ON REMOVABILITY PROPERTIES OF ψ-UNIFORM DOMAINS IN BANACH SPACES

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Abstract. Suppose that E and E' denote real Banach spaces with the same dimension at least 2. The main aim of this paper is to show that a domain D in E is a ψ-uniform domain if and only if $D\backslash P_D$ is a ψ_1-uniform domain, and a domain D in E is a uniform domain if and only if $D\backslash P_D$ is also a uniform domain, where P_D denotes a countable set in D with the property that the quasihyperbolic distance between each pair of distinct points in it has a lower bound greater than or equal to $\frac{1}{2}$.

1. Introduction and main results

The quasihyperbolic metric of a domain in a metric space was introduced by F. W. Gehring and his students B. Palka and B. Osgood in the 1970's [2, 3] in the setup of the Euclidean space $\mathbb{R}^n, n \geq 2$. Since its first appearance, the quasihyperbolic metric has become an important tool in geometric function theory and in its generalizations to metric spaces and to Banach spaces [18]. Yet, some basic questions of the quasihyperbolic geometry in Banach spaces and even in Euclidean spaces are open. For instance, only recently the convexity of balls of the quasihyperbolic metric has been studied in [7, 8, 12, 20].

In this paper, we study the classes of uniform domains [11] and the wider class of ψ-uniform domains [22] in Banach spaces and the stability of these classes of domains under the removal of a countable set of points. The motivation for this study stems from the discussions in [6, 16]. In [6], similar removability questions were studied for the class of John domains. We begin with some basic definitions and the statements of our results. The proofs and necessary supplementary notation and terminology will be given thereafter.

Throughout the paper, we always assume that E and E' denote real Banach spaces with the same dimension at least 2. The norm of a vector z in E is written as $|z|$, and for each pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. We always use $\mathbb{B}(x_0, r)$ to denote the open ball $\{x \in E : |x - x_0| < r\}$ centered at x_0 with radius $r > 0$. Similarly, for the closed balls and spheres, we employ the usual notations

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$\overline{\mathbb{B}}(x_0, r)$ and $\partial\mathbb{B}(x_0, r)$, respectively. We adopt some basic terminology following closely [10, 14, 15, 16].

Definition 1. A domain D in E is said to be *c-uniform* if there exists a constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying

(1) $\min\limits_{j=1,2} \ell(\gamma[z_j, z]) \le c\, d_D(z)$ for all $z \in \gamma$, and

(2) $\ell(\gamma) \le c\,|z_1 - z_2|$,

where $\ell(\gamma)$ denotes the arc length of γ, $\gamma[z_j, z]$ the part of γ between z_j and z, and $d_D(z)$ the distance from z to the boundary ∂D of D [11]. Also we say that γ is a *double c-cone arc*.

Definition 2. Let $\psi : [0, \infty] \to [0, \infty]$ be a homeomorphism. A domain D in E is called *ψ-uniform* if

$$k_D(z_1, z_2) \le \psi\Big(\frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big)$$

for all z_1, $z_2 \in D$ [22].

In [16], Väisälä obtained the following result concerning the removability property of uniform domains. In Section 2 we shall discuss the connection between these two notions of uniformity.

Theorem A. ([16, Theorem 6.5]) *For $x_0 \in E$ and $r > 0$, the domains $\mathbb{B}(x_0, r)$, $\mathbb{B}(x_0, r)\backslash\{x_0\}$ and $E\backslash\{x_0\}$ are c-uniform domains with universal c.*

Theorem B. ([16, Lemma 6.7]) *Suppose that G is a c-uniform domain in E and that $x_0 \in G$. Then $G_0 = G\backslash\{x_0\}$ is c_0-uniform with $c_0 = c_0(c)$.*

In [6], the authors discussed the removability property of John domains.

Theorem C. ([6, Lemma 6.7]) *A domain $D \subset \mathbb{R}^n$ $(n \ge 2)$ is a John domain if and only if $G = D\backslash P$ is also a John domain, where $P = \{p_1, p_2, \cdots, p_m\} \subset D$.*

In general, when P is a countable set in a John domain D, the domain $D\backslash P$ need not be a John domain (cf. [6, Example 1.5]). This motivates us to study countable subsets of a domain satisfying the following quasihyperbolic separation condition.

For a domain D in E and a fixed sequence $\{x_j : j = 1, 2, \dots\}$ of points in D with $k_D(x_i, x_j) \ge \frac{1}{2}$ for $i \ne j$, we always use the convenient notation

$$P_D = \{x_j \in D : j = 1, 2, \dots\}.$$

The purpose of this paper is to discuss the following two problems.

Problem 1. Suppose that D is a ψ-uniform domain in E. Is it true that $G = D\backslash P_D$ is a ψ_1-uniform domain, where ψ_1 depends only on ψ?

Problem 2. Suppose that D is a c-uniform domain in E. Is it true that $G = D\backslash P_D$ is a c_1-uniform domain, where $c_1 = c_1(c)$?

We are now in a position to formulate our results.

Theorem 1. *A domain D in E is a ψ-uniform domain if and only if $G = D\backslash P_D$ is a ψ_1-uniform domain, where $\psi = 3\psi_1(128t)$ and $\psi_1 = 2^{12}\psi(t)$ for $t > 0$.*

As a corollary of Theorem 1, we have

Corollary 1. *A domain D in E is a c-uniform domain if and only if $G = D\backslash P_D$ is a c_1-uniform domain, where the constants c and c_1 depend only on each other.*

The proofs of Theorem 1 and Corollary 1 will be given in Section 3, and some preliminaries will be introduced in Section 2.

