NONPOSITIVELY CURVED MANIFOLDS CONTAINING A PRESCRIBED NONPOSITIVELY CURVED HYPERSURFACE

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Abstract. We use pinched smooth hyperbolization to show that every closed, nonpositively curved n-dimensional manifold M can be embedded as a totally geodesic submanifold of a closed, nonpositively curved $(n+1)$-dimensional manifold $\widehat{M}$ of geometric rank one.

Ralf Spatzier asked the author the following interesting question: for a closed manifold M with sectional curvature ≤ 0, is there a closed manifold $\widehat{M}$ of one dimension higher with sectional curvature ≤ 0 and has geometric rank 1 (and thus is not a product) that contains M as a totally geodesic submanifold? The answer to this question is yes thanks to recent technology of pinched smooth hyperbolization ([3]). In this paper we give a construction of such a manifold $\widehat{M}$.

Theorem 1. *Let (M, g_M) be a closed, Riemannian manifold of dimension n with sectional curvature $\kappa(M) \leq 0$. There exist a closed, Riemannian $(n+1)$-dimensional manifold $\widehat{M}$ of geometric rank 1 with sectional curvature $\kappa(\widehat{M}) \leq 0$ and a isometric embedding $f\colon M \longrightarrow \widehat{M}$.*

Proof. Let $\triangle$ be a triangulation of M. We extend $\triangle$ to a triangulation of $M \times [0,1]$. We cone off the boundary of $M \times [0,1]$ (which has two components) and denote the resulting simplicial complex by X. Then X is a manifold with one singular cone point $*$, that is, $X \setminus \{*\}$ is a manifold. Let $h(X)$ be a strict hyperbolization of X ([2]). Then $h(X)$ is a manifold with one singularity $h(*)$. We pick $h(X)$ such that the faces of each Charney-Davis hyperbolization piece have large enough width as in [3, Lemma 9.1.1] so that pinched smooth hyperbolization can be applied to $h(X) \setminus \{h(*)\}$.

Let $W = h(X) \setminus \{h(*)\}$. Then W is a noncompact manifold with two ends, each of which is homeomorphic to $M \times (0, \infty)$. By [3, Corollary 8.5.1] there is a Riemannian metric g on W with sectional curvature < 0, and the metric on each end is

$$dt^2 + e^{-2t} g_M,$$

where $-1 < t < 1$ (the actual values of -1 and 1 are not crucial in this argument). We truncate each of end of W at $t = 0$ and glue the two boundary components of the resulting manifold together. We then get a closed manifold $\widehat{M}$ with a Riemannian metric $\overline{g}$ that is not smooth at the gluing. The metric $\overline{g}$ is a warped product $dt^2 + e^{-2|t|} g_M$, for $-1 < t < 1$. Therefore, in order to smooth out the metric $\overline{g}$, we just need to smooth out the warping function $e^{-2|t|}$ around $t = 0$ without altering the nonpositivity of the curvature.

Observe that since the metric g_M is nonpositively curved, the warped product metric $dt^2 + \phi^2(t) g_M$ on $\mathbb{R} \times M$ has nonpositive curvature if $\phi(t)$ is a convex function by the Bishop-O'Neill curvature formula ([1]). Thus we can pick ϕ to be a convex, smooth, even function that agrees with $e^{-2|t|}$ outside a small neighborhood of $t = 0$ and assumes a minimum at $t = 0$. We then obtain a Riemannian metric $\widehat{g}$ on $\widehat{M}$ that has sectional curvature $\kappa \leq 0$.

It is not hard to see that map $f\colon M \longrightarrow \widehat{M}$ defined by identifying M with cross section $t = 0$ is a isometric embedding due to the evenness of $\phi(t)$. $\square$

Remark. The theorem holds if we replace "$\leq$" by "$<$".

References

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