LARGE SELF-INJECTIVE RINGS AND THE GENERATING HYPOTHESIS

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Abstract. We construct a number of different examples of non-Noetherian graded rings that are injective as modules over themselves (or have some related but weaker properties). We discuss how these are related to the theory of triangulated categories, and to Freyd's Generating Hypothesis in stable homotopy theory.

1. Introduction

In this paper we study graded commutative rings R that are large in various senses (in particular, not Noetherian) and self-injective (meaning that R is injective as an R-module). We use graded rings because they are relevant for our applications, but ungraded rings are covered as well because they can be regarded as graded rings concentrated in degree zero. The graded setting is assumed everywhere, so "element" means "homogeneous element" and "ideal" means "homogeneous ideal" and so on. Our rings will be commutative in the graded sense, so that $ba = (-1)^{|a||b|}ab$.

It is not hard to prove that any Noetherian self-injective ring is Artinian. In particular, if R is a finitely-generated algebra over a field K that is self-injective then we must have $\dim_K(R) < \infty$ and it turns out that $R \simeq \mathrm{Hom}(R, K)$ as R-modules. Examples of this situation include $R = K[x_1, \ldots, x_n]/(r_1, \ldots, r_n)$ for any regular sequence $r_1, \ldots, r_n$, or the cohomology ring $R = H^*(M; K)$ for any closed orientable manifold M. These are the most familiar examples of self-injective rings, and they are all very small. We will be looking for examples that are much larger.

Our motivation comes from a question in stable homotopy theory, which we briefly recall. In stable homotopy theory we study a certain triangulated category $\mathcal{F}$, the Spanier-Whitehead category of finite spectra. For any $X, Y \in \mathcal{F}$ the set $\mathrm{Hom}_{\mathcal{F}}(X, Y)$ is a finitely generated abelian group. It turns out that most methods for studying $\mathrm{Hom}_{\mathcal{F}}(X, Y)$ treat the p-primary parts separately for different primes p. We will thus fix a prime p and define $[X, Y] = \mathbb{Z}_p \otimes \mathrm{Hom}_{\mathcal{F}}(X, Y)$, where $\mathbb{Z}_p$ is the ring of p-adic integers. These are the morphism sets in a new triangulated category which we call $\mathcal{F}_p$. This has a canonical tensor structure, with the tensor product of X and Y written as $X \wedge Y$. The unit for this structure is called S, so $S \wedge X \simeq X$. As part of the triangulated structure we have a suspension functor $\Sigma\colon \mathcal{F}_p \to \mathcal{F}_p$, and we write S^n for $\Sigma^n S$. We put $R_n = [S^n, S]$. These sets form a graded commutative ring, whose structure is extremely intricate. A great deal of partial information is known, but it seems clear that there will never be a usable complete description. Some highlights are as follows.

- $R_n = 0$ for $n < 0$, and $R_0 = \mathbb{Z}_p$, and R_n is a finite abelian p-group for $n > 0$.
- Both the ranks and the exponents of the groups R_n can be arbitrarily large.
- All elements in R_n with $n > 0$ are nilpotent. Thus, the reduced quotient is $R/\sqrt{0} = \mathbb{Z}_p$.
- Various results are available describing most or all of the structure of R_n for $n < f(p)$, where $f(x)$ is a polynomial of degree at most three. The simplest of these says that $R_n = 0$ for $0 < n < 2p - 3$, and $R_{2p-3} = \mathbb{Z}/p$.

Now consider an arbitrary object $X \in \mathcal{F}_p$. We define $\pi_n(X) = [S^n, X]$ for all $n \in \mathbb{Z}$. This defines a graded abelian group $\pi_*(X)$, which has a natural structure as an R-module.

Conjecture 1.1 (Freyd's Generating Hypothesis). *The functor $\pi_*\colon \mathcal{F}_p \to \mathrm{Mod}_R$ is faithful.*

This is actually a technical modification of Freyd's conjecture [9], because Freyd did not tensor with the p-adics. This causes various trouble in the development of the theory, which Freyd avoided in *ad hoc* ways. Much later Hovey redeveloped the theory in the p-adic setting [10], which involves only minor modifications to Freyd's arguments but works much more smoothly.

Nearly half a century after Freyd made his conjecture, there is still no hint of a proof or a counterexample. However, there has been a certain amount of indirect progress; for example, various authors have settled the analogous questions in other triangulated categories where computations are easier [1–4].

On the other hand, it is known that the Generating Hypothesis would have some very strong and surprising consequences, as we now explain.

Definition 1.2.

(a) A graded ring R is *coherent* if every finitely generated ideal is finitely presented.
(b) A graded ring R is *totally incoherent* if the only finitely presented ideals are 0 and R.

Theorem 1.3 (Freyd [9], Hovey [10])**.** *Suppose that the Generating Hypothesis is true.*

(a) *The functor $\pi_*\colon \mathcal{F}_p \to \mathrm{Mod}_R$ is automatically full as well as being faithful, so it is an embedding of categories.*
(b) *For every object $X \in \mathcal{F}_p$, the image $\pi_*(X)$ is an injective R-module. In particular (by taking $X = S$) the ring R is self-injective.*
(c) *The ring R is totally incoherent.*

Note in particular that (a) gives a full subcategory of Mod_R that has a natural triangulation. This is very unusual; in almost all known triangulated categories, the morphisms are equivalence classes of homomorphisms under some nontrivial equivalence relation, and this equivalence structure is tightly connected to the definition of the triangulation.

Our aim in this paper is to shed light on the Generating Hypothesis by finding examples of self-injective rings that share some of the known or conjectured properties of the stable homotopy ring R.

Our main results are as follows. Firstly, one cannot disprove self-injectivity by looking only in a finite range of degrees:

Theorem 1.4. *Let R be a graded-commutative ring such that*

(a) *$R_k = 0$ for $k < 0$*
(b) *R_k is finite for all $k \geq 0$.*

Suppose given $N > 0$. Then there is an injective map $\phi\colon R \to R'$ of graded rings such that

(1) *R' also has properties (a) and (b).*
(2) *$\phi\colon R_k \to R'_k$ is an isomorphism for $k < N$*
(3) *R' is self-injective.*

This result was a great surprise to the authors at least, although the proof is not too hard. We will restate and prove it as Theorem 6.6. We conjecture that the theorem remains true if we allow R_0 to be $\mathbb{Z}_p$, but we have not proved this.

Most of our remaining results relate to specific examples. We have aimed to give a wide spread of examples, rather than formulating each example with maximum possible generality. We will write $\mathbb{F}$ for $\mathbb{Z}/2$.

Proposition 1.5. *Let E be the exterior algebra over $\mathbb{F}$ with a generator $x_i \in E_{2^i}$ for all $i \in \mathbb{N}$. Then E is self-injective and coherent. The reduced quotient is $E/\sqrt{0} = \mathbb{F}$.*

This will be proved as Example 4.7 and Proposition 5.4 (apart from the fact that $E/\sqrt{0} = \mathbb{F}$, which is clear).

Theorem 1.6. *Consider the ring*

$$C = \mathbb{F}[y_0, y_1, \dots]/(y_i^3 + y_i y_{i+1} \mid i \geq 0),$$

with the grading given by $|y_i| = 2^i$. Then C is self-injective and coherent. The reduced quotient is

$$C/\sqrt{0} = \mathbb{F}[x_0, x_1, \dots]/(x_i x_j \mid i \neq j) = \mathbb{F} \oplus \bigoplus_{n>0} x_n \mathbb{F}[x_n]$$

where $x_n = \sum_{i=0}^{n} y_{n-i}^{2^i}$.

This will be proved as Propositions 7.18, 7.25 and 7.26.

Definition 1.7. Let p be an odd prime, and define a graded ring J as follows. There are isomorphisms $\eta\colon \mathbb{Z}_{(p)} \to J_0$ and $\zeta\colon \mathbb{Q}/\mathbb{Z}_{(p)} \to J_{-2}$. Next, for each nonzero integer k there is a generator $\alpha_k \in J_{2(p-1)k-1}$ generating a cyclic group of order $p^{v_p(k)+1}$, where $v_p(k)$ is the p-adic valuation of k. For the product structure, we have

- $\eta(a)\eta(b) = \eta(ab)$ and $\eta(a)\zeta(b) = \zeta(ab)$ and $\eta(a)\alpha_k = a\,\alpha_k$.
- $\zeta(a)\zeta(b) = 0$ and $\zeta(a)\alpha_k = 0$ for all k.
- If $k > 0$ we have
$$\alpha_k\alpha_{-k} = -\alpha_{-k}\alpha_k = \zeta\left(p^{-1-v_p(k)} + \mathbb{Z}_{(p)}\right).$$
- $\alpha_j\alpha_k = 0$ whenever $j + k \neq 0$.

Remark 1.8. It is known that J is the homotopy ring of the Bousfield localisation of the sphere spectrum with respect to p-local K-theory [14, Section 8], but we will not need to use that fact.

Theorem 1.9. *The ring $\widehat{J} = \mathbb{Z}_p \otimes J$ is self-injective and totally incoherent. The reduced quotient is $\widehat{J}/\sqrt{0} = \mathbb{Z}_p$.*

This will be proved as Corollary 8.4 and Proposition 8.7 (apart from the fact that $\widehat{J}/\sqrt{0} = \mathbb{Z}_p$, which is clear).

Remark 1.10. Tensoring with $\mathbb{Z}_p$ here just has the effect of replacing $\mathbb{Z}_{(p)}$ in degree zero with $\mathbb{Z}_p$. Note that this is not the same as the p-completion of J, because $(\mathbb{Q}/\mathbb{Z}_{(p)})_p = 0$. Moreover, a derived version of p-completion would replace $\mathbb{Q}/\mathbb{Z}_{(p)}$ by a copy of $\mathbb{Z}_p$ shifted by one degree, which is different again. The ring J itself is not self-injective.

Definition 1.11. Let K be a field. For any map $a\colon [0,1] \to K$ we put $\operatorname{supp}(a) = \{q \in [0,1] \mid a(q) \neq 0\}$. We say that a is an *infinite root series* if every nonempty subset of $\operatorname{supp}(a)$ has a smallest element (so $\operatorname{supp}(a)$ is well-ordered). We let P denote the set of infinite root series, and call this the *infinite root algebra*.

Theorem 1.12. *The formula*
$$(ab)(q) = \sum_{0 \leq r \leq q} a(r)\, b(q-r)$$
gives a well-defined ring structure on P. With this structure, P is self-injective and totally incoherent. The reduced quotient is $P/\sqrt{0} = K$.

This will be proved in Propositions 9.20 and 9.21, and Corollary 9.13.

We will also discuss two rings that are not self-injective, but have a related property that we now explain.

Definition 1.13. Let R be a graded commutative ring, and let J be an ideal in R. We put $\operatorname{ann}_R(J) = \{a \in R \mid aJ = 0\}$. It is tautological that the ideal $\operatorname{ann}^2_R(J) = \operatorname{ann}_R(\operatorname{ann}_R(J))$ contains J. We say that R satisfies the *double annihilator condition* if $\operatorname{ann}^2_R(J) = J$ for all finitely generated ideals J.

Proposition 1.14. *If R is self-injective then it satisfies the double annihilator condition. Conversely, if R is Noetherian and satisfies the double annihilator condition, then it is self-injective.*

This is proved in Remark 2.4 and Theorem 4.1.

Definition 1.15. For any integer n we let $B(n)$ be the set of exponents i such that 2^i occurs in the binary expansion of n, so $B(n)$ is the unique finite subset of $\mathbb{N}$ such that $n = \sum_{i \in B(n)} 2^i$.

The *Rado graph* has vertex set $\mathbb{N}$, with an edge from i to j if ($i \in B(j)$ or $j \in B(i)$). The *Rado ideal* in the exterior algebra E has a generator x_ix_j for each pair (i,j) such that there is no edge from i to j in the Rado graph. The *Rado algebra* Q is the quotient of E by the Rado ideal.

Remark 1.16. Although this looks like a very specialised definition, the appearance is deceptive. Roughly speaking, any countable random graph is isomorphic to the Rado graph with probability one. See [5, 6] for discussion of the Rado graph. As far as we know, the corresponding algebra has not been considered before.

Theorem 1.17. *The Rado algebra is totally incoherent (and in particular, not Noetherian). It satisfies the double annihilator condition, but is not self-injective. The reduced quotient is $Q/\sqrt{0} = \mathbb{F}$.*

This will be proved as Propositions 10.5, 10.6 and 10.8 (apart from the fact that $Q/\sqrt{0} = \mathbb{F}$, which is clear).

For our final example, we need to recall some theory of ordinals. There is an exponentiation operation for ordinals (different from the usual one for cardinals). There is a countable ordinal called ϵ_0 such that $\epsilon_0 = \omega^{\epsilon_0}$, and no ordinal $\alpha < \epsilon_0$ satisfies $\alpha = \omega^\alpha$. Any ordinal $\alpha < \epsilon_0$ has a unique Cantor normal form

$$\alpha = \omega^{\beta_1} n_1 + \cdots + \omega^{\beta_r} n_r$$

where the n_i are positive integers and $\alpha > \beta_1 > \cdots > \beta_r$.

Definition 1.18. We write $\mu_0(\alpha, \beta)$ for the coefficient of ω^β in the Cantor normal form of α. We then put

$$\mu(\alpha, \beta) = \max(\mu_0(\alpha, \beta), \mu_0(\beta, \alpha)).$$

We then put

$$A = \mathbb{F}[x_\alpha \mid \alpha < \epsilon_0]/(x_\alpha x_\beta^{1+\mu(\alpha,\beta)} \mid \alpha, \beta < \epsilon_0, \alpha \neq \beta).$$

We call A the ϵ_0-*algebra*.

Given any function $\delta \colon \epsilon_0 \to \mathbb{N}$, we can give A a grading such that $|x_\alpha| = \delta(\alpha)$. In Section 11 we will describe a particular function δ with the property that $\delta(\alpha) > 0$ for all α, and all the sets $\delta^{-1}\{n\}$ are finite. This will ensure that the homogeneous pieces A_d are finite for all d.

Theorem 1.19. *If J is any ideal in A that is generated by a finite set of monomials, then $J = \operatorname{ann}_A^2(J)$. However, there are non-monomial ideals J with $J \neq \operatorname{ann}_A^2(J)$, so A does not satisfy the double annihilator condition, and is not self-injective. Moreover, A is totally incoherent, and the reduced quotient is*

$$A/\sqrt{0} = \mathbb{F}[x_\alpha \mid \alpha < \epsilon_0]/(x_\alpha x_\beta \mid \alpha \neq \beta).$$

This will be proved as Propositions 11.17, 11.21 and 11.22, and Corollary 11.19.

2. General theory of self-injective rings

Let R be a graded commutative ring, and let Mod_R be the category of graded R-modules. Suppose that R is self-injective. For $M \in \mathrm{Mod}_R$ we put $DM = \mathrm{Hom}_R(M, R)$ (regarded as a graded R-module in the usual way). This construction defines a functor $D \colon \mathrm{Mod}_R \to \mathrm{Mod}_R^{\mathrm{op}}$, which is exact because R is self-injective. It follows that D^2 gives an exact covariant functor from Mod_R to itself. There is a natural map $\kappa \colon M \to D^2 M$ given by $\kappa(m)(u) = u(m)$.

Definition 2.1. We let $\mathcal{U} = \mathcal{U}_R$ denote the full subcategory of Mod_R consisting of the modules M for which $\kappa \colon M \to D^2 M$ is an isomorphism.

Proposition 2.2. *The category $\mathcal{U}$ is closed under finite direct sums, suspensions and desuspensions, kernels, cokernels, images and extensions. It also contains R itself.*

Proof. This is clear from the exactness of the functor D^2 and the five lemma. ☐

Corollary 2.3. *If $J \leq R$ is a finitely generated ideal, then J and R/J lie in $\mathcal{U}$.*

Proof. They are the image and cokernel of some map $\bigoplus_{i=1}^n \Sigma^{d_i} R \to R$. ☐

Remark 2.4. If J is an ideal in R then

$$D(R/J) \simeq \{a \in R \mid aJ = 0\} = \operatorname{ann}_R(J).$$

By dualising the sequence $J \to R \to R/J$, we see that $D(J) = R/\operatorname{ann}_R(J)$. It follows that $D^2(J) = \operatorname{ann}_R(\operatorname{ann}_R(J)) = \operatorname{ann}_R^2(J)$. Thus, we have $J \in \mathcal{U}$ iff $J = \operatorname{ann}_R^2(J)$. In particular, if J is finitely generated then $J = \operatorname{ann}_R^2(J)$.