2. Preliminaries

2.1. **Quasihyperbolic distance and neargeodesics.** The *quasihyperbolic length* of a rectifiable arc or a path α in the norm metric in D is the number [3, 19]:

$$\ell_{k_D}(\alpha) = \int_\alpha \frac{|dz|}{d_D(z)}.$$

For each pair of points z_1, z_2 in D, the *distance ratio metric* $j_D(z_1, z_2)$ between z_1 and z_2 is defined by

$$j_D(z_1, z_2) = \log\Big(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big).$$

The *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf\{\ell_{k_D}(\alpha)\},$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in D. For all z_1, z_2 in D, we have [19]

$$k_D(z_1, z_2) \geq \inf_\alpha \Big\{\log\Big(1 + \frac{\ell(\alpha)}{\min\{d_D(z_1), d_D(z_2)\}}\Big)\Big\} \geq \Big|\log \frac{d_D(z_2)}{d_D(z_1)}\Big|, \tag{2.1}$$

where the infimum is taken over all rectifiable curves α in D connecting z_1 and z_2. Since $\ell(\alpha) \geq |z_1 - z_2|$ in (2.1), for all z_1, z_2 in D, we have

$$k_D(z_1, z_2) \geq j_D(z_1, z_2). \tag{2.2}$$

The following observation easily follows from (2.2) and Definition 2.

Proposition 1. *If D is ψ-uniform, then the homeomorphism ψ satisfies*

$$\psi(t) \geq \log(1 + t)$$

for $t > 0$.

Next, if $|z_1 - z_2| \leq d_D(z_1)$, then we have [15], [21, Lemma 2.11]

$$k_D(z_1, z_2) \leq \log\Big(1 + \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|}\Big). \tag{2.3}$$

In [16], Väisälä characterized uniform domains by the quasihyperbolic metric.

Theorem D. ([16, Theorem 6.16]) *For a domain D, the following are quantitatively equivalent:*

(1) *D is a c-uniform domain;*
(2) *$k_D(z_1, z_2) \leq c'\, j_D(z_1, z_2)$ for all $z_1, z_2 \in D$;*
(3) *$k_D(z_1, z_2) \leq c'_1\, j_D(z_1, z_2) + d$ for all $z_1, z_2 \in D$.*

Gehring and Palka [3] introduced the quasihyperbolic metric of a domain in $\mathbb{R}^n$ and it has been recently used by many authors in the study of quasiconformal mappings [1, 5, 20, 22] etc and related questions [4]. In the case of domains in $\mathbb{R}^n$, the equivalence of items (1) and (3) in Theorem D is due to Gehring and Osgood [2] and the equivalence of items (2) and (3) due to Vuorinen [22]. Many of the basic properties of this metric may be found in [2, 15, 16]. By Theorem D, we see that uniformity implies ψ-uniformity.

Recall that an arc α from z_1 to z_2 is a *quasihyperbolic geodesic* if $\ell_{k_D}(\alpha) = k_D(z_1, z_2)$. Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between every pair of points in E exists if the dimension of E is finite, see [2, Lemma 1]. This is not true in Banach spaces [17, Example 2.9]. In order to remedy this shortage, Väisälä introduced the following concepts [16].

Definition 3. Let D be a domain in E. An arc $\alpha \subset D$ is *ν-neargeodesic* if $\ell_{k_D}(\alpha[x, y]) \leq \nu\, k_D(x, y)$ for all $x, y \in \alpha$.

Obviously, a ν-neargeodesic is a quasihyperbolic geodesic if and only if $\nu = 1$.

In [17], Väisälä proved the following property concerning the existence of neargeodesics in E.

Theorem E. ([17, Theorem 3.3]) *Let $\{z_1, z_2\} \subset D$ and $\nu > 1$. Then there is a ν-neargeodesic in D joining z_1 and z_2.*

2.2. **Quasiconvexity.**

Definition 4. We say that an arc γ in $D \subset E$ is *c-quasiconvex* in the norm metric if it satisfies the condition

$$\ell(\gamma[z_1, z_2]) \leq c\, |z_1 - z_2|$$

for every z_1, z_2 in γ.

The following result is due to Schäffer [13].

Theorem F. ([13, 4.4]) *Suppose that S is a sphere, that T is a 2-dimensional linear subspace in E and that the intersection $S \cap T$ contains at least two points. For every pair $\{z_1, z_2\} \subset T \cap S$, if $\gamma \subset T \cap S$ is the minor arc or a half circle with the endpoints z_1 and z_2, then γ is 2-quasiconvex.*

3. The proofs of Theorem 1 and Corollary 1

We recall that D denotes a domain in E and $G = D \backslash P_D$, where $P_D \subset D$ is a countable set satisfying the quasihyperbolic separation condition. Before the proof of Theorem 1, we prove several lemmas.

Lemma 1. (1) *If $x\in D$, $\log\frac{1}{1-\lambda}\leq\frac{1}{4}$ then $\mathbb{B}(x,\lambda d_D(x))$ contains at most one point of P_D;*

(2) *If $x\in G$, $\frac{d_G(x)}{d_D(x)}<\lambda$, λ as in (1), then $\mathbb{B}(x,\lambda d_D(x))$ contains exactly one point of P_D;*

(3) *If $x\in G$, $\frac{d_G(x)}{d_D(x)}<\lambda$, $\lambda\leq\frac{1}{16}$, then $\mathbb{B}(x,\lambda d_D(x))$ contains exactly one point x_i in P_D which satisfies*

$$d_G(z)=|x_i-z|$$

for all $z\in\overline{\mathbb{B}}(x,\frac{1}{16}d_D(x))$.

Proof. (1) Let $x\in D$. It follows from (2.3) that

$$k_D(\mathbb{B}(x,\lambda d_D(x)))=\sup\{k_D(u,v):u,v\in\mathbb{B}(x,\lambda d_D(x))\}\leq 2\log\frac{1}{1-\lambda}<\frac{1}{2}.$$

Then the hypotheses imply that (1) is true.

