IV. The Solow Growth Model (continued)

(III) Shocks and Policies

The Solow model can be interpreted also as a primitive Real Business Cycle (RBC) model. We can use the model to predict the response of the economy to productivity or taste shocks, or to shocks in government policies.

Productivity (or Taste) Shocks

Suppose output is given by

$$Y_t = A_t F(K_t, L_t)$$

or in intensive form

$$y_t = A_t f(k_t)$$

where A_t denotes total factor productivity.

Consider a permanent negative shock in productivity. The $G(k)$ and $\gamma(k)$ functions shift down, as illustrated in **Figure 4**. The new steady state is lower. The economy transits slowly from the old steady state to the new.

If instead the shock is transitory, the shift in $G(k)$ and $\gamma(k)$ is also temporary. Initially, capital and output fall towards the ow steady state. But when productivity reverts to the initial level, capital and output start to grow back towards the old high steady state.

The effect of a productivity shock on k_t and y_t is illustrated in **Figure 5**. The solid lines correspond to a transitory shock, the dashed lines correspond to a permanent shock.

Unproductive Government Spending

Let us now introduce a *government* in the competitive market economy. The government spends resources without contributing to production or capital accumulation.

The resource constraint of the economy now becomes

$$c_t + i_t + g_t = y_t = f(k_t),$$

where g_t denotes government consumption. It follows that the dynamics of capital are given by

$$k_{t+1} - k_t = f(k_t) - (\delta + n)k_t - c_t - g_t.$$

Government spending is financed with proportional income taxation, at rate $\tau \geq 0$. The government thus absorbs a fraction τ of aggregate output:

$$g_t = \tau y_t.$$

Disposable income for the representative household is $(1 - \tau)y_t$. We continue to assume that consumption and ivestment absorb fractions $1 - s$ and s of disposable income:

$$c_t = (1 - s)(y_t - g_t)$$

$$i_t = s(y_t - g_t).$$

Combining the above, we conclude that the dynamics of capital are now given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1 - \tau)\phi(k_t) - (\delta + n).$$

where $\phi(k) \equiv f(k)/k$. Given s and k_t, the growth rate γ_t decreases with τ.

A steady state exists for any $\tau \in [0, 1)$ and is given by

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s(1 - \tau)}\right).$$

Given s, k^* decreases with τ.

Productive Government Spending

Suppose now that the production is given by

$$y_t = f(k_t, g_t) = k_t^\alpha g_t^\beta,$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. Government spending can thus be interpreted as infrastructure or other productive services. The resources constraint is

$$c_t + i_t + g_t = y_t = f(k_t, g_t).$$

We assume again that government spending is financed with proportional income taxation at rate τ, and that private consumption and investment are fractions $1 - s$ and s of disposable household income:

$$\begin{aligned} g_t &= \tau y_t \\ c_t &= (1-s)(y_t - g_t) \\ i_t &= s(y_t - g_t). \end{aligned}$$

Substituting $g_t = \tau y_t$ into $y_t = k_t^\alpha g_t^\beta$ and solving for y_t, we infer

$$y_t = k_t^{\frac{\alpha}{1-\beta}} \tau^{\frac{\beta}{1-\beta}} \equiv k_t^a \tau^b$$

where $a \equiv \alpha/(1-\beta)$ and $b \equiv \beta/(1-\beta)$.

We conclude that the growth rate is given by

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = s(1-\tau)\tau^b k_t^{a-1} - (\delta + n).$$

The steady state is

$$k^* = \left(\frac{s(1-\tau)\tau^b}{\delta + n} \right)^{1/(1-\alpha)}.$$

Consider the rate τ that maximizes either k^*, or γ_t for any given k_t. This is given by

$$\frac{d}{d\tau}[(1-\tau)\tau^b] = 0 \quad \Leftrightarrow \quad b\tau^{b-1} - (1+b)\tau^b = 0 \quad \Leftrightarrow \quad \tau = b/(1+b) = \beta.$$

That is, the growth-maximization τ equals the elasticity of production with respect to government services. The more productive government services are, the higher their "optimal" provision.

