STACKS ASSOCIATED TO ABELIAN TENSOR CATEGORIES

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ABSTRACT. For an abelian tensor category a stack is constructed. As an application we show that our construction can be used to recover a quasi-compact separated scheme from the category of its quasi-coherent sheaves. In another application, we show how the "dual stack" of the classifying stack BG of a finite group G can be obtained by altering the tensor product on the category $G-\mathrm{rep}$ of G-representations. Using glueing techniques we show that the dual pair of a G-gerbe, in the sense of [TT10], can be constructed by glueing local dual stacks.

1. INTRODUCTION

Let X be a Noetherian scheme over an algebraically closed fields k of characteristic 0. It is known that the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X uniquely determines X; see [Gab62, page 447], and [Rou10, Corollary 4.4] for the version with coherent sheaves. Motivated by this, in this paper we study the (re)construction of geometry from abelian categories. Our approach is motivated by the following consideration. Let Aff_k be the category of affine k-schemes. A scheme X is equivalent to its *functor of points*

$$\mathrm{Aff}_k \to \mathrm{Sets}, \quad S \mapsto \mathrm{Hom}(S, X).$$

Given a morphism $\phi : S \to X$ of schemes, one can consider the pull-back functor $\phi^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$. ϕ^* is a *symmetric monoidal functor*, i.e. ϕ^* is compatible with tensor products of sheaves. A version of the Tannakian duality theorem (see [Lur]) states that when X is a quasi-compact separated scheme, $\phi \mapsto \phi^*$ defines an equivalence

$$\mathrm{Hom}(S, X) \simeq \mathrm{Hom}_{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(S))$$

where the right-hand side denotes the category of symmetric monoidal functors which carry flat objects to flat objects and preserve colimits.

The discussion above suggests the following general construction. Let $(A, \otimes)$ be an abelian tensor category. Define a functor

$$\underline{(A,\otimes)} : \mathrm{Aff}_k \to \mathrm{Groupoids}, \quad \underline{(A,\otimes)}(S) := \mathrm{Hom}_{\otimes}(A, \mathrm{QCoh}(S)).$$

Theorem 1.1. *The functor $\underline{(A,\otimes)}$ is represented by a stack.*

The above Theorem is a special case of our main Theorem, which is described and proved in Section 2.3 as Theorem 2.4.

We illustrate our construction in several examples. Suppose that $(A, \otimes) = (\mathrm{QCoh}(X), \otimes)$ is the category of quasi-coherent sheaves on a quasi-compact separated k-scheme X with $\otimes$ being the tensor product of sheaves. In this case we have

Date: June 4, 2012.

Theorem 1.2. *There is an isomorphism*

$$\underline{(\mathrm{QCoh}(X), \otimes)} \simeq X.$$

This is proved in Section 3 as Theorem 3.1 in case X is affine, and Theorem 3.4 for general quasi-compact separated X. We expect that this Theorem also holds true for X being a geometric stack (in the sense of [Lur, Definition 3.1]).

The stack $\underline{(A, \otimes)}$ we construct is sensitive to the tensor structure $\otimes$. To illustrate this, we consider the example $A = G-\mathrm{rep}$, the category of finite dimensional representations of a finite group G over k. $G-\mathrm{rep}$ may be interpreted as the category $\mathrm{Coh}(BG)$ of coherent sheaves on the classifying stack BG. In this point of view tensor product of sheaves gives $G-\mathrm{rep}$ a tensor structure denoted by $\otimes_G$, and we have

Theorem 1.3 (see Theorem 4.4)**.** *There is an isomorphism*

$$\underline{(G-\mathrm{rep}, \otimes_G)} \simeq BG.$$

We expect that the above Theorem to hold also for not necessarily finite groups G, such as linear algebraic groups.

On the other hand, Schur's Lemma implies that $G-\mathrm{rep}$ is equivalent (as abelian categories) to the direct sum of $\#\hat{G}$ copies[1] of the category vect_k of finite dimensional k-vector spaces. Componentwise tensor product of vector spaces then gives a different tensor structure on $G-\mathrm{rep}$, which is denoted by $\otimes_Z$. In this case we have

Theorem 1.4 (See Section 4.1)**.** $\underline{(G-\mathrm{rep}, \otimes_Z)}$ *is isomorphic to a disjoint union of* $\#\hat{G}$ *points.*

Certainly BG is very different from a disjoint union of $\#\hat{G}$ points. However they are closely related. In fact the correspondence between them is the simplest example of the *gerbe duality* studied in [TT10]. For a G-gerbe $\mathcal{Y} \to \mathcal{B}$ the gerbe duality asserts that various geometric properties of $\mathcal{Y}$ are equivalent to geometric properties of a "dual space" $\widehat{\mathcal{Y}}$ *twisted by* a $\mathbb{C}^*$-valued 2-cocycle c. In particular, it is shown in [TT10] that the category of sheaves on $\mathcal{Y}$ is equivalent to the category of c-twisted sheaves on $\widehat{\mathcal{Y}}$. The BG example suggests that it may be possible to realize the G-gerbe $\mathcal{Y}$ and its dual $(\widehat{\mathcal{Y}}, c)$ using stacks associated to the category of sheaves on $\mathcal{Y}$ with different tensor structures. In Section 5 we carry out a construction of this nature. Locally on $\mathcal{Y}$, the stacks we construct realize the G-gerbe and its dual. We then obtain $\mathcal{Y}$ and $(\widehat{\mathcal{Y}}, c)$ by glueing local duals.

Outlook. It is very interesting to study the stacks constructed in this paper for other examples of abelian tensor categories. For instance, it will be very desirable to describe the stacks associated to the category of rational Hodge structures, or the category of representations of a quiver.

One may hope that properties of the stack $\underline{(A, \otimes)}$ can reflect properties of the category $(A, \otimes)$. For this reason it is interesting to study geometric properties of $\underline{(A, \otimes)}$. For example, it will be interesting to find criteria for algebraicity of $\underline{(A, \otimes)}$. We plan to pursue this elsewhere.

The rest of this paper is organized as follows. In section 2 we define the category $\underline{(A, \otimes)}$ of tensor functors from a fixed abelian tensor category A into a varying family $\mathcal{B}$ of abelian tensor

[1] $\hat{G}$ denotes the group of characters on G.

categories, and we show that if the target categories $\mathcal{B}$ form a fibred category or (pre-)stack, then so does $\underline{(A, \otimes)}$. We also explain how objects in A naturally give rise to sheaves on $\underline{(A, \otimes)}$.

In section 3 we explain how schemes can be reconstructed as stacks of the form $\underline{(A, \otimes)}$ when A is taken to be the category of quasi-coherent sheaves on the scheme. In section 4 we consider the example $A = G-\mathrm{rep}$ where G is a finite group, and we study $\underline{(A, \otimes)}$ equipped with different tensor structures.

In Section 5 we construct stacks locally of the form $\underline{(A, \otimes)}$ using 2–descent. As an example we show how the gerbe duality of [TT10] can be understood via this construction: Indeed, a G-gerbe $\mathcal{Y}$ and the underlying space of its dual $\hat{\mathcal{Y}}$ can both locally be realized as stacks of the form $\underline{(A, \otimes)}$ with the same abelian category $A = G-\mathrm{rep}$ and "dual" choices of tensor products.

In Appendix A we give a brief review of the construction of stacks using 2–descent.

2. STACK ASSOCIATED TO AN ABELIAN TENSOR CATEGORY

2.1. **Category fibred in abelian tensor categories.**

2.1.1. Let $\mathfrak{S}$ be a category, and let $\pi : \mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories. This means first of all that π is a functor making $\mathcal{B}$ a fibred category; in particular for every morphism $f : S' \to S$ in $\mathfrak{S}$ there is a *canonical* functor between the fibre categories

$$f^* : \mathcal{B}_S \to \mathcal{B}_{S'}.$$

Every fibre category $\mathcal{B}_S$ is required to be an abelian tensor category, which means that it is an abelian category along with a symmetric monoidal category structure $\otimes_S$ [ML98, Chapter XI.1], such that the functor $x \mapsto y \otimes_S x$ is additive and preserves (small) colimits for every $y \in \mathcal{B}_S$. We denote by u_S the unit object in $\mathcal{B}_S$.

The assumption above in particular implies that the functor $x \mapsto y \otimes_S x$ is right exact, and that the zero object $0 \in \mathcal{B}_S$ satisfies $0 \otimes_S x \cong 0$ for every $x \in \mathcal{B}_S$.

Lastly, the pull-back functors f^* are required to be symmetric strong monoidal [ML98, Chapter XI.2] and preserves colimits; in particular they preserve direct sums. The condition of being symmetric strong monoidal means that there are isomorphisms

$$f^*(y) \otimes_{S'} f^*(x) \xleftarrow{\cong} f^*(x) \otimes_{S'} f^*(y) \xrightarrow{\cong} f^*(x \otimes_S y)$$

and

$$u_{S'} \xrightarrow{\cong} f^*(u_S)$$

for every $x, y \in \mathcal{B}_S$ which satisfy some compatibility, or coherence conditions in the form of commutative diagrams; to simplify notations we will often suppress these isomorphisms and canonically identify these objects in $\mathcal{B}_{S'}$.

We will call symmetric strong monoidal functors preserving colimits simply *tensor functors*. In particular they are right exact functors.

2.1.2. *Example.* Fix a commutative ring Λ with identity. Let $\mathfrak{S}$ be the category of affine Λ-schemes. Let $\mathcal{B}$ be the category of pairs $(S, \mathcal{F})$ with $S \in \mathfrak{S}$ and $\mathcal{F}$ a quasi-coherent sheaf on S. The obvious forgetful functor $\pi : \mathcal{B} \to \mathfrak{S}$ is a category fibred in abelian tensor categories with the sheaf tensor product on each $\mathcal{B}_S = \mathrm{QCoh}(S)$.

2.1.3. *Example.* Fix a field k. Let $\mathfrak{S}$ be the category of finite quivers (without relations). Let $\mathcal{B}$ be the category of pairs (Q, V) with $Q \in \mathfrak{S}$ and V a finite dimensional Q-representation over k. The obvious forgetful functor $\pi : \mathcal{B} \to \mathfrak{S}$ is a category fibred in abelian tensor categories with the vertex-wise tensor product on each $\mathcal{B}_Q = Q{-}\mathrm{rep}_k$.

2.1.4. *Example.* Fix a field k. Let $\mathfrak{S}$ be the category of finite groups. Let $\mathcal{B}$ be the category of pairs (G, V) with $G \in \mathfrak{S}$ and V a finite dimensional G-representation over k. The obvious forgetful functor $\pi : \mathcal{B} \to \mathfrak{S}$ is a category fibred in abelian tensor categories with the representation tensor product on each $\mathcal{B}_Q = G{-}\mathrm{rep}_k$.

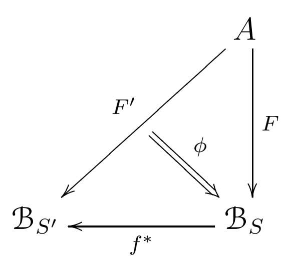
2.2. **Category fibred in groupoids associated to an abelian tensor category.**

2.2.1. Let $(A, \otimes)$ be an abelian tensor category with unit object u, and let $\pi : \mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories as above.

Define a category $\underline{(A, \otimes)}(\mathcal{B})$, sometimes just $\underline{A}(\mathcal{B})$, $\underline{(A, \otimes)}$, or $\underline{A}$ whenever the tensor product and/or the category $\mathcal{B}$ in question are clear, whose objects are pairs (S, F) where $S \in \mathfrak{S}$ and $F : A \to \mathcal{B}_S$ is a *tensor functor*. A morphism $(S', F') \to (S, F)$ is a pair (f, ϕ) where $f : S' \to S$ is a morphism in $\mathfrak{S}$ and ϕ is a *symmetric monoidal* natural *isomorphism* between tensor functors

$$\phi : F' \to f^* \circ F$$

from A to $\mathcal{B}_{S'}$:



$$S' \xrightarrow{\ f\ } S.$$

To say ϕ is *monoidal* means that if for every pair a, b of objects in A the following diagram commutes:

$$\begin{array}{ccc} F'(a \otimes b) & \xrightarrow{\phi_{a\otimes b}} & f^*F(a \otimes b) \\ \downarrow{\scriptstyle \sigma'_{a,b}} & & \downarrow{\scriptstyle f^*\sigma_{a,b}} \\ F'(a) \otimes_{S'} F'(b) & \xrightarrow[\phi_a \otimes_{S'} \phi_b]{} & f^*F(a) \otimes_{S'} f^*F(b), \end{array}$$

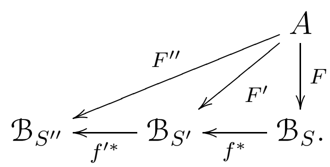
and so does

$$\begin{array}{ccc} u_{S'} & = & u_{S'} \\ {\scriptstyle \sigma_0'}\downarrow & & \downarrow{\scriptstyle f^*\sigma_0} \\ F'(u) & \xrightarrow[\phi_u]{} & f^*F(u). \end{array}$$

Here $\sigma_{a,b}$ and σ_0 are the isomorphisms that come with the monoidal functor F; similarly $\sigma'_{a,b}$ and σ'_0 are for F'. To say ϕ is symmetric means that it satisfies some further conditions in the form of commutative diagrams [ML98, page 257].

2.2.2. To define compositions in $\underline{A}$, suppose we have morphisms $(f, \phi) : (S', F') \to (S, F)$ and $(f', \phi') : (S'', F'') \to (S', F')$. We define

$$(f, \phi) \circ (f', \phi') = (f \circ f', f'^*(\phi) \circ \phi') : (S'', F'') \to (S, F).$$



$$S'' \xrightarrow{f'} S' \xrightarrow{f} S$$

This way $\underline{A}$ is a category with a covariant functor $p : \underline{A} \to \mathfrak{S}$ sending

$$(S, F) \mapsto S$$

and

$$(f, \phi) \mapsto f.$$

The category $\underline{A}$ should be thought of as the category of "representations" of the abelian tensor category A with values in $\mathcal{B}$.

Lemma 2.1. *The functor $p : \underline{A} \to \mathfrak{S}$ defined above is a category fibred in groupoids.*

Proof. Given any morphism $f : S' \to S$ in $\mathfrak{S}$ and $(S, F) \in \underline{A}$, the object $(S', f^* \circ F)$ in $\underline{A}$ admits a morphism

$$(f, \mathrm{Id}_{f^* \circ F}) : (S', f^* \circ F) \to (S, F).$$

This is a lifting of f.