(2) Let $x\in G$ with $\frac{d_G(x)}{d_D(x)}<\lambda$. Then there exists some i such that $|x-x_i|<\lambda d_D(x)$. Hence $x_i\in\mathbb{B}(x,\lambda d_D(x))$ and by (1) $\mathbb{B}(x,\lambda d_D(x))$ cannot contain points in $P_D\backslash\{x_i\}$.

Next we prove (3). By (1), $\mathbb{B}(x,\frac{1}{5}d_D(x))$ contains at most one point x_i of P_D. We see from (2) that $x_i\in\mathbb{B}(x,\lambda d_D(x))$ and x_i satisfies

$$|x-x_i|=d_G(x),$$

which shows for each $z\in\overline{\mathbb{B}}(x,\frac{1}{16}d_D(x))$,

$$d_G(z)=|x_i-z|.$$

The proof is complete. □

Lemma 2. *For w_1, $w_2\in G$ and $0<\mu\leq\frac{1}{32}$, if $w_2\in\overline{\mathbb{B}}(w_1,\mu d_D(w_1))$ and $\min\{d_G(w_1),d_G(w_2)\}\leq\frac{\mu}{2}d_D(w_1)$, then*

$$k_G(w_1,w_2)\leq\frac{13}{2}j_G(w_1,w_2).$$

Proof. Clearly, we have

$$\begin{aligned}\max\{d_G(w_1),d_G(w_2)\} &\leq \min\{d_G(w_1),d_G(w_2)\}+|w_1-w_2|\\ &\leq \frac{3\mu}{2}d_D(w_1).\end{aligned}\tag{3.1}$$

By Lemma 1 (3), we see that there exists some point $x_i\in\mathbb{B}(w_1,\frac{3\mu}{2}d_D(w_1))\cap P_D$ satisfying

$$|w_2-x_i|=d_G(w_2)\ \text{ and } |w_1-x_i|=d_G(w_2).$$

Without loss of generality, we may assume

$$\min\{d_G(w_1),d_G(w_2)\}=d_G(w_2)\tag{3.2}$$

We use $w_{1,1}$ to denote the intersection point of the closed segment $[w_1,x_i]$ with the sphere $\mathbb{S}(x_i,\min\{d_G(w_1),d_G(w_2)\})$. It is possible that $w_{1,1}=w_1$. Let T denote a 2-dimensional linear subspace of E passing thorough the points w_1, w_2 and x_i, and ω_0

the circle $T\cap\mathbb{S}(x_i,\min\{d_G(w_1),d_G(w_2)\})$. Then $w_{1,1}$ and w_2 divide the circle ω_0 into two parts β_i and $\beta_{1,i}$. Without loss of generality, we may assume that $\ell(\beta_i)\leq\ell(\beta_{1,i})$. Then it follows from (3.2) that for each $z\in\beta_i$,

$$|z-w_1|\leq|w_1-x_i|+|x_i-z|\leq 2\mu d_D(w_1),$$

whence

$$\beta_i\subset\overline{\mathbb{B}}\big(w_1,2\mu d_D(w_1)\big),$$

and so Lemma 1 yields that for each $z\in\beta_i$,

$$d_G(z)=d_G(w_2),$$

which, together with (2.3) and Theorem F, shows that

$$\begin{aligned} k_G(w_1,w_2) &\leq k_G(w_1,w_{1,1})+\ell_{k_G}(\beta_i)\\ &\leq \log\Big(1+\frac{|w_1-w_{1,1}|}{d_G(w_2)}\Big)+\frac{2|w_2-w_{1,1}|}{d_G(w_2)}\\ &\leq \log\Big(1+\frac{|w_1-w_{1,1}|}{d_G(w_2)}\Big)+\frac{4}{\log 3}\log\Big(1+\frac{|w_2-w_{1,1}|}{d_G(w_2)}\Big)\\ &< (1+\frac{6}{\log 3})\log\Big(1+\frac{|w_1-w_2|}{d_G(w_2)}\Big)\\ &\leq \frac{13}{2}j_G(w_1,w_2), \end{aligned}$$

since $\frac{|w_2-w_{1,1}|}{d_G(w_2)}\leq 4$ and $|w_2-w_{1,1}|\leq 2|w_1-w_2|$. Hence the proof follows. □

Lemma 3. *For $w_1\in G$, suppose $d_G(w_1)=\frac{1}{128}d_D(w_1)$. If $w_2\in\mathbb{S}(w_1,\frac{1}{32}d_D(w_1))$, then*

$$k_G(w_1,w_2)\leq 2^9k_D(w_1,w_2).$$

Proof. It follows from Lemma 1 that there is a unique element x_i in the intersection $P_D\cap\overline{\mathbb{B}}(w_1,\frac{1}{64}d_D(w_1))$ such that

$$d_G(w_2)=|w_2-x_i|\geq\frac{3}{128}d_D(w_1)=3d_G(w_1).$$

Then Lemma 2 and (2.1) imply

$$\begin{aligned} k_G(w_1,w_2) &\leq \frac{13}{2}\log\Big(1+\frac{|w_1-w_2|}{d_G(w_1)}\Big)\\ &= \frac{13}{2}\log\Big(1+\frac{128|w_1-w_2|}{d_D(w_1)}\Big)\\ &< 2^9\log\Big(1+\frac{|w_1-w_2|}{d_D(w_1)}\Big)\\ &\leq 2^9k_D(w_1,w_2), \end{aligned}$$

which shows that the lemma is true. □

Lemma 4. *Let w_1, $w_2 \in G$ and let γ denote a 2-neargeodesic joining w_1 and w_2 in D. If $d_G(z) \geq \frac{1}{128}d_D(z)$ for each $z \in \gamma$, then*

$$k_G(w_1, w_2) \leq 2^8 k_D(w_1, w_2).$$

Proof. Obviously, we get

$$\begin{aligned} k_G(w_1, w_2) &\leq \ell_{k_G}(\gamma[w_1, w_2]) = \int_{\gamma[w_1,w_2]} \frac{|dx|}{d_G(x)} \leq 128 \int_{\gamma[w_1,w_2]} \frac{|dx|}{d_D(x)} \\ &= 128\ell_{k_D}(\gamma[w_1, w_2]) \leq 2^8 k_D(w_1, w_2). \end{aligned}$$

