V. Models with Overlapping Generations

Environment

- there are agents from different age groups (or cohorts)
- individuals live for two periods only
- each period, a new generation of individuals is born
- the number of individuals born in period t is N_t
- assume all members of a generation are identical

		Time periods						
	$\cdots$	0	1	2	3	4	$\cdots$
	$\vdots$	old					
	0	young-0	old-0				
Generations	1		young-1	old-1			
	2			young-2	old-2		
	3				young-3	old-3	
	$\vdots$					young	

- there is only one consumption good each period (period-t good)
- there is no storage technology
- there is a market (financial market) for private borrowing and lending
- for an agent born in period t we have
  - c_{1t}: consumption in young age
  - c_{2t+1}: consumption in old age
  - utility function $u(c_{1t}, c_{2t+1})$
  - endowment stream $\{y_{1t}, y_{2t+1}\}$

- aggregate quantities are
  - aggregate consumption of the time t good:

$$C_t = N_{t-1}c_{2t} + N_t c_{1t} \quad \forall t$$

  - aggregate endowment of the time t good:

$$Y_t = N_{t-1}y_{2t} + N_t y_{1t} \quad \forall t$$

Agent's Optimization Problem

- The sequence of budget constraints for an individual born in period t

$$\begin{aligned} s_t &= y_{1t} - c_{1t} \\ c_{2t+1} &= y_{2t+1} + (1+r_t)s_t \end{aligned}$$

- Present value budget constraint for an individual born in period t

$$c_{1t} + \frac{c_{2t+1}}{1+r_t} = y_{1t} + \frac{y_{2t+1}}{1+r_t}.$$

- Optimization problem solved by an individual born in period t is to maximize $u(c_{1t}, c_{2t+1})$ subject to the Present Value budget constraint.

- Logarithmic utility example:
  - Maximize

$$u(c_{1t}, c_{2t+1}) = \ln c_{1t} + \beta \ln c_{2t+1}$$

    subject to

$$c_{1t} + \frac{c_{2t+1}}{1+r_t} = y_{1t} + \frac{y_{2t+1}}{1+r_t}.$$

  - Unconstrained maximization problem is

$$\max_{c_{1t}} \quad \ln(c_{1t}) + \beta \ln[y_{2t+1} + (1+r_t)(y_{1t} - c_{1t})].$$

– The first-order condition is

$$\frac{1}{c_{1t}} - \frac{\beta(1+r_t)}{y_{2t+1} + (1+r_t)(y_{1t} - c_{1t})} = 0$$

– which implies

$$c_{1t}(r_t, y_{1t}, y_{2t+1}) = \frac{1}{1+\beta}\left[y_{1t} + \frac{y_{2t+1}}{1+r_t}\right]$$

– Using the Present Value budget constraint

$$c_{2t+1}(r_t, y_{1t}, y_{2t+1}) = \frac{\beta}{1+\beta}[(1+r_t)y_{1t} + y_{2t+1}]$$

– Using the definition of savings

$$s_t(r_t, y_{1t}, y_{2t+1}) = \frac{\beta}{1+\beta}y_{1t} - \frac{y_{2t+1}}{(1+\beta)(1+r_t)}.$$

Competitive Equilibrium

- A competitive equilibrium is a consumption allocation and a price system such that
  1. the quantities that are relevant for a particular person maximize that person's utility subject to the relevant budget constraint, taking prices as given.
  2. the quantities clear all markets at all dates.
- There are only two markets in period t
  – a market (goods market) for the period-t consumption good
  – a market (financial market) for private borrowing and lending
- Only young individuals participate to the borrowing/lending market
  – Let $S_t(r_t)$ represent total savings of the period-t young generation
  – In equilibrium, market clearing condition on the borrowing/lending market in period t is then $S_t(r_t) = 0$.
- The Equilibrium conditions $S_t(r_t) = 0$ takes into account of

- market clearing on the goods market
- market clearing on the financial market
- utility maximization.

- when all members of a generation are identical and have logarithmic utility the aggregate savings function is

$$S_t(r_t) = N_t s_t(r_t) = \frac{\beta}{1+\beta} N_t y_{1t} - \frac{N_t y_{2t+1}}{(1+\beta)(1+r_t)}.$$

