{R}$ is not finite. Anyway, Q^f is uniformly continuous on each Borel set Δ with finite Lebesgue measure.

Now, we show that Q^f is not in general uniformly continuous. We give the details of the following particular case.

Example 5 (Optimal Phase Space Representation)**.** *If we choose*

$$f^2(x) = \frac{1}{l\sqrt{2\pi}}\,e^{(-\frac{x^2}{2l^2})}, \quad l \in \mathbb{R} - \{0\}.$$

in (26), we get an optimal phase space representation of quantum mechanics [37]. In this case,

$$Q^f(\Delta) = \int_{-\infty}^{\infty} \Big(\int_{\Delta} |f(x-y)|^2)\, dy \Big)\, dQ_x$$
$$= \frac{1}{l\sqrt{2\,\pi}} \int_{-\infty}^{\infty} \Big(\int_{\Delta} e^{-\frac{(x-y)^2}{2\,l^2}}\, dy \Big)\, dQ_x = \int_{-\infty}^{\infty} \mu_{\Delta}(x)\, dQ_x$$

where,

$$\mu_{\Delta}(x) = \frac{1}{l\sqrt{2\,\pi}} \int_{\Delta} e^{-\frac{(x-y)^2}{2\,l^2}}\, dy \tag{27}$$

defines a Markov kernel.

In order to prove that Q^f is not uniformly continuous we consider the family of sets $\Delta_i = (-\infty, a_i)$, $\lim_{i\to\infty} a_i = -\infty$ such that $\Delta_i \downarrow \emptyset$, and prove that $\lim_{i\to\infty} \|Q^f(\Delta_i)\| = 1$. For each $i \in \mathbb{N}$,

$$\lim_{x\to -\infty} \mu_{\Delta_i}(x) = \lim_{x\to -\infty} \frac{1}{l\sqrt{2\,\pi}} \int_{\Delta_i} e^{-\frac{(x-y)^2}{2\,l^2}}\, dy$$
$$= \lim_{x\to -\infty} \frac{1}{l\sqrt{2\,\pi}} \int_{(-\infty,\, a_i - x)} e^{-\frac{y^2}{2\,l^2}}\, dy = \frac{1}{l\sqrt{2\,\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\,l^2}}\, dy = 1.$$

Now, we prove that $\|F(\Delta_i)\| = 1$, $i \in \mathbb{N}$. Indeed, if

$$\psi_n = \chi_{[-n,-n+1]}(x),$$

$$\lim_{n\to\infty} \langle \psi_n, Q^f(\Delta_i)\psi_n \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \mu_{\Delta_i}(x) |\psi_n(x)|^2\, dx \tag{28}$$
$$= \lim_{n\to\infty} \int_{[-n,-n+1]} \mu_{\Delta_i}(x)\, dx = 1. \tag{29}$$

Since, for each $\Delta \in \mathcal{B}(\mathbb{R})$, $\|Q^f(\Delta)\| \le 1$, equation (28) implies that $\|Q^f(\Delta_i)\| = 1$, for each $i \in \mathbb{N}$. Hence, $\lim_{i\to\infty} \|Q^f(\Delta_i)\| = 1$ and Q^f cannot be uniformly continuous.

It is worth noticing that although Q^f is not uniformly continuous, μ_{Δ} is continuous for each interval $\Delta \in \mathcal{B}(\mathbb{R})$. Indeed,

$$|\mu_{\Delta}(x) - \mu_{\Delta}(x')| = \frac{1}{l\sqrt{2\,\pi}} \Big| \int_{\Delta} e^{-\frac{(x-y)^2}{2\,l^2}}\, dy - \int_{\Delta} e^{-\frac{(x'-y)^2}{2\,l^2}}\, dy \Big|$$
$$= \frac{1}{l\sqrt{2\,\pi}} \Big| \int_{\Delta_x} e^{-\frac{(y)^2}{2\,l^2}} - \int_{\Delta_{x'}} e^{-\frac{(y)^2}{2\,l^2}}\, dy \Big| \le \frac{1}{l\sqrt{2\,\pi}} \Big| \int_{\overline{\Delta}} e^{-\frac{(y)^2}{2\,l^2}}\, dy \Big|$$

where,

$$\Delta_x = \{z \in \mathbb{R} \,|\, z = y - x,\, y \in \Delta\}, \quad \Delta_{x'} = \{z \in \mathbb{R} \,|\, z = y - x',\, y \in \Delta\}$$

and,

$$\overline{\Delta} = (\Delta_x - \Delta_{x'}) \cup (\Delta_{x'} - \Delta_x).$$

Therefore, $|x - x'| \leq \epsilon$ *implies,*

$$|\mu_\Delta(x) - \mu_\Delta(x')| \leq \frac{1}{l\,\sqrt{2\,\pi}} \Big| \int_{\overline{\Delta}} e^{-\frac{(y)^2}{2\,l^2}}\, dy \Big| \leq \frac{1}{l\,\sqrt{2\,\pi}} \int_{\overline{\Delta}} dy = \frac{\sqrt{2}}{l\,\sqrt{\pi}}\,\epsilon.$$