Now suppose

$$(f, \phi_1) : (S', F_1) \to (S, F)$$

is another lifting of f. Then we have a commutative diagram

$$\begin{array}{ccc} (S', f^*\circ F) & & \\ \uparrow{\scriptstyle (\mathrm{Id}_{S'},\phi_1)} & \searrow^{(f,\mathrm{Id}_{f^*\circ F})} & \\ & & (S,F) \\ & \nearrow_{(f,\phi_1)} & \\ (S', F_1) & & \end{array}$$

where the dashed arrow is the unique morphism making the diagram commutative and has image under p equal to $\mathrm{Id}_{S'}$. □

2.2.3. *Remark.* We can define a larger category $\underline{A}^+$ with the same objects as $\underline{A}$ but morphisms (f,ϕ) where ϕ is a symmetric monoidal natural *transformation* between monoidal functors. Essentially the same proof above shows that $\underline{A}^+ \to \mathfrak{S}$ is a fibred category: The morphism $(f, \mathrm{Id}_{f^*\circ F})$ is easily seen to be strongly cartesian and is a lifting of morphism $f : S' \to S$ with a given target (S,F) in $\underline{A}^+_S$.

2.2.4. Suppose we have a morphism between categories fibred in abelian tensor categories:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{q} & \mathcal{B}' \\ \downarrow & & \downarrow \\ \mathfrak{S} & \xrightarrow{q_0} & \mathfrak{S}'. \end{array}$$

Let A be an abelian tensor category, then we have an induced morphism

$$q_* : \underline{A}(\mathcal{B}) \to \underline{A}(\mathcal{B}')$$

between categories fibred in groupoids over the functor q_0 defined by sending

$$q_* : (S,F) \mapsto (q_0(S), q_S \circ F),$$

where $q_S : \mathcal{B}_S \to \mathcal{B}'_{q_0(S)}$ is the restriction of q to the fibre category $\mathcal{B}_S$.

A special case of this is when $\mathcal{B}$ is the fibre product $\mathfrak{S} \times_{\mathfrak{S}'} \mathcal{B}' = q_0^{-1}\mathcal{B}'$, then we also have

$$\underline{A}(\mathcal{B}) \cong \mathfrak{S} \times_{\mathfrak{S}'} \underline{A}(\mathcal{B}') = q_0^{-1}\underline{A}(\mathcal{B}').$$

2.2.5. Fix a category $\mathcal{B} \to \mathfrak{S}$ fibred in abelian tensor categories. If $g : A \to A_1$ is a tensor functor between abelian tensor categories, then we have an induced morphism between categories fibred in groupoids

$$g^* : \underline{A_1} \longrightarrow \underline{A}.$$

More explicitly, g^* sends $(S, F_1) \mapsto (S, F_1 \circ g)$, and if $(f,\phi) : (S', F_1') \to (S, F_1)$ is a morphism in $\underline{A}$ then g^* sends $(f,\phi) \mapsto (f, g^*\phi)$ where $g^*\phi : F_1' \circ g \to f^* \circ F_1 \circ g$ is given by

$$(g^*\phi)_a = \phi_{g(a)} : F_1' g(a) \longrightarrow f^* F_1 g(a)$$

for $a \in A$.

2.2.6. Let k be a *perfect* field and let A_1 and A_2 be two abelian k-*linear* tensor categories with unit objects u_1 and u_2, and tensor products $\otimes_1$ and $\otimes_2$ respectively. Assume moreover that every object in A_i, $i = 1, 2$ is of finite length and every Hom space has finite dimension over k.

Then by [Del90, 5.13(i)], the tensor product $A = A_1 \otimes A_2$ exists and is a k-linear abelian category satisfying the same finiteness conditions above.

This category admits a functor

$$\boxtimes : A_1 \times A_2 \longrightarrow A = A_1 \otimes A_2$$

that is exact in each variable and satisfies

$$\mathrm{Hom}_A(a_1 \boxtimes a_2, b_1 \boxtimes b_2) \cong \mathrm{Hom}_{A_1}(a_1, b_1) \otimes_\Lambda \mathrm{Hom}_{A_2}(a_2, b_2)$$

for every $a_i, b_i \in A_i$.

By [Del90, 5.16] the category A is moreover an abelian *tensor* category with a tensor product $\otimes$ right exact in each variable and satisfying

$$(a_1 \boxtimes a_2) \otimes (b_1 \boxtimes b_2) \cong (a_1 \otimes_1 b_1) \boxtimes (a_2 \otimes_2 b_2). \tag{2.1}$$

Its unit object is $u = u_1 \boxtimes u_2$.

In particular we have tensor functors $q_i : A_i \to A$ defined by $q_1 : a_1 \mapsto a_1 \boxtimes u_2$ and $q_2 : a_2 \mapsto u_1 \boxtimes a_2$. Moreover (2.1) implies

$$a_1 \boxtimes a_2 \cong q_1(a_1) \otimes q_2(a_2) \tag{2.2}$$

in A.

Lemma 2.2. *Let k be a perfect field, and let A_1 and A_2 be abelian k-linear tensor categories satisfying the finiteness conditions above, and let $A = A_1 \otimes A_2$. Let $\mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories. Then we have an isomorphism of categories fibred in groupoids over $\mathfrak{S}$:*

$$Q : \underline{A} \longrightarrow \underline{A_1} \times_{\mathfrak{S}} \underline{A_2}.$$

Proof. The functor Q is given by the universal property of the fibre product $\underline{A_1} \times_{\mathfrak{S}} \underline{A_2}$ along with the functors $q_1^* : \underline{A} \to \underline{A_1}$ and $q_2^* : \underline{A} \to \underline{A_2}$.

More precisely,

$$Q : (S, F) \mapsto ((S, F_1), (S, F_2), \mathrm{Id}_S)$$

where $F_i : A_i \to \mathcal{B}_S$ sends $a_i \mapsto F(q_i(a_i))$. It is straightforward to show that F_i are tensor functors.

Let $(f, \phi) : (S', F') \to (S, F)$ be a morphism in $\underline{A}$ then Q sends

$$(f, \phi) \mapsto ((f, \phi_1), (f, \phi_2))$$

where $\phi_i : F_i' \to f^* \circ F_i$ is give by

$$\phi_{i,a_i} = \phi_{q_i(a_i)} : F_i'(a_i) = F(q_i(a_i)) \longrightarrow f^* F(q_i(a_i)) = f^* F_i(a_i).$$

Since ϕ is a *symmetric monoidal* natural isomorphism, so are ϕ_i.

We define a quasi-inverse Q' as follows: Let $((S, G_1), (S', G_2), g)$ be an object in $\underline{A_1} \times_{\mathfrak{S}} \underline{A_2}$ over S where $g : S \to S'$ is an *isomorphism*. Then Q' sends it to $(S, G) \in \underline{A}$ where

$$G(a_1 \boxtimes a_2) = G_1(a_1) \otimes_S g^* G_2(a_2) \in \mathcal{B}_S.$$

Then we have

$$Q' \circ Q : (S, F) \mapsto (S, F')$$

where

$$F'(a_1 \boxtimes a_2) = F_1(a_1) \otimes_S F_2(a_2) = F(q_1(a_1)) \otimes_S F(q_2(a_2)) \cong F(a_1 \boxtimes a_2)$$

by (2.2) since F is a tensor functor. This implies that $Q' \circ Q \cong \mathrm{Id}$.

Conversely, we have

$$Q \circ Q' : ((S, G_1), (S', G_2), g) \mapsto ((S, G_1'), (S, G_2'), \mathrm{Id}_S)$$

where

$$\phi_{a_1} : G_1'(a_1) \cong G_1(a_1)$$

and

$$\psi_{a_2} : G_2'(a_2) \cong g^* G_2(a_2).$$

These isomorphisms are natural isomorphisms.

It is straightforward to verify that we have an isomorphism

$$((\mathrm{Id}_S, \phi), (g, \psi)) : ((S, G_1'), (S, G_2'), \mathrm{Id}_S) \xrightarrow{\cong} ((S, G_1), (S', G_2), g).$$

This shows $Q \circ Q' \cong \mathrm{Id}$ and so we are done. □

2.2.7. The basic example of the tensor product of two abelian categories is as follows. Let R_1 and R_2 be two (not necessarily commutative) k-algebras which are right coherent; this means that every finitely generated right ideal is finitely presented.

Denote by A_i the abelian category of finitely presented right R_i-modules, and denote by A the abelian category of finitely presented right $(R_1 \otimes_k R_2)$-modules. Then by [Del90, 5.3] the tensor product over k

$$\boxtimes = \otimes_k : A_1 \times A_2 \to A$$

makes A the tensor product of A_1 with A_2. In this case we only need to assume k to be a commutative ring.

As an example:

Proposition 2.3. *Let $\mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories, and let H_1 and H_2 be finite groups. Let k be a field, then we have an isomorphism of categories fibred in groupoids over $\mathfrak{S}$:*

$$\underline{(H_1 \times H_2){-}\mathrm{rep}_k} \xrightarrow{\cong} \underline{H_1{-}\mathrm{rep}_k} \times_{\mathfrak{S}} \underline{H_2{-}\mathrm{rep}_k}.$$

2.3. **Stack and descent.**

2.3.1. Now let $\mathfrak{S}$ be a site. Let $\pi : \mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories.

Let S be an object in $\mathfrak{S}$ and let $\mathcal{U} = \{u_i : T_i \to S\}$ be a covering. Denote by T_{ij} the product $T_i \times_S T_j$ and p_ℓ the ℓ-th projection from T_{ij}.

We have canonical natural isomorphisms

$$can_{ij} : p_1^* u_i^* \longrightarrow p_2^* u_j^*$$

between functors from $\mathcal{B}_S$ to $\mathcal{B}_{T_{ij}}$, and this gives a functor δ from the category $\mathcal{B}_S$ to the category $DD(\mathcal{U})$ of descent data with respect to $\mathcal{U}$. More precisely, we have

$$\delta : x \mapsto (u_i^*(x), can_{ij,x}) \in DD(\mathcal{U}).$$

Notice that the category $DD(\mathcal{U})$ is naturally an abelian tensor category, and the functor δ is a tensor functor.

Theorem 2.4. *Let $\pi : \mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories over a category $\mathfrak{S}$ with a given Grothendieck topology. Let A be an abelian tensor category.*

(i) If the functor δ is fully faithful for every $S \in \mathfrak{S}$ and covering $\mathcal{U}$ then $p : \underline{A} \to \mathfrak{S}$ is a prestack.
(ii) If the functor δ is fully faithful and essentially surjective for every $S \in \mathfrak{S}$ and covering $\mathcal{U}$, then $p : \underline{A} \to \mathfrak{S}$ is a stack.

This theorem will be proved in the next few paragraphs; its slogan is: "If $\mathcal{B}$ is a (pre)stack, then so is $\underline{A}$". We remark that the converse statements also hold.

The conditions hold in the cases of Example 2.1.2 (the descent of quasi-coherent sheaves in the fpqc topology) and Example 2.1.3 (see [Liu12]). On the other hand, in the case of finite groups Example 2.1.4 (with covering families given by collections of subgroups whose union is equal to G), only the conditions in (i) hold, and so we get only a prestack.

2.3.2. *Remarks.* The novelty of Theorem 2.4 is that we consider categories of *tensor* functors and *monoidal* natural isomorphisms between them. An analogous result for arbitrary cartesian functors is a very special case of [Gir71, II, Corollaire 2.1.5], which states that if $\pi_A : \mathcal{A} \to \mathfrak{S}$ and $\pi_B : \mathcal{B} \to \mathfrak{S}$ are fibred categories over a site $\mathfrak{S}$, and $\mathcal{B}$ is a (pre)stack, then the fibred category

$$\mathrm{CART}(\mathcal{A}, \mathcal{B})$$

over $\mathfrak{S}$ is also a (pre)stack.

Here the fibred category $\mathrm{CART}(\mathcal{A}, \mathcal{B})$ is defined so that the fibre category $\mathrm{CART}(\mathcal{A}, \mathcal{B})_S$ over $S \in \mathfrak{S}$ is the category

$$\mathbf{Cart}_{\mathfrak{S}_{/S}}(\mathcal{A}_{/S}, \mathcal{B}_{/S})$$

of cartesian functors between the fibred categories $\mathcal{A}_{/S}$ and $\mathcal{B}_{/S}$ over S. Here $\mathcal{A}_{/S}$ is the category whose objects are pairs (a, f) where $a \in \mathcal{A}$ and $f : \pi_A(a) \to S$ is a morphism in $\mathfrak{S}$.

To compare Giraud's theorem above with our situation, let A be a category, and let $\mathcal{A} = A \times \mathfrak{S}$. With the identity functor Id_A as the pull-back functor, $\mathcal{A}$ is a fibred category over $\mathfrak{S}$. In this case we have $\mathcal{A}_{/S} \cong A \times (\mathfrak{S}_{/S})$.

Then we have an equivalence between fibre categories

$$\mathrm{Fun}(A, \mathcal{B}_S) \longrightarrow \mathbf{Cart}_{\mathfrak{S}_{/S}}(A \times (\mathfrak{S}_{/S}), \mathcal{B}_{/S})$$

given by

$$F \mapsto ((a,f) \mapsto (f^*F(a), f))$$

with quasi-inverse

$$\tilde{F} \mapsto \tilde{F}|_{A \times \mathrm{Id}_S}.$$

It seems possible to modify the proof of [Gir71, II, Corollaire 2.1.5] to give the proof of a more general version of Theorem 2.4. In the following we give a direct proof; by the remarks above, the main point is to show that tensor functors glue to tensor functors, and symmetric monoidal isomorphisms glue to symmetric monoidal isomorphisms.

2.3.3. Let $x = (S_0, F)$ and $y = (S_0, G)$ be objects in $\underline{A}$ over the same object $S_0 \in \mathfrak{S}$. If $f : S \to S_0$ is a morphism, then we have objects $f^*x = (S, f^* \circ F)$ and $f^*y = (S, f^* \circ G)$ over S, and the usual definition

$$\mathcal{I}som_{\underline{A}}(x,y)(S) = \{\phi : f^* \circ F \xrightarrow{\cong} f^* \circ G\}$$

gives a presheaf of sets on the category $\mathfrak{S}_{S_0}$ of objects over S_0. With x and y fixed and understood we simply denote this set by $\mathcal{I}(S)$. For every $\phi \in \mathcal{I}(S)$ and morphism $h : a \to b$ in A, we have a commutative diagram

$$\begin{array}{ccc} f^*F(a) & \xrightarrow{\phi_a} & f^*G(a) \\ {\scriptstyle f^*F(h)}\downarrow & & \downarrow{\scriptstyle f^*G(h)} \\ f^*F(b) & \xrightarrow{\phi_b} & f^*G(b) \end{array} \tag{2.3}$$

in $\mathcal{B}_S$ whose rows are isomorphisms.