Hence the proof is complete. □

Lemma 5. *Let w_1, $w_2 \in D$. If $|w_1 - w_2| \geq \frac{1}{c}d_D(w_1)$ $(c \geq 2)$, then*

$$|w_1 - w_2| \geq \frac{1}{c+1}d_D(w_2).$$

Proof. Suppose on the contrary that

$$|w_1 - w_2| < \frac{1}{c+1}d_D(w_2).$$

Then

$$d_D(w_1) \geq d_D(w_2) - |w_1 - w_2| > \frac{c}{c+1}d_D(w_2),$$

which shows that

$$|w_1 - w_2| \geq \frac{1}{c}d_D(w_1) > \frac{1}{c+1}d_D(w_2).$$

This is the desired contradiction. □

3.1. **The proof of Theorem 1.** First we prove the sufficiency. Suppose that G is a ψ_1-uniform domain. Then we shall prove that for z_1, $z_2 \in D$,

$$k_D(z_1, z_2) \leq \psi\Big(\frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big),$$

where $\psi(t) = 3\psi_1(128t)$ for $t > 0$, which implies that D is a ψ-uniform domain. Without loss of generality, we assume that

$$\min\{d_G(z_1), d_G(z_2)\} = d_G(z_1).$$

We divide the proof into two cases.

3.1.1. *We first suppose that* $|z_1 - z_2| \leq \frac{1}{2}d_D(z_1)$.

Then it follows from (2.3) that

$$k_D(z_1, z_2) \leq \log\Big(1 + \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|}\Big) \leq 2\log\Big(1 + \frac{|z_1 - z_2|}{d_D(z_1)}\Big). \tag{3.3}$$

3.1.2. *We then suppose that* $|z_1 - z_2| > \frac{1}{2}d_D(z_1)$.

Hence by Lemma 5, we have

$$|z_1 - z_2| \geq \frac{1}{3}d_D(z_2). \tag{3.4}$$

Case 1. $d_G(z_1) \geq \frac{1}{64}d_D(z_1)$.

Under this assumption, we have

$$k_D(z_1, z_2) \leq k_G(z_1, z_2) \leq \psi_1\Big(\frac{|z_1 - z_2|}{d_G(z_1)}\Big) \leq \psi_1\Big(\frac{64|z_1 - z_2|}{d_D(z_1)}\Big). \tag{3.5}$$

Case 2. $d_G(z_1) < \frac{1}{64}d_D(z_1)$.

For a proof in this case, we let $u_1 \in \mathbb{S}(z_1, \frac{1}{32}d_D(z_1))$. Then we know

$$k_D(z_1, u_1) \leq \int_{[z_1,u_1]} \frac{|dz|}{d_D(z)} \leq \frac{32|z_1 - u_1|}{31d_D(z_1)} = \frac{1}{31} < \log\Big(1 + \frac{|z_1 - z_2|}{d_D(z_1)}\Big), \tag{3.6}$$

since $d_D(z) \geq d_D(z_1) - |z_1 - z| \geq \frac{31}{32}d_D(z_1)$ for each $z \in [z_1, u_1]$. Moreover, we get

$$|u_1 - z_2| \leq |z_1 - z_2| + |z_1 - u_1| \leq \frac{17}{16}|z_1 - z_2|, \tag{3.7}$$

and it follows from Lemma 1 and the assumption "$d_G(z_1) < \frac{1}{64}d_D(z_1)$" that there exists only one element x_i in $\overline{\mathbb{B}}(z_1, \frac{1}{64}d_D(z_1)) \cap P_D$ such that $d_G(u_1) = |u_1 - x_i|$ and so

$$d_D(u_1) > d_G(u_1) = |u_1 - x_i| \geq |u_1 - z_1| - |z_1 - x_i| \geq \frac{1}{64}d_D(z_1). \tag{3.8}$$

Subcase 1. $d_G(z_2) \geq \frac{1}{120}d_D(z_2)$.

Then by (3.6), (3.7) and (3.8), we have

$$\begin{aligned} k_D(z_1, z_2) &\leq k_D(z_1, u_1) + k_D(u_1, z_2) \\ &\leq \log\Big(1 + \frac{|z_1 - z_2|}{d_D(z_1)}\Big) + k_G(u_1, z_2) \\ &\leq j_D(z_1, z_2) + \psi_1\Big(\frac{|u_1 - z_2|}{\min\{d_G(u_1), d_G(z_2)\}}\Big) \\ &\leq j_D(z_1, z_2) + \psi_1\Big(120\frac{|u_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big) \\ &\leq j_D(z_1, z_2) + \psi_1\Big(128\frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big) \\ &< 2\psi_1\Big(128\frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big). \end{aligned} \tag{3.9}$$

Subcase 2. $d_G(z_2) < \frac{1}{120}d_D(z_2)$.