Appendices

A Semispectral measures with spectrum in $\mathbb{R}$

We show that the results proved in the preceding sections hold in the general case of a POV measure with spectrum in $\mathbb{R}$. It is sufficient to show that theorems 3 and 8 hold for POV measures with unbounded spectrum.
In the following f will denote a continuous, one-to-one function from $(0,1)$ to $\mathbb{R}$. For example, f can be the function $f(x) = \tan(\pi x - \frac{\pi}{2})$. Anyway, the results we are going to prove do not depend on the choice of f.

Definition 20. *Given a POV measure* $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$*, we introduce the POV measure*

$$\begin{aligned} \overline{F} :& \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H}) \\ \Delta \in& \mathcal{B}([0,1]) \mapsto \overline{F}(\Delta) := F[f(\Delta \cap (0,1))] \end{aligned}$$

where, $f : (0,1) \to \mathbb{R}$ *is a continuous one-to-one function.*

Notice that the POV measure $\overline{F}$ has spectrum in $[0,1]$ and

$$F(\Delta) = \overline{F}(f^{-1}(\Delta)), \quad \Delta \in \mathcal{B}(\mathbb{R}).$$

Theorem A1. *For any real commutative POV measure F with spectrum in $\mathbb{R}$, there exist a bounded self-adjoint operator A with spectrum $\sigma(A) \subset [0,1]$, a strong Markov Kernel $\mu : \sigma(A) \times \mathcal{B}(\mathbb{R}) \to [0,1]$ and a ring R of subset of $\mathbb{R}$ which generates $\mathcal{B}(\mathbb{R})$, such that:*

1) $\mu_\Delta(\cdot)$ *is continuous for each* $\Delta \in R$,

2) $\mu_{(\cdot)}(\lambda)$ *is additive on* R,

3) $F(\Delta) = \mu_\Delta(A), \quad \Delta \in \mathcal{B}(\mathbb{R})$,

4) there is a set $\Gamma \subset \sigma(A)$, $E^A(\Gamma) = \mathbf{1}$, *such that* $\mu_{(\cdot)}(\lambda) \neq \mu_{(\cdot)}(\lambda')$, $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Gamma$.

Proof. Let $\overline{F}$ be as in definition 20. Since $\overline{F}$ has spectrum in $[0, 1]$, theorem 6 applies. Therefore, there exists a bounded self-adjoint operator A with spectrum $\sigma(A) \subset [0, 1]$, a strong Markov kernel $\overline{\mu}_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}([0, 1]) \to [0, 1]$, and a ring $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}([0, 1])$, such that

i) $\overline{\mu}_\Delta(\cdot)$ is continuous for each $\Delta \in \mathcal{R}(\mathcal{S})$,

ii) $\overline{\mu}_{(\cdot)}(\lambda)$ is additive on $\mathcal{R}(\mathcal{S})$,

iii) $F(\Delta) = \overline{\mu}_\Delta(A), \quad \Delta \in \mathcal{B}([0, 1])$,

iv) there is a set $\Gamma \subset \sigma(A)$, $E^A(\Gamma) = \mathbf{1}$, such that $\overline{\mu}_{(\cdot)}(\lambda) \neq \overline{\mu}_{(\cdot)}(\lambda')$, $\lambda \neq \lambda'$, $\lambda, \lambda' \in \Gamma$,

v) $\mathcal{R}(\mathcal{S})$ generates the Borel σ-algebra $\mathcal{B}([0, 1])$.