Proof of Theorem 2.4(i). Let $\mathcal{U} = \{u_i : T_i \to S\}$ be a covering in $\mathfrak{S}$, then we have the functor

$$\delta : \mathcal{B}_S \longrightarrow DD(\mathcal{U}).$$

Let a be an object in A and let ϕ be in $\mathcal{I}(S)$; then ϕ_a is a morphism in $\mathcal{B}_S$ from $f^*F(a)$ to $f^*G(a)$. By the faithfulness assumption, we see that ϕ_a is determined by its pull-backs

$$u_i^*\phi_a : u_i^* f^* F(a) \longrightarrow u_i^* f^* G(a)$$

in $\mathcal{B}_{T_i}$ since $\{u_i^*\phi_a\} = \delta(\phi_a)$. Hence we conclude that the natural map

$$\mathcal{I}(S) \longrightarrow \prod_i \mathcal{I}(T_i)$$

induced by u_i is an injection.

Now consider the next natural maps

$$\prod_i \mathcal{I}(T_i) \rightrightarrows \prod_{i,j} \mathcal{I}(T_{ij})$$

induced by the two projections p_1, p_2 from T_{ij}. Suppose

$$\phi_i' : u_i^* \circ f^* \circ F \longrightarrow u_i^* \circ f^* \circ G$$

are in $\prod \mathcal{I}(T_i)$ with the same image under the two arrows into $\prod \mathcal{I}(T_{ij})$. To be precise, this means that ϕ_i' are monoidal natural isomorphisms such that the diagram of functors from A into $\mathcal{B}_{T_{ij}}$:

$$\begin{array}{ccc} p_1^*u_i^*f^*F & \xrightarrow{p_1^*\phi_i'} & p_1^*u_i^*f^*G \\ {\scriptstyle can_{ij}}\downarrow & & \downarrow{\scriptstyle can_{ij}} \\ p_2^*u_j^*f^*F & \xrightarrow[p_2^*\phi_j']{} & p_2^*u_j^*f^*G \end{array}$$

is commutative. (Recall that can_{ij} are the canonical natural isomorphism between $p_1^*u_i^*$ and $p_2^*u_j^*$.)

Then for any $a \in A$ we get a isomorphisms

$$\phi'_{i,a} : u_i^*f^*F(a) \longrightarrow u_i^*f^*G(a),$$

and the commutative diagram above means precisely that $\{\phi'_{i,a}\}$ is a morphism in $DD(\mathcal{U})$ from $(u_i^*f^*F(a), can_{ij,f^*F(a)})$ to $(u_i^*f^*G(a), can_{ij,f^*G(a)})$.

By the fullness assumption we get a (unique) isomorphism ϕ_a from $f^*F(a)$ to $f^*G(a)$ such that $u_i^*\phi_a = \phi'_{i,a}$ for every i.

The association $a \mapsto \phi_a$ is a natural transformation: To this end we need to show that the diagram (2.3) is commutative for every morphism $h : a \to b$ in A. But this diagram is commutative when applied with u_i^* since ϕ_i' is a natural transformation, for every i, hence we conclude that ϕ is a natural transformation by the faithfulness of δ.

Finally, we need to verify that ϕ is a *symmetric monoidal* natural isomorphism. We verify that it is monoidal, while its being symmetric can be shown in the same way. That is, we need to verify that the following diagrams

$$\begin{array}{ccc} f^*F(a\otimes b) & \xrightarrow{\phi_{a\otimes b}} & f^*G(a\otimes b) \\ {\scriptstyle f^*\sigma_{a,b}}\downarrow & & \downarrow{\scriptstyle f^*\tau_{a,b}} \\ f^*F(a)\otimes_S f^*F(b) & \xrightarrow[\phi_a\otimes_S\phi_b]{} & f^*G(a)\otimes_S f^*F(b) \end{array}$$

and

$$\begin{array}{ccc} u_S & = & u_S \\ {\scriptstyle f^*\sigma_0}\downarrow & & \downarrow{\scriptstyle f^*\tau_0} \\ f^*F(u) & \xrightarrow[\phi_u]{} & f^*G(u) \end{array}$$

are commutative. But this follows once again from the faithfulness assumption and the fact that these diagrams are commutative when applied with u_i^* for every i, since ϕ_i' are monoidal transformations.

Hence we conclude that the sequence

$$\mathcal{I}(S) \to \prod_i \mathcal{I}(T_i) \rightrightarrows \prod_{i,j} \mathcal{I}(T_{ij})$$

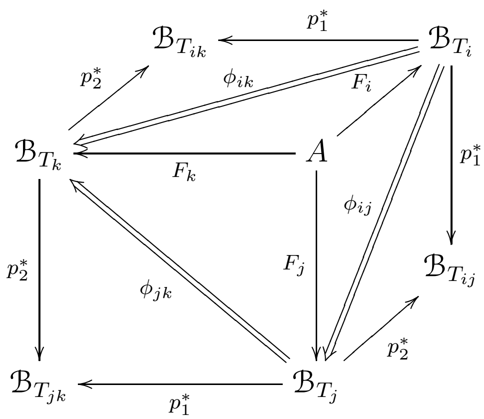
is exact, as required. □

2.3.4. Here we finish the proof of Theorem 2.4.

Proof of Theorem 2.4(ii). Let $\mathcal{U} = \{u_i : T_i \to S\}$ be a covering in $\mathfrak{S}$. Suppose we have objects (T_i, F_i) in $\underline{A}$ along with isomorphisms $\phi_{ij} : p_1^* \circ F_i \to p_2^* \circ F_j$ satisfying the cocycle condition

$$p_{13}^*\phi_{ik} = (p_{23}^*\phi_{jk}) \circ (p_{12}^*\phi_{ij})$$

for every i, j, k, where $p_{\ell,\ell'}$ are projections from $T_{ijk} = T_i \times_S T_j \times_S T_k$.



Then for every object a in A, we have isomorphisms $\phi_{ij,a} : p_1^*F_i(a) \to p_2^*F_j(a)$ satisfying the cocyle condition. That is, $(F_i(a), \phi_{ij,a})$ is an object in $DD(\mathcal{U})$.

Hence by the essential surjectivity assumption we have an object denoted suggestively as $F(a)$ in $\mathcal{B}_S$ along with isomorphisms $\lambda_i(a) : u_i^*F(a) \to F_i(a)$ satisfying

$$\begin{array}{ccc} p_1^*u_i^*F(a) & \xrightarrow{p_1^*\lambda_i(a)} & p_1^*F_i(a) \\ {\scriptstyle can_{ij,F(a)}}\downarrow & & \downarrow{\scriptstyle \phi_{ij,a}} \\ p_2^*u_j^*F(a) & \xrightarrow{p_2^*\lambda_j(a)} & p_2^*F_j(a). \end{array} \tag{2.4}$$

If $h : a \to b$ is a morphism in A, then we define $F(h) : F(a) \to F(b)$ to be the unique morphism pulling-back via u_i (for each i) to the composition

$$u_i^*F(a) \xrightarrow{\lambda_i(a)} F_i(a) \xrightarrow{F_i(h)} F_i(b) \xrightarrow{\lambda_i(b)^{-1}} u_i^*F(b).$$

To be more careful, we need to show that the composition above indeed is a morphism in $DD(\mathcal{U})$. That is, we need to verify that the following diagram is commutative:

$$\begin{array}{ccccccc} p_1^*u_i^*F(a) & \xrightarrow{p_1^*\lambda_i(a)} & p_1^*F_i(a) & \xrightarrow{p_1^*F_i(h)} & p_1^*F_i(b) & \xrightarrow{p_1^*\lambda_i(b)^{-1}} & p_1^*u_i^*F(b) \\ {\scriptstyle can_{ij,F(a)}}\downarrow & & \downarrow{\scriptstyle \phi_{ij,a}} & & \downarrow{\scriptstyle \phi_{ij,b}} & & \downarrow{\scriptstyle can_{ij,F(b)}} \\ p_2^*u_j^*F(a) & \xrightarrow[p_2^*\lambda_j(a)]{} & p_2^*F_j(a) & \xrightarrow[p_2^*F_j(h)]{} & p_2^*F_j(b) & \xrightarrow[p_2^*\lambda_j(b)^{-1}]{} & p_2^*u_j^*F(b). \end{array} \tag{2.5}$$

The commutativity of the first and the last squares is the compatibility condition (2.4), and the commutativity of the second square follows from the fact that ϕ_{ij} is a natural transformation.

The association $a \mapsto F(a)$ from A to $\mathcal{B}_S$ is then a functor: the fact that it respects composition of morphisms follows from the faithfulness assumption and that F_i are functors. Moreover, using the faithfulness and fullness assumptions it is straightforward to show that F preserves colimits since the F_i do.

It remains to show that F is a symmetric monoidal functor, in particular we need to define for every $a, b \in A$ an isomorphism

$$\sigma_{a,b} : F(a \otimes b) \longrightarrow F(a) \otimes_S F(b)$$

in $\mathcal{B}_S$ as well as

$$\sigma_0 : u_S \longrightarrow F(u).$$

We have isomorphisms in $\mathcal{B}_{T_i}$:

$$u_i^* F(a \otimes b) \xrightarrow{\lambda_i(a \otimes b)} F_i(a \otimes b) \xrightarrow{\sigma_{i,a,b}} F_i(a) \otimes_{T_i} F_i(b) \longrightarrow u_i^* F(a) \otimes_{T_i} u_i^* F(b),$$

where $\sigma_{i,a,b}$ is the isomorphism for the tensor functor F_i, and the last arrow is the inverse of $\lambda_i(a) \otimes_{T_i} \lambda_i(b)$.

By considering a diagram similar to (2.5) we see that there is a unique isomorphism

$$\sigma_{a,b} : F(a \otimes b) \to F(a) \otimes_S F(b)$$

for every $a, b \in A$ such that $u_i^* \sigma_{a,b} = \sigma_{i,a,b}$. (Here we used the fact that the isomorphisms ϕ_{ij} are *monoidal* natural isomorphisms.)

The isomorphisms $\sigma_{a,b}$ are required to satisfy coherence conditions in the form of commutativity diagrams. These follow from the commutativity of diagrams after pulling-back via u_i using the faithfulness assumption. The isomorphism σ_0 can be constructed in the same way. Details are omitted.

Hence we conclude that (S, F) is an object in $\underline{A}$ over S pulling-back to F_i via u_i^*, as required. □

2.4. **Sheaves of modules.**

2.4.1. In this section we work under the conditions of Theorem 2.4. Namely, $\mathcal{B} \to \mathfrak{S}$ is a category fibred in abelian tensor categories over a site $\mathfrak{S}$ where all the functors $\delta : \mathcal{B}_S \to DD(\mathcal{U})$ for all covering $\mathcal{U}$ of $S \in \mathfrak{S}$ are equivalences.

In this case we define a sheaf of rings on $\mathfrak{S}$ as follows. Recall that every fibre category $\mathcal{B}_S$ has a unit object u_S. Let $\mathcal{O}$ be the presheaf on $\mathfrak{S}$ given by

$$S \mapsto \mathcal{O}(S) := \mathrm{End}_{\mathcal{B}_S}(u_S),$$

where if $f : S' \to S$ is a morphism in $\mathfrak{S}$ then the restriction map $\mathcal{O}(S) \to \mathcal{O}(S')$ is induced by the functor $f^* : \mathcal{B}_S \to \mathcal{B}_{S'}$.

The assumption that the descent functors δ are fully faithful implies that this is indeed a sheaf. This is a sheaf of *commutative* rings if we impose the additional assumption on the categories $\mathcal{B}_S$ that the two isomorphisms $u_S \otimes_S x \cong x$ and $x \otimes_S u_S \cong x$ for any $x \in \mathcal{B}_S$ give the same isomorphism $u_S \otimes_S u_S \cong u_S$ when x is taken to be u_S [Bal02, Lemma 9.6].

2.4.2. Let A be an abelian tensor category, then we have a stack $p : \underline{A} \to \mathfrak{S}$ by Theorem 2.4. Composing with the functor p defines a ring-valued *presheaf* on $\underline{A}$ which we will denote by $\mathcal{O}_{\underline{A}}$:

$$(S, F) \mapsto \mathrm{End}_{\mathcal{B}_S}(u_S).$$

There is an inherited Grothendieck topology on the category $\underline{A}$: A family of morphisms

$$\mathcal{V} = \{v_i : y_i \to x\}$$

with a fixed target $x \in \underline{A}$ is by definition a covering if $p(\mathcal{V}) = \{p(v_i) : p(y_i) \to p(x)\}$ is a covering in $\mathfrak{S}$. In this topology the presheaf defined above is a sheaf on $\underline{A}$.

We denote by $\mathcal{O}_{\underline{A}}-\mathrm{Mod}$ the category of sheaves of right $\mathcal{O}_{\underline{A}}$-modules on $\underline{A}$. With $\mathcal{O}_{\underline{A}}$ on $\underline{A}$ as the *structure sheaf*, we have the category $\mathrm{QCoh}(\underline{A})$ of quasi-coherent sheaves on $\underline{A}$.

2.4.3. Now we define a natural covariant functor

$$\mathcal{F} : A \longrightarrow \mathcal{O}_{\underline{A}}-\mathrm{Mod}.$$

For every $a \in A$ let $\mathcal{F}_a$ be the contravariant functor on $\underline{A}$ defined by

$$\mathcal{F}_a : (S, F) \mapsto \mathrm{Hom}_{\mathcal{B}_S}(u_S, F(a)).$$

This is an object in $\mathcal{O}_{\underline{A}}-\mathrm{Mod}$. The functor $a \mapsto \mathcal{F}_a$ is additive but probably not left or right exact. It is not clear under what conditions on the category A is the functor $a \mapsto \mathcal{F}_a$ full, faithful, or a tensor functor.

2.4.4. We say an object $a \in A$ is *locally of finite presentation (with respect to* $\pi : \mathcal{B} \to \mathfrak{S}$*)* if for every $x = (S, F) \in \underline{A}$ there exists a covering $\mathcal{V} = \{v_i : y_i \to x\}$ where $y_i = (T_i, F_i)$ and $v_i = (u_i, \phi_i)$ such that for every i the object $F_i(a)$ in $\mathcal{B}_{T_i}$ admits a finite presentation

$$u_{T_i}^p \longrightarrow u_{T_i}^q \longrightarrow F_i(a) \longrightarrow 0. \tag{2.6}$$

More generally, we say a is *locally finite* if such a covering exists with a surjection

$$u_{T_i}^q \longrightarrow F_i(a) \longrightarrow 0.$$

Lemma 2.5. *Suppose for every $S \in \mathfrak{S}$, that u_S is a projective object in $\mathcal{B}_S$.*

(i) The functor $a \mapsto \mathcal{F}_a$ is right exact.
(ii) Let $a \in A$ be locally of finite presentation, then the sheaf $\mathcal{F}_a$ on $\underline{A}$ is locally of finite presentation.