(IV) Continuous Time and Convergence Rate

The Solow Model in Continuous Time

Recall that the basic growth equation in the discrete-time Solow model is

$$\frac{k_{t+1} - k_t}{k_t} = \gamma(k_t) \equiv s\phi(k_t) - (\delta + n).$$

We would expect a similar condition to hold under continuous time. We verify this below.

The resource constraint of the economy is

$$C + I = Y = F(K, L).$$

In per-capita terms,

$$c + i = y = f(k).$$

Population growth is now given by

$$\frac{\dot{L}}{L} = n$$

and the law of motion for aggregate capital is

$$\dot{K} = I - \delta K.$$

Let $k \equiv K/L$. Then,

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}.$$

Substituting from the above, we infer

$$\dot{k} = i - (\delta + n)k.$$

Combining this with

$$i = sy = sf(k),$$

we conclude

$$\dot{k} = sf(k) - (\delta + n)k.$$

Equivalently, the growth rate of the economy is given by

$$\frac{\dot{k}}{k} = \gamma(k) \equiv s\phi(k) - (\delta + n). \tag{1}$$

The function $\gamma(k)$ thus gives the growth rate of the economy in the Solow model, whether time is discrete or continuous.

Log-linearization and the Convergence Rate

Define $z \equiv \ln k - \ln k^*$. We can rewrite the growth equation (1) as

$$\dot{z} = \Gamma(z),$$

where

$$\Gamma(z) \equiv \gamma(k^* e^z) \equiv s\phi(k^* e^z) - (\delta + n).$$

Note that $\Gamma(z)$ is defined for all $z \in \mathcal{R}$. By definition of k^*, $\Gamma(0) = s\phi(k^*) - (\delta + n) = 0$. Similarly, $\Gamma(z) > 0$ for all $z < 0$, $\Gamma(z) < 0$ for all $z > 0$. Finally, $\Gamma'(z) = s\phi'(k^* e^z)k^* e^z < 0$ for all $z \in \mathcal{R}$.

We next (log)linearize $\dot{z} = \Gamma(z)$ around $z = 0$ (first-order Taylor-series approximation of $\Gamma(z)$ around $z = 0$):

$$\dot{z} = \Gamma(0) + \Gamma'(0) \cdot z$$

or equivalently

$$\dot{z} = \lambda z$$

where we substituted $\Gamma(0) = 0$ and let $\lambda \equiv \Gamma'(0)$.

Straightforward algebra gives

$$\begin{aligned} \Gamma'(z) &= s\phi'(k^* e^z)k^* e^z < 0 \\ \phi'(k) &= \frac{f'(k)k - f(k)}{k^2} = -\left[1 - \frac{f'(k)k}{f(k)}\right] \frac{f(k)}{k^2} \\ sf(k^*) &= (\delta + n)k^* \end{aligned}$$

We infer

$$\Gamma'(0) = -(1 - \varepsilon_K)(\delta + n) < 0$$

where $\varepsilon_K \equiv F_K K/F = f'(k)k/f(k)$ is the elasticity of production with respect to capital, evaluated at the steady-state k.

We conclude that

$$\frac{\dot{k}}{k} = \lambda \ln\left(\frac{k}{k^*}\right)$$

where

$$\lambda = -(1-\varepsilon_K)(\delta+n) < 0.$$

The quantity $-\lambda$ is called the *convergence rate.*

In the Cobb-Douglas case, $y = k^\alpha$, the convergence rate is simply

$$-\lambda = (1-\alpha)(\delta+n),$$

where α is the capital income share. Note that as $\lambda \to 0$ as $\alpha \to 1$. That is, convergence becomes slower and slower as the capital income share becomes closer and closer to 1. Indeed, if it were $\alpha = 1$, the economy would be on a balanced growth path.