We take $u_2 \in \mathbb{S}(z_2, \frac{1}{32}d_D(z_2))$. It follows from Lemma 1 and (3.4) that there is only one element x_i in $\overline{\mathbb{B}}(z_2, \frac{1}{32}d_D(z_2)) \cap P_D$ such that $d_G(u_2) = |u_2 - x_i|$ and so

$$d_D(u_2) \geq d_G(u_2) = |u_2 - x_i| \geq |u_2 - z_2| - |z_2 - x_i| \geq \frac{11}{480}d_D(z_2) \tag{3.10}$$

and by (3.4), we have

$$|u_1 - u_2| \leq |z_1 - z_2| + |z_1 - u_1| + |u_2 - z_2| \leq \frac{37}{32}|z_1 - z_2|. \tag{3.11}$$

It follows from (3.4) and a similar argument as in (3.6) that

$$k_D(z_2, u_2) \leq \int_{[z_2,u_2]} \frac{|dz|}{d_D(z)} \leq \frac{32|z_2 - u_2|}{31d_D(z_2)} < \log\Big(1 + \frac{|z_1 - z_2|}{d_D(z_2)}\Big). \tag{3.12}$$

Then we infer from (3.6), (3.8), (3.10), (3.11), (3.12) and Proposition 1 that

$$\begin{aligned} k_D(z_1, z_2) &\leq k_D(z_1, u_1) + k_G(u_1, u_2) + k_D(u_2, z_2) \\ &\leq 2j_D(z_1, z_2) + k_G(u_1, u_2) \\ &\leq 2j_D(z_1, z_2) + \psi_1\Big(\frac{|u_1 - u_2|}{\min\{d_G(u_1), d_G(u_2)\}}\Big) \\ &\leq 2j_D(z_1, z_2) + \psi_1\Big(64\frac{|u_1 - u_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big) \\ &\leq 2j_D(z_1, z_2) + \psi_1\Big(74\frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big) \\ &< 3\psi_1\Big(74\frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\Big). \end{aligned} \tag{3.13}$$

So the proof of the sufficiency follows from the inequalities (3.3), (3.5), (3.9) and (3.13).

Next we prove the necessity. Suppose that D is a ψ-uniform domain. Then we shall prove that for z_1, $z_2 \in G$,

$$k_G(z_1, z_2) \leq 2^{12}\psi\Big(\frac{|z_1 - z_2|}{\min\{d_G(z_1), d_G(z_2)\}}\Big),$$

which implies that G is a ψ_1-uniform domain with $\psi_1 = 2^{13}\psi$.

Without loss of generality, we may assume that $\min\{d_G(z_1), d_G(z_2)\} = d_G(z_1)$. In the following, we consider the two cases where $d_G(z_1) \leq \frac{1}{64}d_D(z_1)$ and $d_G(z_1) > \frac{1}{64}d_D(z_1)$, respectively.

3.1.3. *We first suppose that* $d_G(z_1) \leq \frac{1}{64}d_D(z_1)$.

Let γ be a 2-neargeodesic joining z_1 and z_2 in D.

Case 3. $|z_1 - z_2| \leq \frac{1}{32}d_D(z_1)$.

Then by Lemma 2 and Proposition 1, we have

Claim 1. $k_G(z_1, z_2) \leq \frac{13}{2}\log\Big(1 + \frac{|z_1-z_2|}{d_G(z_1)}\Big) \leq \frac{13}{2}\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)$.

Case 4. $|z_1-z_2|>\frac{1}{32}d_D(z_1)$.

Then Lemma 5 implies

$$|z_1-z_2|\geq \frac{1}{33}d_D(z_2). \tag{3.14}$$

Obviously, there exists some point $v_1\in\gamma\cap\mathbb{S}(z_1,\frac{1}{32}d_D(z_1))$ such that

$$\gamma[z_2,v_1]\subset D\backslash\mathbb{B}(z_1,\frac{1}{32}d_D(z_1)).$$

By Lemma 1, there exists some point $x_{i,1}\in\overline{\mathbb{B}}(z_1,\frac{1}{64}d_D(z_1))\cap P_D$ such that

$$d_G(v_1)=|v_1-x_{i,1}|\geq\frac{1}{64}d_D(z_1)\geq\frac{1}{66}d_D(v_1), \tag{3.15}$$

since $d_D(v_1)\leq d_D(z_1)+|z_1-v_1|\leq\frac{33}{32}d_D(z_1)$.

It follows from Lemma 2 and (3.15) that

$$k_G(z_1,v_1)\leq\frac{13}{2}\log\Big(1+\frac{|z_1-v_1|}{d_G(z_1)}\Big)\leq\frac{13}{2}\log\Big(1+\frac{|z_1-z_2|}{d_G(z_1)}\Big). \tag{3.16}$$

Next, we divide the proof into two subcases.

Subcase 3. $d_G(z)\geq\frac{1}{128}d_D(z)$ for each $z\in\gamma[v_1,z_2]$.

By Lemma 4, we know

$$\begin{aligned} k_G(v_1,z_2) &\leq 2^8k_D(v_1,z_2)\leq 2^8\ell_{k_D}(\gamma[v_1,z_2])\leq 2^8\ell_{k_D}(\gamma[z_1,z_2])\\ &\leq 2^9k_D(z_1,z_2)\leq 2^9\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big), \end{aligned}$$

since D is ψ-uniform, and so the following inequality easily follows from (3.16)

Claim 2. $k_G(z_1,z_2)\leq k_G(z_1,v_1)+k_G(v_1,z_2)\leq 519\psi\Big(\frac{|z_1-z_2|}{d_D(z_1)}\Big)$.

Subcase 4. There exists some point $z\in\gamma[v_1,z_2]$ such that $d_G(z)<\frac{1}{128}d_D(z)$.

It follows from (3.15) that there exists some point y_1 which is the first point in γ along the direction from v_1 to z_2 such that

$$d_G(y_1)=\frac{1}{128}d_D(y_1).$$

Then Lemma 4 shows

$$\begin{aligned} k_G(v_1,y_1)&\leq 2^8k_D(v_1,y_1)\leq 2^8\ell_{k_D}(\gamma[z_1,z_2])\\ &\leq 2^9k_D(z_1,z_2)\leq 2^9\psi\Big(\frac{|z_1-z_2|}{d_D(z_1)}\Big). \end{aligned} \tag{3.17}$$

Subsubcase 1. $|z_2-y_1|\leq\frac{1}{32}d_D(y_1)$.

Then we see from Lemma 2 that

$$k_G(y_1,z_2)\leq\frac{13}{2}\log\Big(1+\frac{|y_1-z_2|}{\min\{d_G(y_1),d_G(z_2)\}}\Big). \tag{3.18}$$

Claim 3. $k_G(z_1,z_2)\leq 564\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)$.