Proof. (i) Clear.

(ii) Choose the covering $\mathcal{V}$ for a as in the definition above. Applying the *exact* functor $\mathrm{Hom}_{\mathcal{B}_{T_i}}(u_{T_i}, -)$ to the sequence (2.6) gives the result. □

2.4.5. *Remarks.* We should point out that we could replace u_S in the constructions above with any system of objects $v_S \in \mathcal{B}_S$ admitting a morphism $f^*v_S \to v_{S'}$ in $\mathcal{B}_{S'}$ for every morphism $f : S' \to S$ in $\mathfrak{S}$. This results in a different "structure sheaf", which may be more useful. For instance, in the case when $\mathfrak{S}$ is the category of quivers, the vertex-wise tensor unit object carries less information than the path algebra.

Slightly more generally, consider a bi-functor $A \times A \to \mathcal{O}_{\underline{A}}{-}\mathrm{Mod}$ defined by

$$(a, b) \mapsto \mathcal{F}_{a,b}$$

where

$$\mathcal{F}_{a,b}((S, F)) = \mathrm{Hom}_{\mathcal{B}_S}(F(a), F(b)).$$

Then the sheaves $\mathcal{F}_{a,b}$ on $\underline{A}$ have the additional structure of composition

$$\circ : \mathcal{F}_{a,b} \times \mathcal{F}_{b,c} \to \mathcal{F}_{a,c}.$$

In particular every $\mathcal{F}_{a,a}$ is a sheaf of rings over which $\mathcal{F}_{a,b}$ is a right module and $\mathcal{F}_{b,a}$ is a left module. Moreover, there is a natural morphism

$$\mathcal{F}_{a,b} \otimes_{\mathcal{O}_{\underline{A}}} \mathcal{F}_{c,d} \to \mathcal{F}_{a\otimes c, b\otimes d}$$

3. Example: Schemes

In this Section we show that our construction in Section 2 can be used to recover schemes from their category of quasi-coherent sheaves.

3.1. **Reconstruction of affine schemes.**

3.1.1. Fix a commutative ring Λ with identity, and all schemes considered in this section will be over Λ.

Let $\mathfrak{S}$ be the category of affine schemes (over Λ) with the étale topology, and $\mathcal{B} \to \mathfrak{S}$ as in Example 2.1.2; in particular for every affine scheme S we have $\mathcal{B}_S = \mathrm{QCoh}(S)$.

3.1.2. Let X be a scheme; denote by $\underline{X}$ the associated stack over $\mathfrak{S}$. Consider the abelian tensor category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves with its sheaf tensor product. Then we have a morphism of stacks

$$\underline{X} \xrightarrow{\alpha} \underline{\mathrm{QCoh}(X)},$$

sending $\alpha : (f : S \to X) \mapsto (S, f^*)$ and $\alpha : (g : S' \to S) \mapsto (g, \phi) \in \mathrm{Hom}((S', f'^*), (S, f^*))$ where ϕ denotes the natural isomorphism $f'^* \to g^*f^*$.

3.1.3. *Example.* Suppose $\Lambda = k$ is a field. Let A be the abelian tensor category Vect_k of (possibly infinite dimensional) vector spaces over k with the usual tensor product. Then we have $A = \mathrm{QCoh}(\mathrm{Spec}(k))$ and a stack morphism

$$\underline{\mathrm{Spec}(k)} \longrightarrow \underline{A}.$$

If $(S, F) \in \underline{A}$ then we must have $F(k) \cong \mathcal{O}_S$, and so $\underline{A}_S$ has only one object (up to isomorphism) for every $S \in \mathfrak{S}$, since every object in A is a direct sum of copies of k, and F preserves direct sums.

Now suppose (Id_S, ϕ) is an automorphism of (S, F) in $\underline{A}$; in particular $\phi : F \to F$ is a *monoidal* natural isomorphism. Then ϕ_k is an automorphism of $\mathcal{O}_S$. But then we have

$$\phi_k = \phi_{k\otimes k} = \phi_k \otimes \phi_k = (\phi_k)^2$$

as elements in $\mathrm{Aut}(\mathcal{O}_S)$, hence we must have $\phi_k = \mathrm{Id}_{\mathcal{O}_S}$.

Similarly we see that there is always a unique lifting $(f, \phi) : (S', F') \to (S, F)$ in $\underline{A}$ for every scheme morphism $f : S' \to S$. And we conclude that the morphism

$$\alpha : \underline{\mathrm{Spec}(k)} \longrightarrow \underline{A}$$

is an isomorphism.

3.1.4. More generally, we have

Theorem 3.1. *Let X be an affine scheme, then the morphism $\alpha : \underline{X} \to \underline{\mathrm{QCoh}(X)}$ is an isomorphism.*

For any tensor functor $F : A \to B$ between abelian tensor categories with unit objects u_A and u_B respectively, we denote by $F(1) : \mathrm{End}(u_A) \to \mathrm{End}(F(u_A)) \cong \mathrm{End}(u_B)$ the homomorphism given by F; here the last two endomorphism rings are identified using the isomorphism $\sigma_0 : u_B \to F(u_A)$.

Lemma 3.2. *Let X be an affine scheme, and let S be a scheme. If $\psi : \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_S)$ is a ring homomorphism inducing a scheme morphism $f : S \to X$, then the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{End}(\mathcal{O}_X) & \xrightarrow{f^*(1)} & \mathrm{End}(\mathcal{O}_S) \\ \Gamma\downarrow & & \downarrow\Gamma \\ \Gamma(\mathcal{O}_X) & \xrightarrow[\psi]{} & \Gamma(\mathcal{O}_S). \end{array}$$

Here $f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ is the pull-back functor, and the vertical arrows are natural isomorphisms.

Lemma 3.3. *Let X be an affine scheme, and let S be a scheme. Let $F, G : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ be two tensor functors. If $F(1) = G(1)$ then $F \cong G$ via a symmetric monoidal natural isomorphism.*

Proof. Let $a \in \mathrm{QCoh}(X)$, then it admits a (possibly infinite) presentation

$$\mathcal{O}_X^p \xrightarrow{m} \mathcal{O}_X^q \longrightarrow a \longrightarrow 0,$$

where m is a matrix with entries in $\mathrm{End}(\mathcal{O}_X)$. Applying the right exact functors F and G we get exact sequences

$$\begin{array}{ccccccc} F(\mathcal{O}_X^p) & \xrightarrow{F(1)(m)} & F(\mathcal{O}_X^q) & \longrightarrow & F(a) & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & & & \\ G(\mathcal{O}_X^p) & \xrightarrow[G(1)(m)]{} & G(\mathcal{O}_X^q) & \longrightarrow & G(a) & \longrightarrow & 0, \end{array}$$

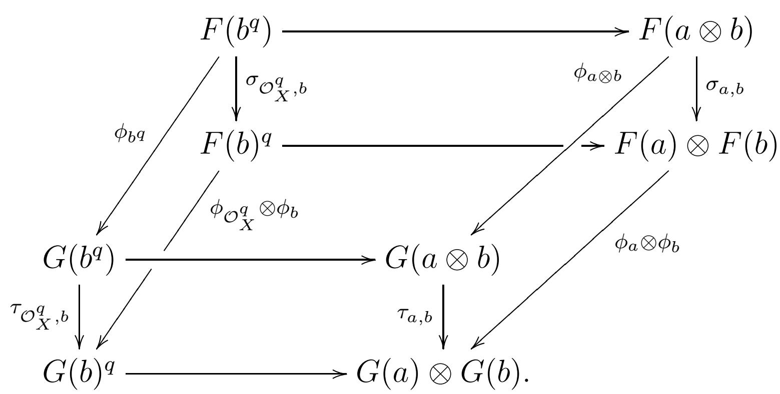
where $F(1)(m)$ and $G(1)(m)$ are respectively the matrix with $F(1)$ and $G(1)$ applied to the entries of m, and the vertical isomorphisms are given by

$$F(\mathcal{O}_X) \xrightarrow{\sigma_0^{-1}} \mathcal{O}_S \xrightarrow{\tau_0} G(\mathcal{O}_X).$$

By the assumption $F(1) = G(1)$ the square in this diagram commutes, and so we have an isomorphism $\phi_a : F(a) \to G(a)$.

It is straightforward to verify that this is independent of the choice of the presentation of a and moreover gives a natural isomorphism $\phi : F \to G$. It remains to show that it is a *symmetric monoidal* natural transformation. We show that it is monoidal, leaving the symmetry to the reader.

So let $b \in \mathrm{QCoh}(X)$. Consider the following diagram:



(Here b^q stands for $\mathcal{O}_X^q \otimes b$, $F(b)^q$ stands for $\mathcal{O}_S^q \otimes F(b)$, etc.)

The commutativity of the right side is the condition we need to verify.

The top side is obtained by first applying $- \otimes b$ to the surjection $\mathcal{O}_X^q \to a$, and then the functors F and G; the bottom side is obtained by applying these functor in the other order. These two sides are commutative since ϕ is a natural transformation.

The front side and the back side are commutative by the compatibility conditions on the isomorphisms σ and τ. Therefore we are reduced to prove the commutativity of the left side.

By considering a presentation $\mathcal{O}_X^s \to \mathcal{O}_X^t \to b \to 0$ of b, this is in turn reduced to the commutativity of the following diagram:

$$\begin{array}{ccc}
F(\mathcal{O}_X^t \otimes \mathcal{O}_X^q) & \xrightarrow{\phi_{\mathcal{O}_X^t \otimes \mathcal{O}_X^q}} & G(\mathcal{O}_X^t \otimes \mathcal{O}_X^q) \\
\Big\downarrow{\scriptstyle \sigma_{\mathcal{O}_X^t, \mathcal{O}_X^q}} & & \Big\downarrow{\scriptstyle \tau_{\mathcal{O}_X^t, \mathcal{O}_X^q}} \\
F(\mathcal{O}_X^t) \otimes F(\mathcal{O}_X^q) & \xrightarrow[\phi_{\mathcal{O}_X^t} \otimes \phi_{\mathcal{O}_X^q}]{} & G(\mathcal{O}_X^t) \otimes G(\mathcal{O}_X^q).
\end{array}$$

It suffices to show that this diagram is commutative in the special case $q = t = 1$. Using the left diagram of (4) in [ML98, page 256] (applied to both F and G) we are reduced to showing the commutativity of the following diagram:

$$\begin{array}{ccc} F(\mathcal{O}_X)\otimes\mathcal{O}_S & \xrightarrow{\phi_{\mathcal{O}_X}\otimes\mathrm{Id}} & G(\mathcal{O}_X)\otimes\mathcal{O}_S \\ \downarrow{\scriptstyle \mathrm{Id}\otimes\sigma_0} & & \downarrow{\scriptstyle \mathrm{Id}\otimes\tau_0} \\ F(\mathcal{O}_X)\otimes F(\mathcal{O}_X) & \xrightarrow{\phi_{\mathcal{O}_X}\otimes\phi_{\mathcal{O}_X}} & G(\mathcal{O}_X)\otimes G(\mathcal{O}_X). \end{array}$$

But by construction we have $\phi_{\mathcal{O}_X} = \tau_0 \circ \sigma_0^{-1}$, and so we are done. ☐

Proof of Theorem 3.1. We define a morphism

$$\alpha' : \underline{\mathrm{QCoh}(X)} \longrightarrow \underline{X}$$

by sending (S, F) to $f : S \to X$ where $f = \mathrm{Spec}(F(1))$.

We first show that $\alpha' \circ \alpha \cong \mathrm{Id}$. So suppose $f : S \to X$ is an affine scheme over X. Let $\psi = f_X^{\#} : \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_S)$, then

$$\mathrm{Spec}(f^*(1)) = \mathrm{Spec}(\psi) = f,$$

where we identified the rings using Γ in Lemma 3.2.

Conversely, we need to show that $\alpha \circ \alpha' \cong \mathrm{Id}$. So let (S, F) be an object in $\underline{\mathrm{QCoh}(X)}$ over S. Let $\psi = \Gamma(F(1)) : \Gamma(\mathcal{O}_X) \to \Gamma(\mathcal{O}_S)$ as in Lemma 3.2, then we have

$$F(1) = \mathrm{Spec}(\psi)^*(1),$$

hence we conclude $F \cong \mathrm{Spec}(\psi)^*$ by Lemma 3.3. ☐

3.1.5. *Remarks.* Recall from 2.4.3 that there is a functor from $\mathrm{QCoh}(X)$ into the category $\mathcal{O}_X-\mathrm{Mod}$ on the stack $\underline{\mathrm{QCoh}(X)} \cong X$. It is easily seen to be isomorphic to the inclusion functor.

The proof above also shows that we have an isomorphism $X \to \underline{A}$ whenever $A \subset \mathrm{QCoh}(X)$ is an abelian tensor subcategory; any such A must contain the unit object $\mathcal{O}_X$ and hence every finitely presented objects in $\mathrm{QCoh}(X)$. For example, when X is a noetherian affine scheme, we may take A to be $\mathrm{Coh}(X)$. (Here the noetherian assumption is used to guarantee that the tensor product of two coherent sheaves is still coherent.)

3.1.6. *Example.* Here we show that it is crucial to use the sheaf tensor product in the proof of Theorem 3.1.

Let k be a field of characteristic not equal to 2, and let $X = \mathrm{Spec}(k[t]/(t^2-1))$ be the affine scheme of two reduced points. Let A be the abelian category $\mathrm{Coh}(X)$ on which we have the sheaf tensor product $\otimes_X$, then we have

$$\underline{X} \cong \underline{(A, \otimes_X)}$$

as stacks, as remarked in 3.1.5.

There is, however, a different tensor product on the category A by identifying the ring $k[t]/(t^2-1)$ with the group algebra kG, where $G = \{1, t\}$ is the cyclic group of order two. This realizes A as the category of finite dimensional G-representations; denote the representation tensor product on A by $\otimes_G$. Notice that this is indeed a different tensor product from $\otimes_X$ since they have different unit objects.

Every object V in A then decomposes as $V_+ \oplus V_-$ where V_+ is the trivial representation and t acts as the multiplication by -1 on V_-.