Lemma 2.5. *For any $a \in R_d$ there is an isomorphism $D(Ra) \simeq \Sigma^{-d} Ra$.*

Proof. Given $u \in D(Ra)_e$ we put $\alpha(u) = u(a) \in R_{d+e}$. This defines a map $\alpha \colon D(Ra) \to \Sigma^{-d} R$, which is clearly injective. Note that if $b \in \operatorname{ann}_R(a)$ then $\alpha(a)b = \alpha(ab) = \alpha(0) = 0$. This proves that $\alpha(a) \in \operatorname{ann}_R^2(Ra)_{d+e} = (Ra)_{d+e}$. In the opposite direction, if $c \in (Ra)_{d+e}$ then we have $c = ma$ for some $m \in R_e$, and the rule $\mu_m(x) = mx$ defines an element $\mu_m \in D(Ra)_e$ with $\alpha(\mu_m) = c$. This proves that the image of α is $\Sigma^{-d} Ra$, as required. ☐

Proposition 2.6. *If R is self-injective and $a \in R$ then $R/\operatorname{ann}(a)$ is also self-injective.*

Proof. Put $Q = R/\operatorname{ann}(a)$, and let $i\colon Q \to R$ be induced by $x \mapsto xa$, so i is injective, with image Ra. For $M \in \operatorname{Mod}_Q$ we write $D_Q(M) = \operatorname{Hom}_Q(M, Q) = \operatorname{Hom}_R(M, Q)$ and $D_R(M) = \operatorname{Hom}_R(M, R)$. We are given that D_R is exact, and we must show that D_Q is exact. The map $i\colon Q \to R$ gives a natural monomorphism $i\colon D_Q(M) \to D_R(M)$, and it will suffice to show that this is also an epimorphism. For any $\phi\colon M \to R$ we see that $\operatorname{ann}(a).\phi(M) = \phi(\operatorname{ann}(a)M) = \phi(0) = 0$, so $\phi(M) \leq \operatorname{ann}_R^2(a) = Ra$, and $i\colon Q \to Ra$ is an isomorphism, so $\phi = i(\psi)$ for some $\psi \in D_Q(M)$, as required. □

Proposition 2.7. *If R is self-injective and I and J are ideals in R then $\operatorname{ann}_R(I + J) = \operatorname{ann}_R(I) \cap \operatorname{ann}_R(J)$ and $\operatorname{ann}_R(I \cap J) = \operatorname{ann}_R(I) + \operatorname{ann}_R(J)$.*

Proof. There is a short exact sequence

$$R/(I \cap J) \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} R/I \oplus R/J \xrightarrow{[1\ -1]} R/(I + J).$$

By applying the exact functor D, we get a short exact sequence

$$\operatorname{ann}_R(I \cap J) \xleftarrow{[1\ 1]} \operatorname{ann}_R(I) \oplus \operatorname{ann}_R(J) \xleftarrow{\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]} \operatorname{ann}_R(I + J).$$

The claim follows. □

Corollary 2.8. *If R is local and self-injective and I and J are nontrivial ideals, then $I \cap J$ is also nontrivial.*

Proof. Let $\mathfrak{m}$ be the maximal ideal. As I and J are nontrivial we have $\operatorname{ann}(I) < R$ and $\operatorname{ann}(J) < R$, so $\operatorname{ann}(I) \leq \mathfrak{m}$ and $\operatorname{ann}(J) \leq \mathfrak{m}$, so $\operatorname{ann}(I \cap J) = \operatorname{ann}(I) + \operatorname{ann}(J) \leq \mathfrak{m} < R$, so $I \cap J$ is nontrivial. □

3. Criteria for self-injectivity

We first record a graded version of the standard Baer criterion for injectivity.

Definition 3.1. Let R be a graded ring, and let I be a graded R-module. We say that I satisfies the *Baer condition* if for every graded ideal $J \leq R$, every integer d and every R-module homomorphism $\phi\colon \Sigma^d J \to I$, there exists $m \in I_d$ such that $\phi(a) = am$ for all $a \in I$. We say that I satisfies the *finite Baer condition* if the same condition holds for all finitely generated graded ideals J.

Proposition 3.2. *In the above context, the module I is injective if and only if it satisifes the Baer condition.*

Proof. This was originally done in the ungraded context in [7], as an application of Zorn's Lemma. The proof is also given in many textbooks such as [12, page 63]. It can be modified in an obvious way to keep track of gradings, which gives our statement above. □

Proposition 3.3. *Suppose that I_d is finite for all d, and that I satisfies the finite Baer condition. Then I also satisfies the full Baer condition and so is injective.*

Proof. Consider a graded ideal $J \leq R$ and a homomorphism $\phi\colon \Sigma^d J \to I$. For each finitely generated ideal $K \subseteq J$ we put

$$M(K) = \{m \in I_d \mid \phi(a) = am \text{ for all } a \in K\}.$$

The finite Baer condition means that this is a nonempty subset of the finite set I_d. Choose K such that $|M(K)|$ is as small as possible, and choose $m \in M(K)$. For $a \in J$ it is clear that $M(K + Ra) \subseteq M(K)$, so by the minimality property we must have $M(K + Ra) = M(K)$, so $m \in M(K + Ra)$, so $\phi(a) = am$. This proves the full Baer condition. □

Definition 3.4. Let R be a graded ring, and let I be an R-module. A *test pair* of length r and degree d is a pair (u, v) where $u \in R^r$ and $v \in I^r$ such that the entries u_i and v_i are homogeneous with $|v_i| = |u_i| + d$ for all i. A *block* for such a pair is a vector $b \in R^r$ such that $b.u = 0$ but $b.v \neq 0$ (where $b.x = \sum_i b_i x_i$). A *transporter* is an element $m \in I_d$ such that $v_i = m u_i$ for all i.

Remark 3.5. We implicitly formulate the theory of graded groups in such a way that the zero elements in different degrees are distinct. Thus, the notation $|u|$ is meaningful even if $u = 0$.

Proposition 3.6. *The module I satisfies the finite Baer condition iff every test pair has either a block or a transporter.*

Proof. Suppose that every test pair has either a block or a transporter. Consider a finitely generated graded ideal $J \leq R$, and a homomorphism $\phi\colon \Sigma^d J \to R$. Choose a list $u = (u_1, \ldots, u_r)$ of homogeneous elements that generates J, and put $v_i = \phi(u_i) \in I$. Note that if $b \in R^r$ with $b.u = 0$ then we can apply ϕ to see that $b.v = 0$. It follows that the pair (u, v) has no block, so it must have a transporter. This means that there is an element $m \in I_d$ with $\phi(u_i) = u_i m$ for all i, and it follows easily that $\phi(a) = am$ for all $a \in J$, as required.

Conversely, suppose that I satisfies the finite Baer condition. Consider a test pair (u, v) of degree d with no block, and let J be the ideal generated by the entries u_i. Define $\phi\colon \Sigma^d J \to I$ by $\phi(\sum_i b_i u_i) = \sum_i b_i v_i$ (the absence of a block means that this is well-defined). The finite Baer condition means that there is an element $m \in I_d$ with $\phi(a) = am$ for all $a \in J$, and this m is clearly a transporter for (u, v). □

Corollary 3.7. *Let R be a graded commutative ring such that R_k is finite for all k. Suppose also that there are subrings*

$$R(0) \leq R(1) \leq R(2) \leq \cdots \leq R$$

such that each $R(n)$ is self-injective and $R = \bigcup_n R(n)$. Then R is self-injective.

Proof. Any test pair $(u, v) \in R^r \times R^r$ can be regarded as a test pair over $R(n)$ for sufficiently large n. As $R(n)$ is self-injective, there must be a block in $R(n)^r$ or a transporter in $R(n)$. It is clear from the definitions that such a block or transporter still qualifies as a block or transporter over R, so we see that R satisfies the finite Baer condition. As we have assumed that R_k is finite for all k, we can use Proposition 3.3 to see that R is injective as an R-module. □

Theorem 3.8. *Let R be a graded commutative ring such that R_k is finite for all k. Then the following are equivalent:*

(a) *R is self-injective.*

(b) *For all finitely generated ideals $J, K \leq R$ we have $\operatorname{ann}_R^2(J) = J$ and*

$$\operatorname{ann}_R(J \cap K) = \operatorname{ann}_R(J) + \operatorname{ann}_R(K).$$

(c) *For all elements $a \in R$ and every finitely generated ideal $J \leq R$ we have $\operatorname{ann}_R^2(a) = Ra$ and*

$$\operatorname{ann}_R(J \cap Ra) = \operatorname{ann}_R(J) + \operatorname{ann}_R(a).$$

Proof. It follows from Remark 2.4 and Proposition 2.7 that (a) implies (b). If (b) holds, then (c) follows immediately. Now suppose (c) holds. As we have assumed that R_k is finite for all k, we may use the theory of blocks and transporters. We proceed by induction on the length of a test pair to show that every test pair over the ring R has either a block or a transporter. Let $(u; v)$ be a test pair of length 1 and degree d. Suppose this test pair has neither block nor transporter. Then $\operatorname{ann}_R(u) \leq \operatorname{ann}_R(v)$ and by assumption we have $Rv = \operatorname{ann}_R^2(v) \leq \operatorname{ann}_R^2(u) = Ru$, that is, $v = um$ for some $m \in R_d$. Since m is a transporter for this test pair, we have a contradiction.

Now suppose each test pair of length $\leq k$ and arbitrary degree has either a block or a transporter. A test pair of length $k+1$ and degree d takes the form $(u, u_{k+1}; v, v_{k+1})$ where $(u; v)$ is a test pair of length k and degree d and (u_{k+1}, v_{k+1}) is a test pair of length 1 and degree d. By the inductive hypothesis, both the test pairs $(u; v)$ and (u_{k+1}, v_{k+1}) have either a block or a transporter. If $(u; v)$ has block r, then $(r, 0)$ is a block for the test pair $(u, u_{k+1}; v, v_{k+1})$. Similarly, if (u_{k+1}, v_{k+1}) has block r_{k+1}, then $(0, \ldots, 0, r_{k+1})$ is a block for the test pair $(u, u_{k+1}; v, v_{k+1})$. Otherwise, $(u; v)$ must have transporter $m \in R_d$ and (u_{k+1}, v_{k+1}) must have transporter $n \in R_d$. In this situation, suppose the test pair $(u, u_{k+1}; v, v_{k+1})$ has neither block nor transporter and let J be the ideal generated by the entries of u. The absence of a block implies that there is a well defined map $\phi : \Sigma^d(J + Ru_{k+1}) \to R$ defined by $\phi(\sum_{i=1}^{k+1} b_i u_i) = \sum_{i=1}^{k+1} b_i v_i$. Now let s be an element in the intersection $J \cap Ru_{k+1}$. Then we must have $s = \sum_{i=1}^{k} s_i u_i = s_{k+1} u_{k+1}$ for elements $s_i \in R$ for each i. Applying the map ϕ to the zero element $(\sum_{i=1}^{k} s_i u_i) - s_{k+1} u_{k+1}$ gives

$$0 = \left(\sum_{i=1}^{k} s_i v_i \right) - s_{k+1} v_{k+1} = \left(\sum_{i=1}^{k} s_i u_i m \right) - s_{k+1} u_{k+1} n = s(m - n).$$

Thus it follows that the element $m - n$ is in the annihilator ideal $\mathrm{ann}_R(J \cap Ru_{k+1})$. By assumption, we have $\mathrm{ann}_R(J \cap Ru_{k+1}) = \mathrm{ann}_R(J) + \mathrm{ann}_R(u_{k+1})$. Now let $m - n = x - y$ where $x \in \mathrm{ann}_R(J)$ and $y \in \mathrm{ann}_R(u_{k+1})$ and put $z = m - x = n - y$. Since $u_i z = u_i(m - x) = u_i m = v_i$ for each $i \leq k$ and $u_{k+1} z = u_{k+1}(n - y) = u_{k+1} n = v_{k+1}$ it follows that z is a transporter for the test pair $(u, u_{k+1}; v, v_{k+1})$. As this gives a contradiction, it follows that every test pair of length $k+1$ and arbitrary degree must have either a block or transporter. We deduce that every test pair in the ring R must have either a block or transporter, and since R_k is finite for each k, we can use Proposition 3.6 to show that R is injective as an R-module. □

4. The Noetherian case

Theorem 4.1. *Let R be a Noetherian graded commutative ring. Then the following are equivalent:*

(a) *R is self-injective.*
(b) *For every ideal $J \leq R$ we have $\mathrm{ann}_R^2(J) = J$.*
(c) *R is Artinian (and thus is a finite product of Artinian local rings), and each of the local factors has one-dimensional socle.*

The proof will be given after some lemmas.

Lemma 4.2. *Let R be an Artinian local graded ring, with maximal ideal $\mathfrak{m}$, and put $K = R/\mathfrak{m}$. Suppose that the socle $\mathrm{soc}(R) = \mathrm{ann}_R(\mathfrak{m})$ has dimension one over K. Then every nonzero ideal in R contains $\mathrm{soc}(R)$.*

Proof. Let I be a nonzero ideal. By the Artinian condition, we can choose an ideal J that is minimal among nonzero ideals contained in I. Recall that every Artinian ring is Noetherian (see for example [13, Theorem 3.2]), so we can use Nakayama's Lemma to see that $\mathfrak{m}J < J$ and thus (by minimality) that $\mathfrak{m}J = 0$. This means that J is a nontrivial K-subspace of $\mathrm{soc}(R)$, but $\mathrm{soc}(R)$ has dimension one, so $J = \mathrm{soc}(R)$, so $\mathrm{soc}(R) \leq I$. □

Lemma 4.3. *Suppose that R is as in Lemma 4.2. Then for all ideals $J \leq R$ we have $\mathrm{ann}_R^2(J) = J$.*

Proof. First, it is standard that we can fit together a composition series for J with a composition series for R/J to get a chain

$$0 = I_0 < I_1 < \cdots < I_r = R$$

with $I_i/I_{i-1} \simeq K$ for all i, and $J = I_t$ for some t. Now let A_j be the annihilator of I_j, so we have

$$R = A_0 \geq A_1 \geq \cdots \geq A_r = 0.$$

Now $\mathfrak{m}A_iI_{i+1} = A_i(\mathfrak{m}I_{i+1}) \leq A_iI_i = 0$, so $A_iI_{i+1} \leq \mathrm{soc}(R)$. On the other hand, we have $A_iI_i = 0$ and $A_{i+1}I_{i+1} = 0$. We therefore have a natural map

$$\xi_i \colon A_i/A_{i+1} \to \mathrm{Hom}_K(I_{i+1}/I_i, \mathrm{soc}(R))$$

given by $\xi_i(a + A_{i+1})(b + I_i) = ab$. It is clear from the definitions that this is injective, and the codomain is isomorphic to K, so A_i/A_{i+1} is either 0 or K. It is standard that any two composition series have the same length, so we must have $A_i/A_{i+1} \simeq K$ for all i, so A_i has length $r - i$. After applying the same logic to the composition series $\{A_{r-i}\}_{i=0}^r$ we see that the ideal $\mathrm{ann}(A_i) = \mathrm{ann}^2(I_i)$ has length i. We also know that $I_i \leq \mathrm{ann}^2(I_i)$ and that I_i also has length i; it follows that $I_i = \mathrm{ann}^2(I_i)$, as required. □

Corollary 4.4. *Suppose that R is as in Lemma 4.3. Then R is self-injective.*

Proof. Consider an ideal $I \leq R$ and an R-module map $f \colon I \to R$. Choose a composition series $0 = J_0 < J_1 < \cdots < J_r = I$. We have $J_i/J_{i-1} \simeq K$ so we can find $a_i \in J_i \setminus J_{i-1}$ such that $J_i = J_{i-1} + Ra_i$ with $\mathfrak{m}a_i \leq J_{i-1}$.

