V. Models with Overlapping Generations (Continued)

(II) Government Borrowing

- Government issues one-period riskless bonds.
- B_t = number of units of bonds sold in period t.
- p_t = price of a bond.
- Government budget constraint in period t is

$$N_t t_{1t} + N_{t-1} t_{2t} + p_t B_t = B_{t-1}.$$

- A member of generation t faces the sequence of budget constraints

$$s_t = y_{1t} - c_{1t} - t_{1t} - p_t b_t$$

$$c_{2t+1} = y_{2t+1} - t_{2t+1} + (1 + r_t) s_t + b_t$$

where b_t = quantity of government bond purchased by a member of generation t.

- The present-value budget constraint is

$$c_{1t} + \frac{c_{2t+1}}{1 + r_t} = y_{1t} - t_{1t} + \frac{y_{2t+1} - t_{2t+1}}{1 + r_t} - b_t \left[p_t - \frac{1}{1 + r_t} \right].$$

- Since 1-period government bonds have the same risk characteristics as the 1-period private borrowing and lending, arbitrage implies $1 + r_t = 1/p_t$.
- Present-value budget constraint is unaffected by the addition of government borrowing in the model $\Rightarrow$ consumption and saving functions are also unaffected.
- One consequence of having two identical assets in the economy (government bonds and private borrowing and lending) is that we are unable to identify the particular mix of private borrowing/lending and government bonds that an individual chooses (*i.e.* we cannot write out individual demand functions for particular assets).

- Competitive equilibrium condition is

$$S_t(r_t) = p_t B_t \quad \text{or} \quad S_t(r_t) = \frac{B_t}{1+r_t}.$$

- Note that $p_t B_t$ = number of units of the time-t good the government wishes to borrow.

A Numerical Example

- $N_t = 100$ for all t.
- Utility of an agent born in period t:

$$u(c_{1t}, c_{2t+1}) = \ln c_{1t} + 0.95 \ln c_{2t+1}.$$

- Endowment stream $\{y_{1t}, y_{2t+1}\} = \{1, 1.25\}$.
- Government policy:
  - In period 1, the government borrows 5 units of the time 1 good and transfers those units to the old individuals alive in period 1.
  - The government will payoff its debt by taxing the members of generation 2 in their young age and will issue no debt after that.
  - Therefore we have

$$p_1 B_1 = 5, \quad t_{21} = \frac{-5}{100} = -0.05, \quad t_{12} = \frac{B_1}{100}.$$

- **Period 1**
  - Aggregate savings function is

$$S_1(r_1) = 48.72 - \frac{64.10}{1+r_1}$$

  - Equilibrium condition is

$$48.72 - \frac{64.10}{1+r_1} = 5 \quad \Rightarrow \quad r_1 = 0.47.$$

- Since $B_1/(1+r_1) = 5$ we have $B_1 = 5 \times 1.47 = 7.35$.
- Using $r_1 = 0.47$ in the consumption functions we get $c_{11} = 0.95$ and $c_{22} = 1.33$.
- An old agent alive in period 1 consumes $c_{21} = 1.25 + 0.05 = 1.3$.

- **Period 2**
  - Taxes imposed on period 2 young are $t_{12} = 7.35/100 = 0.0735$.
  - Aggregate savings function is

$$S_2(r_2) = 45.14 - \frac{64.10}{1+r_2}$$

  - No government borrowing in period 2 implies

$$45.14 - \frac{64.10}{1+r_2} = 0 \quad \Rightarrow \quad r_2 = 0.42$$

  - Using $r_2 = 0.42$ in the consumption functions we get $c_{12} = 0.93$ and $c_{23} = 1.25$.

- **Period $t \geq 3$**
  - No government involvement in the economy.
  - All cohort members are identical so

$$c_{1t} = 1, \quad c_{2t+1} = 1.25, \qquad t \geq 3.$$

- Competitive equilibrium prices and consumption allocation are

$$\{r_t\}_{t=1}^{\infty} = \{0.47, 0.42, 0.32, 0.32, \ldots\}$$

$$\{c_{1t}\}_{t=1}^{\infty} = \{0.95, 0.93, 1, 1, \ldots\}$$

$$\{c_{2t}\}_{t=1}^{\infty} = \{1.30, 1.33, 1.25, 1.25, \ldots\}.$$

Ricardian Equivalence

Given an initial equilibrium under some pattern of lump-sum taxation and government borrowing, alternative (intertemporal) patterns of lump-sum taxation that **keep the present value (at the initial equilibrium's interest rate) of each individual's total tax liability equal to that in the initial equilibrium** are equivalent in the following sense. Corresponding to each alternative taxation pattern is a pattern of government borrowing such that the initial equilibrium's consumption allocation, including consumption of the government, and the initial equilibrium's gross interest rates are an equilibrium under the alternative taxation pattern.