Denote by k the one dimensional trivial G-representation, and let M be the one dimensional G-representation on which t acts as the multiplication by -1. Then any tensor functor F from $(A, \otimes_G)$ into an abelian tensor category B is determined by the object $F(M)$, which must satisfy

$$F(M) \otimes F(M) \cong F(k) \cong u_B.$$

And conversely, any object in B whose tensor square is isomorphic to u_B gives rise to such a functor. Therefore we conclude that, if S is an affine scheme, for instance, then the fibre category $\underline{(A, \otimes_G)}_S$ is isomorphic to the group of order two elements in $\mathrm{Pic}(S)$, namely,

$$\underline{(A, \otimes_G)}_S \cong \mathrm{Pic}(S)[2].$$

In particular we have

$$\underline{(A, \otimes_X)} \not\cong \underline{(A, \otimes_G)}.$$

See also Section 4 for a class of examples which illustrates the dependence on tensor structures.

3.2. **Reconstruction of schemes.**

3.2.1. Now we generalize Theorem 3.1 to more general schemes.

Theorem 3.4. *Let X be a quasi-compact and separated scheme, then the comparison morphism $\alpha : \underline{X} \longrightarrow \underline{\mathrm{QCoh}(X)}$ is an isomorphism.*

The point is to construct a quasi-inverse

$$\alpha' : \underline{\mathrm{QCoh}(X)} \longrightarrow \underline{X},$$

which means that for every affine scheme S and a tensor functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$, we need to define a scheme morphism $f : S \to X$.

We must do this locally on S and X: The idea is to glue open affine subschemes, but *a priori* we do now even know if the stack $\underline{\mathrm{QCoh}(X)}$ is representable, and so, for example, if $U \subset X$ is an open affine subscheme, then it is not clear why the induced morphism

$$U \cong \underline{\mathrm{QCoh}(U)} \longrightarrow \underline{\mathrm{QCoh}(X)}$$

should be an *open immersion*.

So let $a_1, a_2, \ldots$ be finitely presented quasi-coherent sheaves on X. Then in particular each a_i has a closed support. Suppose that the intersection $\bigcap_i \mathrm{supp}(a_i)$ is empty on X. Then the complements of $\mathrm{supp}(a_i)$ form an open covering of X. Each $F(a_i)$ is finitely presented on S, and in particular each $\mathrm{supp}(F(a_i))$ is closed.

Lemma 3.5. *With notations and assumptions as above, we have $\bigcap_i \mathrm{supp}(F(a_i)) = \emptyset$ on S. In particular the complements of $\mathrm{supp}(F(a_i))$ form an open covering of S.*

Proof. Suppose on the contrary that the intersection is non-empty on S. Then there is a point $s \in S$ such that $s^*F(a_i)$ is non-zero in $\mathrm{Vect}_{k(s)}$ for every i. Note that $s^* \circ F : \mathrm{QCoh}(X) \to \mathrm{Vect}_{k(s)}$ is a tensor functor, and so by Lemma 3.6 below, there is a point $x \in X$ such that $x^*(a_i)$ is non-zero for every i. This means that the point x lies in the intersection of $\mathrm{supp}(a_i)$, a contradiction. $\square$

Lemma 3.6. *Let Let X be a quasi-compact and separated scheme, and let K be a field. If $G : \mathrm{QCoh}(X) \to \mathrm{Vect}_K$ is a tensor functor, then there is a K-point x on X such that G is isomorphic to x^* as tensor functors.*

Proof. We first reduce to the affine case. Let U be an open affine subscheme of X, and let $M \subset \mathrm{QCoh}(X)$ be the kernel of the restriction functor $-|_U : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U)$. That is, M is the full subcategory consisting of objects $b \in \mathrm{QCoh}(X)$ such that $b|_U \cong 0$; then we have

$$M = \bigcap_{u \in U} \ker(u^*).$$

We claim that there is an open affine subscheme U of X such that M is contained in $\ker(G)$. Suppose this were not the case. Then for every $x \in X$ there is an object $a_x \in \ker(x^*)$ not in $\ker(G)$. We may choose a_x to be of finite type: Indeed, a_x is a quasi-coherent sheaf and hence a direct limit of finite type subsheaves. All of these finite type subsheaves must be in $\ker(x^*)$ in order to have $x^*(a_x) = 0$, and at least one of these finite type subsheaves is not in $\ker(G)$ since otherwise we would have $G(a_x) \cong 0$, since G commutes with direct limits.

So for every $x \in X$ we choose and fix a sheaf a_x of finite type satisfying $a_x \in \ker(x^*)$ and $a_x \notin \ker(G)$. Since a_x is of finite type, its support $\mathrm{supp}(a_x)$ is closed in X; let $U_x = X - \mathrm{supp}(a_x)$. Then $\{U_x \mid x \in X\}$ form an open covering of X. Since X is quasi-compact we have a finite subcovering $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$. Let

$$a := \bigotimes_{i=1}^{n} a_{x_i}.$$

Then we have $x^*(a) = \bigotimes x^*(a_{x_i}) = 0$ since x lies in one of U_{x_i}, for every $x \in X$. This means that we must have $a \cong 0$ in $\mathrm{QCoh}(X)$. But on the other hand, we have $G(a) \neq 0$ since every $G(a_{x_i}) \neq 0$ in Vect_K, a contradiction.

Therefore there is an open affine subscheme U of X such that $M = \ker(-|_U)$ is contained in $\ker(G)$. Then we have a diagram

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{G} & \mathrm{Vect}_K \\ \downarrow & \nearrow_{\bar{G}} & \\ \mathrm{QCoh}(X)/M & & \\ \downarrow{\scriptstyle\cong} & & \\ \mathrm{QCoh}(U), & & \end{array}$$

where $\mathrm{QCoh}(X)/M$ is the localization [Gab62]. The lower vertical arrow is an equivalence by [Gab62, Chapter 3, proposition 5].

More precisely, the equivalence follows from the fact that the push-forward functor induced by the open immersion $U \to X$ sends quasi-coherent sheaves on the *affine* scheme U to quasi-coherent sheaves on X, and this functor is a section functor of the restriction functor. The fact that quasi-coherent sheaves are preserved under push-forward follows from our assumptions on the scheme X: Indeed, we need the open immersion to be quasi-compact and separated [Har77, Chapter 2, Proposition 5.8]. The separatedness follows from the fact that affine schemes are separated and

[Har77, Chapter 2, Corollary 4.6]; the open immersion $U \to X$ is quasi-compact since for any open affine subscheme Y of X, the intersection $Y \cap U$ is affine, and hence quasi-compact.

We then have a tensor functor $H : \mathrm{QCoh}(U) \to \mathrm{Vect}_K$ where U is an open affine subscheme of X. Thus we are reduced to the affine case.

The functor H induces a ring homomorphism

$$H(1) : \Gamma(U, \mathcal{O}_U) \cong \mathrm{End}_{\mathrm{QCoh}(U)}(\mathcal{O}_U) \longrightarrow \mathrm{End}_{\mathrm{Vect}_K}(K) \cong K.$$

This gives a point $u : \mathrm{Spec}(K) \to U$. Lemma 3.3 with $F = H$ and $G = u^*$ shows $H \cong u^*$. □

3.2.2. Suppose now we choose the objects $a_i \in \mathrm{QCoh}(X)$ with sufficiently large support so that each $U_i := X - \mathrm{supp}(a_i)$ is an open *affine* subscheme. The complement V_i of $\mathrm{supp}(F(a_i))$ in S is open but not necessarily affine. We need to define a scheme morphism $f_i : V_i \longrightarrow U_i$, which will glue to give $f : S \to X$.

Since U_i is affine, it suffices to give a ring homomorphism

$$f_i^{\#} : \Gamma(U_i, \mathcal{O}_{U_i}) \longrightarrow \Gamma(V_i, \mathcal{O}_{V_i}).$$

This can be constructed at the categorical level using the functor F and localization as follows.

Consider the restriction tensor functor $-|_{U_i} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(U_i)$; denote by $M_i = \ker(-|_{U_i})$ the subcategory of objects in $\mathrm{QCoh}(X)$ consisting of those objects b satisfying $b|_{U_i} \cong 0$ in $\mathrm{QCoh}(U_i)$; or in other words object b with $\mathrm{supp}(b) \subset \mathrm{supp}(a_i)$ as sets.

Lemma 3.7. *If $b \in K$ then $F(b)|_{V_i} \cong 0$ in $\mathrm{QCoh}(V_i)$.*

Proof. It suffices to prove the following statement: If $c, c' \in \mathrm{QCoh}(X)$ are such that $\mathrm{supp}(c') \subset \mathrm{supp}(c)$ as sets, then $\mathrm{supp}(F(c')) \subset \mathrm{supp}(F(c))$ as sets. To see that this is enough, take $c = a_i$ and $c' = b$ in the situation above, this statement then implies $\mathrm{supp}(F(b)) \subset \mathrm{supp}(F(a_i)) = S - V_i$.

To prove the statement above, let $s \in S$ be a point in $\mathrm{supp}(F(c'))$. Then $s^*F(c') \neq 0$. By Lemma 3.6 there is a point $x \in X$ such that $s^* \circ F \cong x^*$, hence $x^*(c') \neq 0$. In other words $x \in \mathrm{supp}(c') \subset \mathrm{supp}(c)$. Therefore $0 \neq x^*(c) \cong s^*F(c)$; that is, $s \in \mathrm{supp}(F(c))$. □

Then we have a diagram similar to the one in the proof of Lemma 3.6:

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{F} & \mathrm{QCoh}(S) \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X)/M_i & \overset{F_i}{\dashrightarrow} & \mathrm{QCoh}(V_i) \\ \downarrow \cong & \nearrow_{\tilde{F}_i} & \\ \mathrm{QCoh}(U_i). & & \end{array} \tag{3.1}$$

The induced functor F_i is a tensor functor, hence we have a ring homomorphism

$$\Gamma(U_i, \mathcal{O}_{U_i}) = \mathrm{End}(\mathcal{O}_{U_i}) \xrightarrow{\cong} \mathrm{End}_{\mathrm{QCoh}(X)/M_i}(\mathcal{O}_X) \xrightarrow{F_i} \mathrm{End}(\mathcal{O}_{V_i}) = \Gamma(V_i, \mathcal{O}_{V_i}),$$

as desired.

3.2.3. We show that the scheme morphisms $f_i : V_i \to U_i$ coincide on intersections $V_{ij} := V_i \cap V_j$. It then follows that there is a scheme morphism $f : S \to X$ for every given tensor functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$. This defines a morphism

$$\alpha' : \underline{\mathrm{QCoh}(X)} \longrightarrow \underline{X}.$$

Notice that $U_{ij} := U_i \cap U_j$ is equal to $X - \mathrm{supp}(a_i) \cup \mathrm{supp}(a_j) = X - \mathrm{supp}(a_i \oplus a_j)$, and $V_{ij} = S - \mathrm{supp}(F(a_i \oplus a_j))$. Since X is quasi-compact and separated, and both U_i and U_j are affine, U_{ij} is affine. So by the same construction above we have a scheme morphism $g : V_{ij} \to U_{ij}$, and it only remains to show that this is equal to the restriction of $f_i : V_i \to U_i$. This follows from the commutativity of the following diagram on the endomorphism rings of the identity objects:

$$\begin{array}{ccc}
\mathrm{QCoh}(X) & \xrightarrow{F} & \mathrm{QCoh}(S) \\
\downarrow & & \downarrow \\
\mathrm{QCoh}(X)/M_i & \dashrightarrow & \mathrm{QCoh}(V_i) \\
\downarrow \cong & & \Big\downarrow \\
\mathrm{QCoh}(U_i) & & \\
\downarrow & & \\
\mathrm{QCoh}(U_i)/M_{ij} & \dashrightarrow & \mathrm{QCoh}(V_{ij}) \\
\downarrow \cong & & \\
\mathrm{QCoh}(U_{ij}). & &
\end{array}$$

3.2.4. Now we can finish the

Proof of Theorem 3.4. It suffices show that the functor α' defined above is a quasi-inverse to α.

First we prove $\alpha \circ \alpha' \cong \mathrm{Id}$. So suppose $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$ is a tensor functor, which induces a scheme morphism $f : S \to X$ as above by choosing an open affine covering $X = \cup U_i$, $U_i = X - \mathrm{supp}(a_i)$. Denote by $S = \cup V_i$ the corresponding open covering as in the construction above. Let $h_i : U_i \hookrightarrow X$ and $g_i : V_i \hookrightarrow S$ be the open immersions, and let $f_i : V_i \to U_i$ be the restrictions of f.

We need to show that (S, F) and (S, f^*) are isomorphic objects in $\underline{\mathrm{QCoh}(X)}$. Since this is a stack it suffices to show that their associated descent data with respect to the covering $\{h_i : V_i \to S\}$ are isomorphic. That is, we need to show that $g_i^* \circ F$ and $g_i^* \circ f^*$ are isomorphic as tensor functors from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(V_i)$. (We also need to show that the isomorphisms we construct commute with the canonical natural transformations in the descent data; we leave this part to the reader.)

By the commutative diagram of schemes

$$\begin{array}{ccc}
V_i & \xrightarrow{f_i} & U_i \\
{\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle h_i} \\
S & \xrightarrow[f]{} & X
\end{array}$$

we have natural isomorphisms

$$g_i^* \circ f^* \cong f_i^* \circ h_i^*.$$

On the other hand, denote by $\tilde{F}_i : \mathrm{QCoh}(U_i) \to \mathrm{QCoh}(V_i)$ the tensor functor constructed in diagram (3.1), we have

$$g_i^* \circ F \cong \tilde{F}_i \circ h_i^*.$$

Hence it suffices to prove $f_i^* \cong \tilde{F}_i$. But $\tilde{F}_i$ is constructed from f_i and satisfies $\tilde{F}_i(1) = f_i^*(1)$, hence we conclude by Lemma 3.3.

Conversely, we need to prove that $\alpha' \circ \alpha \cong \mathrm{Id}$. So let $f : S \to X$ be a scheme morphism. Cover X with open affine subschemes U_i of the form $X - \mathrm{supp}(a_i)$. Let $V_i = f^{-1}(U_i)$ and let $f_i : V_i \to U_i$ be the restrictions of f. Let $F = f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(S)$, then we have as in diagram (3.1) tensor functors

$$F_i : \mathrm{QCoh}(X)/M_i \longrightarrow \mathrm{QCoh}(V_i).$$

It suffices to prove that the scheme morphism $f_i' : V_i \to U_i$ induced by F_i is equal to f_i. In other words, we need to prove $F_i(1) = f_i^*(1)$ as ring homomorphisms from $\Gamma(U_i, \mathcal{O}_{U_i}) \cong \mathrm{End}(\mathcal{O}_{U_i})$ to $\Gamma(V_i, \mathcal{O}_{V_i}) \cong \mathrm{End}(\mathcal{O}_{V_i})$:

$$\begin{array}{ccc} \mathrm{QCoh}(X)/M_i & \overset{F_i}{\dashrightarrow} & \mathrm{QCoh}(V_i) \\ \downarrow \cong & \overset{f_i^*}{\nearrow} & \\ \mathrm{QCoh}(U_i). & & \end{array}$$

This follows from the fact that the functor F_i is induced by the scheme morphism $f : S \to X$ which restricts to f_i. □

3.2.5. *Remark.* The idea of considering supports of objects in a possibly abstract category is the starting point of tensor triangular geometry. See [Bal02] and [Bal10].