We will construct elements $x_0, \dots, x_r \in R$ such that $f(a) = ax_i$ for all $a \in J_i$. We start with $x_0 = 0$. Now suppose we have found x_{i-1}. Put $u_i = f(a_i) - x_{i-1}a_i$. Using the fact that $\mathfrak{m}a_i \leq I_{i-1}$ we find that $\mathfrak{m}u_i = 0$, so $u_i \in \mathrm{soc}(R)$. Next, we have $a_i \notin I_{i-1} = \mathrm{ann}^2(I_{i-1})$, so $\mathrm{ann}(I_{i-1})a_i \neq 0$. As every nontrivial ideal contains the socle, we see that $u_i \in \mathrm{ann}(I_{i-1})a_i$, so we can write $u_i = y_ia_i$ for some y_i with $y_iI_{i-1} = 0$. We now put $x_i = x_{i-1} + y_i$. By construction we have $f(a) = ax_i$ for $a \in I_{i-1}$ or for $a = a_i$, and it follows that this equation holds for all $a \in I_i$ as required. At the end of the induction we have an element x_r which fulfils Baer's criterion. □

Proof of Theorem 4.1. It follows from Remark 2.4 that (a) implies (b). Now suppose that (b) holds. Consider a descending chain of ideals $I_0 \geq I_1 \geq I_2 \geq \cdots$ in R. The ideals $\text{ann}(I_k)$ then form an ascending chain, which must eventually stabilise because R is Noetherian. We can thus take annihilators again to see that the original chain also stabilises. This shows that R is Artinian. It follows in a standard way that there are only finitely many maximal ideals, and that R is the product of its maximal localisations. We thus have a splitting $R = \prod_{i=1}^{n} R_i$ say, where each factor R_i an Artinian local ring. It follows that the lattice of ideals in R is the product of the corresponding lattices for the factors R_i, and thus that each R_i satisfies condition (b). We can thus reduce to the case where R is local, with maximal ideal $\mathfrak{m}$ say. Recall that the socle is $\text{soc}(R) = \{a \in R \mid a\mathfrak{m} = 0\} = \text{ann}_R(\mathfrak{m})$, which is naturally a vector space over the field $K = R/\mathfrak{m}$. If $\text{soc}(R)$ were zero we would have $\mathfrak{m} = \text{ann}^2(\mathfrak{m}) = \text{ann}(\text{soc}(R)) = \text{ann}(0) = R$, which is a contradiction. We can therefore choose a nonzero element $u \in \text{soc}(R)$. We find that $Ku = Ru$ is a nonzero ideal in R, so $\text{ann}(Ku)$ is a proper ideal containing $\text{ann}(\text{soc}(R)) = \mathfrak{m}$, so $\text{ann}(Ku) = \mathfrak{m}$ by maximality. We can now take annihilators again to see that $Ku = \text{ann}(\mathfrak{m}) = \text{soc}(R)$, so $\text{soc}(R)$ is one-dimensional. This proves (c).

Finally, we will assume (c) and prove (a). It is again easy to reduce to the case where R is local, and the local case is covered by Corollary 4.4. □

Definition 4.5. Let K be a field. A *Poincaré duality algebra* over K is a graded commutative K-algebra R equipped with a K-linear map $\theta\colon R_d \to K$ for some $d \geq 0$ such that

- For $i < 0$ or $i > d$ we have $R_i = 0$
- $R_0 = K$.
- For $0 \leq i \leq d$ we have $\dim_K(R_i) < \infty$, and the map $(a, b) \mapsto \theta(ab)$ defines a perfect pairing between R_i and R_{d-i}.

Proposition 4.6. *Every Poincaré duality algebra is self-injective.*

Proof. Let R be a Poincaré duality algebra of top dimension d, and put $\mathfrak{m} = \bigoplus_{i>0} R_i$. It is clear that $R/\mathfrak{m} = K$ and $\mathfrak{m}^{d+1} = 0$, and it follows that $\mathfrak{m}$ is the unique maximal ideal. As R has finite total dimension over K it is clearly Artinian. The perfect pairing condition implies that $\text{soc}(R) = R_d$ and that this has dimension one. It follows by Theorem 4.1 that R is self-injective.

Alternatively, for any R-module M we can define a natural map

$$\tau\colon \text{Hom}_R(M, R) \to \text{Hom}_K(M_d, K)$$

by $\tau(\phi) = \theta \circ \phi_d$. Using the perfectness of the pairing we see that this is an isomorphism. As K is a field, the functor $M \mapsto \text{Hom}_K(M_d, K)$ is exact, and it follows that the functor $M \mapsto \text{Hom}_R(R, R)$ is also exact, or in other words that R is injective as an R-module. □

Example 4.7. Put

$$E = \mathbb{F}[x_0, x_1, x_2, \ldots]/(x_i^2 \mid i \geq 0),$$

with $|x_i| = 2^i$. For any finite set $I \subset \mathbb{N}$ we put $x_I = \prod_{i\in I} x_i$, so $|x_I| = \sum_{i\in I} 2^i$ and the elements x_I form a basis for E over $\mathbb{F}$. It follows that $E_k \simeq \mathbb{F}$ for all $k \geq 0$, and $E_k = 0$ for $k < 0$. Let $E(n)$ be the subalgebra of E generated by $x_0, \ldots, x_{n-1}$. This is a Poincaré duality algebra, with socle generated by the element $\prod_{i<n} x_i$, and it is clear that $E = \bigcup_n E(n)$. Corollary 3.7 therefore tells us that E is self-injective.

5. Coherence

We now briefly recall some standard ideas about finite presentation.

Proposition 5.1. *Let R be a graded commutative ring, and let M be a graded R-module. Then the following are equivalent:*

(a) *There exists an exact sequence*

$$P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0,$$

where P_0 and P_1 are finitely generated free modules.

(b) *M is finitely generated, and for every epimorphism $g\colon P_0 \to M$ (with P_0 a finitely generated free module) the module* $\ker(g)$ *is also finitely generated.*

If these conditions hold, we say that M is finitely presented.

Remark 5.2. By a *finitely generated free module* we mean one of the form $\bigoplus_{i=1}^{r} \Sigma^{d_i} R$; we do not assume that the degree shift d_i is zero.

Proof. First suppose that (b) holds. Then any finite system of generators $(m_1, \dots, m_r)$ for M will give rise to an epimorphism $g\colon P_0 \to M$, where $P_0 = \bigoplus_i \Sigma^{|m_i|} R$. By assumption $\ker(g)$ will be finitely generated, and any finite system of generators will give an epimorphism $f\colon P_1 \to \ker(g) \leq P_0$ with P_1 free and finitely generated, and this proves (a).

Conversely, suppose that (a) holds. The existence of g means that M is finitely generated. Consider another epimorphism $g'\colon P_0' \to M$, where P_0' is free and finitely generated. As P_0 and P_0' are projective, we can choose maps $P_0 \xrightarrow{p} P_0' \xrightarrow{q} P_0$ with $g'p = g$ and $gq = g'$. Put

$$K = \text{image}(P_0' \xrightarrow{1-pq} P_0') + \text{image}(pf\colon P_1 \to P_0').$$

It is easy to see that this is finitely generated and contained in $\ker(g')$, so it will suffice to show that $\ker(g') \leq K$. Consider an element $x \in \ker(g')$. We then have $gq(x) = g'(x) = 0$, but $\ker(g) = \text{image}(f)$ by assumption, so $q(x) = f(y)$ for some $y \in P_1$. We now have $x = (1 - pq)(x) + pf(y) \in K$ as required. ☐

Corollary 5.3. *If R is Noetherian, then every finitely generated ideal is finitely presented.*

Proof. Condition (b) is clearly satisfied. ☐

As we stated in Definition 1.2, a graded ring R is said to be *coherent* if every finitely generated ideal is finitely presented, and *totally incoherent* if the only finitely presented ideals are 0 and R. It is clear that every Noetherian ring is coherent. We mention as background that if R is coherent, then the category of finitely generated modules is closed under images, kernels, cokernels and extensions, so it is an abelien category. The following example is standard:

Proposition 5.4. *The infinite exterior algebra E (as in Example 4.7) is coherent.*

Proof. Let $E(n)$ be the subalgebra generated by $x_0, \dots, x_{n-1}$, and let $E'(n)$ be generated by the remaining variables, so $E = E(n) \otimes_{\mathbb{F}} E'(n)$. Any finitely generated ideal is the image of some E-linear map $g\colon E^r \to E$, which will have the form $g(u) = u.v$ for some vector $v \in E^r$. We must show that the module $K = \ker(g)$ is finitely generated. Choose n large enough that $v_i \in E(n)$ for all i. Now v gives a map $g'\colon E(n)^r \to E(n)$ of $E(n)$-modules, and $E(n)$ is Noetherian, so the module $K' = \ker(g')$ is finitely generated over $E(n)$. We can identify g with $g' \otimes 1$ with respect to the splitting $E = E(n) \otimes E'(n)$, and it follows that $K = K' \otimes E(n)'$, and thus that any finite generating set for K' over $E(n)$ also generates K over E. ☐

The following result will be our main tool for proving incoherence results.

Lemma 5.5. *Let A be a local graded ring, with maximal ideal $\mathfrak{m}$, and let I be a finitely presented ideal in A. Then for each $u \in I \setminus \mathfrak{m}I$, the image of $\text{ann}_A(u)$ in $\mathfrak{m}/\mathfrak{m}^2$ has finite dimension over $A/\mathfrak{m}$.*

Note here that as $u \notin \mathfrak{m}I$ we have $u \neq 0$, so $\text{ann}_A(u) \leq \mathfrak{m}$ and it is meaningful to talk about the image in $\mathfrak{m}/\mathfrak{m}^2$.

Proof. As I is finitely generated over A, we see that $I/\mathfrak{m}I$ is a finite-dimensional vector space over $A/\mathfrak{m}$. We can choose a basis for this space containing the image of u, and then choose elements of I lifting these basis elements. This gives a list $v_1, \dots, v_n \in I$ with $v_1 = u$ such that the corresponding map $g\colon A^n \to I$ induces an isomorphism $\overline{g}\colon (A/\mathfrak{m})^n \to I/\mathfrak{m}I$. Now $\text{cok}(g)$ is a finitely generated module with $\mathfrak{m}.\text{cok}(g) = \text{cok}(g)$, so $\text{cok}(g) = 0$ by Nakayama's Lemma, so g is an epimorphism. As I is assumed to be finitely presented, we see that $\ker(g)$ is also finitely generated over A. Moreover, as $\overline{g}$ is an isomorphism we see that $\ker(g) \leq \mathfrak{m}^n$. It follows that the image of $\ker(g)$ in $(\mathfrak{m}/\mathfrak{m}^2)^n$ is finite-dimensional. The intersection of $\ker(g)$ with the first copy of A in A^n is just the annihilator of u, so we see that the image of $\text{ann}_A(u)$ in $\mathfrak{m}/\mathfrak{m}^2$ is finite-dimensional. ☐

Corollary 5.6. *Let A be a local graded ring, with maximal ideal $\mathfrak{m}$. Suppose that for all $u \in A$ we have either*

(a) $u = 0$*; or*
(b) *the image of* $\text{ann}_A(u)$ *in* $\mathfrak{m}/\mathfrak{m}^2$ *has infinite dimension; or*
(c) u *is invertible.*

Then A is totally incoherent.

Proof. Let I be a finitely presented ideal. If $\mathfrak{m}I = I$ then $I = 0$ by Nakayama's Lemma. Otherwise, we can choose $u \in I \setminus \mathfrak{m}I$. As $u \notin \mathfrak{m}I$ we have $u \neq 0$. By the lemma, the image of $\mathrm{ann}_A(u)$ in $\mathfrak{m}/\mathfrak{m}^2$ must have finite dimension. Thus, possibilities (a) and (b) are excluded, so u must be invertible. As $u \in I$ we conclude that $I = A$. □

Next we record a graded version of Chase's Theorem for coherent rings.

Theorem 5.7. *Let R be a graded commutative ring. Then the following are equivalent:*

(a) *R is coherent.*
(b) *For all elements $a \in R$ and for every finitely generated ideal $J \leq R$, the conductor ideal*
$$(J : a) = \{r \in R \mid ra \in J\}$$
is finitely generated.
(c) *For all elements $a \in R$, the annihilator ideal $\mathrm{ann}_R(a)$ is finitely generated, and for all finitely generated ideals $J, K \leq R$, the intersection $J \cap K$ is finitely generated.*

Proof. The ungraded version of the proof is given in many textbooks such as [12, page 142]. It can be modified in an obvious way to keep track of gradings, which gives our statement above. □

Theorem 5.8. *Let R be a graded commutative ring such that R_k is finite for all k. Then the following are equivalent:*

(a) *R is coherent and self-injective.*
(b) *R is coherent and for all finitely generated ideals $J \leq R$ we have $\mathrm{ann}^2_R(J) = J$.*
(c) *For every finitely generated ideal $J \leq R$, the ideal $\mathrm{ann}_R(J)$ is finitely generated and $\mathrm{ann}^2_R(J) = J$.*
(d) *R is self injective and for all finitely generated ideals $J \leq R$, the ideal $\mathrm{ann}_R(J)$ is finitely generated.*

Proof. It follows from Remark 2.4 that (a) implies (b). To show that (b) implies (c) we need to show that the ideal $\mathrm{ann}_R(J)$ is finitely generated for each finitely generated ideal $J \leq R$. If we let $(r_1, \dots, r_n)$ be generators for the ideal J, then we can take the annihilator of J to give $\mathrm{ann}_R(J) = \bigcap_i \mathrm{ann}_R(r_i)$. Since R is assumed to be coherent, it follows from part (c) of Theorem 5.7 that $\mathrm{ann}_R(r_i)$ is finitely generated for each i and that a finite intersection of finitely generated ideals is also finitely generated. Thus $\mathrm{ann}_R(J)$ is finitely generated as claimed. Now suppose that part (c) holds. To prove that (c) implies (d), we need to show that R is injective as an R-module. For all ideals $J, K \leq R$ we have
$$\mathrm{ann}_R(\mathrm{ann}_R(J) + \mathrm{ann}_R(K)) = \mathrm{ann}^2_R(J) \cap \mathrm{ann}^2_R(K) = J \cap K.$$
By assumption, the ideal sum $\mathrm{ann}_R(J) + \mathrm{ann}_R(K)$ must be finitely generated. Thus we can take double annihilators to give
$$\mathrm{ann}_R(J) + \mathrm{ann}_R(K) = \mathrm{ann}_R(J \cap K).$$
Since R_k is finite for each k, we can use part (b) of Theorem 3.8 to complete the claim. We now conclude by showing that (d) implies (a). By assumption, the annihilator ideal $\mathrm{ann}_R(a)$ is finitely generated for all elements $a \in R$. Then for all ideals $J, K \leq R$ we know that the ideal sum $\mathrm{ann}_R(J) + \mathrm{ann}_R(K)$ is finitely generated by assumption. By taking annihilators we then have
$$\mathrm{ann}_R(\mathrm{ann}_R(J) + \mathrm{ann}_R(K)) = \mathrm{ann}^2_R(J) \cap \mathrm{ann}^2_R(K) = J \cap K$$
where the double annihilator condition holds by Remark 2.4. However, by assumption, the annihilator of a finitely generated ideal is also finitely generated. Thus the intersection $J \cap K$ must be finitely generated. It follows from part (c) of Theorem 5.7 that the ring R is coherent as claimed. □

6. Self-injective adjustment

Definition 6.1. We write $\mathcal{R}$ for the category of commutative graded $\mathbb{F}$-algebras such that

(a) $R_k = 0$ for all $k < 0$.
(b) $R_0 = \mathbb{F}$.
(c) R_k is finite for all $k > 0$.

Proposition 6.2. *Let R be a ring in $\mathcal{R}$, and let $\mathcal{P}$ be a finite set of test pairs in R that have no transporters. Let m be a positive integer. Then there is an extension $R' \geq R$ of graded rings such that*

(a) *R' is also in $\mathcal{R}$.*
(b) *$R'_k = R_k$ for all $k < m$.*
(c) *Each test pair in $\mathcal{P}$ has a block in R'.*

Proof. List the elements of $\mathcal{P}$ as $(u_0, v_0), \dots, (u_{p-1}, v_{p-1})$ say. Suppose that (u_t, v_t) has length r_t, and let d_t be the maximum of the degrees of the entries $u_{t,j}$ for $0 \leq j < r_t$. Let P be the polynomial ring obtained from R by adjoining variables $b_{t,j}$ for $0 \leq t < p$ and $0 \leq j < r_t$, with $|b_{t,j}| = m + d_t - |u_{t,j}| \geq m > 0$. Put $w_t = \sum_{j=0}^{r_t - 1} b_{t,j} u_{t,j} \in P$ and $R' = P/(w_0, \dots, w_{p-1})$. There is an evident ring map $\eta \colon R \to R'$, and also a ring map $\pi \colon R' \to R$ given by $\pi(b_{t,j}) = 0$ for all t and j. It is clear that $\pi\eta = 1$, so η is injective, and we can use it to regard R' as an extension of R. As $|b_{t,j}| \geq m > 0$, it is easy to see that $R' \in \mathcal{R}$ and that the map $R_k \to R'_k$ is surjective (and therefore bijective) for $k < m$. By construction we have $b_t.u_t = 0$ in R'. We claim that $b_t.v_t \neq 0$ in R', or equivalently that $b_t.v_t$ cannot be written as $\sum_s c_s w_s$ in P. To see this, let c^* denote the constant term in the polynomial c_t. By examining the coefficient of $b_{t,j}$ in the equation $b_t.v_t = \sum_s c_s w_s$ we obtain $v_{t,j} = c^* u_{t,j}$ for all j, which means that c^* is a transporter for (u_t, v_t), contrary to assumption. Thus, b_t is a block for (u_t, v_t) in R', as required. □

Definition 6.3. Let R be a ring in $\mathcal{R}$, and let (u, v) be a test pair for R. We say that (u, v) is *good* if it has either a block or a transporter, and *bad* otherwise. We say that (u, v) is *nondegenerate* if $u_i \neq 0$ for all i. For any homogeneous element $x \in R$ we put $|x|_+ = \max(0, |x|)$. The *weight* of (u, v) is $\sum_i (1 + |u_i|_+ + |v_i|_+)$.