Diamond's Model (OLG Model with Production)

- Production economy where aggregate output is given by

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}, \qquad 0 < \alpha < 1$$

  The difference between Endowment and Production economies:

  ∗ Individuals are endowed with time rather than units of goods.

  ∗ There is still only one good but it is storable, so it can be used for consumption and investment.

  ∗ The model includes firms who employ workers and capital to produce.

- No capital depreciation.

- Agents are endowed with one unit of labour time when young and 0 units when old.

- Labour time supplied inelastically. Wage rate is w_t.

- Households savings are invested in physical capital. Rate of return (real interest rate) is r_t.

- Household's sequence of budget constraint

$$\begin{aligned} c_{1t} &= w_t - s_t \\ c_{2t+1} &= (1+r_t)s_t \end{aligned}$$

- Present-value budget constraint

$$c_{2t+1} = (1+r_t)(w_t - c_{1t})$$

- Assuming log utility, household problem is

$$\max \quad \ln c_{1t} + \beta \ln[(1+r_t)(w_t - c_{1t})]$$

$$\Rightarrow \qquad c_{1t} = \frac{1}{1+\beta} w_t, \quad s_t(w_t) = \frac{\beta}{1+\beta} w_t$$

- Since production technology has CRS, the number of firms is indeterminate. Assume there is a single firm.

- Firm maximizes (aggregate) profits

$$\Pi_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - r_t K_t$$

$$\Rightarrow \qquad r_t = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha}, \quad w_t = (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha}$$

- Population grows at rate $\eta \geq 0$

$$N_{t+1} = (1+\eta) N_t \quad \Rightarrow \quad N_t = (1+\eta)^t N_0$$

- Technology grows at rate $\gamma \geq 0$

$$A_{t+1} = (1+\gamma) A_t \quad \Rightarrow \quad A_t = (1+\gamma)^t A_0$$

- In summary we have

$$S_t(w_t) = \frac{\beta}{1+\beta} N_t w_t \tag{1}$$

$$r_t = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{2}$$

$$w_t = (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{3}$$

$$K_{t+1} = S_t(w_t) \tag{4}$$

- Nonlinear difference equation in K
  - Combine (1) and (4)
$$K_{t+1} = \frac{\beta}{1+\beta} N_t w_t$$
  - Plug in (3)
$$K_{t+1} = \frac{\beta}{1+\beta} N_t (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha}$$
  - Simplifying, we get the transition equation
$$K_{t+1} = \frac{(1-\alpha)\beta}{1+\beta} A_t K_t^{\alpha} N_t^{1-\alpha}$$

Case 1: No population Growth, No Technological Growth

- $\eta = \gamma = 0$; $N_t = N$ and $A_t = A$ for all t.
- Transition equation becomes
$$K_{t+1} = \left[\frac{(1-\alpha)\beta}{1+\beta} A N^{1-\alpha} \right] K_t^{\alpha} \equiv z K_t^{\alpha}.$$
- Steady-state capital stock K^* is found by setting $K_{t+1} = K_t$ in transition equation
$$\Rightarrow \qquad K^* = z^{\frac{1}{1-\alpha}}$$
- The model without growth in technology and in population eventually reaches a steady-state where all variables are constant over time.
- In the model with logarithmic utility, the saving rate is $\beta/(1+\beta)$.
- To see the effect of a change in the savings rate, suppose agents become less patient ($\beta' < \beta$).
$$\frac{\partial z}{\partial \beta} = \frac{(1-\alpha) A N^{1-\alpha}}{(1+\beta)^2} \geq 0$$

Case 2: Population Growth, No Technological Growth

- $\eta > 0$ and $\gamma = 0$
- Transition equation becomes

$$K_{t+1} = \left[\frac{(1-\alpha)\beta}{1+\beta}A\right] N_t^{1-\alpha} K_t^{\alpha}$$

$$K_{t+1} = \left[\frac{(1-\alpha)\beta}{1+\beta}AN_0^{1-\alpha}\right] (1+\eta)^{(1-\alpha)t} K_t^{\alpha}$$

$$K_{t+1} \equiv z_n (1+\eta)^{(1-\alpha)t} K_t^{\alpha}.$$

- Aggregate capital (and output) grows at rate η along a balanced growth path (BGP)

$$\begin{aligned} K_{t+1} &= z_n(1+\eta)^{(1-\alpha)t} K_t^{\alpha} \\ K_t &= z_n(1+\eta)^{(1-\alpha)(t-1)} K_{t-1}^{\alpha} \\ \Rightarrow \quad \frac{K_{t+1}}{K_t} &= (1+\eta)^{1-\alpha}\left(\frac{K_t}{K_{t-1}}\right)^{\alpha} \end{aligned}$$