3.2.6. *Remarks.* Theorem 3.4 is a stronger version of a special case of Lurie's recontruction of geometric stacks ([Lur, Theorem 5.11] and [Lur11, Theorem 3.0.1]) which describes the essential image of the natural functor

$$\mathrm{Hom}(S, X) \longrightarrow \mathrm{Fun}(\mathrm{QCoh}(X), \mathrm{QCoh}(S))$$

sending $f \mapsto f^*$. More precisely, Lurie's theorem applies to geometric stack X and states that this functor is fully faithful with essential image consisting of tensor functors which carry flat objects to flat objects.

In the case when X is a quasi-compact separated scheme (which is an example of a geometric stack), Theorem 3.4 states that this essential image consists of tensor functors. Therefore every tensor functor is isomorphic (via a symmetric monoidal natural isomorphism) to a tensor functor which moreover carries flat objects to flat objects, and in this special case of Lurie's theorem we may remove the condition that the tensor functor preserves flat objects.

4. Example: Finite group representations

In this Section we consider the category $G-\mathrm{rep}$ of finite dimensional linear representations of a finite group G. We show that our construction in Section 2 applied to $G-\mathrm{rep}$ equipped with tensor product of representations give the classifying stack BG. We also observe that a different tensor structure can be given to $G-\mathrm{rep}$ and our construction produces a stack quite different from BG.

4.1. **The set of characters.**

4.1.1. Let k be an algebraically closed field of characteristic zero, and let G be a finite group. The abelian category $G-\mathrm{rep}$ of finite dimensional G-representations over k is equivalent to the category $kG-\mathrm{mod}$.

Let $Z(kG)$ be the center of the group algebra kG, then $Z(kG)$ is a commutative subalgebra. It is isomorphic as an algebra to the direct sum $k^{\#\hat{G}}$ of $\#\hat{G}$ copies of k, where $\hat{G}$ is the set of characters on G.

We have a Morita equivalence:

$$kG-\mathrm{mod} \xrightarrow{\cong} G-\mathrm{rep} \xrightarrow{\chi} Z(kG)-\mathrm{mod},$$

where the second arrow sends an irreducible representation to the one dimensional k-vector space spanned by its character.

More explicitly, denote the irreducible G-representation by $\rho_1, \dots, \rho_{\#\hat{G}}$. Then we have

$$\chi : \rho \cong \bigoplus_i \rho_i^{\oplus m_i} \mapsto \prod_i k^{m_i}$$

where the k-vector space on the right is a $Z(kG)$-module with the "diagonal" action. More intrinsically, we have

$$\rho \cong \bigoplus_i \mathrm{Hom}_G(\rho_i, \rho) \otimes_k \rho_i,$$

then we have

$$\chi(\rho) = \prod_i \mathrm{Hom}_G(\rho_i, \rho).$$

If $\lambda : \rho_i \to \rho_i$ is the morphism given by multiplication by $\lambda \in k$, then

$$\chi(\lambda) : \chi(\rho_i) \to \chi(\rho_i)$$

is the multiplication by λ.

A quasi-inverse of the equivalence χ is given by

$$\chi^{-1} : \prod_i W_i \mapsto \bigoplus_i W_i \otimes_k \rho_i,$$

where each W_i is a finite dimensional k-vector space.

4.1.2. Denote by A the abelian category $G-\mathrm{rep} \cong Z(kG)-\mathrm{mod}$. Let $\otimes_G$ be the representation tensor product on $G-\mathrm{rep}$, and let $\otimes_Z$ be the module tensor product on $Z(kG)-\mathrm{mod}$. Consider abelian tensor categories $A_G := (A, \otimes_G)$ and $A_Z := (A, \otimes_Z)$ with identical underlying abelian categories.

By Theorem 3.1 we have a stack isomorphism

$$\underline{\mathrm{Spec}(k^{\#\hat{G}})} \cong \underline{A_Z}$$

over the étale site $\mathfrak{S}$ of affine k-schemes. So $\underline{A_Z}$ is the disjoint union of $\#\hat{G}$ points.

4.2. **The representation tensor product.**

4.2.1. Now we consider A_G. Consider the stack BG over $\mathfrak{S}$, whose objects are pairs $(S, \mathcal{M})$ where $\mathcal{M}$ is a sheaf on the affine scheme S (in its étale topology) which is a G-torsor. We define a natural functor

$$\beta : BG \longrightarrow \underline{A_G}$$

as follows.

Let $(S, \mathcal{M})$ be an object in BG, then the G-torsor gives an element in $\check{H}^1(S, G)$. If

$$\rho : G \to GL(V_\rho)$$

is a representation, then we have an induced map

$$\rho_* : \check{H}^1(S, G) \longrightarrow \check{H}^1(S, GL(V_\rho)).$$

The element $\rho_*(\mathcal{M})$ is a $GL(V_\rho)$-torsor over S, which gives a flat vector bundle over S; we denote this vector bundle also by $\rho_*(\mathcal{M})$. Then we define β by sending

$$\beta : (S, \mathcal{M}) \mapsto (S, \beta(\mathcal{M})),$$

where $\beta(\mathcal{M}) : A_G \to \mathrm{QCoh}(S)$ sends $\rho \mapsto \rho_*(\mathcal{M})$.

More explicitly, given $\mathcal{M}$ we can find a covering $\mathcal{U} = \{u_i : T_i \to S\}$ so that the torsor $\mathcal{M}$ is glued from the trivial torsors $\{G \times T_i\}$ via the transition elements $\{g_{ij}\}$ satisfying the cocycle condition, where each g_{ij} is an element in G, and multiplication by g_{ij} gives the isomorphisms

$$g_{ij} : p_1^*(G \times T_i) \longrightarrow p_2^*(G \times T_j)$$

over $T_{ij} = T_i \times_S T_j$.

For any $\rho \in G-\mathrm{rep}$, the vector bundle $\beta(\mathcal{M})(\rho) = \rho_*(\mathcal{M})$ is then glued from the trivial vector bundles $\{V_\rho \otimes_\Lambda \mathcal{O}_{T_i}\}$ from the transition elements $\{\rho(g_{ij})\}$. From this description it is clear that the functor $\beta(\mathcal{M}) : \rho \mapsto \rho_*(\mathcal{M})$ is indeed a tensor functor from A_G into $\mathrm{QCoh}(S)$.

Lemma 4.1. *The functor $\beta : BG \to \underline{A_G}$ is faithful.*

Proof. It suffices to show that if $\mathcal{M}$ is a G-torsor over S, then β induces an injection from $\mathrm{Aut}(\mathcal{M})$ to $\mathrm{Aut}(\beta(\mathcal{M}))$; the latter automorphism group consists of symmetric monoidal natural automorphisms of the functor $\beta(\mathcal{M})$.

Let $\mathcal{M}$ be given by $\{g_{ij}\}$ as above, then any automorphism of $\mathcal{M}$ is given by $\{h_i\}$, $h_i \in G$, satisfying

$$h_j g_{ij} = g_{ij} h_i,$$

and the automorphism $\beta(\{h_i\})_\rho : \rho_*(\mathcal{M}) \to \rho_*(\mathcal{M})$ is given by $\{\rho(h_i)\}$. Therefore we conclude by taking ρ to be any faithful representation. $\square$

4.2.2. Let S be a connected affine scheme, and take the trivial G-torsor $\mathcal{M} = G \times S$ as an example; notice that we have $\mathrm{Aut}(\mathcal{M}) \cong G$. Then all the transition elements g_{ij} are the identity element in G, and for every $\rho \in G-\mathrm{rep}$ we have

$$\rho_*(\mathcal{M}) = V_\rho \otimes_k \mathcal{O}_S.$$

We claim that the composition

$$G \xrightarrow{\cong} \mathrm{Aut}(\mathcal{M}) \xrightarrow{\beta} \mathrm{Aut}(\beta(\mathcal{M}))$$

is an isomorphism. By Lemma 4.1 it only remains to prove that it is a surjection.

For every point $p \in S$ denote by $k(p)$ its residue field, which is then a field extension of k. Consider the composition tensor functor

$$p^* \circ \beta(\mathcal{M}) : A_G \xrightarrow{\beta(\mathcal{M})} \mathrm{QCoh}(S) \xrightarrow{p^*} \mathrm{Vect}_{k(p)};$$

this is a fibre functor in the sense of [Del90, 1.9].

Since $\mathcal{M}$ is the pull-back of the trivial G-torsor $\mathcal{M}_0 \to \mathrm{Spec}(k)$ via the structural morphism $S \to \mathrm{Spec}(k)$, we have a commutative diagram

$$\begin{array}{ccccc} G & \xrightarrow{\cong} & \mathrm{Aut}(\mathcal{M}) & \longrightarrow & \mathrm{Aut}(\beta(\mathcal{M})) \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\ & & \mathrm{Aut}(\mathcal{M}_0) & \xrightarrow[\cong]{} & \mathrm{Aut}(p^* \circ \beta(\mathcal{M})), \end{array}$$

where the lower horizontal arrow is an isomorphism by the classical Tannakian duality, or the high-powered [Del90, 1.12].

Let $\phi \in \mathrm{Aut}(\beta(\mathcal{M}))$, then the association given by the right vertical arrow above

$$p \mapsto p^* \phi$$

gives a map $S \to G$, under which the preimage of every group element in G is closed. Since S is connected, this map must be a constant map, and so ϕ lies in the image of G.

Lemma 4.2. *The functor $\beta : BG \to \underline{A_G}$ is full.*

Proof. We need to show that if $\mathcal{M}$ is a G-torsor over S, then β induced a surjection from $\mathrm{Aut}(\mathcal{M})$ to $\mathrm{Aut}(\beta(\mathcal{M}))$. The case when $\mathcal{M}$ is the trivial torsor is treated above.

Fix a covering $\mathcal{U} = \{u_i : T_i \to S\}$ such that each $u_i^*\mathcal{M}$ is trivial. Then since both BG and $\underline{A_G}$ are stacks, we have a commutative diagram with exact rows:

$$\begin{array}{ccccc} \mathrm{Aut}(\mathcal{M}) & \longrightarrow & \prod_i \mathrm{Aut}(u_i^*\mathcal{M}) & \rightrightarrows & \prod_{i,j} \mathrm{Aut}(u_{ij}^*\mathcal{M}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Aut}(\beta(\mathcal{M})) & \longrightarrow & \prod_i \mathrm{Aut}(u_i^* \circ \beta(\mathcal{M})) & \rightrightarrows & \prod_{i,j} \mathrm{Aut}(u_{ij}^* \circ \beta(\mathcal{M})). \end{array}$$

Since the second and the third vertical arrows are isomorphisms, so is the first. □

4.2.3. Now consider the essential image of the functor $\beta : BG \to \underline{A_G}$. Let $(S, F) \in \underline{A_G}$, then by [Del90, 2.7] we know that $F(\rho)$ is a vector bundle of finite rank on S for every affine scheme $S \in \mathfrak{S}$ and every $\rho \in G{-}\mathrm{rep}$. Moreover, by specializing to a closed point as in the argument before Lemma 4.2 we see that the rank of $F(\rho)$ is equal to the dimension of V_ρ using the fact that there is essentially only one fibre functor into the category of k-vector spaces, namely the forgetful functor [Del90, 1.10].

Since there are only finitely many isomorphism classes of irreducible representation, we can find a covering $\mathcal{U} = \{u_i : T_i \to S\}$ such that each $u_i^* F(\rho)$ is a trivial vector bundle on T_i for every $\rho \in G{-}\mathrm{rep}$. In particular, this means that the functor $u_i^* \circ F : A_G \to \mathrm{QCoh}(T_i)$ is isomorphic to $\beta(\mathcal{M}_i)$ where $M_i = G \times T_i$ is the trivial G-torsor over T_i.

Denote by $h_{ij,\rho}$ the transition isomorphism $p_1^* u_i^* F(\rho) \to p_2^* u_j^* F(\rho)$ on T_{ij} of the vector bundle $F(\rho)$. This gives a natural isomorphism $h_{ij} : p_1^* u_i^* F \to p_2^* u_j^* F$ between functors from A to $\mathrm{QCoh}(T_{ij})$. Therefore we have isomorphisms

$$h_{ij} : p_1^* \beta(\mathcal{M}_i) \longrightarrow p_2^* \beta(\mathcal{M}_j).$$

Identifying these functors with $\beta(\mathcal{M}_{ij})$, where $\mathcal{M}_{ij} = G \times T_{ij}$ is the trivial G-torsor, we see that each h_{ij} is an element in $\mathrm{Aut}(\beta(\mathcal{M}_{ij}))$, which is isomorphic to a product of copies of G (the number of copies is equal to the number of connected components of T_{ij}).

Thus the datum $\{\beta(\mathcal{M}_i), h_{ij}\}$ glues in $\underline{A}$ to an object $\beta(\mathcal{M})$ in the image of β, and the local isomorphisms $u_i^* F \cong \beta(\mathcal{M}_i)$ gives an isomorphism $F \cong \beta(\mathcal{M})$. Hence we conclude:

Lemma 4.3. *The functor $\beta : BG \to \underline{A_G}$ is essentially surjective.*

Combining Lemmas 4.1, 4.2, and 4.3 we have proven

Theorem 4.4. *The functor $\beta : BG \to \underline{A_G}$ is an equivalence.*

4.2.4. *Remark.* In the case when G is finite and *abelian* the equivalence

$$\beta : BG \to \underline{A_G} = \underline{(A, \otimes_G)}$$

can be described more explicitly.

First notice that if H_1 and H_2 are finite groups, then we have

$$B(H_1 \times H_2) \cong BH_1 \times_{\mathfrak{S}} BH_2.$$

Combining this with Proposition 2.3 and choosing any decomposition

$$G \cong \prod_i (\mathbb{Z}/n_i\mathbb{Z})$$

we may reduce to the case when $G \cong \mathbb{Z}/n\mathbb{Z} \cong \mu_n$ is finite and cyclic.

Fix any embedding $\rho : G \to k^\times = GL(k)$, viewed as a one-dimensional representation. Then $\{\rho, \rho^{\otimes 2}, \rho^{\otimes 3}, \dots, \rho^{\otimes n}\}$ is a full list of irreducible representations. Therefore every tensor functor $F : G-\mathrm{rep} \to \mathrm{QCoh}(S)$ is determined by the line bundle $F(\rho)$ on S.