Lemma 6.4. *Let R be a ring in $\mathcal{R}$, and suppose that all nondegenerate test pairs are good. Then R is self-injective.*

Proof. Consider an arbitrary test pair $(u, v) \in R^r \times R^r$. If there exists i such that $u_i = 0$ but $v_i \neq 0$, then the basis vector $e_i \in R^r$ is a block for (u, v). Otherwise, let (u', v') be the test pair obtained by removing all zeros from u and the corresponding zeros from v. This is nondegenerate, so it has a block or a transporter. If b' is a block for (u', v'), then we can construct a block for (u, v) by inserting some zeros. If m' is a transporter for (u', v'), then it is also a transporter for (u, v). We therefore see that all test pairs for R are good, so R is self-injective. □

Lemma 6.5. *There are only finitely many nondegenerate bad test pairs of any given weight.*

Proof. Consider an integer $N \geq 0$. Any nondegenerate bad test pair (u, v) of weight N must have length at most N. Moreover, as (u, v) is nondegenerate we must have $u_i \neq 0$ for all i, and as $R \in \mathcal{R}$ this means that $|u_i| \geq 0$. We also have $\sum_i |u_i| \leq \text{weight}(u, v) = N$. It is clear from this (and the finiteness of R_k) that there are only finitely many possibilities for u. Next, let d be the degree of (u, v), so $|v_i| = |u_i| + d$. From this it is clear that $d \leq N$. If d is sufficiently negative then we will have $v_i = 0$ for all i, so 0 is a transporter for (u, v), contradicting the assumption that (u, v) is bad. We therefore see that there are only finitely many possibilities for d. Given u and d, it is clear that there are only finitely many possibilities for v. □

Theorem 6.6. *Suppose that $R \in \mathcal{R}$, and that $m \geq 0$. Then there is an extension $R' \geq R$ such that*

(a) *R' is also in R.*
(b) *$R'_k = R_k$ for all $k < m$.*
(c) *R' is self-injective.*

Proof. We define rings $R'(0) \leq R'(1) \leq \cdots$ as follows. We start with $R'(0) = R$. For each $k \geq 0$, we let $R'(k+1)$ be an extension of $R'(k)$ that agrees with $R'(k)$ in degrees less than $k + m$, such that every nondegenerate bad test pair of weight at most k in $R'(k)$ has a block in $R'(k+1)$. This can be constructed by Proposition 6.2 and Lemma 6.5. Now take R' to be the colimit of the rings $R'(k)$. By construction we have $R'_i = R'(k)_i$ for sufficiently large k, and using this it is clear that $R' \in \mathcal{R}$. Consider a nondegenerate test pair $(u, v) \in R'$. For sufficiently large k we can assume that $k \geq \text{weight}(u, v)$ and that $u_i, v_i \in R'(k)$ for all i. If (u, v) is good in $R'(k)$ then it is good in R'. If it is bad in $R'(k)$ then by construction it becomes good in $R'(k+1)$ and therefore in R'. □

7. The cube algebra

Recall that in the statement of Theorem 1.6 we introduced the ring

$$C = \mathbb{F}[y_0, y_1, \dots]/(y_i^3 + y_i y_{i+1} \mid i \geq 0),$$

with the grading given by $|y_i| = 2^i$. We now investigate the structure of this ring (which we call the *cube algebra*).

Definition 7.1. We also put

$$\begin{aligned} C[n,\infty] &= \mathbb{F}[y_n, y_{n+1}, \dots]/(y_i^3 + y_i y_{i+1} | n \leq i < \infty) \\ C[n,m] &= \mathbb{F}[y_n, \dots, y_m]/(y_i^3 + y_i y_{i+1} | n \leq i < m) \\ \overline{C}[n,m] &= C[n,m]/y_m. \end{aligned}$$

Lemma 7.2. *The evident maps*

$$\begin{array}{ccccc} C[n+1,m] & \longrightarrow & C[n+1,m+1] & \longrightarrow & C[n+1,\infty] \\ \downarrow & & \downarrow & & \downarrow \\ C[n,m] & \longrightarrow & C[n,m+1] & \longrightarrow & C[n,\infty] \\ \downarrow & & \downarrow & & \downarrow \\ C[0,m] & \longrightarrow & C[0,m+1] & \longrightarrow & C[0,\infty] = C \end{array}$$

are all split injective, so all the rings mentioned can be considered as subrings of C.

Proof. There is a graded ring map $\tau_0 \colon \mathbb{F}[y_0, y_1, \dots] \to C[n,m]$ given by

$$\tau_0(y_i) = \begin{cases} 0 & \text{if } i < n \\ y_i & \text{if } n \leq i \leq m \\ y_m^{2^{i-m}} & \text{if } m \leq i. \end{cases}$$

It is straightforward to check that $\tau_0(y_i^3 + y_i y_{i+1}) = 0$ for all $i \geq 0$, so there is an induced map $\tau \colon C \to C[n,m]$. It is clear that the composite $C[n,m] \to C \xrightarrow{\tau} C[n,m]$ is the identity, so the map $C[n,m] \to C$ is injective for all m and n. The other claims follow from this. □

Definition 7.3. We write P for the polynomial ring $\mathbb{F}[y_0, y_1, \dots]$, so that C is a quotient of P. A *multiindex* is a sequence $\alpha = (\alpha_0, \alpha_1, \dots)$ of natural numbers with $\alpha_i = 0$ for $i \gg 0$. We write MP for the set of all multiindices. Given $\alpha \in MP$ we write $y^\alpha = \prod_i y_i^{\alpha_i}$ and $|\alpha| = |y^\alpha| = \sum_i \alpha_i 2^i$. It is clear that the set $BP = \{y^\alpha \mid \alpha \in MP\}$ is a basis for P over $\mathbb{F}$.

Definition 7.4. We put

$$\begin{aligned} M'C[n,m] &= \{\alpha \in MP \mid \alpha_i = 0 \text{ for } i < n \text{ or } i > m \text{ and } \alpha_i < 3 \text{ for } n \leq i < m\} \\ M\overline{C}[n,m] &= \{\alpha \in MP \mid \alpha_i = 0 \text{ for } i < n \text{ or } i \geq m\} \\ B'C[n,m] &= \{y^\alpha \mid \alpha \in M'C[n,m]\} \\ B\overline{C}[n,m] &= \{y^\alpha \mid \alpha \in M\overline{C}[n,m]\}. \end{aligned}$$

Note that in the definition of $M'C[n,m]$ the constraint $\alpha_i < 3$ does not apply when $i = m$, so in particular $M'C[n,m]$ is infinite.

Proposition 7.5. *$B'C[n,m]$ is a basis for $C[n,m]$, and $B\overline{C}[n,m]$ is a basis for $\overline{C}[n,m]$. Moreover, $\overline{C}[n,m]$ is a Poincaré duality algebra over $\mathbb{F}$.*

The proof depends on the following result:

Lemma 7.6. *Let A be a commutative algebra over $\mathbb{F}$, let $f(t) \in A[t]$ be a monic polynomial of degree d, and put $B = A[x]/f(x)$. Then $\{1, x, \dots, x^{d-1}\}$ is a basis for B over A. Moreover, if A is finite-dimensional over $\mathbb{F}$ and has Poincaré duality, then the same is true of B.*

Proof. We first claim that any polynomial $g(x) \in A[x]$ can be expressed uniquely in the form $g(x) = q(x)f(x) + r(x)$ with $\deg(r(x)) < d$. This can easily be proved by induction on the degree of $g(x)$, and it follows directly that $\{1, \dots, x^{d-1}\}$ is a basis for B over A. Now suppose that A has Poincaré duality, so there is a linear map $\theta \colon A \to \mathbb{F}$ such that the bilinear form $(u, v) \mapsto \theta(u, v)$ is perfect. This means that there exist bases $\{u_0, \dots, u_{n-1}\}$ and $\{v_0, \dots, v_{n-1}\}$ for A such that $\theta(u_i v_j) = \delta_{ij}$. Now define $\phi \colon B \to \mathbb{F}$ by $\phi(\sum_{i=0}^{d-1} a_i x^i) = \theta(a_{d-1})$. We define bases $\{s_0, \dots, s_{nd-1}\}$ and $\{t_0, \dots, t_{nd-1}\}$ for B by $s_{ni+j} = x^i u_j$ and $t_{ni+j} = x^{d-1-i} v_j$ for $0 \leq i < d$ and $0 \leq j < n$. It is clear that $\phi(s_k t_k) = 1$. Suppose we have $0 \leq k < k' < nd$. Write $k = ni + j$ and $k' = ni' + j'$ as before; we must have either $i < i'$, or ($i = i'$ and $j < j'$). In either case, we find that $\phi(s_i t_j) = 0$. Thus, the matrix of ϕ with respect to our bases is triangular, with ones on the diagonal, proving that ϕ gives a perfect pairing on B. □

Proof of Proposition 7.5. From the definitions we have $C[m, m] = \mathbb{F}[y_n]$ and $B'C[m, m] = \{y_n^{\alpha_n} \mid \alpha_n \in \mathbb{N}\}$ so it is clear that $B'C[m, m]$ is a basis for $C[m, m]$. Similarly, it is clear that the set $\overline{C}[m, m] = \{1\}$ is a basis for the ring $\overline{C}[m, m] = C[m, m]/y_m = \mathbb{F}$, and that this has Poincaré duality.

Next, $C[n, m]$ can be described as $C[n + 1, m][y_n]/f(y_n)$, where $f(t) = t^3 + y_{n+1}t$ is a monic polynomial of degree three with coefficients in $C[n + 1, m]$. It also follows that $\overline{C}[n, m] = \overline{C}[n + 1, m][y_n]/f(y_n)$. All claims in the proposition now follow by downwards induction on n using Lemma 7.6. □

Remark 7.7. Note that the algebra

$$\overline{C}[n, m] = \frac{\mathbb{F}[y_n, y_{n+1}, \dots, y_{m-1}]}{(y_n^3 + y_n y_{n+1}, \dots, y_{m-1}^3)}$$

has the same number of relations as generators, and has finite dimension over $\mathbb{F}$. It is known that in this situation the sequence of relations is necessarily regular, and that the algebra automatically has Poincaré duality. (This can be extracted from [13, Section 17], for example.) This would give another approach to Proposition 7.5.

Definition 7.8. Let α be a multiindex. We say that

(a) α is *flat* if $\alpha_i < 3$ for all i;
(b) α is *n-truncated* if $\alpha_i = 0$ for all $i < n$;
(c) α is *m-solid* if it is flat and whenever $m \leq p \leq q$ and $\alpha_q > 0$ we also have $\alpha_p > 0$.

We consider all flat multiindices to be ∞-solid. For $0 \leq n \leq m \leq \infty$ we put

$$MC[n, m] = \{\alpha \in MP \mid \alpha \text{ is } n\text{-truncated and } m\text{-solid }\},$$

and $BC[n, m] = \{y^\alpha \mid \alpha \in MC[n, m]\}$. We also write MC for the set $MC[0, \infty]$ of all flat multiindices.

Proposition 7.9. *$BC[n, \infty]$ is a basis for $C[n, \infty]$.*

Proof. We must show that for each degree $d \in \mathbb{N}$, the set $BC[n, \infty]_d$ is a basis for $C[n, \infty]_d$. Choose $m > n$ such that $2^m > d$. It is then clear that $BC[n, \infty]_d = B'C[n, m]_d$ and $C[n, \infty]_d = C[n, m]_d$ so the claim follows from Proposition 7.5. □

It is also true that $BC[n, m]$ is a basis for $C[n, m]$ when $m < \infty$, but it is convenient to leave the proof until later.

Proposition 7.10. *For any multiindex $\alpha \in MP$, there is a multiindex $\beta \in MC$ such that $y^\alpha = y^\beta$.*

Proof. If $\alpha \notin MC$, we let k denote the smallest index such that $\alpha_k > 2$, and define $\alpha' \in MP$ by

$$\alpha'_i = \begin{cases} \alpha_i & \text{if } i < k \\ \alpha_k - 2 & \text{if } i = k \\ \alpha_{k+1} + 1 & \text{if } i = k + 1 \\ \alpha_i & \text{if } i > k + 1. \end{cases}$$

Because $y_k^3 = y_k y_{k+1}$ we have $y^\alpha = y^{\alpha'}$. Moreover, α' has the same degree as α, and is lexicographically lower than α. There are only finitely many monomials of any given degree, so the claim follows by induction over the lexicographic order. □

Definition 7.11.

(a) We put $x_0 = y_0$, and $x_n = y_n + y_{n-1}^2$ for all $n > 0$.
(b) For $n \geq m \geq 0$ we put $x_{[m,n]} = \prod_{i=n}^m x_i$ and $y_{[m,n]} = \prod_{i=n}^m y_i$.

Proposition 7.12. *For all $n \geq 0$ we have $y_n = \sum_{i=0}^n x_{n-i}^{2^i}$ and $y_n x_{n+1} = 0$. Thus, the ring C can also be presented as*

$$C = \mathbb{F}[x_0, x_1, x_2, \dots]/(x_{n+1} \sum_{i=0}^n x_{n-i}^{2^i} \mid n \geq 0).$$

Proof. Once we recall that $(a+b)^2 = a^2 + b^2 \pmod 2$, the equation $y_n = \sum_{i=0}^n x_{n-i}^{2^i}$ is easily checked by induction. Note that this already holds in the polynomial ring P. As the elements x_i can be expressed in terms of the y_j and vice-versa, we see that $P = \mathbb{F}[x_0, x_1, \dots]$. The defining relations $y_n^3 + y_n y_{n+1} = 0$ for C can clearly be rewritten as $y_n x_{n+1} = 0$ and thus as $x_{n+1} \sum_{i=0}^n x_{n-i}^{2^i} = 0$. □

Lemma 7.13. *Whenever $m \leq n$ we have $y_m y_{[m,n]}^2 = y_{[m,n+1]}$.*

Proof. The inductive step is

$$y_m y_{[m,n+1]}^2 = y_m y_{[m,n]}^2 y_{n+1}^2 = y_{[m,n+1]} y_{n+1}^2 = y_{[m,n]} y_{n+1}^3 = y_{[m,n]} y_{n+1} y_{n+2} = y_{[m,n+2]}.$$

□

Corollary 7.14. *For $k \geq 0$ we have $y_m^{2^k - 1} = y_{[m,m+k-1]}$.*

Proof. The induction step is

$$y_m^{2^{k+1}-1} = y_m \left(y_m^{2^k - 1} \right)^2 = y_m y_{[m,m+k-1]}^2 = y_{[m,m+k]}.$$

□

Lemma 7.15. *Fix $m \in \mathbb{N}$, and put*

$$U = \{\alpha \in MC \mid \alpha \text{ is } m\text{-solid and } \alpha_i = 0 \text{ for } i < m\}.$$

Then there is a bijection $\mathbb{N} \to U$ written as $k \mapsto \theta[m,k]$ such that $y^{\theta[m,k]} = y_m^k$ in C.

Proof. First, if $\alpha \in U$ it is clear that $|\alpha|$ is divisible by 2^m, so we can define $\delta \colon U \to \mathbb{N}$ by $\delta(\alpha) = |\alpha|/2^m$.

Now consider $k \in \mathbb{N}$. There is a unique $r \in \mathbb{N}$ such that $2^r - 1 \leq k < 2^{r+1} - 1$. This means that $0 \leq k - (2^r - 1) < 2^r$, so there is a unique set $J \subseteq \{0, 1, \dots, r-1\}$ with $k - (2^r - 1) = \sum_{j \in J} 2^j$. We put

$$\theta[m,k]_i = \begin{cases} 0 & \text{if } i < m \\ 1 & \text{if } m \leq i < m+r \text{ and } i - m \notin J \\ 2 & \text{if } m \leq i < m+r \text{ and } i - m \in J \\ 0 & \text{if } m + r \leq i. \end{cases}$$

This is clearly in U. Next, we claim that $y^{\theta[m,k]} = y_m^k$. To see this, put $z = y_m^{2^r - 1}$, which is the same as $y_{[m,m+r-1]}$ by Corollary 7.14. We have

$$y^{\theta[m,k]} = y_{[m,m+r-1]} \prod_{j \in J} y_{m+j} = z \prod_{j \in J} y_{m+j}$$
$$y_m^k = y_m^{2^r - 1 + \sum_{j \in J} 2^j} = z \prod_{j \in J} y_m^{2^j}$$

Now, for $0 \leq j < r$ we have $y_{m+j}(y_{m+j}^2 + y_{m+j+1}) = 0$ and z is divisible by y_{m+j} so $z(y_{m+j}^2 + y_{m+j+1}) = 0$, so $y_{m+j+1} = y_{m+j}^2$ modulo $\text{ann}(z)$. It follows inductively that $y_{m+j} = y_m^{2^j} \pmod{\text{ann}(z)}$, so $\prod_{j \in J} y_{m+j} = \prod_{j \in J} y_m^{2^j} \pmod{\text{ann}(z)}$, so $y^{\theta[m,k]} = y_m^k$ as claimed. It also follows that $\delta(\theta[m,k]) = |y^{\theta[m,k]}|/2^m = |y_m^k|/2^m = k$.