Along a BGP, all variables grow at constant (possibly different) rates

$$\begin{aligned} \text{i.e.} \quad \frac{K_{t+1}}{K_t} &= \frac{K_t}{K_{t-1}} \quad \text{along a BGP} \\ \Rightarrow \quad \frac{K_{t+1}}{K_t} &= (1+\eta)^{1-\alpha}\left(\frac{K_{t+1}}{K_t}\right)^{\alpha} \\ \Rightarrow \quad \frac{K_{t+1}}{K_t} &= (1+\eta) \end{aligned}$$

- Since K grows at same rate as N, the capital-labour ratio does not grow.
- Define the capital-labour ratio

$$k_t = \frac{K_t}{N_t}$$

- We can write the transition equation in terms of k. Divide both sides by N_t

$$\begin{aligned} \frac{N_{t+1}}{N_t}\frac{K_{t+1}}{N_{t+1}} &= \left[\frac{(1-\alpha)\beta A}{1+\beta}\right]\frac{N_t^{1-\alpha}}{N_t^{1-\alpha}}\frac{K_t^{\alpha}}{N_t^{\alpha}} \\ (1+\eta)k_{t+1} &= \left[\frac{(1-\alpha)\beta A}{1+\beta}\right] k_t^{\alpha} \\ k_{t+1} &= \left[\frac{(1-\alpha)\beta A}{(1+\beta)(1+\eta)}\right] k_t^{\alpha} \end{aligned}$$

- The capital-labour ratio in steady-state is

$$k^* = \left[\frac{(1-\alpha)\beta A}{(1+\beta)(1+\eta)}\right]^{\frac{1}{1-\alpha}}$$

- K, L, Y grow at the same rate (η) in the long run, k, w, r will not grow in the long run.

Case 3: Technological Growth, No Population Growth

- $\eta = 0$ and $\gamma > 0$
- Transition equation becomes

$$\begin{aligned} K_{t+1} &= \frac{(1-\alpha)\beta}{1+\beta} A_t N^{1-\alpha} K_t^{\alpha} \\ K_{t+1} &= \left[\frac{(1-\alpha)\beta N^{1-\alpha} A_0}{1+\beta}\right] (1+\gamma)^t K_t^{\alpha} \end{aligned}$$

- Or using capital-labour ratio

$$k_{t+1} = \left[\frac{(1-\alpha)\beta A_0}{1+\beta}\right] (1+\gamma)^t k_t^{\alpha}$$

- In case 3, both K and k grow at a gross rate of $(1+\gamma)^{\frac{1}{1-\alpha}}$ along a BGP

$$\begin{aligned} \frac{k_{t+1}}{k_t} &= \frac{(1+\gamma)^t}{(1+\gamma)^{t-1}} \left(\frac{k_t}{k_{t-1}}\right)^{\alpha} \\ \Rightarrow \quad \frac{k_{t+1}}{k_t} &= (1+\gamma)^{\frac{1}{1-\alpha}} \end{aligned}$$

- Growth in other variables
  - r_t constant along a BGP
  - w_t grows at rate $(1+\gamma)^{\frac{1}{1-\alpha}}$
  - consumption per worker (c_{1t}) grows at a gross rate $(1+\gamma)^{\frac{1}{1-\alpha}}$
  - consumption per retired individual (c_{2t}) grows at a gross rate $(1+\gamma)^{\frac{1}{1-\alpha}}$

Why Do We Observe Growth in Per Capita Variables in Case 3?

- For a given capital stock and population, an increase in A_t increases the rate of return in capital (r_t).
- This increase in r_t makes investment in capital more interesting. As a result, households save more.
- Higher savings imply higher aggregate capital stock.
- Since population is constant, the capital-labour ratio increases also (which depresses r_t.)

A Final Note

- In all 3 cases, the initial capital stock have no effects on the growth rate and level of capital stock in the long run.
- Whatever initial capital, K always end up on the same BGP in the long run.
- Also true for k.
- Cross-country differences in the initial condition A_0 do have long-run implications.