The line bundle $F(\rho)$ admits an isomorphism $F(\rho)^{\otimes n} \cong F(\rho^{\otimes n}) \cong \mathcal{O}_S$. Let

$$m_F : F(\rho) \longrightarrow F(\rho)^{\otimes n} \cong \mathcal{O}_S$$

be the n-th tensor power morphism. Denote by $1 \in \Gamma(S, \mathcal{O}_S)$ the nowhere vanishing global section. Then the preimage $m_F^{-1}(1)$ is a subsheaf of $F(\rho)$ which is easily seen to be a μ_n-torsor over S. The association

$$(S, F) \mapsto (S, m_F^{-1}(1))$$

gives a quasi-inverse to the functor β.

4.2.5. *Remarks.* (1) Theorem 4.4 above is very similar to Lurie's result applied to the geometric stack $X = BG$: Indeed, recalling that the category $G-\mathrm{Rep}$ of possibly infinite dimensional G-representations is equivalent to the category of G-equivariant sheaves on $\mathrm{Spec}(k)$, which in turn is equivalent to the category $\mathrm{QCoh}(BG)$. Lurie's theorem [Lur, Theorem 5.11] states that the natural functor

$$BG(S) \longrightarrow \mathrm{Fun}(G-\mathrm{Rep}, \mathcal{O}_S-\mathrm{Mod})$$

has its essential image consisting of tensor functors which carry flat objects to flat objects. Compare this with Remarks 3.2.6.

(2) More generally, if G acts on a scheme W and we take A to be the abelian tensor category of G-equivariant quasi-coherent sheaves on W, then $\underline{A}$ is isomorphic to the stack $[W/G]$.

(3) If G and H are finite groups with the same number of conjugacy classes, then on the abelian category

$$A = G-\mathrm{rep} \cong Z(kG)-\mathrm{mod} \cong Z(kH)-\mathrm{mod} \cong H-\mathrm{rep}$$

there are tensor products $\otimes_G$, $\otimes_H$, and $\otimes_Z$ so that

$$\underline{(A, \otimes_G)} \cong BG,$$

$$\underline{(A, \otimes_H)} \cong BH,$$

and

$$\underline{(A, \otimes_Z)} \cong \hat{G} \cong \hat{H}.$$

4.2.6. By 2.4.3 there are functors from A_G into the category of sheaves on $\underline{A_G} \cong BG$, and from A_Z into the category of sheaves on $\underline{A_Z} \cong \hat{G}$. Here we describe them with both underlying abelian categories A_G and A_Z realized as $G-\mathrm{rep}$:

$$\begin{array}{ccc} & & Sh(\underline{A_Z}) \\ & \nearrow & \\ G-\mathrm{rep} & & \\ & \searrow & \\ & & Sh(\underline{A_G}) \end{array} \qquad\qquad \begin{array}{ccc} & & \mathcal{G}_\rho \\ & \nearrow & \\ \rho & & \\ & \searrow & \\ & & \mathcal{F}_\rho \end{array}$$

We identify $\underline{A_Z} = \hat{G}$ with the set $\{\chi_1, \chi_2, \ldots, \chi_{\#\hat{G}}\}$ of irreducible characters; denote by ρ_i the irreducible representation with character χ_i. Then $\mathcal{G}_{\rho_i}$ corresponds to the one-dimensional $k^{\#\hat{G}}$-module supported at the point χ_i.

On the other hand, let $\rho : G \to GL(V_\rho)$ be a representation. The corresponding sheaf $\mathcal{F}_\rho$ on $\underline{A_G} \cong BG$ sends an object $(S, \mathcal{M}) \in BG$ to the global section of the vector bundle $\rho_*(\mathcal{M})$ with fibres isomorphic to V_ρ on S corresponding to the G-torsor $\mathcal{M}$. In particular, this sheaf restricts to the sheaf $\mathcal{M}$ on the category $BG_{/(S,\mathcal{M})} \cong \mathfrak{S}_{/S}$ over BG.

5. Example: the dual of G-gerbes

In this Section we apply the framework of 2-descent of stacks recalled in Appendix A to realize G-gerbes and their duals as stacks which locally are of the form $\underline{A}$.

5.1. **Comparison.** This subsection contains some preparatory results, to be used in the main construction.

5.1.1. Consider the equivalences

$$kG-\mathrm{mod} \xrightarrow{\cong} G-\mathrm{rep} \xrightarrow{\cong} Z(kG)-\mathrm{mod},$$

where $kG-\mathrm{mod}$ is given the representation tensor $\otimes_G$, while $Z(kG)-\mathrm{mod}$ is given $\otimes_Z$. Denote the second equivalence by χ, then it is *not* a tensor functor: for instance, it does not send the $\otimes_G$-unit object to the $\otimes_Z$-unit object.

Suppose $\phi : G \to H$ is a group homomorphism, then we have the following diagram:

$$\begin{array}{ccc} H-\mathrm{rep} & \xrightarrow{\chi_H} & Z(kH)-\mathrm{mod} \\ {\scriptstyle \phi^*}\downarrow & & \downarrow {\scriptstyle \chi_G \circ \phi^* \circ \chi_H^{-1}} \\ G-\mathrm{rep} & \xrightarrow[\chi_G]{} & Z(kG)-\mathrm{mod}. \end{array}$$

Here ϕ^* is a tensor functor. It is an interesting question to understand the functor given by the dashed arrow

$$F := \chi_G \circ \phi^* \circ \chi_H^{-1}.$$

Notice that the functor F is in general *not* a tensor functor: For example, let $G = \{1\}$ and let H be any group with $\#\hat{H} \geq 2$; let $\phi : G \to H$ be the inclusion of the identity element. If σ_1 and σ_2 are non-isomorphic irreducible H-representations, then we have

$$\chi_H(\sigma_1) \otimes \chi_H(\sigma_2) = 0 \in Z(kH)\text{−mod}$$

but $\phi^*(\sigma_1)$ and $\phi^*(\sigma_2)$ are both trivial G-representations, and we have

$$\chi_G\phi^*(\sigma_1) \otimes \chi_G\phi^*(\sigma_2) \neq 0 \in Z(kG)\text{−mod}.$$

5.1.2. Consider now the special case when we have a group *automorphism* $\phi : G \to G$. Then we have

$$\begin{array}{ccc} G\text{−rep} & \xrightarrow{\chi} & Z(kG)\text{−mod} \\ \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \chi\circ\phi^*\circ\chi^{-1}=F} \\ G\text{−rep} & \xrightarrow[\chi]{} & Z(kG)\text{−mod}. \end{array}$$

Since ϕ sends any conjugacy class of G into a conjugacy class, it defines an automorphism

$$\Phi : \hat{G} \cong \mathrm{Spec}(Z(kG)) \longrightarrow \mathrm{Spec}(Z(kG)) \cong \hat{G}$$

by pre-composing characters with ϕ.

Let ρ_i be an irreducible G-representation. Then $F(\rho_i)$ is the character of the representation $\rho_i \circ \phi$, and so we have

$$\Phi(\chi(\rho_i)) = \chi(\rho_i \circ \phi).$$

Therefore we have

$$F = \Phi^*.$$

In particular we see that F is a tensor functor. Hence we conclude:

Lemma 5.1. *Let $\phi : G \to G$ be a group automorphism of a finite group G. Then:*

(i) ϕ induces a tensor functor

$$(G\text{−rep}, \otimes_G) \longrightarrow (G\text{−rep}, \otimes_G)$$

defined by $\rho \mapsto \rho \circ \phi$.

(ii) ϕ induces a tensor functor

$$(G\text{−rep}, \otimes_Z) \longrightarrow (G\text{−rep}, \otimes_Z)$$

defined as the functor F above.

5.2. **G-gerbes and their duals.**

5.2.1. Let k be a field, let G be a finite group, and let $A = G-\mathrm{rep}$. Recall from Section 4.1.2 that we have two abelian tensor categories $A_G = (A, \otimes_G)$ and $A_Z = (A, \otimes_Z)$ with identical underlying abelian categories.

Then we have maps

$$\begin{array}{ccccc} & & & & \mathrm{Aut}^{\otimes}(A_G) \\ & & & \overset{\kappa}{\nearrow} & \\ G & \xrightarrow{\iota} & \mathrm{Aut}(G) & & \\ & & & \underset{\kappa'}{\searrow} & \\ & & & & \mathrm{Aut}^{\otimes}(A_Z), \end{array}$$

where the first arrow is the inner automorphism map

$$\iota(\beta) : x \mapsto \beta^{-1} x \beta.$$

The map κ sends an automorphism $\alpha : G \to G$ to $(\rho \mapsto \rho \circ \alpha)$ for every representation $\rho : G \to GL(V_\rho)$ in A; notice that it is an *anti*-homomorphism. Finally, κ' factors through κ, and is defined using Lemma 5.1 by choosing $H = G$ and $\phi = \alpha \in \mathrm{Aut}(G)$.

Lemma 5.2. *Let $\beta \in G$, and let $\alpha, \alpha' \in \mathrm{Aut}(G)$.*

(i) There is a natural isomorphism $\mu : \kappa(\alpha \circ \iota(\beta)) \to \kappa(\alpha)$. More precisely, $\mu_\rho = \rho\alpha(\beta)$.
(ii) $\mu \circ \kappa(\alpha') : \kappa(\alpha' \circ \alpha \circ \iota(\beta)) \to \kappa(\alpha' \circ \alpha)$ is given by $(\mu \circ \kappa(\alpha'))_\rho = \rho\alpha'\alpha(\beta)$.
(iii) $\kappa(\alpha') \circ \mu : \kappa(\alpha \circ \iota(\beta) \circ \alpha') \to \kappa(\alpha \circ \alpha')$ is given by $(\kappa(\alpha') \circ \mu)_\rho = \rho\alpha(\beta)$.
(iv) There is a natural isomorphism $\mu' : \kappa'(\alpha \circ \iota(\beta)) \to \kappa'(\alpha)$. More precisely, $\mu'_\rho = \chi_\rho(\alpha(\beta))$ where χ_ρ is the character of ρ.

Proof. (i) For every $\rho \in A$, we need an isomorphism

$$\mu_\rho : \rho \circ \alpha \circ \iota(\beta) \longrightarrow \rho \circ \alpha$$

of representations. The following commutative diagram gives the result:

$$\begin{array}{ccc} V & \xrightarrow{\rho\alpha\iota(\beta)(x)} & V \\ {\scriptstyle \rho\alpha(\beta)}\downarrow & & \downarrow{\scriptstyle \rho\alpha(\beta)} \\ V & \xrightarrow[\rho\alpha(x)]{} & V, \end{array}$$

for every $x \in G$, where $V = V_\rho = V_{\rho\circ\alpha} = V_{\rho\circ\alpha\circ\iota(\beta)}$.

(ii) and (iii) are straightforward.

(iv) Recall that we have an equivalence between abelian tensor categories

$$\chi : A_Z = (G-\mathrm{rep}, \otimes_Z) \longrightarrow (k^{\#\hat{G}}-\mathrm{mod}, \otimes)$$

where c is the number of conjugacy classes in G. The map χ sends any irreducible representation ρ to the one-dimensional vector space spanned by its character χ_ρ.

Applying χ to the natural transformation μ gives the result. □

5.2.2. Suppose now we are given elements $\alpha_{ij} \in \mathrm{Aut}(G)$ and $\beta_{ijk} \in G$ satisfying the 2-cocycle conditions:

$$\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik} \circ \iota(\beta_{ijk}), \tag{5.1}$$

and

$$\alpha_{ij}^{-1}(\beta_{jkl})\beta_{ijl} = \beta_{ijk}\beta_{ikl}. \tag{5.2}$$

Let $\lambda_{ij} = \kappa(\alpha_{ij}) \in \mathrm{Aut}^{\otimes}(A_G)$; similarly, let $\lambda'_{ij} = \kappa'(\alpha_{ij}) \in \mathrm{Aut}^{\otimes}(A_Z)$.

The first of the 2-cocycle conditions gives for every i, j, k a natural isomorphism

$$\mu_{ijk} : \lambda_{ij} \circ \lambda_{jk} = \kappa(\alpha_{ik} \circ \iota(\beta_{ijk})) \Longrightarrow \lambda_{ik}$$

between tensor autoequivalences on A_G, by taking $\alpha = \alpha_{ik}$ and $\beta = \beta_{ijk}$ in Lemma 5.2(i).

Similarly, by Lemma 5.2(iv) we have natural isomorphism

$$\mu'_{ijk} : \lambda'_{ij} \circ \lambda'_{jk} \Longrightarrow \lambda'_{ik}.$$

5.2.3. With notations as above, we claim that the following diagram is commutative:

$$\begin{array}{ccccc} (\lambda_{ij} \circ \lambda_{jk}) \circ \lambda_{kl} & \xRightarrow{\mu_{ijk} \circ \lambda_{kl}} & \lambda_{ik} \circ \lambda_{kl} & \xRightarrow{\mu_{ikl}} & \lambda_{il} \\ \| & & & & \| \\ \lambda_{ij} \circ (\lambda_{jk} \circ \lambda_{kl}) & \xRightarrow[\lambda_{ij} \circ \mu_{jkl}]{} & \lambda_{ij} \circ \lambda_{jl} & \xRightarrow[\mu_{ijl}]{} & \lambda_{il}. \end{array}$$

By Theorem A.1, we have the following consequence:

Corollary 5.3. *Let G be a finite group, let $A_G = (G{-}\mathrm{rep}, \otimes_G)$. Let $\mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories over a site with final object S_0. Let $\mathcal{U} = \{u_i : T_i \to S_0\}$ be a covering.*

Let $\alpha_{ij} \in \mathrm{Aut}(G)$ and $\beta_{ijk} \in G$ be chosen so that they satisfy the 2-cocycle conditions (5.1) *and* (5.2)*. Then*

$$(\underline{A_G}(\mathcal{B}_{/T_i}), \lambda_{ij}, \mu_{ijk})$$

defined above is a 2-descent datum of stacks over $\mathfrak{S}$.

In particular there is a stack $\mathcal{Y}$ over $\mathfrak{S}$ satisfying $u_i^{-1}\mathcal{Y} \cong \underline{A_G}(\mathcal{B}_{/T_i})$ by Theorem A.1.