- Imposing the equilibrium condition $S_t(r_t) = 0$ we find an expression for the equilibrium interest rate

$$r_t = \frac{y_{2t+1}}{\beta y_{1t}} - 1.$$

A Numerical Example

- $N_t = 75$ for $t \geq 0$.
- utility is logarithmic

$$u(c_{1t}, c_{2t+1}) = \ln c_{1t} + 0.9 \ln c_{2t+1}$$

- endowment stream $\{y_{1t}, y_{2t+1}\} = \{2.2, 2\} \quad \forall t$
- present value budget constraint

$$c_{1t} + \frac{c_{2t+1}}{1+r_t} = 2.2 + \frac{2}{1+r_t}$$

- unconstrained optimization problem

$$\max_{c_{1t}} \quad \ln c_{1t} + 0.9 \ln[2.2(1+r_t) + 2 - (1+r_t)c_{1t}]$$

- first-order condition

$$\frac{1}{c_{1t}} = \frac{0.9(1+r_t)}{2.2(1+r_t) + 2 - (1+r_t)c_{1t}}$$

- consumption function in young age

$$c_{1t}(r_t) = 1.1579 + \frac{1.0526}{1+r_t}$$

- savings function

$$s_t(r_t) = 2.2 - c_{1t}(r_t) = 1.0421 - \frac{1.0526}{1+r_t}$$

- using the present value budget constraint we have

$$c_{2t+1} = 0.9474 + 1.0421(1+r_t)$$

- Aggregate savings are

$$S_t(r_t) = N_t s_t(r_t) = 78.1575 - \frac{78.945}{1+r_t}$$

- The equilibrium condition $S_t(r_t) = 0$ implies

$$r_t = \frac{78.945}{78.1575} - 1 = 0.01$$

- and therefore

$$c_{1t} = 2.2, \qquad c_{2t+1} = 2.$$

- Since all periods are identical, the competitive equilibrium prices and consumption allocation are

$$\{r_t\}_{t=1}^{\infty} = \{0.01\}_{t=1}^{\infty}, \qquad \{c_{1t}, c_{2t}\}_{t=1}^{\infty} = \{2.2, 2\}_{t=1}^{\infty}.$$

Two Extensions

(I) Population Growth

- cohort size not constant over time

$$N_t = (1+\eta)N_{t-1}$$

- population grows at rate η
- so does aggregate consumption (C_t) and aggregate endowment (Y_t)
- the equilibrium interest rate and individual quantities (c_{1t}, s_t, c_{2t+1}) are unaffected.

(II) Heterogeneity within Cohorts

- Heterogeneity creates incentives for trade.
- Assume there are two types of agents born in each period.
- Type 1 (group 1)
  - N_t individuals of type 1 are born in period t
  - they have utility function $u(c_{1t}, c_{2t+1})$
  - and an endowment stream $\{y_{1t}, y_{2t+1}\}$.
- Type 2 (group 2)
  - N_t^* individuals of type 2 are born in period t
  - they have utility function $u^*(c_{1t}^*, c_{2t+1}^*)$
  - and an endowment stream $\{y_{1t}^*, y_{2t+1}^*\}$.
- The two types of agents can differ in three respects:
  - N_t does not necessarily equal N_t^*
  - $u(-)$ doest not necessarily equal $u(-)$, for example, $\beta \neq \beta^*$
  - endowment streams can differ across countries

Heterogeneity — Numerical Example

- Type 1 (Group 1)

- $N_t = 75$ for $t \geq 0$
- $u(c_{1t}, c_{2t+1}) = \ln c_{1t} + 0.9 \ln c_{2t+1}$
- endowment stream $\{y_{1t}, y_{2t+1}\} = \{2.2, 2\}$

- Type 2 (Group 2)
  - $N_t^* = 50$ for $t \geq 0$
  - $u^*(c_{1t}^*, c_{2t+1}^*) = \ln c_{1t}^* + 0.95 \ln c_{2t+1}^*$
  - endowment stream $\{y_{1t}^*, y_{2t+1}^*\} = \{1, 1.25\}$

- individual savings functions are

$$s_t(r_t) = 1.0421 - \frac{1.0526}{1 + r_t}, \qquad s_t^*(r_t) = 0.4872 - \frac{0.6410}{1 + r_t}$$

- aggregate savings function

$$S_t(r_t) = 75 s_t(r_t) + 50 s_t^*(r_t) = 102.5175 - \frac{110.995}{1 + r_t}$$

- in equilibrium, $S_t(r_t) = 0$ which implies

$$r_t = \frac{110.995}{102.5175} - 1 = 0.0827$$

- Plug $r_t = 0.0827$ in consumption and savings functions

$$s_t = 0.07, \qquad s_t^* = -0.1048$$

- there is trade in equilibrium ($c \neq y$).

- Is trade welfare improving?
  - Type 1 in autarky: $u = \ln 2.2 + 0.9 \ln 2 = 1.4123$
  - Type 1 in free trade: $u = \ln 2.13 + 0.9 \ln 2.08 = 1.415 > 1.4123$
  - Type 2 in autarky: $u = \ln 1 + 0.95 \ln 1.25 = 0.2120$
  - Type 2 in free trade: $u = \ln 1.1048 + 0.95 \ln 1.1364 = 0.2211 > 0.2120$