Now let α be an arbitrary element of U. By the definition of solidity, there is an integer $s \geq 0$ such that when $m \leq i < m+s$ we have $\alpha_i \in \{1, 2\}$ and for $i \geq m+s$ we have $\alpha_i = 0$. It is then clear that

$$\sum_{m \leq i < m+s} 2^i \leq |\alpha| \leq 2 \sum_{m \leq i < m+s} 2^i,$$

or in other words $2^s - 1 \leq \delta(\alpha) < 2^{s+1} - 1$. It follows easily that $\alpha = \theta[m, \delta(\alpha)]$, so we have a bijection as claimed. □

Proposition 7.16. *For $0 \leq n \leq m \leq \infty$, the set $BC[n,m]$ is a basis for $C[n,m]$.*

Proof. The case $m = \infty$ was covered by Proposition 7.9, so we may assume that $m < \infty$, so $B'C[n,m]$ is a basis for $C[n,m]$ by Proposition 7.5. However, Lemma 7.15 implies that $B'C[n,m]$, considered as a system of elements in $C[n,m]$, is just the same as $BC[n,m]$. □

Proposition 7.17. *Suppose that $0 \leq n < k \leq m \leq \infty$ and $k < \infty$. Then $\mathrm{ann}_{C[n,m]}(x_k) = C[n,m]y_{k-1}$.*

Proof. The $m = \infty$ case will follow from the $m < \infty$ case, because $C[n,m]_d = C[n,\infty]_d$ when m is large relative to d. We will thus assume that $m < \infty$.

We have already observed that $x_k y_{k-1} = 0$, so $\mathrm{ann}_{C[n,m]}(x_k) \geq C[n,m]y_{k-1}$, and multiplication by x_k gives a well-defined map $f\colon C[n,m]/(C[n,m]y_{k-1}) \to C[n,m]$. It will suffice to show that f is injective.

For this, we put

$$\begin{aligned} N &= \{\alpha \in MC[n,m] \mid \alpha_{k-1} = 0\} \\ A &= \{y^\alpha \mid \alpha \in N\} \subseteq C[n,m] \\ Z &= \mathrm{span}(A) \leq C[n,m]. \end{aligned}$$

By inspecting the generators and relations on both sides, we see that

$$C[n,m]/(C[n,m]y_{k-1}) = \overline{C}[n,k-1] \otimes C[k,m].$$

Propositions 7.5 and 7.9 show that A also gives a basis for $C[n,m]/(C[n,m]y_{k-1})$, so $C[n,m] = Z \oplus (C[n,m]y_{k-1})$. Now let g denote the composite

$$Z \xrightarrow{\simeq} C[n,m]/(C[n,m]y_{k-1}) \xrightarrow{f} C[n,m] \xrightarrow{\text{proj}} C[n,m]/Z.$$

It will certainly be enough to show that g is injective. It is not hard to see that $y_k Z \leq Z$, and $x_k = y_{k-1}^2 + y_k$, so $g(z) = x_k z + Z = y_{k-1}^2 z + Z$, so g gives an injective map from A to $BC[n,m] \setminus A$. These sets are bases for the domain and codomain of g, so g is injective as required. □

Proposition 7.18. *C is self-injective.*

Proof. As C is finite in each degree, it will suffice (by Propositions 3.3 and 3.6) to show that every test pair (u,v) in C has either a block or a transporter. Let d be the degree of (u,v), so $|v_i| = |u_i| + d$. Note that some of the entries u_i and v_i may be zero, in which case $|u_i|$ or $|v_i|$ can be negative. Choose m such that $2^m > d$ and also $2^m > |u_i|$ and $2^m > |v_i|$ for all i. Now (u,v) can be regarded as a test pair in $C[n,m]$. Let π be the projection $C[n,m] \to \overline{C}[n,m] = C[n,m]/y_m$. As $\overline{C}[n,m]$ has Poincaré duality, it is self-injective, so the test pair $(\pi(u), \pi(v))$ has either a block or a transporter. First, suppose that there is a transporter $\pi(t)$, so $\pi(v_i) = \pi(tu_i)$ for all i. This is an equation between elements of degree $|v_i| < 2^m$, and $\pi\colon C[n,m] \to \overline{C}[n,m]$ is an isomorphism in this degree, so $v_i = tu_i$, so we have a transporter for the original pair (u,v).

Suppose instead that there is a block for $(\pi(u), \pi(v))$, say $\pi(b)$. This means that $\pi(b.u) = 0$ but $\pi(b.v) \neq 0$, so $b.u \in C[n,m]y_m$ but $b.v \notin C[n,m]y_m$. Using our bases for the various rings under consideration, we see that $C[n,m]y_m = (Cy_m) \cap C[n,m]$, and thus that $b.v \notin Cy_m$. It now follows from Proposition 7.17 that $(x_{m+1}b).u = 0$ and $(x_{m+1}b).v \neq 0$, so $x_{m+1}b$ is a block for the original pair (u,v). □

We now wish to prove that C is coherent, which turns out to involve substantial work. It will be convenient to regard the set $B\overline{C}[n,m] = \{y^\alpha \mid \alpha \in M\overline{C}[n,m]\}$ as a subset of $C[n,m]$ rather than a subset of $\overline{C}[n,m]$. We write $\widetilde{C}[n,m]$ for the span of this set, so the projection $C[n,m] \to \overline{C}[n,m]$ restricts to give an isomorphism $\widetilde{C}[n,m] \to \overline{C}[n,m]$.

Lemma 7.19. *For $p \geq 3$ we have*

$$y^2_{[0,p-3]} y^2_{[0,p-1]} y_1 y_{p-1} y_p = y^2_{[0,p]}$$

(and in particular, this is nonzero modulo y_{p+1}).

Proof. Put $A = C[0,p]/\operatorname{ann}(y_{[0,p]})$. We claim that in A we have

$$y^2_{[0,p-3]} y_{[0,p-1]} y_1 y_{p-1} = y_{[0,p]}.$$

Assuming this, we can just multiply by $y_{[0,p]}$ to recover the statement in the lemma.

For $0 \leq i < p$ we have $y_i(y_i^2 + y_{i+1}) = 0$ so $y_{[0,p]}(y_i^2 + y_{i+1}) = 0$ so $y_{i+1} = y_i^2$ in A. We thus have $y_k = y_0^{2^k}$ in A for $0 \leq k \leq p$, and so $A = \mathbb{F}[y_0]$. It is thus enough to show that the two sides of the claimed equation have the same degree, which is a straightforward calculation. □

Lemma 7.20. *For any $p \geq 3$ we have*

$$B\overline{C}[0,p-2]\ B\overline{C}[0,p] \subseteq \coprod_{i=0}^{3} B\overline{C}[0,p-1] y^i_{p-1}.$$

Proof. Consider $\alpha \in M\overline{C}[0,p-2]$ and $\beta \in M\overline{C}[0,p]$. We note that $y^\alpha, y^\beta \in C[0,p-1]$ so we can rewrite $y^{\alpha+\beta}$ as an element of the basis $B'C[0,p-1]$, which means $y^{\alpha+\beta} = y^\gamma$ for some $\gamma \in M'C[0,p-1]$. It will be enough to show that $\gamma_{p-1} \leq 3$.

Note that y^α divides $y^2_{[0,p-3]}$ and y^β divides $y^2_{[0,p-1]}$ so y^γ divides $y^2_{[0,p-3]} y^2_{[0,p-1]}$. It follows using Lemma 7.19 that $y^\gamma y_{p-1} y_p \neq 0 \pmod{y_{p+1}}$. However,

$$y^4_{p-1} y_{p-1} y_p = y^5_{p-1} y_p = y^3_{p-1} y_p^2 = y_{p-1} y_p^3 = y_{p-1} y_p y_{p+1} = 0 \pmod{y_{p+1}},$$

so y^γ cannot be divisible by y^4_{p-1}, as required. □

Definition 7.21. For any vector $u \in C^n$ and $p \geq 0$, we put

$$K(u,p) = \{v \in C[0,p]^n \mid u.v = 0\}$$
$$\overline{K}(u,p) = \{\overline{v} \in \overline{C}[0,p]^n \mid \pi(u).\overline{v} = 0\}.$$

More precisely, $K(u,p)$ is the graded group where

$$K(u,p)_d = \{v \in C[0,p]^n \mid |v_i| = d - |u_i| \text{ for all } i \text{ and } \sum_i u_i v_i = 0\},$$

and $\overline{K}(u,p)$ is graded in a similar way.

Lemma 7.22. *If $u_i \in \widetilde{C}[0,p-2]$ for all i, then the map $\pi\colon K(u,p+1) \to \overline{K}(u,p+1)$ is surjective.*

Proof. Consider an element $\overline{v} \in \overline{K}(u,p+1)$. This can be written as $\pi(v)$ for a unique element $v \in \widetilde{C}[0,p+1]^n$, which must satisfy $u.v = 0 \pmod{y_{p+1}}$. We can write v as $\sum_{k=0}^{2} v_k y_p^k$ with $v_k \in \widetilde{C}[0,p]^n$. Using Lemma 7.20 we see that $u.v_k$ can be written as $\sum_{j=0}^{3} w_{jk} y^j_{p-1}$ for some elements $w_{jk} \in \widetilde{C}[0,p-1]$. This gives $u.v = \sum_{j=0}^{3}\sum_{k=0}^{2} w_{jk} y^j_{p-1} y^k_p$. After reducing the terms $y^j_{p-1} y^k_p$ using the defining relations for C, we obtain

$$\begin{aligned} u.v =& w_{00} + w_{01} y_p + w_{02} y_p^2 + w_{10} y_{p-1} + (w_{11} + w_{30}) y_{p-1} y_p + (w_{12} + w_{31}) y_{p-1} y_p^2 + \\ & w_{20} y^2_{p-1} + w_{21} y^2_{p-1} y_p + w_{22} y^2_{p-1} y^2_p + w_{32} y_{p-1} y_p y_{p+1}. \end{aligned}$$

By hypothesis, this maps to zero in $\overline{C}[0,p+1] = C[0,p+1]/y_{p+1}$. However, $\overline{C}[0,p+1]$ splits as the direct sum of subgroups $\widetilde{C}[0,p-1] y^i_{p-1} y^j_p$ for $0 \leq i,j < 3$, so we must have

$$w_{00} = w_{01} = w_{02} = w_{10} = w_{20} = w_{21} = w_{22} = 0$$

and $w_{11} = w_{30}$ and $w_{12} = w_{31}$, so $u.v = w_{32} y_{p-1} y_p y_{p+1}$.

Now put $d = |u.v|$, so $|w_{jk}| = d - j2^{p-1} - k2^p$. In particular, we have $|w_{32}| = d - 2^{p-1} - 2^p - 2^{p+1}$. If $d < 2^{p-1} + 2^p + 2^{p+1}$ then $|w_{32}| < 0$ so $w_{32} = 0$ so $u.v = 0$. This means that $v \in K(u, p+1)$ with $\pi(v) = \overline{v}$, as required. Suppose instead that $d \geq 2^{p-1} + 2^p + 2^{p+1}$. We have

$$|w_{11}| = |w_{30}| = d - 2^{p-1} - 2^p \geq 2^{p+1}$$
$$|w_{12}| = |w_{31}| = d - 2^{p-1} - 2^{p+1} \geq 2^p.$$

However, the elements w_{jk} lie in $\widetilde{C}[0, p-1]$, which is zero in degrees larger than $2^p - 2$. We therefore have $w_{11} = w_{12} = w_{30} = w_{31} = 0$, which means that $u.v_0 = 0$ and $u.v_1 = 0$ and $u.v_2 = w_{32}y_{p-1}^3 = w_{32}y_{p-1}y_p$. Put

$$v' = v_0 + v_1 y_p + v_2(y_p^2 + y_{p+1}),$$

so $\pi(v') = \pi(v) = \overline{v}$ and

$$u.v' = u.v_0 + u.v_1 y_p + u.v_2(y_p^2 + y_{p+1}) = w_{32}y_{p-1}y_p(y_p^2 + y_{p+1}) = 0.$$

Thus, v' is the required lift of $\overline{v}$ in $\overline{K}(u, p+1)$. ☐

Lemma 7.23. *For all $p \geq 0$ we have a splitting*

$$C[0, p+1] = C[0, p] \oplus \bigoplus_{k>0} \overline{C}[0, p]x_{p+1}^k.$$

Proof. By definition we have $C[0, p+1] = C[0, p][y_{p+1}]/(x_{p+1}y_p)$, where $x_{p+1} = y_{p+1} + y_p^2$ as usual. From this it is clear that

$$C[0, p][y_{p+1}] = C[0, p][x_{p+1}] = C[0, p] \oplus \bigoplus_{k>0} C[0, p]x_{p+1}^k.$$

The ideal generated by $y_p x_{p+1}$ in this ring clearly has a compatible splitting

$$C[0, p][y_{p+1}].y_p x_{p+1} = \bigoplus_{k>0} C[0, p]y_p x_{p+1}^k.$$

We can thus pass to the quotient to get

$$C[0, p+1] = C[0, p] \oplus \bigoplus_{k>0} \frac{C[0, p]}{C[0, p]y_p} x_{p+1}^k = C[0, p] \oplus \bigoplus_{k>0} \overline{C}[0, p]x_{p+1}^k$$

as claimed. ☐

Corollary 7.24. *If $u_i \in \widetilde{C}[0, p-2]$ for $i = 0, \ldots, n-1$, then $K(u, p+1) = C[0, p+1].K(u, p)$.*

Proof. It is clear that $C[0, p+1].K(u, p) \leq K(u, p+1)$. For the converse, consider an element $v \in K(u, p+1) \leq C[0, p+1]^n$. Using Lemma 7.23, we can write v as $v_0 + \sum_{k>0} \overline{v}_k x_{p+1}^k$, with $v_0 \in C[0, p]^n$ and $\overline{v}_k \in \overline{C}[0, p]^n$ (with $\overline{v}_k = 0$ for $k \gg 0$). It follows that $u.v_0 \in C[0, p]$ and $u.\overline{v}_k \in \overline{C}[0, p]$ and

$$u.v_0 + \sum_{k>0} (u.\overline{v}_k)x_{p+1}^k = u.v = 0.$$

As the sum in Lemma 7.23 is direct, we must have $u.v_0 = 0$ and $u.\overline{v}_k = 0$, so $v_0 \in K(u, p)$ and $\overline{v}_k \in \overline{K}(u, p)$. By Lemma 7.22, we can choose $v_k \in K(u, p)$ for $k > 0$ lifting $\overline{v}_k$. If $\overline{v}_k = 0$ we choose $v_k = 0$; this ensures that $v_k = 0$ for $k \gg 0$. We now have $v = \sum_{k \geq 0} v_k x_{p+1}^k \in C[0, p+1].K(u, p)$, as required. ☐

Proposition 7.25. *The ring C is coherent.*

Proof. Let $I \leq C$ be a finitely generated ideal. Choose elements $u_0, \ldots, u_{n-1}$ generating I. These give an epimorphism $g \colon \bigoplus_i \Sigma^{|u_i|} C \to I$, with $\ker(g) = K(u, \infty)$, so it will suffice to show that $K(u, \infty)$ is finitely generated as a C-module. Now choose p large enough that $u_i \in \widetilde{C}[0, p-2]$ for all i. As $C[0, p]$ is Noetherian, we can choose a finite subset $T \subseteq C[0, p]^n$ that generates $K(u, p)$ as a $C[0, p]$-module. Corollary 7.24 tells us that T also generates $K(u, p+1)$ as a $C[0, p+1]$-module. In fact, we can apply the same corollary inductively to see that T generates $K(u, q)$ as a $C[0, q]$-module for all $q \geq p$. As $C = \bigcup_q C[0, q]$ we conclude that T generates $K(u, \infty)$ as required. ☐

Proposition 7.26. *The reduced quotient of C is*

$$C/\sqrt{0} = \mathbb{F}[x_i \mid i \geq 0]/(x_i x_j \mid i \neq j).$$

Proof. Put $C' = C/\sqrt{0}$. We first claim that for all p, q with $0 \leq p < q$ we have $x_p x_q = 0$ in C'. We may assume inductively that $x_i x_j = 0$ in C' whenever $0 \leq i < j < q$. By a nested downward induction over p, we may assume that $x_k x_q = 0$ in C' whenever $p < k < q$. As in Proposition 7.12, we have $x_q \sum_{k=0}^{q-1} x_k^{2^{q-1-k}} = 0$. We can multiply this by x_p and use the inner and outer inductive assumptions to see that $x_p x_q x_p^{2^{q-1-p}} = 0$, or in other words $x_p^m x_q = 0$ for some $m > 0$. This gives $(x_p x_q)^m = 0$ in C', but C' is reduced by construction so $x_p x_q = 0$ in C' as claimed.