We now prove this claim. Since

$$d_D(z_2)\geq d_D(y_1)-|z_2-y_1|\geq \frac{31}{32}d_D(y_1)\geq 31|z_2-y_1|,$$

we infer from (3.14) that

$$|z_1-z_2|\geq \frac{31}{33}|y_1-z_2|, \tag{3.19}$$

and by (3.18), we get

$$k_G(y_1,z_2)\leq \frac{13}{2}\log\Big(1+\frac{33|z_1-z_2|}{31\min\{d_G(y_1),d_G(z_2)\}}\Big).$$

Further, we have

$$k_G(y_1,z_2)\leq 35\log\Big(1+\frac{|z_1-z_2|}{d_G(z_2)}\Big). \tag{3.20}$$

To prove this estimate, obviously, we only need to consider the case $d_G(z_2)\geq d_G(y_1)$. Since

$$d_G(z_2)\leq d_G(y_1)+|z_2-y_1| \text{ and } |z_2-y_1|\leq \frac{1}{32}d_D(y_1)=4d_G(y_1),$$

we see from (3.18) that

$$k_G(y_1,z_2)\leq \frac{13}{2}\log\Big(1+\frac{|y_1-z_2|}{d_G(y_1)}\Big)\leq \frac{13}{2}\log\Big(1+\frac{5|y_1-z_2|}{d_G(z_2)}\Big)<35\log\Big(1+\frac{|z_1-z_2|}{d_G(z_2)}\Big).$$

Hence (3.20) is true.

We come back to the proof of Claim 3. It follows from (3.16), (3.17) and (3.20) that

$$\begin{aligned} k_G(z_1,z_2) &\leq k_G(z_1,v_1)+k_G(v_1,y_1)+k_G(y_1,z_2)\\ &\leq \frac{13}{2}\log\Big(1+\frac{|z_1-z_2|}{d_G(z_1)}\Big)+2^9\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)+35\log\Big(1+\frac{|z_1-z_2|}{d_G(z_2)}\Big)\\ &\leq 42\log\Big(1+\frac{|z_1-z_2|}{d_G(z_1)}\Big)+2^9\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)\\ &\leq 564\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big), \end{aligned}$$

which shows that Claim 3 is true.

Subsubcase 2. $|z_2-y_1|>\frac{1}{32}d_D(y_1)$.

Obviously, there exists some point $v_2\in\gamma\cap\mathbb{S}(y_1,\frac{1}{32}d_D(y_1))$ such that

$$\gamma[z_2,v_2]\subset D\backslash\mathbb{B}(y_1,\frac{1}{32}d_D(y_1)).$$

By Lemma 1, we see that there exists some point $x_{i,2} \in P_D \cap \overline{\mathbb{B}}(y_1, \frac{1}{128}d_D(y_1))$ such that

$$d_G(v_2) = |v_2 - x_{i,2}| \geq |v_2 - y_1| - |y_1 - x_{i,2}| \geq \frac{3}{128}d_D(y_1) \geq \frac{1}{44}d_D(v_2), \tag{3.21}$$

since $d_D(v_2) \leq d_D(y_1) + |y_1 - v_2| \leq \frac{33}{32}d_D(y_1)$.

Moreover, by Lemma 3, we have

$$k_G(y_1, v_2) \leq 2^9 k_D(y_1, v_2) \leq 2^{10} k_D(z_1, z_2). \tag{3.22}$$

If $d_G(z) \geq \frac{1}{128}d_D(z)$ for each $z \in \gamma[v_2, z_2]$, then Lemma 4 implies

$$k_G(v_2, z_2) \leq 2^8 k_D(v_2, z_2) \leq 2^9 k_D(z_1, z_2), \tag{3.23}$$

and thus we infer from (3.16), (3.17), (3.22), (3.23) that

$$k_G(z_1, z_2) \leq \frac{13}{2} \log\Big(1 + \frac{|z_1 - z_2|}{d_G(z_1)}\Big) + 3 \cdot 2^{10} k_D(z_1, z_2).$$

Hence we have the following estimate.

Claim 4. $k_G(z_1, z_2) \leq 2^{12}\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)$.

For the remaining case, that is, when there exists some point $z \in \gamma[v_2, z_2]$ such that $d_G(z) < \frac{1}{128}d_D(z)$, similar discussions as in Subcase 3 show that there exists some point $y_2 \in \gamma[v_2, z_2]$ satisfying

$$d_G(y_2) = \frac{1}{128}d_D(y_2),$$

and

$$k_G(v_2, y_2) \leq 2^8 k_D(v_2, y_2). \tag{3.24}$$

Now, if $|z_2 - y_2| \leq \frac{1}{32}d_D(y_2)$, then the similar reasoning as in the proof of (3.20) shows that

$$k_G(y_2, z_2) \leq 35 \log\Big(1 + \frac{|z_1 - z_2|}{d_G(z_2)}\Big). \tag{3.25}$$

Then it follows from (3.16), (3.17), (3.22), (3.24) and (3.25) that

$$\begin{aligned} k_G(z_1, z_2) &\leq k_G(z_1, v_1) + k_G(v_1, y_1) + k_G(y_1, v_2) + k_G(v_2, y_2) + k_G(y_2, z_2) \\ &\leq 2^{10} k_D(z_1, z_2) + 42 \log\Big(1 + \frac{|z_1 - z_2|}{d_G(z_1)}\Big). \end{aligned}$$

Hence we reach the following estimate.

Claim 5. $k_G(z_1, z_2) \leq 2^{112}\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)$.