Government Policy

(I) A Government with a Balanced Budget

- government is infinitely-lived
- government commits to its policies
- for simplicity, no government consumption
- a member of generation t must pay lump-sum taxes t_{1t} when young and t_{2t+1} when old.
- for now, no government deficits

$$N_t t_{1t} + N_{t-1} t_{2t} = 0, \quad \forall t$$

- modified budget constraints

$$s_t = y_{1t} - t_{1t} - c_{1t} \qquad \text{(young)}$$

$$c_{2t+1} = y_{2t+1} - t_{2t+1} + (1 + r_t)s_t \qquad \text{(old)}$$

$$c_{1t} + \frac{c_{2t+1}}{1 + r_t} = y_{1t} - t_{1t} + \frac{y_{2t+1} - t_{2t+1}}{1 + r_t}. \qquad \text{(PV)}$$

- A **competitive equilibrium** with a government is a government policy, a consumption allocation and a price system such that
  1. the quantities that are relevant for a particular person maximize that person's utility subject to the relevant budget constraint, taking prices as given
  2. the quantities clear all markets at all dates
  3. the government satisfies its budget constraint.
- To solve for the competitive equilibrium interest rate, we still impose $S_t(r_t) = 0$.

A first application: Social Security

- There is heterogeneity within each cohort.
- Type 1 (Poor)
  - $N_t = 25$ for $t \geq 0$
  - $u(c_{1t}, c_{2t+1}) = \ln c_{1t} + 0.95 \ln c_{2t+1}$
  - endowment stream $\{y_{1t}, y_{2t+1}\} = \{1, 1\}$
- Type 2 (Rich)
  - $N_t^* = 50$ for $t \geq 0$
  - $u^*(c_{1t}^*, c_{2t+1}^*) = \ln c_{1t}^* + 0.95 \ln c_{2t+1}^*$
  - endowment stream $\{y_{1t}^*, y_{2t+1}^*\} = \{3, 2\}$
- Equilibrium consumption allocation

$$(c_{1t}, c_{1t}^*, c_{2t}, c_{2t}^*) = (1.1949, 2.9026, 0.8535, 2.0733) \quad \forall t$$

lifetime utility of a poor 0.0275 lifetime utility of a rich 1.7583

- Suppose in period 1 the government announces the policy

$$\{t_{1t}, t_{2t}\}_{t=1}^{\infty} = \{-1, -1\}_{t=1}^{\infty}, \qquad \{t_{1t}^*, t_{2t}^*\}_{t=1}^{\infty} = \{1, 0\}_{t=1}^{\infty}$$

- The resulting equilibrium consumption allocation is

$$\{c_{1t}, c_{2t}\}_{t=1}^{\infty} = \{2, 2\}_{t=1}^{\infty}, \qquad \{c_{1t}^*, c_{2t}^*\}_{t=1}^{\infty} = \{2, 2\}_{t=1}^{\infty}$$

- Once the policy is introduced all agents born in $t \geq 1$ are identical in terms of preferences and after-tax endowments. No trade in equilibrium among agents born in $t \geq 1$.
- Welfare improving policy?

- utility of a poor born in period $t \geq 1$ without the policy:
  $\ln 1.1949 + 0.95 \ln 0.8535 = 0.0275$.
- utility of a poor born in period $t \geq 1$ with the policy:
  $\ln 2 + 0.95 \ln 2 = 1.3516$.
- utility of a rich born in period $t \geq 1$ without the policy:
  $\ln 2.9026 + 0.95 \ln 2.0733 = 1.7583$.
- utility of a rich born in period $t \geq 1$ with the policy:
  $\ln 2 + 0.95 \ln 2 = 1.3516$.
- consumption of an old poor alive in period 1 without the policy: 0.8535.
- consumption of an old poor alive in period 1 with the policy: 0.8535+1.
- consumption of an old rich alive in period 1 without the policy: 2.0733.
- consumption of an old rich alive in period 1 with the policy: $2.0733 - 0$.

A Second Application: Optimal Taxation

- All members of a cohort are identical.
- No population growth.
- Introduce a symmetric tax-transfer scheme in period 1

$$t_{1t} = -t_{2t} = t^*, \quad t \geq 1.$$

- Identical cohort members ⇒ no trade in equilibrium

$$c_{1t} = y_{1t} - t^*, \quad c_{2t+1} = y_{2t+1} + t^*.$$

- The government chooses t^* to maximize the indirect utility function of a member of generation t

$$u(t^*) = u(y_{1t} - t^*, y_{2t+1} + t^*).$$

- With log utility the solution is

$$t^* = \frac{1}{1+\beta}(\beta y_{1t} - y_{2t+1}).$$

- By construction,

$$u(y_{1t} - t^*, y_{2t+1} + t^*) > u(y_{1t}, y_{2t+1}) \quad \text{when } t^* \neq 0.$$

- Therefore, the optimal tax-transfer scheme makes agents born in period $t \geq 1$ better when $t^* \neq 0$ and leaves them as well off when $t^* = 0$.

- However, old agents alive in period 1 might be worse off.
  A period 1 old consumes y_{21} without tax-transfer,
  A period 1 old consumes $y_{21} + t^*$ with tax-transfer.
  Period 1 old agents as well off with optimal tax-transfer scheme only when $t^* \geq 0$.