Now put

$$C'' = C/(x_i x_j \mid i, j,\ i < j) = \mathbb{F}[x_i \mid i \geq 0]/(x_i x_j \mid i, j \geq 0,\ i < j).$$

We now see that C'' is a quotient of C by nilpotent elements, so C' can also be described as $C''/\sqrt{0}$. However, there is an obvious splitting

$$C'' = \mathbb{F} \oplus \bigoplus_{i \geq 0} x_i \mathbb{F}[x_i],$$

and using this we see that C'' is reduced. It follows that $C' = C''$ as claimed. □

8. The image of J

We now study the graded ring J described by Definition 1.7, and the tensor product $\widehat{J} = \mathbb{Z}_p \otimes J$. It is standard that $\mathbb{Z}_p \otimes \mathbb{Z}/p^r = \mathbb{Z}/p^r$. Moreover, the group $\mathbb{Q}/\mathbb{Z}_{(p)}$ can be written as the colimit of the evident sequence

$$\mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p^3 \to \dots,$$

and we can tensor with $\mathbb{Z}_p$ to get $\mathbb{Z}_p \otimes (\mathbb{Q}/\mathbb{Z}_{(p)}) = \mathbb{Q}/\mathbb{Z}_{(p)}$. Thus, the only difference between J and $\widehat{J}$ is that $J_0 = \mathbb{Z}_{(p)}$ whereas $\widehat{J}_0 = \mathbb{Z}_p$.

Definition 8.1. For each $k \in \mathbb{Z}$, we define

$$\xi_k \colon \widehat{J}_k \to \operatorname{Hom}_{\mathbb{Z}}(\widehat{J}_{-k-2}, \mathbb{Q}/\mathbb{Z}_{(p)})$$

by $\xi_k(a)(b) = \zeta^{-1}(ab)$ (where ζ is the isomorphism $\mathbb{Q}/\mathbb{Z}_{(p)} \to \widehat{J}_{-2} = J_{-2}$ that is given as part of the definition of J.)

Lemma 8.2. *The maps ξ_k are isomorphisms for all k.*

Proof. For $k \neq -2$ this is a straightforward calculation. For $k = -2$ we use the description $\mathbb{Q}/\mathbb{Z}_{(p)} = \varinjlim_j \mathbb{Z}/p^j$ to get

$$\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}_{(p)}, \mathbb{Q}/\mathbb{Z}_{(p)}) = \varprojlim_j \operatorname{Hom}(\mathbb{Z}/p^j, \mathbb{Q}/\mathbb{Z}_{(p)}) = \varprojlim_j \mathbb{Z}/p^j = \mathbb{Z}_p,$$

as required. □

Corollary 8.3. *For $\widehat{J}$-modules M there is a natural isomorphism*

$$\operatorname{Hom}_{\widehat{J}}(M, \widehat{J}) = \operatorname{Hom}_{\mathbb{Z}}(M_{-2}, \mathbb{Q}/\mathbb{Z}_{(p)}).$$

Proof. Given $\phi \in \operatorname{Hom}_{\widehat{J}}(M, \widehat{J})$, we put

$$\tau(\phi) = \zeta^{-1} \circ \phi_{-2} \colon M_{-2} \to \mathbb{Q}/\mathbb{Z}_{(p)}.$$

This defines a map $\tau \colon \operatorname{Hom}_{\widehat{J}}(M, \widehat{J}) \to \operatorname{Hom}_{\mathbb{Z}}(M_{-2}, \mathbb{Q}/\mathbb{Z}_{(p)})$.

Now suppose we have a map $\psi \colon M_{-2} \to \mathbb{Q}/\mathbb{Z}_{(p)}$. For any $k \in \mathbb{Z}$ we have a map

$$\phi'_k \colon M_k \to \operatorname{Hom}_{\mathbb{Z}}(\widehat{J}_{-k-2}, \mathbb{Q}/\mathbb{Z}_{(p)})$$

given by $\phi'_k(m)(a) = (-1)^k \psi(am)$. (The exponent of -1 here is morally $|a||m| = -k(k+2)$, but that is congruent to k modulo 2.) Lemma 8.2 tells us that there is a unique element $\phi_k(m) \in \widehat{J}_k$ such that

$$\phi'_k(m)(a) = \zeta^{-1}(\phi_k(m)a)$$

for all $a \in \widehat{J}_{-k-2}$. We leave it to the reader to check that this gives a map $\phi\colon M \to \widehat{J}$ of $\widehat{J}$-modules, and that this is the unique such map with $\tau(\phi) = \psi$. □

Corollary 8.4. *The ring $\widehat{J}$ is self-injective.*

Proof. We need to show that the functor $M \mapsto \mathrm{Hom}_{\widehat{J}}(M, \widehat{J})$ is exact, but it is isomorphic to the functor $M \mapsto \mathrm{Hom}_{\mathbb{Z}}(M_{-2}, \mathbb{Q}/\mathbb{Z}_{(p)})$, which is exact because $\mathbb{Q}/\mathbb{Z}_{(p)}$ is divisible and therefore injective as an abelian group. □

Remark 8.5. The ring J itself is not self-injective. To see this, note that J_{-2} is an ideal in J and is a module over $\mathbb{Z}_p$. Choose any element $a \in \mathbb{Z}_p \setminus \mathbb{Z}_{(p)}$ and define $u\colon J_{-2} \to J$ by $u(x) = ax$. This cannot be extended to give a J-linear endomorphism of J.

Lemma 8.6. *The ring $\widehat{J}$ is local (in the graded sense). The unique maximal graded ideal is given by $\mathfrak{m}_0 = p\mathbb{Z}_p$ and $\mathfrak{m}_k = \widehat{J}_k$ for all $k \neq 0$. Moreover, the elements α_k together with the element p give a basis for $\mathfrak{m}/\mathfrak{m}^2$ over $\mathbb{Z}/p$.*

Proof. It is straightforward to check that the graded group $\mathfrak{m}$ described above is an ideal in $\widehat{J}$, and the quotient $\widehat{J}/\mathfrak{m}$ is the field $\mathbb{Z}/p$, so it is a maximal ideal. Let $\mathfrak{m}'$ be an arbitrary maximal graded ideal. Put $\mathfrak{a} = \bigoplus_{k\neq 0} \widehat{J}_k$. Every homogeneous element $a \in \mathfrak{a}$ satisfies $a^2 = 0$, and it follows that $\mathfrak{a} \leq \mathfrak{m}'$ This means that $\mathfrak{m}'$ corresponds to a maximal ideal in the quotient $\widehat{J}/\mathfrak{a} \simeq \mathbb{Z}_p$, and the only such ideal is $p\mathbb{Z}_p$. It follows that $\mathfrak{m}' = \mathfrak{m}$ as claimed. The description of $\mathfrak{m}/\mathfrak{m}^2$ is a straightforward calculation. □

Proposition 8.7. *The ring $\widehat{J}$ is totally incoherent.*

Proof. Put $V = \{\alpha_k \mid k \neq 0 \pmod{p}\} \subset J$, so V is infinite and $pV = 0$ and V and remains linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. By inspecting the multiplication rules, we see that every non-invertible element of $\widehat{J}$ annihilates all elements of V with at most one exception. It follows using Corollary 5.6 that $\widehat{J}$ is totally incoherent. □

9. The infinite root algebra

In this section we fix a field K and study the infinite root algebra P over K, which was introduced in Definition 1.11. We first recall the details.

Definition 9.1. We say that a subset $U \subseteq [0,1]$ is *well-ordered* if the usual order inherited from $\mathbb{R}$ is a well-ordering, so every nonempty subset of U has a smallest element. It is equivalent to say that every infinite nonincreasing sequence in U is eventually constant, or that there are no infinite, strictly decreasing sequences.

An *infinite root series* is a function $a\colon [0,1] \to K$ such that the set $\mathrm{supp}(a) = \{q \mid a(q) \neq 0\}$ is well-ordered. The infinite root algebra is the set P of all infinite root series. We regard this as an ungraded object, or equivalently as a graded object concentrated in degree zero.

Remark 9.2. It is clear that any subset of a well-ordered set is well-ordered, and that the union of any two well-ordered sets is well-ordered. Now if $a, b \in P$ we have $\mathrm{supp}(a+b) \subseteq \mathrm{supp}(a) \cup \mathrm{supp}(b)$, so P is closed under addition. It is clearly also closed under multiplication by elements of K.

Lemma 9.3. *Any well-ordered subset of $[0,1]$ is countable. Moreover, for any countable ordinal α, there is a well-ordered subset $U \subseteq [0,1]$ that is order-isomorphic to α.*

Proof. Firstly, we can regard rational numbers in $[0,1]$ as coprime pairs of integers and this gives a lexicographic ordering on $\mathbb{Q} \cap [0,1]$, which is a well-ordering.

Next, let U be a well-ordered subset of $[0,1]$. We define $f\colon U \to \mathbb{Q}$ as follows. If u is maximal in U, we put $f(u) = 1$. Otherwise, the set $\{v \in U \mid v > u\}$ has a smallest element v_0, and we define $f(u)$ to be the lexicographically smallest element of $\mathbb{Q} \cap [u, v_0)$. It is clear that f is injective, so U is countable.

Let α be any countable ordinal; we claim that there is an order-embedding $g\colon \alpha \to [0,1]$. To see this, choose an injective map $p\colon \alpha \to \mathbb{N}$ and then put

$$g(\beta) = \sum_{\gamma<\beta} 2^{-p(\gamma)-1}.$$

It is clear that this has the required properties. □

Lemma 9.4. *If $U, V \subseteq [0,1]$ are well-ordered and $w \in [0,1]$ then $\{(u,v) \in U \times V \mid u + v = w\}$ is finite.*

Proof. Put $U' = \{u \in U \mid w - u \in V\}$. This is well-ordered (because it is a subset of U) and it will suffice to show that it is finite. If not, we can define an infinite sequence $u_0 < u_1 < u_2 < \cdots$ in U' as follows: we take u_0 to be the smallest element in U', then take u_1 to be the smallest element in $U' \setminus \{u_0\}$, and so on. We then note that $w - u_0, w - u_1, w - u_2, \dots$ is an infinite strictly decreasing sequence in V, contradicting the assumption that V is well-ordered. □

Lemma 9.5. *Let U be a well-ordered subset of $[0,1]$, and let (u_n) be a sequence in U. Then there exists an infinite nondecreasing subsequence.*

Proof. Put $v_0 = \min\{u_j \mid j \geq 0\}$ (which is meaningful because U is well-ordered) and then $n_0 = \min\{j \mid u_j = v_0\}$. For $i > 0$ we define recursively $v_i = \min\{u_j \mid j > n_{i-1}\}$ and $n_i = \min\{j > n_{i-1} \mid u_j = v_i\}$. We find that $n_0 < n_1 < n_2 < \cdots$ and $v_0 \leq v_1 \leq v_2 \leq \cdots$, or equivalently $u_{n_0} \leq u_{n_1} \leq u_{n_2} \leq \cdots$ as required. □

Lemma 9.6. *Let U and V be well-ordered subsets of $[0,1]$, and put $U * V = \{u + v \mid u \in U \text{ and } v \in V\}$. Then $U * V$ is also well-ordered.*

Proof. Suppose not. We can then find an infinite strictly descending chain in $U * V$, so we can choose a sequence (u_n, v_n) in $U \times V$ with $u_i + v_i > u_{i+1} + v_{i+1}$ for all i. Lemma 9.5 tells us that after passing to a subsequence, we may assume that $u_j \leq u_{j+1}$ for all j. After passing again to a sparser subsequence, we may also assume that $v_k \leq v_{k+1}$ for all k. This is clearly impossible. □

Proposition 9.7. *We can make P into a commutative ring by the rule*

$$ab(w) = \sum_{w=u+v} a(u)b(v).$$

Proof. Lemma 9.4 shows that the sum is essentially finite, so there is no problem with convergence. It is clear that $\mathrm{supp}(ab) \subseteq \mathrm{supp}(a) * \mathrm{supp}(b)$, and Lemma 9.6 shows that $\mathrm{supp}(a) * \mathrm{supp}(b)$ is well-ordered, so $ab \in P$. It is straightforward to check that the multiplication operation is commutative, associative and bilinear. Moreover, if we define $e(0) = 1$ and $e(q) = 0$ for $q \neq 0$, then e is a multiplicative identity element for P. □

Definition 9.8. For $a \in P \setminus \{0\}$, we put $\delta(a) = \min(\mathrm{supp}(a))$. We also put $\delta(0) = \infty$.

Remark 9.9. Note that if $\delta(a) + \delta(b) \leq 1$ we have

$$(ab)(\delta(a) + \delta(b)) = a(\delta(a))\, b(\delta(b)) \neq 0,$$

so $ab \neq 0$ and $\delta(ab) = \delta(a) + \delta(b)$. On the other hand, if $\delta(a) + \delta(b) > 1$ then $ab = 0$.

Definition 9.10. For $q \in \mathbb{R} \cup \{\infty\}$ with $q \geq 0$, we define $x^q \in P$ by

$$x^q(u) = \begin{cases} 1 & \text{if } u = q \\ 0 & \text{otherwise.} \end{cases}$$

Remark 9.11. We note that

(a) x^0 is the multiplicative identity element e.
(b) If $q > 1$ then $x^q = 0$.
(c) If $0 \leq q \leq 1$ then $\delta(x^q) = q$.
(d) For all $q, r \geq 0$ we have $x^q x^r = x^{q+r}$.

Lemma 9.12. *Consider an element $a \in P \setminus \{0\}$. If $a(0) = 0$ (or equivalently, $\delta(a) > 0$) then a is nilpotent, but if $\delta(a) = 0$ then a is invertible.*

Proof. If $\delta(a) > 0$ then we can find a positive integer n with $\delta(a) > 1/n$, and using Remark 9.9 we see that $a^n = 0$. Suppose instead that $\delta(a) = 0$. We can then write $a = ue + b = u(e + b/u)$ where $u \in K \setminus 0$ and $e = x^0$ is the multiplicative identity of P and $\delta(b) > 0$, so $b^n = 0$ for some n. Now a has inverse $\sum_{i=0}^{n-1} u^{-1}(-b/u)^i$. □

Corollary 9.13. *The map $a \mapsto a(0)$ induces an isomorphism $P/\sqrt{0} \to K$.*

Proof. Clear. ☐

Definition 9.14. For $a \in P$ with $\delta(a) \geq t$, we define $\lambda_t(a) \in P$ by

$$\lambda_t(a)(r) = \begin{cases} a(r+t) & \text{if } 0 \leq r \leq 1-t \\ 0 & \text{if } 1-t < r \leq 1. \end{cases}$$

Corollary 9.15. *If $\delta(a) \geq t$ then $a = x^t\, \lambda_t(a)$ and $\delta(\lambda_t(a)) = \delta(a) - t$. Moreover, if $\delta(a) = t$ then $\lambda_t(a)$ is invertible, so $Pa = Px^t$.*

Proof. The first two claims are clear from the definitions, and the third then follows using Lemma 9.12. ☐

Definition 9.16. For $t \in [0, 1]$ we put

$$J_t = \{a \in P \mid \delta(a) > t\}$$
$$\overline{J}_t = \{a \in P \mid \delta(a) \geq t\} = Px^t.$$

Proposition 9.17. *Every ideal in P has the form J_t or $\overline{J}_t$.*

Proof. Let I be an ideal in P. If $I = 0$ then $I = J_1$. Otherwise, we put $t = \inf\{\delta(a) \mid a \in I\}$. If $t = \delta(a)$ for some $a \in I$ then Corollary 9.15 shows that $x^t \in I$, and it follows easily that $I = \overline{J}_t$. Suppose instead that there is no element $a \in I$ with $\delta(a) = t$. It is then clear that $I \leq J_t$. Moreover, if $b \in J_t$ then $\delta(b) > t$ so (by the infimum condition) there exists $a \in I$ with $\delta(b) > \delta(a) > t$. After applying Corollary 9.15 to a and b, we see that b is a multiple of a, and so $b \in I$. We now see that $I = J_t$, as required. ☐

Proposition 9.18. *For all $t \in [0, 1]$ we have $\text{ann}_P(J_t) = \overline{J}_{1-t}$ and $\text{ann}_P(\overline{J}_t) = J_{1-t}$.*

Proof. This follows easily from the fact that $ab = 0$ iff $\delta(a) + \delta(b) > 1$. ☐

Corollary 9.19. *For any ideal $I \leq P$ we have $\text{ann}^2_P(I) = I$.*

Proof. Immediate from the last two propositions. ☐

Proposition 9.20. *P is self-injective.*

Proof. As we have classified all ideals in P, we can use Baer's criterion. Consider a number $t \in [0, 1]$ and a P-module map $f\colon \overline{J}_t = (x^t) \to P$. If $f(x^t) = a$ then we must have $J_{1-t}a = f(J_{1-t}x^t) = f(0) = 0$, so $a \in \text{ann}(J_{1-t}) = \overline{J}_t$, so $a = x^t\lambda_t(a)$. We can now define $f'\colon P \to P$ extending f by $f'(p) = p\,\lambda_t(a)$, so Baer's criterion is satisfied in this case.