Now, let us consider the map

$$\begin{aligned} \mu_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}(\mathbb{R}) \to [0, 1] \\ (\lambda, \Delta) \mapsto \mu_\Delta(\lambda) := \overline{\mu}_{f^{-1}(\Delta)}(\lambda) \end{aligned} \tag{30}$$

Since $\cup_{i=1}^\infty f^{-1}(\Delta_i) = f^{-1}(\cup_{i=1}^\infty \Delta_i)$, $\Delta_i \cap \Delta_j = \emptyset$, and $\overline{\mu}$ is a strong Markov kernel, the map μ must be a strong Markov kernel. Since $\overline{\mu}$ is additive on $\mathcal{R}(\mathcal{S})$, μ is additive on the ring $R = \{f(\Delta \cap (0, 1)), \Delta \in \mathcal{R}(\mathcal{S})\}$ which generates (see page 63 in Ref. [34]) the Borel σ-algebra of the reals $\mathcal{B}(\mathbb{R})$. Moreover,

$$\int_{\sigma(A)} \mu_\Delta(\lambda)\, dE_\lambda^A = \int_{\sigma(A)} \overline{\mu}_{f^{-1}(\Delta)}(\lambda)\, dE_\lambda^A = \overline{F}(f^{-1}(\Delta)) = F(\Delta)$$

so that (F, A, μ) is a von Neumann triplet.
Since $\overline{\mu}$ is continuous on $\mathcal{R}(\mathcal{S})$, all the functions $\mu_\Delta(\cdot)$, $\Delta \in R$, are continuous. Item 4 comes directly from item iv) above. □

Now, we extend theorem 8 to the case of POVM with spectrum in $\mathbb{R}$.

Theorem A2. *A commutative semispectral measure $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ admits a strong Feller Markov kernel if and only if it is uniformly continuous.*

Proof. First, we prove that F is uniformly continuous if and only if $\overline{F}$ is uniformly continuous. Suppose F uniformly continuous and $\Delta, \Delta_i \in \mathcal{B}([0,1])$, $\Delta_i \downarrow \Delta$. We have,

$$\lim_{i\to\infty} \|\overline{F}(\Delta_i) - \overline{F}(\Delta)\| = \lim_{i\to\infty} \|F[f(\Delta_i \cap (0,1)] - F[f(\Delta \cap (0,1)]\| = 0$$

Conversely, suppose $\overline{F}$ uniformly continuous and $\Delta, \Delta_i \in \mathcal{B}(\mathbb{R})$, $\Delta_i \downarrow \Delta$. We have,

$$\lim_{i\to\infty} \|F(\Delta_i) - F(\Delta)\| = \lim_{i\to\infty} \|\overline{F}[f^{-1}(\Delta_i)] - \overline{F}[f^{-1}(\Delta)]\| = 0$$

Now, we can proceed with the proof of the theorem. Suppose F uniformly continuous. Then, $\overline{F}$ is uniformly continuous and, by theorem 8, there exist a bounded self-adjoint operator A and a strong Feller Markov kernel $\overline{\mu}_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}([0,1]) \to \mathcal{F}(\mathcal{H})$ such that, for each $\Delta \in \mathcal{B}([0,1])$, $\overline{\mu}_\Delta(A) = \overline{F}(\Delta)$. Let us consider the following Markov kernel,

$$\begin{aligned} \mu_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}(\mathbb{R}) &\to [0,1] \\ (\lambda, \Delta) &\mapsto \mu_\Delta(\lambda) := \overline{\mu}_{f^{-1}(\Delta)}(\lambda) \end{aligned}$$

We have

$$\int_{\sigma(A)} \mu_\Delta(\lambda)\, dE^A_\lambda = \int_{\sigma(A)} \overline{\mu}_{f^{-1}(\Delta)}(\lambda)\, dE^A_\lambda = \overline{F}(f^{-1}(\Delta)) = F(\Delta)$$

so that (F, A, μ) is a von Neumann triplet.
Moreover [34], $\mathcal{B}(\mathbb{R}) = \{f(\Delta \cap (0,1)),\, \Delta \in \mathcal{B}([0,1])\}$. By definition, $\mu_B(\cdot) = \overline{\mu}_{\Delta\cap(0,1)}(\cdot)$ for each Borel set $B := f(\Delta \cap (0,1))$, $\Delta \in \mathcal{B}([0,1])$. Since, $\overline{\mu}$ is a strong Feller Markov kernel, the function $\mu_B(\cdot) = \overline{\mu}_{\Delta\cap(0,1)}(\cdot)$ is continuous. Therefore, μ is a strong Feller Markov kernel.
Conversely, suppose that there exists a strong Feller Markov kernel $\mu_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}(\mathbb{R}) \to [0,1]$ such that $\mu_\Delta(A) = F(\Delta)$. Then, thanks to the bijectivity of f, the map