Note that, around the steady state

$$\frac{\dot{y}}{y} = \varepsilon_K \cdot \frac{\dot{k}}{k} \qquad \text{and} \qquad \ln\left(\frac{y}{y^*}\right) = \varepsilon_K \cdot \ln\left(\frac{k}{k^*}\right).$$

It follows that

$$\frac{\dot{y}}{y} = \lambda \ln\left(\frac{y}{y^*}\right).$$

Thus, $-\lambda$ is the convergence rate for either capital or output.

In the example with productive government spending, $y = k^\alpha g^\beta = k^{\alpha/(1-\beta)}\tau^{\beta/(1-\beta)}$, we get

$$-\lambda = \left(1 - \frac{\alpha}{1-\beta}\right)(\delta+n)$$

The convergence rate thus decreases with β, the productivity of government services. And $\lambda \to 0$ as $\beta \to 1-\alpha$.

Calibration: If $\alpha = 35\%$, $n = 3\%$ (=1% population growth +2% exogenous technological process), and $\delta = 5\%$, then $-\lambda = 5.2\%$. This contradicts the data. But if $\alpha = 70\%$, then $-\lambda = 2.4\%$, which matches the data.

(V) The Golden Rule and Dynamic Inefficiency

The Golden Rule: Consumption at the steady state is given by

$$c^* = (1-s)f(k^*) = f(k^*) - (\delta + n)k^*$$

Suppose the social planner chooses s so as to maximize c^*. Since k^* is a monotonic function of s, this is equivalent to choosing k^* so as to maximize c^*. Note that

$$c^* = f(k^*) - (\delta + n)k^*$$

is strictly concave in k^*. The FOC is thus both necessary and sufficient. c^* is thus maximized if and only if $k^* = k_{gold}$, where k_{gold} solves

$$f'(k_{gold}) - \delta = n.$$

Equivalently, $s = s_{gold}$, where s_{gold} solves

$$s_{gold} \cdot \phi(k_{gold}) = (\delta + n).$$

The above is called the "*golden rule*" for savings, after Phelps.

Dynamic Inefficiency: If $s > s_{gold}$ (equivalently, $k^* > k_{gold}$), the economy is dynamically inefficient: If the saving rate is lowered to $s = s_{gold}$ for all t, then consumption in all periods will be higher!

On the other hand, if $s < s_{gold}$ (equivalently, $k^* < k_{gold}$), then raising s towards s_{gold} will increase consumption in the long run, but at the cost of lower consumption in the short run. Whether such a trade-off between short-run and long-run consumption is desirable will depend on how the social planner weight the short run versus the long run.

(VI) Cross-Country Differences

The Solow model implies that steady-state capital, productivity, and income are determined primarily by technology (f and δ), the national saving rate (s), and population growth (n).

Suppose that countries share the same technology in the long run, but differ in terns of saving behavior and fertility rates. If the Solow model is correct, observed cross-country income and productivity differences should be "explained" by observed cross-country differences in s and n.

Mankiw, Romer and Weil tests this hypothesis against the data. In its simple form, the Solow model fails to predict the large cross-country dispersion of income and productivity levels.

Mankiw, Romer and Weil then consider an extension of the Solow model, that includes two types of capital, physical capital (k) and human capital (h). Output is given by

$$y = k^{\alpha} h^{\beta},$$

where $\alpha > 0, \beta > 0$, and $\alpha + \beta < 1$. The dynamics of capital accumulation are now given by

$$\dot{k} = s_k y - (\delta + n)k$$

$$\dot{h} = s_h y - (\delta + n)h$$

where s_k and s_h are the investment rates in physical capital and human capital, respectively. The steady-state levels of k, h, and y then depend on both s_k and s_h, as well as δ and n.

Proxying s_h by education attainment levels in each country, Mankiw, Romer and Weil find that the Solow model extended for human capital does a pretty good job in "explaining" the cross-country dispersion of output and productivity levels.