Now consider instead a P-module map $f\colon J_t \to P$. If $t = 1$ then $J_t = 0$ and the zero map $P \to P$ extends f. We suppose instead that $t < 1$. For $s \in (t, 1]$ we put $a_s = \lambda_s(f(x^s))$, so the first case shows that $f(p) = pa_s$ for all $p \in \overline{J}_s < J_t$. Now suppose that $t < r \leq s \leq 1$. As $x^s \in \overline{J}_s \leq \overline{J}_r$ we have $x^s(a_r - a_s) = f(x^s) - f(x^s) = 0$, so $a_r(q) = a_s(q)$ for all $q \leq 1 - s$. Moreover, from the definition of the λ operation we have $a_s(q) = 0$ for $q > 1 - s$, and thus certainly for $q \geq 1 - t$. We now see that there is a unique map $a\colon [0, 1] \to K$ with $a = a_s$ on $[0, 1 - s]$ (for all $s \in (t, 1]$) and $a = 0$ on $[1 - t, 1]$. It follows easily from these properties that $\text{supp}(a)$ is well-ordered, so $a \in P$. We also see from the first property that f agrees with multiplication by a on $\overline{J}_s$ for all $s \in (t, 1]$. It follows that the same is true on $\bigcup_{s\in(t,1]} \overline{J}_s = J_t$, as required. ☐

Proposition 9.21. *P is totally incoherent.*

Proof. Let I be a finitely generated ideal, say $I = (a_1, \ldots, a_r)$, where we can assume that the generators a_i are nonzero. If $r = 0$ then $I = 0$, and this is finitely presented. If $r > 0$ we can use Corollary 9.15 to see that $I = \overline{J}_t$, where $t = \min(\delta(a_1), \ldots, \delta(a_r))$.

Now suppose that I is nonzero and finitely presented. We must have $I = \overline{J}_t$ for some t, so we have an epimorphism $g\colon P \to I$ given by $g(a) = ax^t$. Proposition 5.1 tells us that $\ker(g)$ must also be finitely generated, but $\ker(g) = \text{ann}_P(x^t) = J_{1-t}$, and this is only finitely generated when $t = 0$ and so $\ker(g) = J_1 = 0$ and $I = \overline{J}_0 = P$. ☐

Remark 9.22. Put $P' = \{a \in P \mid \operatorname{supp}(a) \subseteq \mathbb{Q}\}$. This is a subring of P, and one can adapt the above arguments to show that it is again self-injective and totally incoherent. Every ideal in P' has the form $J_t \cap P'$ or $\overline{J}_t \cap P'$ for some $t \in [0,1]$, and these are all distinct except for the fact that $J_t \cap P' = \overline{J}_t \cap P'$ when t is irrational.

10. The Rado algebra

In this section we study the Rado algebra Q, which was defined in Definition 1.15. We will write Γ for the Rado graph.

We first clarify the kinds of graphs that we will consider.

Definition 10.1. A *graph* is a pair (V, E), where V is a set and E is a subset of $V \times V$ such that

(a) For all $v \in V$ we have $(v, v) \notin E$.
(b) For all $v, w \in V$ we have $((v, w) \in E$ iff $(w, v) \in E)$.

Definition 10.2. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A *full embedding* of G in G' is an injective map $f\colon V \to V'$ such $E = (f \times f)^{-1}(E')$ (so vertices $v_0, v_1 \in V$ are linked by an edge in G iff the images $f(v_0)$ and $f(v_1)$ are linked by an edge in G'). Similarly, a *full subgraph* of G' is a graph of the form $G = G'|_V = (V, E' \cap V^2)$ for some subset $V \subseteq V'$, so the inclusion map gives a full embedding $G \to G'$.

Lemma 10.3. *Suppose we have a finite graph G', a full subgraph G, and a full embedding $f\colon G \to \Gamma$. Then there is a full embedding $f'\colon G' \to \Gamma$ extending f.*

Proof. It is easy to reduce to the case where G' has only one more vertex than G, say $V' = V \amalg \{x\}$. Put $A = \{v \in V \mid (v, x) \in E'\}$ and $N = \max\{f(v) \mid v \in V\} + 1$, then let $f'\colon V' \to \mathbb{N}$ be the map extending f with $f'(x) = 2^N + \sum_{v \in A} 2^{f(v)}$. It is straightforward to check that this has the required properties. □

Remark 10.4. As we mentioned in Example 4.7, each group E_k (for $k \geq 0$) is isomorphic to $\mathbb{F}$. The generator is the element $y_k = x_{B(k)} = \prod_{i \in B(k)} x_i$. We say that a finite subset $I \subseteq \mathbb{N}$ is Γ-*complete* if the full subgraph $\Gamma|_I$ is a complete graph (so every two distinct points are linked by an edge). We say that a natural number n is $B\Gamma$-*complete* if $B(n)$ is Γ-complete. It is clear that the set

$$\{y_n \mid n \text{ is not } B\Gamma\text{-complete }\}$$

is a basis for the Rado ideal, and thus that the set

$$\{y_n \mid n \text{ is } B\Gamma\text{-complete }\}$$

gives a basis for Q.

Proposition 10.5. *For any finitely generated ideal $I \leq Q$, we have $\operatorname{ann}^2(I) = I$. (In other words, Q satisfies the double annihilator condition.)*

Proof. Let $I \leq Q$ be a finitely generated ideal. Because of Remark 10.4, the ideal I must be generated by a finite list of monomials, say $I = (x_{A_1}, \dots, x_{A_r})$, where each A_i is a finite Γ-complete subset of $\mathbb{N}$. Simiilarly, $\operatorname{ann}^2(I)$ is generated by the monomials that it contains.

Let T be another Γ-complete subset of $\mathbb{N}$. If T contains A_i for some i, it is clear that $x_T \in I$. Suppose instead that T does not contain any of the A_i. Let N be strictly larger than any of the elements of $T \cup \bigcup_i A_i$, and put $n = 2^N + \sum_{t \in T} 2^t$, so $B(n) = \{N\} \cup T$. It is clear that $n \notin T$ and $T \cup \{n\}$ is Γ-complete so $x_n x_T \neq 0$. However, we claim that $x_n x_{A_i} = 0$ for all i. Indeed, as $T \not\supseteq A_i$ we can choose $k \in A_i \setminus T$. As N is so large we cannot have $n \in B(k)$, and also $k \notin \{N\} \cup T = B(n)$, so $x_n x_k = 0$, so $x_n x_{A_i} = 0$ as claimed. We now see that $x_n \in \operatorname{ann}(I)$, but $x_n x_T \neq 0$, so $x_T \notin \operatorname{ann}^2(I)$. It follows that $\operatorname{ann}^2(I) = I$ as claimed. □

Proposition 10.6. *Q is not self-injective.*

Proof. Take any pair $p, q \in \mathbb{N}$ with $p \neq q$ and $x_p x_q = 0$ (say $p = 0$ and $q = 2$). Put $u = (x_p, x_q)$ and $v = (0, x_q)$, and consider the test pair (u, v). Any transporter would have to be an element $t \in Q_0 = \{0, 1\}$ with $t x_p = 0$ and $t x_q = x_q$. It is clear from this that there is no transporter. A block would be a pair (a, b) with $b x_q \neq 0$ but $a x_p + b x_q = 0$ (so $a x_p = b x_q \neq 0$). This means that a and b are nonzero homogeneous elements, say $a = x_A$ and $b = x_B$ for some Γ-complete sets A and B. As $a x_p \neq 0$ we see that $p \notin A$, and that $A \cup \{p\}$ is again Γ-complete. Similarly, we have $q \notin B$ and $B \cup \{q\}$ is Γ-complete. The equation $a x_p = b x_q$

means that $A \cup \{p\} = B \cup \{q\}$, so we have $A = C \cup \{q\}$ and $B = C \cup \{p\}$ for some set C. This now gives $bx_q = x_C x_p x_q$ but $x_p x_q = 0$ so $bx_q = 0$, contrary to assumption. This shows that we have neither a block nor a transporter, so Q is not self-injective. □

Remark 10.7. We could give Q a different grading with such that there are some pairs (i, j) with $i \neq j$ but $|x_i| = |x_j|$, so $x_i + x_j$ becomes homogeneous. One can check that if $x_i x_j = 0$ then $\operatorname{ann}^2(x_i + x_j) = (x_i, x_j) \neq (x_i + x_j)$, so the double annihilator condition no longer holds. We will discuss a similar situation with more details in Lemma 11.18. We believe that the self-injectivity condition is similarly sensitive to the choice of grading, but we do not have an example to prove this.

Proposition 10.8. *Q is totally incoherent.*

Proof. First, it is clear that Q is local, with maximal ideal $\mathfrak{m} = (x_i \mid i \in \mathbb{N}) = \bigoplus_{k>0} Q_k$. The generators x_i form a basis for $\mathfrak{m}/\mathfrak{m}^2$. Note that if $A \subset \mathbb{N}$ is nonempty and Γ-complete, then infinitely many of the variables x_i will satisfy $x_i x_A = 0$, so the image of $\operatorname{ann}(x_A)$ in $\mathfrak{m}/\mathfrak{m}^2$ will have infinite dimension. The claim therefore follows by Corollary 5.6. □

11. The ϵ_0-algebra

The ϵ_0 algebra A was introduced in Definition 1.18. We now explain the definition in more detail, and prove some properties.

Definition 11.1. Suppose we have a sequence $\underline{\beta} = (\beta_1 > \beta_2 > \cdots > \beta_r)$ of ordinals, and a sequence $\underline{n} = (n_1, \ldots, n_r)$ of positive integers. We write

$$C(\underline{\beta}, \underline{n}) = \omega^{\beta_1} n_1 + \ldots + \omega^{\beta_r} n_r.$$

Note that this uses ordinal exponentiation, defined in the usual recursive way by $\alpha^{\beta+1} = \alpha\alpha^{\beta}$ and $\alpha^{\lambda} = \bigcup_{\beta<\lambda} \alpha^{\beta}$ when λ is a limit ordinal.

The following fact is standard (and not hard to prove by transfinite induction).

Proposition 11.2. *For any ordinal α there is a unique pair $(\underline{\beta}, \underline{n})$ such that $\alpha = C(\underline{\beta}, \underline{n})$. (This is the* Cantor normal form *for α.)*

Proof. See [11, Exercise 6.10], for example. □

Definition 11.3. We put $\pi_0 = \omega$ and define π_n recursively by $\pi_{n+1} = \omega^{\pi_n}$, and then put $\epsilon_0 = \bigcup_n \pi_n$.

One can check that $\epsilon_0 = \omega^{\epsilon_0}$, and that ϵ_0 is the smallest ordinal with this property. Note that the expression $\epsilon_0 = \omega^{\epsilon_0}$ is the Cantor normal form of ϵ_0. For $\alpha < \epsilon_0$ we find that the exponents β_t in the Cantor normal form of α are strictly less than α, so in this case one can do induction or recursion based on the Cantor normal form.

Definition 11.4. We define $\delta \colon \epsilon_0 \to \mathbb{N}$ recursively by $\delta(0) = 1$ and $\delta(\alpha) = (\sum_t (\delta(\beta_t) + 2) n_t) - 1$ if $\alpha = \omega^{\beta_1} n_1 + \cdots + \omega^{\beta_r} n_r$.

We will give enough examples to show that δ is not injective, which will be needed later.

Example 11.5.

$$\begin{aligned}
\delta(1) &= \delta(\omega^0) = (\delta(0) + 2) - 1 = 2 \\
\delta(2) &= \delta(\omega^0\, 2) = (\delta(0) + 2)2 - 1 = 5 \\
\delta(\omega) &= \delta(\omega^1) = (\delta(1) + 2) - 1 = 3 \\
\delta(\omega + 1) &= \delta(\omega^1 + \omega^0) = (\delta(1) + 2) + (\delta(0) + 2) - 1 = 6 \\
\delta(\omega^2) &= (\delta(2) + 2) - 1 = 6.
\end{aligned}$$

In order to analyse δ, it is helpful to modify the Cantor normal form slightly.

Lemma 11.6. *If $\alpha < \epsilon_0$ then there is a unique way to write*

$$\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_m}$$

with $\alpha > \beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$. (This is the expanded Cantor normal form.*)*

Proof. Just take the ordinary Cantor normal form and replace $\omega^{\beta_t} n_t$ by n_t copies of ω^{β_t}. □

Lemma 11.7. *For any $d \in \mathbb{N}$ there are only finitely many ordinals $\alpha \in \epsilon_0$ with $\delta(\alpha) = d$.*

Proof. Let A denote the alphabet $\{0, \pi, +\}$. For each $\alpha < \epsilon_0$ we define a word $\phi(\alpha)$ in A as follows. We start with $\phi(0) = 0$. If $\theta > 0$ has expanded Cantor normal form $\theta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ we put

$$\phi(\theta) = \phi(\beta_1)\pi\phi(\beta_2)\pi \cdots \phi(\beta_m)\pi + \cdots +$$

(with $m - 1$ plusses at the end). For example we have

$$\phi(3) = \phi(\omega^0 + \omega^0 + \omega^0) = 0\pi0\pi0\pi + +$$
$$\phi(\omega^\omega + \omega) = 0\pi\pi\pi0\pi\pi + .$$

It is clear from the definitions that $\delta(\theta)$ is the length of $\phi(\theta)$, and there are only 3^d words in A of length d, so it will suffice to show that ϕ is injective. If we interpret π as the operator $x \mapsto \omega^x$ then $\phi(\theta)$ is a reverse polish expression that evaluates to θ, and this implies injectivity. □

Corollary 11.8. *ϵ_0 is countable.* □

Definition 11.9. Let $\widetilde{A}$ be the graded polynomial algebra over $\mathbb{F}$ generated by elements x_α for each ordinal $\alpha < \epsilon_0$, with $|x_\alpha| = \delta(\alpha)$.

Using Lemma 11.7 we see that $\widetilde{A}_d$ is finite for all d.

Definition 11.10. For ordinals $\alpha, \beta < \epsilon_0$ with $\alpha \neq \beta$ we define $\mu_0(\alpha, \beta)$ to be the coefficient of ω^β in α. More explicitly, if the Cantor normal form of α involves a term $\omega^\beta n$, then $\mu_0(\alpha, \beta) = n$; if there is no such term then $\mu_0(\alpha, \beta) = 0$. One can check that if $\mu_0(\alpha, \beta) > 0$ then $\mu_0(\beta, \alpha) = 0$. We put $\mu(\alpha, \beta) = \max(\mu_0(\alpha, \beta), \mu_0(\beta, \alpha))$.

Proposition 11.11. *For any finite set $J \subset \epsilon_0$ and map $\nu \colon J \to \mathbb{N}$ there exists $\alpha \in \epsilon_0 \setminus J$ such that $\mu(\alpha, \beta) = \nu(\beta)$ for all $\beta \in J$. (We will call this the* extension property.*)*

Proof. Write J in order as $J = \{\beta_1 > \beta_2 > \cdots > \beta_r\}$ and then take

$$\alpha = \omega^{\beta_1 + 1} + \omega^{\beta_1}.\nu(\beta_1) + \cdots + \omega^{\beta_r}.\nu(\beta_r).$$

It is visible that $\mu_0(\alpha, \beta_t) = \nu(\beta_t)$ for all t. Also, because of the initial term ω^{β_1+1} we have $\omega^\alpha > \alpha > \beta_t$ for all t and so $\mu_0(\beta_t, \alpha) = 0$. It follows that $\mu(\alpha, \beta_t) = \nu(\beta_t)$ for all t, as required. □

From now on we will only need the fact that our index set ϵ_0 is countable and that the extension property holds. It will therefore be notationally convenient to write $I = \epsilon_0$ and ignore the fact that the elements of I are ordinals, and to write i instead of α for a typical element of I. We also put $I_2 = \{(i, j) \in I^2 \mid i \neq j\}$.

Definition 11.12. For each $(i, j) \in I_2$ we put $\rho(i, j) = x_i x_j^{\mu(i,j)+1}$. We then let A be the quotient of $\widetilde{A}$ by all such elements $\rho(i, j)$. We call this the *ϵ_0-algebra*.

Definition 11.13. Given a map $\alpha \colon I \to \mathbb{N}$, we write $\text{supp}(\alpha) = \{i \mid \alpha(i) > 0\}$. Let $M\widetilde{A}$ be the set of all such maps α for which $\text{supp}(\alpha)$ is finite. For $\alpha \in M\widetilde{A}$ we put $x^\alpha = \prod_i x_i^{\alpha(i)} \in \widetilde{A}$. We write $B\widetilde{A}$ for the set of all such monomials x^α, so $B\widetilde{A}$ is a basis for $\widetilde{A}$. Next, put

$$MA = \{\alpha \in M\widetilde{A} \mid \forall i \neq j \; \alpha(i) > 0 \Rightarrow \alpha(j) \leq \mu(i, j)\}$$

and $BA = \{x^\alpha \mid \alpha \in MA\}$. One can check that BA gives a basis for A.