$$\overline{\mu}_{(\cdot)}(\cdot) : \sigma(A) \times \mathcal{B}([0,1]) \to [0,1]$$
$$(\lambda, \Delta) \mapsto \overline{\mu}_\Delta(\lambda) := \mu_{[f(\Delta \cap (0,1)]}(\lambda)$$

is a strong Feller Markov kernel. Indeed, for each $\lambda \in \sigma(A)$,

$$\begin{aligned}\overline{\mu}_\Delta(\lambda) &= \overline{\mu}_{\cup_{i=1}^\infty \Delta_i}(\lambda) = \mu_{f[\cup_{i=1}^\infty \Delta_i \cap (0,1)]}(\lambda)\\ &= \mu_{\cup_{i=1}^\infty [f(\Delta_i \cap (0,1)]}(\lambda) = \sum_{i=1}^\infty \mu_{f(\Delta_i \cap (0,1))}(\lambda) = \sum_{i=1}^\infty \overline{\mu}_{\Delta_i}(\lambda).\end{aligned}$$

where, $\Delta = \cup_{i=1}^\infty \Delta_i$, $\Delta_i \cap \Delta_j = \emptyset$. Moreover,

$$\int_{\sigma(A)} \overline{\mu}_\Delta(\lambda)\, dE_\lambda^A = \int_{\sigma(A)} \mu_{[f(\Delta)\cap(0,1)]}(\lambda)\, dE_\lambda^A = F[f(\Delta) \cap (0,1)) = \overline{F}(\Delta).$$

Therefore, theorem 8 implies the uniform continuity of $\overline{F}$ and then the uniform continuity of F. □

B Sequences of continuous functions

The following theorem is due to Dini. We give a proof based on the use of sequences.

Theorem B1. *Let $\{f_n(\lambda)\}_{n\in\mathbb{N}}$ be a non increasing sequence of continuous functions defined on a compact set $B \subset [0,1]$ with values in $[0,1]$ and such that $f_n(\lambda) \to 0$ point-wise. Then, $f_n(\lambda) \to 0$ uniformly.*

Proof. Since $f_{n+1}(\lambda) \leq f_n(\lambda)$ for each $\lambda \in B$, we have $\|f_{n+1}\|_\infty \leq \|f_n\|_\infty$. If $\|f_n\|_\infty \to 0$ clearly $f_n(\lambda) \to 0$ uniformly.
Then, suppose $\|f_n\|_\infty \to a > 0$. Since $\|f_{n+1}\|_\infty \leq \|f_n\|_\infty$, we have $\|f_n\|_\infty \geq a$, for each $n \in \mathbb{N}$.
Let λ_n be such that $f_n(\lambda_n) = \|f_n\|_\infty$. Since $\{\lambda_n\}$ is a bounded sequence of real numbers, there exists a convergent subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Let β be its limit, i.e., $\beta := \lim_{k\to\infty} \lambda_{n_k}$. The compactness of B assures that $\beta \in B$. Moreover, $\lim_{k\to\infty} f_{n_k}(\lambda_{n_k}) = a$.
Let us consider the sequence of numbers $f_{n_k}(\beta)$. We prove that $f_{n_k}(\beta) \geq a$ for each $k \in \mathbb{N}$. We proceed by contradiction. Suppose that there exists $\bar{k} \in \mathbb{N}$

such that $f_{n_{\bar{k}}}(\beta) < a$. Then, there exists a neighborhood $I(\beta)$ of β such that $f_{n_{\bar{k}}}(\lambda) < a$ for each $\lambda \in I(\beta)$. Moreover, since $\lambda_{n_k} \to \beta$, there exists $l \in \mathbb{N}$ such that $k > l$ implies $\lambda_{n_k} \in I(\beta)$. Take $k > \max\{\bar{k}, l\}$. Then, $\lambda_{n_k} \in I(\beta)$ and $f_{n_k}(\lambda) \leq f_{n_{\bar{k}}}(\lambda)$, for each $\lambda \in B$. Therefore,

$$f_{n_k}(\lambda_{n_k}) \leq f_{n_{\bar{k}}}(\lambda_{n_k}) < a$$

which contradicts the fact that $f_{n_k}(\lambda_{n_k}) = \|f_{n_k}\|_\infty \geq a$, for each $k \in \mathbb{N}$.
We have proved that $f_{n_k}(\beta) \geq a$, for each $k \in \mathbb{N}$. This implies that $\lim_{k\to\infty} f_{n_k}(\beta) \geq a$ and contradicts one of the hypothesis of the lemma, i.e., $\lim_{n\to\infty} f_n(\lambda) = 0$ for each $\lambda \in B$. $\square$

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