We assume now that $|z_2 - y_2| > \frac{1}{32}d_D(y_2)$. Then there exists some point $v_3 \in \gamma[y_2, z_2] \cap \mathbb{S}(y_2, \frac{1}{32}d_D(y_2))$ such that

$$\gamma[z_2, v_3] \subset D \backslash \mathbb{B}(y_2, \frac{1}{32}d_D(y_2)),$$

and the similar reasoning as in the proof of (3.22) shows that

$$k_G(y_2, v_3) \le 2^9 k_D(y_2, v_3) \le 2^{10} k_D(z_1, z_2).$$

By repeating the procedure as above, we will reach a finite sequence of points in γ:

(1) $\{z_1, v_1, y_1, v_2, y_2, \cdots, v_t, z_2\}$ such that $d_G(z) \ge \frac{1}{128} d_D(z)$ for each $z \in \gamma[v_t, z_2]$; or

(2) $\{z_1, v_1, y_1, v_2, y_2, \cdots, v_t, y_t, z_2\}$ such that $|z_2 - y_t| \le \frac{1}{32} d_D(y_t)$.

It follows from (2.1) that

$$k_D(y_i, v_{i+1}) \ge \log\left(1 + \frac{|y_i - v_{i+1}|}{d_D(y_i)}\right) \ge \log \frac{33}{32}$$

for each $i \in \{1, \cdots, t\}$, and we see that

$$t \le \frac{k_D(z_1, z_2)}{\log \frac{33}{32}}.$$

For the former case, i.e., when the statement (1) as above holds, we have shown that

(1) $k_G(z_1, v_1) \le \frac{13}{2} \log\left(1 + \frac{|z_1 - v_1|}{d_G(z_1)}\right) \le \frac{13}{2} \log\left(1 + \frac{|z_1 - z_2|}{d_G(z_1)}\right)$;

(2) $k_G(v_i, y_i) \le 2^8 k_D(v_i, y_i)$, where $i \in \{1, \cdots, t-1\}$;

(3) $k_G(y_i, v_{i+1}) \le 2^9 k_D(y_i, v_{i+1})$, where $i \in \{1, \cdots, t\}$; and

(4) $k_G(v_t, z_2) \le 2^8 k_D(v_t, z_2) \le 2^9 k_D(z_1, z_2)$.

Hence we obtain

$$\begin{aligned} k_G(z_1, z_2) &\le k_G(z_1, v_1) + \sum_{i=1}^{t-1} k_G(v_i, y_i) + \sum_{i=2}^{t} k_G(y_{i-1}, v_i) + k_G(v_t, z_2) \\ &\le \frac{13}{2} \log\left(1 + \frac{|z_1 - z_2|}{d_G(z_1)}\right) + 2^8 \sum_{i=1}^{t-1} k_D(v_i, y_i) + 2^9 \sum_{i=2}^{t} k_D(y_{i-1}, v_i) \\ &\quad + 35 \log\left(1 + \frac{|z_1 - z_2|}{d_G(z_2)}\right) \\ &\le \frac{13}{2} \log\left(1 + \frac{|z_1 - z_2|}{d_G(z_1)}\right) + 2^{10} k_D(z_1, z_2) + 35 \log\left(1 + \frac{|z_1 - z_2|}{d_G(z_2)}\right), \end{aligned}$$

which shows

Claim 6. $k_G(z_1, z_2) \le 2^{11} \psi\left(\frac{|z_1 - z_2|}{d_G(z_1)}\right)$.

For the latter case, i.e., when the statement (2) as above holds, we also have shown that

(1) $k_G(z_1, v_1) \le \frac{13}{2} \log\left(1 + \frac{|z_1 - v_1|}{d_G(z_1)}\right) \le \frac{13}{2} \log\left(1 + \frac{|z_1 - z_2|}{d_G(z_1)}\right)$;

(2) $k_G(v_i, y_i) \le 2^8 k_D(v_i, y_i)$, where $i \in \{1, \cdots, t\}$;

(3) $k_G(y_i, v_{i+1}) \le 2^9 k_D(y_i, v_{i+1})$, where $i \in \{1, \cdots, t-1\}$; and

(4) $k_G(y_t, z_2) \le 35 \log\left(1 + \frac{|z_1 - z_2|}{d_G(z_2)}\right)$.

Hence we get

$$\begin{aligned} k_G(z_1,z_2) &\leq k_G(z_1,v_1)+\sum_{i=1}^{t}k_G(v_i,y_i)+\sum_{i=2}^{t}k_G(y_{i-1},v_i)+k_G(y_t,z_2)\\ &\leq \frac{13}{2}\log\Big(1+\frac{|z_1-z_2|}{d_G(z_1)}\Big)+2^8\sum_{i=1}^{t}k_D(v_i,y_i)+2^9\sum_{i=2}^{t}k_D(y_{i-1},v_i)\\ &\quad +35\log\Big(1+\frac{|z_1-z_2|}{d_G(z_2)}\Big)\\ &\leq \frac{13}{2}\log\Big(1+\frac{|z_1-z_2|}{d_G(z_1)}\Big)+2^{10}k_D(z_1,z_2)+35\log\Big(1+\frac{|z_1-z_2|}{d_G(z_2)}\Big), \end{aligned}$$

which implies

Claim 7. $k_G(z_1,z_2)\leq 2^{11}\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)$.

Now we are in a position to conclude that the proof for the case $d_G(z_1)\leq \frac{1}{64}d_D(z_1)$ follows from Claims 1 $\sim$ 7.

3.1.4. *We then suppose that* $d_G(z_1)>\frac{1}{64}d_D(z_1)$.

Case 5. $|z_1-z_2|\leq \frac{1}{2}d_G(z_1)$.

Then we have

$$k_G(z_1,z_2)\leq \int_{[z_1,z_2]}\frac{|dz|}{d_G(z)}\leq \frac{2|z_1-z_2|}{d_G(z_1)},$$

since $d_G(z)\geq d_G(z_1)-|z_1-z|\geq \frac{1}{2}d_G(z_1)$ for each $z\in[z_1,z_2]$. Hence we have

Claim 8. $k_G(z_1,z_2)\leq 3\log\Big(1+\frac{|z_1-z_2|}{d_G(z_1)}\Big)$.