Definition 11.14. A *monomial ideal* is just an ideal in A that is generated by some subset of BA.

Remark 11.15. Let P be a monomial ideal, generated by $\{x^\alpha \mid \alpha \in U\}$ for some subset $U \subseteq MA$. Put

$$U^+ = \{\alpha \in MA \mid \alpha \geq \beta \text{ for some } \beta \in U\}.$$

It is easy to see that $\{x^\alpha \mid \alpha \in U^+\}$ is then a basis for P over $\mathbb{F}$. It follows easily that sums, products, intersections and annihilators of monomial ideals are again monomial ideals.

Lemma 11.16. *If P is a monomial ideal then it is finitely generated if and only if there is a finite list of monomials that generate it.*

Proof. Suppose that P is generated by $a_1, \dots, a_m$, where the elements a_t need not be monomials. We can write $a_t = \sum_{\alpha \in U_t} a_{t,\alpha} x^\alpha$, for some finite set $U_t \subset MA$ and some nonzero coefficients $a_{t,\alpha}$. Using Remark 11.15 we see that the terms x^α (for $\alpha \in U_t$) lie in P. Put $U = \bigcup_t U_t$ (which is finite) and put $P' = (x^\alpha \mid \alpha \in U) \leq P$. Clearly $a_t \in (x^\alpha \mid \alpha \in U_t) \leq P'$ and the elements a_t generate P so $P \leq P'$ so $P = P'$. Thus, P is generated by a finite list of monomials. □

Proposition 11.17. *Let $P \leq A$ be a finitely generated monomial ideal. Then* $\text{ann}^2(P) = P$.

Proof. It is automatic that $P \leq \text{ann}^2(P)$, so it will suffice to prove the opposite inclusion. Note that both P and $\text{ann}^2(P)$ are monomial ideals, so it will suffice to show that they contain the same monomials. Suppose that x^β is a nonzero monomial that does not lie in P; we must find $y \in \text{ann}(P)$ such that $x^\beta y \neq 0$.

We can choose a finite list $\alpha_1, \dots, \alpha_r \in M$ such that $P = (x^{\alpha_1}, \dots, x^{\alpha_r})$. Put $J = \text{supp}(\beta) \cup \bigcup_i \text{supp}(\alpha_i)$, which is a finite subset of I. Put $N = \max\{\beta(j) \mid j \in J\}$.

Next, for each t we note that x^β cannot be divisible by x^{α_t}, so we can choose $i_t \in J$ such that $\alpha_t(i_t) > \beta(i_t)$. Using the extension property we can recursively define distinct elements $k_1, \dots, k_r \in I \setminus J$ such that

(a) $\mu(k_t, i_t) = \alpha_t(i_t) - 1$
(b) $\mu(k_t, j) = N$ for $j \in J \setminus \{i_t\}$
(c) $\mu(k_t, k_s) = 1$ for $s < t$.

Put $y = \prod_t x_{k_t}$. This is nonzero by property (c). Property (a) tells us that $x_{j_t} x^{\alpha_t} = 0$ for all t, which implies that $y \in \text{ann}(A)$. On the other hand, we note that

- Clause (a) above tells us that yx^β is not divisible by any relator $\rho(k_t, i_t)$.
- Clause (b) tells us that yx^β is not divisible by any relator $\rho(k_t, j)$ with $j \in J \setminus \{i_t\}$.
- Clause (c) tells us that yx^β is not divisible by any relator $\rho(k_t, k_s)$.
- Our original assumption $x^\beta \neq 0$ implies that yx^β is not divisible by any relator $\rho(j, j')$ with $j, j' \in J$.

This shows that $yx^\beta \neq 0$, but $y \in \text{ann}(P)$, so $x^\beta \notin \text{ann}^2_(P)$, as claimed. □

Lemma 11.18. *Let i and j be any two distinct indices in I with $|x_i| = |x_j|$ and $\mu(i,j) = 0$. Then* $\text{ann}^2(x_i + x_j) = (x_i, x_j) > (x_i + x_j)$.

Proof. As $\mu(i,j) = 0$ we have $x_i x_j = 0$ and so (using monomial bases) $(x_i) \cap (x_j) = 0$. If $u(x_i + x_j) = 0$ then we have $ux_i = -ux_j$, with the left hand side in (x_i) and the right hand side in (x_j). As $(x_i) \cap (x_j) = 0$ this gives $ux_i = ux_j = 0$. It now follows that $\text{ann}(x_i + x_j) = \text{ann}(x_i, x_j)$ and so $\text{ann}^2(x_i + x_j) = \text{ann}^2(x_i, x_j)$. As (x_i, x_j) is a monomial ideal we also have $\text{ann}^2(x_i, x_j) = (x_i, x_j)$, so $\text{ann}^2(x_i + x_j) = (x_i, x_j) > (x_i + x_j)$ as claimed. □

Corollary 11.19. *Example 11.5 shows that the lemma applies to the pair $(\omega^2, \omega + 1)$, so A does not satisfy the double annihilator condition. Thus, Remark 2.4 shows that A cannot be self-injective.* □

Remark 11.20. We could choose a different grading such that all the generators had different degrees, which would eliminate any examples as in Lemma 11.18. However, we cannot ensure that A_d has dimension at most one for all d, because when $i \neq j$ the elements $x_i^{|x_j|}$ and $x_j^{|x_i|}$ have the same degree and are linearly independent. Thus, there will always be ideals that are not monomial ideals. We suspect that there is no grading for which A satisfies the full double annihilator condition, but we have not proved this.

Proposition 11.21. *A is totally incoherent.*

Proof. Put $\mathfrak{m}_0 = 0$ and $\mathfrak{m}_k = A_k$ for all $k > 0$, so $A/\mathfrak{m} = \mathbb{F}$. It is clear that $\mathfrak{m}$ is an ideal, and that the (homogeneous) elements of $\mathfrak{m}$ are precisely the elements of A that are not invertible. Given this, it follows that $\mathfrak{m}$ is the unique maximal ideal in A, so A is local. From the form of the relations in A we see that $\{x_i \mid i \in I\}$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$.

Now consider an element $a \in A_d$ for some $d > 0$. Put

$$U = \{i \in I \mid \delta(i) \leq d\}$$
$$V = \{\omega^i \mid i \in I \setminus U\}.$$

We find that $x_i x_j = 0$ for all $i \in U$ and $j \in V$. Moreover, we have $a \in (x_i \mid i \in U)$, so $ax_j = 0$ for all $j \in V$, so the image of $\operatorname{ann}(a)$ in $\mathfrak{m}/\mathfrak{m}^2$ has infinite dimension.

Now let P be a finitely presented ideal in A. If $P = \mathfrak{m}P$ then $P = 0$ by Nakayama's Lemma. Otherwise, we can choose $a \in P \setminus \mathfrak{m}P$, and Lemma 5.5 tells us that $\operatorname{ann}(a)$ has finite image in $\mathfrak{m}/\mathfrak{m}^2$. The above remarks show that we must have $|a| = 0$, and $a \notin \mathfrak{m}P$ so $a \neq 0$, so a is invertible, so $P = A$. □

Proposition 11.22. *The reduced quotient is*

$$A/\sqrt{0} = \mathbb{F}[x_i \mid i \in I]/(x_i x_j \mid i \neq j).$$

Proof. In A we have $x_i x_j^{\mu(i,j)+1} = 0$, so $(x_i x_j)^{\mu(i,j)+1} = 0$, so $x_i x_j$ is nilpotent. If we put

$$A' = A/(x_i x_j \mid i \neq j) = \mathbb{F}[x_i \mid i \in I]/(x_i x_j \mid i \neq j),$$

we deduce that $A/\sqrt{0} = A'/\sqrt{0}$. However, it is easy to see that A' is already reduced, so $A/\sqrt{0} = A'$ as claimed. □

12. Triangulation

Recall that a *triangulated category* is a triple $(\mathcal{C}, \Sigma, \Delta)$, where $\mathcal{C}$ is an additive category, and $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an equivalence, and Δ is a class of diagrams of shape

$$X \to Y \to Z \to \Sigma X$$

(called *distinguished triangles*), subject to certain axioms that we will not list here.

Definition 12.1. Let R be a self-injective graded ring, let Mod_R be the category of R-modules, and let $\Sigma\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ be the usual suspension functor so that $(\Sigma M)_i = M_{i-1}$. Let InjMod_R be the full subcategory of injective modules. A *triangulation structure* for R is a pair $(\mathcal{N}, \Delta)$, where

(a) $\mathcal{N}$ is a full subcategory of InjMod_R containing R.
(b) $\mathcal{N}$ is closed under finite direct sums, retracts, suspensions and desuspensions.
(c) Δ is a class of distinguished triangles making $(\mathcal{N}, \Sigma, \Delta)$ into a triangulated category.

We can also make a similar definition for ungraded rings.

Definition 12.2. Let R be a self-injective ungraded ring. An *ungraded triangulation structure* for R is a pair $(\mathcal{N}, \Delta)$, where

(a) $\mathcal{N}$ is a full subcategory of InjMod_R containing R.
(b) $\mathcal{N}$ is closed under finite direct sums, retracts, suspensions and desuspensions.
(c) Δ is a class of distinguished triangles making $(\mathcal{N}, 1, \Delta)$ into a triangulated category.

In [15] we constructed ungraded triangulation structures for $\mathbb{Z}/4$ and for $K[\epsilon]/\epsilon^2$ (where K is any field of characteristic two). If Freyd's Generating Hypothesis is true, then the image of the functor π_* gives a graded triangulation structure for the ring $\pi_*(S)^\wedge_p$. We have not succeeded in constructing any examples of graded triangulation structures by pure algebra. Here we offer only some rather limited and negative results.

Lemma 12.3. *If $(\mathcal{N}, \Delta)$ is a triangulation structure (in the graded or ungraded context) then all distinguished triangles in Δ are exact sequences.*

Proof. The general theory of triangulated categories tells us that all functors of the form $\mathcal{N}(X, -)$ send distinguished triangles to long exact sequences. By assumption we have $R \in \mathcal{N}$, and we can take $X = R$ to prove the claim. □

Lemma 12.4. *If $(\mathcal{N}, \Delta)$ is a triangulation structure then all surjective maps in $\mathcal{N}$ are split.*

Proof. Let $M \xrightarrow{f} N$ be a surjective map in $\mathcal{N}$. This must fit into a distinguished triangle $L \xrightarrow{e} M \xrightarrow{f} N \xrightarrow{g} \Sigma L$. Here $gf = 0$ but f is surjective so $g = 0$. It is standard that the functor $\mathcal{N}(N, -)$ converts our distinguished triangle to an exact sequence, so $f_* \colon \mathcal{N}(N, M) \to \mathcal{N}(N, N)$ is surjective. We can thus find $h \colon N \to M$ with $fh = 1$, so h splits f. □

Corollary 12.5. *If $(\mathcal{N}, \Delta)$ is a triangulation structure then all finitely generated modules in $\mathcal{N}$ are projective. Thus, if R is local then all such modules are free.*

Proof. Let N be a finitely generated module in $\mathcal{N}$. This means that there is a surjective homomorphism $f \colon F \to N$ for some finitely generated free module F. As $\mathcal{N}$ is standard we see that $F \in \mathcal{N}$, so the lemma tells us that N is a retract of F, so it is projective. It is well-known that finitely generated projective modules over local rings are free. □

Proposition 12.6. *Suppose that R is a local graded ring with $R_i = 0$ for $i < 0$, and suppose that R admits a triangulation structure. Then R is totally incoherent.*

Proof. Let $\mathfrak{m}$ be the unique maximal ideal, and let $(\mathcal{N}, \Delta)$ be a triangulation structure. It is not hard to see that $\mathfrak{m}_0$ is the unique maximal ideal in R_0, so R_0 is a local ring in the ungraded sense.

Let J be any finitely generated ideal. We can then find a finitely generated free module Q and an epimorphism $Q \to J$ such that $Q/\mathfrak{m}Q \to J/\mathfrak{m}J$ is an isomorphism. We will write g for the composite map $Q \to J \to R$, so that $J = \text{image}(g)$. If J is finitely presented then $\ker(g)$ is again finitely generated, so we can find a finitely generated free module P and a map $f \colon P \to Q$ with $\text{image}(f) = \ker(g)$ and $P/\mathfrak{m}P \xrightarrow{\simeq} \ker(g)/\mathfrak{m}\ker(g)$. With these minimal choices for P and Q, it is clear that $P_i = Q_i = 0$ when $i < 0$. Next, we can fit g into a distinguished triangle $\Sigma^{-1}R \xrightarrow{d} K \xrightarrow{i} Q \xrightarrow{g} R$. As $gf = 0$, we can find a lift $\tilde{f} \colon P \to K$ with $i\tilde{f} = f$. We can combine this with d to give a map $P \oplus \Sigma^{-1}R \to K$, and a diagram chase shows that this is surjective. Using Lemma 12.4 we deduce that this map is split epi and that K is a finitely generated free module. It follows that $K_i = 0$ for $i < -1$ and that K_{-1} is a retract of R_0. As R_0 is local we must have either $K_{-1} = 0$ or $K_{-1} = R_0$. If $K_{-1} = 0$ then $d \colon \Sigma^{-1}R \to K$ must be zero, which implies that $g \colon Q \to R$ is split epi, which means that $J = R$. If $K_{-1} \neq 0$ then we find that d must induce an monomorphism $\Sigma^{-1}R/\mathfrak{m} \to K$, and as R is local this implies that d is a split monomorphism, and thus that $g = 0$ and so $J = 0$. □

Remark 12.7. As mentioned previously, there is an ungraded triangulation structure for the ring $\mathbb{Z}/4$. The ideal $(2) < \mathbb{Z}/4$ is finitely presented and is neither 0 nor $\mathbb{Z}/4$. It follows that our grading assumptions are playing an essential role in the proof of the above proposition.

Corollary 12.8. *Neither the infinite exterior algebra (as in Example 4.7) nor the cube algebra (as in Section 7) admits a triangulation structure.*

Proof. Both rings are coherent, by Propositions 5.4 and 7.25. □

References

[1] Jon F. Carlson, Sunil K. Chebolu, and Ján Mináč, *Freyd's generating hypothesis with almost split sequences*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2575–2580, DOI 10.1090/S0002-9939-09-09826-8. MR2497468 (2010c:20007)

[2] Mark Hovey, Keir Lockridge, and Gena Puninski, *The generating hypothesis in the derived category of a ring*, Math. Z. **256** (2007), no. 4, 789–800, DOI 10.1007/s00209-007-0103-x. MR2308891 (2008b:18017)

[3] David J. Benson, Sunil K. Chebolu, J. Daniel Christensen, and Ján Mináč, *The generating hypothesis for the stable module category of a p-group*, J. Algebra **310** (2007), no. 1, 428–433, DOI 10.1016/j.jalgebra.2006.12.013. MR2307802 (2007k:16011)

[4] Keir H. Lockridge, *The generating hypothesis in the derived category of R-modules*, J. Pure Appl. Algebra **208** (2007), no. 2, 485–495, DOI 10.1016/j.jpaa.2006.01.018. MR2277690 (2007i:55009)

[5] Peter J. Cameron, *The random graph revisited*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 267–274. MR1905324 (2003j:05110)

[6] R. Rado, *Universal graphs and universal functions*, Acta Arith. **9** (1964), 331–340. MR0172268 (30 #2488)

[7] Reinhold Baer, *Abelian groups that are direct summands of every containing abelian group*, Bull. Amer. Math. Soc. **46** (1940), 800–806. MR0002886 (2,126i)

[8] Peter J. Cameron, *The random graph revisited*, European Congress of Mathematics, Vol. I (Barcelona, 2000), 2001, pp. 267–274.MR1905324 (2003j:05110)

[9] Peter Freyd, *Stable homotopy*, Proc. conf. categorical algebra (la jolla, calif., 1965), 1966, pp. 121–172. MR35#2280

[10] Mark Hovey, *On Freyd's generating hypothesis*, Q. J. Math. **58** (2007), no. 1, 31–45, DOI 10.1093/qmath/hal013. MR2305048 (2008b:55014)

[11] P. T. Johnstone, *Notes on logic and set theory*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1987. MR923115 (89e:03003)

[12] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR1653294 (99i:16001)

[13] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, 1986.

[14] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, American Journal of Mathematics **106** (1984), 351–414.

[15] Fernando Muro, Stefan Schwede, and Neil Strickland, *Triangulated categories without models*, Invent. Math. **170** (2007), no. 2, 231–241, DOI 10.1007/s00222-007-0061-2. MR2342636 (2008g:18016)