Case 6. $|z_1-z_2|>\frac{1}{2}d_G(z_1)$

Since $|z_1-z_2|>\frac{1}{128}d_D(z_1)$, we know from Lemma 5 that

$$|z_1-z_2|>\frac{1}{129}d_D(z_2). \tag{3.26}$$

Let β be a 2-neargeodesic joining z_1 and z_2 in D. We divide the discussions into two subcases.

Subcase 5. $d_G(z)\geq \frac{1}{64}d_D(z)$ for each $z\in\beta$.

In this case, the following inequality easily follows from Lemma 4.

Claim 9. $k_G(z_1,z_2)\leq 2^8k_D(z_1,z_2)\leq 2^8\psi\Big(\frac{|z_1-z_2|}{d_G(z_1)}\Big)$.

Subcase 6. There exists some point $z\in\beta$ such that $d_G(z)<\frac{1}{64}d_D(z)$.

Then it follows from the assumption "$d_G(z_1)>\frac{1}{64}d_D(z_1)$" that there exists point p_1 which is the first point in β along the direction from z_1 to z_2 such that

$$d_G(p_1)=\frac{1}{64}d_D(p_1).$$

Then Lemma 4 shows

$$k_G(z_1, p_1) \le 2^8 k_D(z_1, p_1). \tag{3.27}$$

We consider the case where $|z_2 - p_1| \le \frac{1}{32} d_D(p_1)$ and the case where $|z_2 - p_1| > \frac{1}{32} d_D(p_1)$, respectively.

Subsubcase 3. $|z_2 - p_1| \le \frac{1}{32} d_D(p_1)$.

It follows from (3.26) that

$$|z_2 - p_1| \le \frac{1}{31} d_D(z_2) \le \frac{129}{31} |z_1 - z_2| \tag{3.28}$$

since $d_D(z_2) \ge d_D(p_1) - |z_2 - p_1| \ge \frac{31}{32} d_D(p_1)$.

By Lemma 1, we have

$$d_G(z_2) \le \frac{1}{16} d_D(p_1) \le 4 d_G(p_1).$$

Then we know from Lemma 2 and (3.28) that

$$k_G(p_1, z_2) \le \frac{13}{2} \log\Big(1 + \frac{|z_2 - p_1|}{\min\{d_G(z_2), d_G(p_1)\}}\Big) \le 28 \log\Big(1 + \frac{|z_2 - z_1|}{d_G(z_2)}\Big),$$

which, together with (3.27), implies

Claim 10. $k_G(z_1, z_2) \le 28\psi\Big(\frac{|z_1 - z_2|}{d_G(z_1)}\Big)$.

Subsubcase 4. $|z_2 - p_1| > \frac{1}{32} d_D(p_1)$.

Obviously, there exists some point $q_1 \in \beta[p_1, z_2]$ such that

$$\beta[q_1, z_2] \subset D \backslash \mathbb{B}(p_1, \frac{1}{32} d_D(p_1)).$$

By Lemma 1, we see that there exists some point $x_{i,3} \in P_D \cap \overline{\mathbb{B}}(p_1, \frac{1}{128} d_D(p_1))$ such that

$$\begin{aligned} d_G(q_1) &= |q_1 - x_{i,3}| \ge |q_1 - p_1| - |p_1 - x_{i,3}| \\ &\ge \frac{3}{128} d_D(p_1) \ge \frac{1}{44} d_D(q_1), \end{aligned} \tag{3.29}$$

since $d_D(q_1) \le d_D(p_1) + |p_1 - q_1| \le \frac{33}{32} d_D(p_1)$.

Then the similar reasoning as in Subsubcase 2 in Subsection 3.1.3 implies that we will get a finite sequence of points in β:

(1) $\{z_1, p_1, q_1, \cdots, p_s, z_2\}$ such that $d_G(z) \ge \frac{1}{128} d_D(z)$ for each $z \in \gamma[p_s, z_2]$; or
(2) $\{z_1, p_1, q_1, \cdots, p_s, q_s, z_2\}$ such that $|z_2 - q_s| \le \frac{1}{32} d_D(q_s)$.

It follows from the similar arguments as in Claims 6 and 7 in Subsubsection 2 that

Claim 11. $k_G(z_1, z_2) \le 2^{11}\psi\Big(\frac{|z_1 - z_2|}{d_G(z_1)}\Big)$.

The proof for the case $d_G(z_1) > \frac{1}{64} d_D(z_1)$ follows from Claims 8 $\sim$ 11. Hence the proof of Theorem 1 is complete. □

3.2. **The proof of Corollary 1.** First, we prove the sufficiency. Suppose $G = D\backslash P_D$ is a c_1-uniform domain. Then Theorem D implies that there exists a constant c_1' depending only on c_1 such that for all x and y in G,

$$k_G(x,y) \le c_1'\, j_G(x,y).$$

By Theorem 1, we see that for all z_1 and z_2 in D,

$$k_D(z_1,z_2) \le 3c_1' \log\left(1+\frac{128|z_1-z_2|}{\min\{d_D(z_1),d_D(z_2)\}}\right) \le 384c_1' j_D(z_1,z_2),$$

which, together with Theorem D, shows that D is a c-uniform domain, where c depends only on c_1.

Next, we prove the necessity. Suppose that D is a c-uniform domain. Then Theorem D implies that there exists a constant c' depending only on c such that for all x and y in D,

$$k_D(x,y) \le c'\, j_D(x,y).$$

By Theorem 1, we see that for all z_1 and z_2 in $G = D\backslash P_D$,

$$k_G(z_1,z_2) \le 2^{12} c' j_G(z_1,z_2),$$

which, together with Theorem D, shows that G is a c_1-uniform domain, where c_1 depends only on c. □

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