The proof of the commutativity of the diagram above is a direct calculation using Lemma 5.2(i)-(iii) and the first cocycle condition (5.1):

$$\begin{aligned} (\mu_{ikl} \circ (\mu_{ijk} \circ \lambda_{kl}))_\rho &= \mu_{ikl,\rho} \circ \mu_{ijk,\lambda_{kl}(\rho)} \\ &= \rho\alpha_{il}(\beta_{ikl}) \circ \rho\alpha_{kl}\alpha_{ik}(\beta_{ijk}) \\ &= \rho\alpha_{il}(\beta_{ikl}) \circ \rho\alpha_{il}\iota(\beta_{ikl})(\beta_{ijk}) \\ &= \rho\alpha_{il}(\beta_{ijk}\beta_{ikl}), \end{aligned}$$

$$\begin{aligned}(\mu_{ijl}\circ(\lambda_{ij}\circ\mu_{jkl}))_\rho &= \mu_{ijl,\rho}\circ\lambda_{ij}(\mu_{jkl,\rho})\\ &= \rho\alpha_{il}(\beta_{ijl})\circ\rho\alpha_{jl}\alpha_{ij}\alpha_{ij}^{-1}(\beta_{jkl})\\ &= \rho\alpha_{il}(\beta_{ijl})\circ\rho\alpha_{il}\iota(\beta_{ijl})\alpha_{ij}^{-1}(\beta_{jkl})\\ &= \rho\alpha_{il}(\alpha_{ij}^{-1}(\beta_{jkl})\beta_{ijl}).\end{aligned}$$

Therefore the commutativity follows from the second cocycle condition (5.2).

By applying χ to the calculation above, we see that the analogous diagram with λ replaced with λ' and μ replaced with μ' is also commutative. Hence we have

Corollary 5.4. *Let G be a finite group, let $A_Z = (G-\mathrm{rep}, \otimes_Z)$. Let $\mathcal{B}\to\mathfrak{S}$ be a category fibred in abelian tensor categories over a site with final object S_0. Let $\mathcal{U} = \{u_i : T_i \to S_0\}$ be a covering.*

Let $\alpha_{ij}\in\mathrm{Aut}(G)$ and $\beta_{ijk}\in G$ be chosen so that they satisfy the 2-cocycle conditions (5.1) *and* (5.2)*. Then*

$$(\underline{A_Z}(\mathcal{B}_{/T_i}), \lambda'_{ij}, \mu'_{ijk})$$

defined above is a 2-descent datum of stacks over $\mathfrak{S}$.

In particular there is a a stack $\hat{\mathcal{Y}}$ over $\mathfrak{S}$ satisfying $u_i^{-1}\hat{\mathcal{Y}}\cong\underline{A_Z}(\mathcal{B}_{/T_i})$ by Theorem A.1.

5.2.4. Let $\mathfrak{S}$ be the étale site of affine schemes over a given scheme S_0 and let $\mathcal{B}\to\mathfrak{S}$ be the fibred category of quasi-coherent sheaves.

The stack $\mathcal{Y}$ constructed above is a G-gerbe over $\mathfrak{S}$. Indeed, by the proof of Theorem 4.4 we know that each stack $\underline{A_G}(\mathcal{B}_{/T_i})$ is isomorphic to $BG\times T_i$. The stack $\hat{\mathcal{Y}}$ should be considered as its *dual space* as in [TT10].

5.2.5. *Remarks.* The G-gerbes arising from the 2-descent Theorem A.1 considered here are a special kind. To cover more general G-gerbes one needs to consider 2-cocycles with upper indices

$$(\alpha^r_{ij}, \beta^{rst}_{ijk})$$

as in [Bre94, 2.4 and 2.7], where $T^r_{ij}\to T_{ij}$ is a covering of T_{ij} with index r, $\alpha^r_{ij}\in\mathrm{Aut}(G)$, and $\beta^{rst}_{ijk}\in G$.

The 2-cocycle conditions are

$$\alpha^s_{jk}\circ\alpha^r_{ij} = \alpha^t_{ik}\circ\iota(\beta^{rst}_{ijk}), \tag{5.3}$$

and

$$(\alpha^r_{ij})^{-1}(\beta^{swv}_{jkl})\beta^{rvu}_{ijl} = \beta^{rst}_{ijk}\beta^{twu}_{ikl}. \tag{5.4}$$

We illustrate the construction in this more general setting by considering the G-gerbe

$$BH\to BQ$$

arising from a short exact sequence

$$1\longrightarrow G\longrightarrow H\longrightarrow Q\longrightarrow 1$$

of finite groups.

In this case consider the étale covering $\mathrm{pt} \to BQ$ by a single affine scheme $T_1 := \mathrm{pt} = \mathrm{Spec}(k)$. Then we have, with notations as in 5.2.2,

$$T_{11} = \mathrm{pt} \times_{BQ} \mathrm{pt} \cong Q$$

as a set of points; set T_{ij}^r to be the point corresponding to an element r in Q. Therefore there is only one lower index, namely 1, and the upper indices correspond to elements in Q.

For every $r \in Q$ choose a lifting $\tilde{r} \in H$. This gives a set map

$$Q \to \mathrm{Aut}(G)$$

by sending r to the conjugation automorphism of G by $\tilde{r}$. Denote this automorphism by α_{11}^r.

The condition (5.3) defines β_{111}^{rst} whenever the group elements $r, s, t \in Q$ satisfy the equality $t = rs$ and gives a 2-cocycle; that is, condition (5.4) is satisfied.

These choices of α_{11}^r and β_{111}^{rst} allow us to glue $T_{11} \times BG$, that is, $\#Q$ copies of BG together to get a G-gerbe $\mathcal{Y}$ over BQ.

Recall that each local copy of BG is realized as $\underline{A_G}(\mathcal{B}_{/\mathrm{pt}})$, where pt is a point in T_{11} and $\mathcal{B} \to \mathfrak{S}$ is the fibred category of quasi-coherent sheaves over the site of affine schemes over Λ. Hence the dual $\hat{\mathcal{Y}}$ is glued from $\#Q$ copies of the scheme

$$\hat{G} = \text{the set of isomorphism classes of irreducible representations of } G = \underline{A_Z}(\mathcal{B}_{/\mathrm{pt}})$$

via the isomorphisms induced by α_{11}^r, $r \in Q$. In other words, $\hat{\mathcal{Y}}$ is the quotient of $\hat{G}$ by this Q-action

$$\hat{\mathcal{Y}} = [\hat{G}/Q].$$

5.2.6. Now we construct a twisted sheaf on $\hat{\mathcal{Y}}$ using A.1.4. See [TT10] for the role twisted sheaves played in gerbe duality.

First we consider $\hat{\mathcal{Y}}$, which is glued from $\underline{A_Z}(\mathcal{B}/_{T_i}) = \hat{G} \times T_i$. Consider the regular representation $\tilde{\rho} = \sum V_{\rho_s^*} \otimes \rho_s$, where $\{\rho_s\}$ is a set of representatives of isomorphism classes of irreducible G-representations. Here $V_{\rho_s^*} \otimes \rho_s$ means the direct sum of $\dim V_{\rho_s^*}$ copies of ρ_s. Denote by χ_s the character of ρ_s, then $\hat{G} \cong \{\chi_s\}$.

Let $\mathcal{G}_i$ be the sheaf on $\hat{G} \times T_i$ corresponding to $\tilde{\rho} \in A$ via the construction in 2.4.3. Then $\mathcal{G}_i$ is simply the sheaf on $\hat{G} \times T_i$ whose restriction to $\{\chi_s\} \times T_i$ is the trivial vector bundle $V_{\rho_s^*} \otimes \mathcal{O}_{T_i}$.

We will identify $\hat{G} \times T_i|_{ij}$ with $\hat{G} \times T_{ij}$, then we have an automorphism χ_{ij} on $\hat{G} \times T_{ij}$ induced by the automorphism $\alpha_{ij} \in \mathrm{Aut}(G)$. By the cocycle condition (5.1) and the fact that an inner automorphism acts trivially on the set $\hat{G}$ of characters, we see that the natural isomorphism

$$\phi_{ijk} : \chi_{jk} \circ \chi_{ij} \Rightarrow \chi_{ik}$$

between stack isomorphisms is the identity: In fact every χ_{ij} is the scheme automorphism on $\hat{G} \times T_{ij}$ given by the action of α_{ij} acting on $\hat{G}$.

The sheaf $\chi_{ij}^*(\mathcal{G}_j|_{ij})$ is the the sheaf whose restriction to $\{\chi_s\} \times T_{ij}$ is the trivial vector bundle $V_{\rho_v^*} \otimes \mathcal{O}_{T_{ij}}$, where ρ_v is the representative of the isomorphism class of $\rho_s \circ \alpha_{ij}$.

The isomorphism $\rho_s^* \circ \alpha_{ij} \cong \rho_v^*$ is given by a vector space isomorphism (unique only up to a non-zero scalar)

$$\tau_{ij,s} : V_{\rho_s^*} \xrightarrow{\cong} V_{\rho_v^*}.$$

This defines sheaf isomorphisms

$$\delta_{ij} : \mathcal{G}_i|_{ij} \xrightarrow{\cong} \chi_{ij}^*(\mathcal{G}_j|_{ij})$$

between sheaves on $\hat{G} \times T_{ij}$, and hence we have constructed a twisted sheaf $\mathcal{G}$ on $\hat{\mathcal{Y}}$ associated to the regular representation $\tilde{\rho}$.

To compute the twisting, which is a 2-cycle with values in $\mathcal{O}_{S_0}^\times$, first recall that by the cocycle condition (5.1) we have

$$\alpha_{ki} \circ \alpha_{jk} \circ \alpha_{ij} = \iota(\beta_{ijk})$$

in $\mathrm{Aut}(G)$, where $\beta_{ijk} \in G$, and for any $\beta \in G$, $\iota(\beta)$ denotes the inner automorphism $x \mapsto \beta^{-1}x\beta$ on G. In particular, the isomorphism of vector spaces $\rho_s^*(\beta^{-1})$ gives an isomorphism from ρ_s to $\rho_s \circ \iota(\beta)$. Now consider the following commutative diagram:

$$\begin{array}{ccccccc}
\rho_s^* & & & \text{-- -- -- } c_{s,ijk} \text{ -- -- --}\dashrightarrow & & & \\
\downarrow{\scriptstyle \rho_s^*(\beta_{ijk}^{-1})} & & & & & & \\
\rho_s^* \circ \iota(\beta_{ijk}) & & & & & & \\
\| & & & & & & \\
\rho_s^* \circ \alpha_{ki} \circ \alpha_{jk} \circ \alpha_{ij} & \xrightarrow[\tau_{ki,s}]{} & \rho_t^* \circ \alpha_{jk} \circ \alpha_{ij} & \xrightarrow[\tau_{jk,t}]{} & \rho_u^* \circ \alpha_{ij} & \xrightarrow[\tau_{ij,u}]{} & \rho_s^*,
\end{array}$$

where $c_{s,ijk}$ is a non-zero scalar: Indeed, we can show that the composition of these vector space isomorphisms gives an automorphism of the *representation* ρ_s^*. The sheaf $\mathcal{G}$ is $\{c_{s,ijk}\}$-twisted.

5.2.7. An analogous construction as above gives a (non-twisted) sheaf on $\mathcal{Y}$, which is glued from $\underline{A_G}(\mathcal{B}_{/T_i}) = BG \times_{S_0} T_i$. Again consider the regular representation $\tilde{\rho}$, and let $\mathcal{F}_i$ be the sheaf on $BG \times_{S_0} T_i$ corresponding to $\tilde{\rho}$.

If $\mathcal{M} \to S$ is a G-torsor over a scheme S over T_i, namely an object in $BG \times_{S_0} T_i$, then

$$\mathcal{F}_i : (\mathcal{M} \to S) \mapsto \Gamma(S, \tilde{\rho}_*\mathcal{M})$$

where $\tilde{\rho}_*\mathcal{M}$ is the flat vector bundle on S given by the 1-cocycle with values in $GL(V_{\tilde{\rho}})$ by pushing-forward the 1-cocycle with values in G corresponding to $\mathcal{M}$.

We will identify $BG \times_{S_0} T_i|_{ij}$ with $BG \times_{S_0} T_{ij}$, and then we have an automorphism χ_{ij} on it induced by $\alpha_{ij} \in \mathrm{Aut}(G)$. Since inner automorphisms on G act trivially on BG, the natural isomorphism

$$\phi_{ijk} : \chi_{jk} \circ \chi_{ij} \Rightarrow \chi_{ik}$$

between stack isomorphisms is again the identity.

The sheaf $\chi_{ij}^*(\mathcal{F}_j|_{ij})$ is given by

$$(\mathcal{M} \to S) \mapsto \Gamma(S, (\tilde{\rho} \circ \alpha_{ij})_*\mathcal{M}).$$

Notice that $V_{\tilde{\rho}}$ is isomorphic to kG as a k-vector space, and so any group automorphism $\alpha \in \mathrm{Aut}(G)$ induces a *linear* automorphism of $V_{\tilde{\rho}}$. In particular, we then have a natural isomorphism

$$\delta_{ij} : \mathcal{F}_i|_{ij} \xrightarrow{\cong} \chi_{ij}^*(\mathcal{F}_j|_{ij}).$$

Finally, the vector bundles $\tilde{\rho}_*\mathcal{M}$ and $(\tilde{\rho}\circ\iota(\beta_{ijk}))_*\mathcal{M}$ are identified, as in the case of $\hat{\mathcal{Y}}$, by β_{ijk}^{-1}, which is exactly the inverse of the composition of δ_{ij}'s over T_{ijk}. Therefore the twisting is the identity in this case. In other words the twisted sheaf $\mathcal{F}$ we obtained is actually a sheaf.

APPENDIX A. TWISTING BY A 2-COCYCLE

A.1. 2-descent data.

A.1.1. In this appendix we fix some notations and recall the 2-descent of stacks. Some of the materials below can be found in [Bre94].

A.1.2. Let $\mathfrak{S}$ be a site with final object S_0, then we have an equivalence $\mathfrak{S}\cong\mathfrak{S}_{/S_0}$. Fix any covering $\mathcal{U}=\{u_i:T_i\to S_0\}$ of S_0, then any stack $\mathcal{X}$ over $\mathfrak{S}$ gives rise to a stack

$$\mathcal{X}|_i := u_i^{-1}\mathcal{X}\cong\mathcal{X}\times_{\mathfrak{S}}\mathfrak{S}_{/T_i}$$

over $\mathfrak{S}_{/T_i}$ for every i, along with isomorphisms

$$can_{ij}: p_1^{-1}u_i^{-1}\mathcal{X}\xrightarrow{\cong}p_2^{-1}u_j^{-1}\mathcal{X}$$

of stacks over $\mathfrak{S}_{/T_{ij}}$, where $T_{ij}=T_i\times_{S_0}T_j$, and p_1 and p_2 denote the two projections to T_i and T_j respectively. To simplify notations, we will write $\mathcal{X}|_i|_{ij}$ for $p_1^{-1}u_i^{-1}\mathcal{X}$, etc; in particular we have

$$can_{ij}:\mathcal{X}|_i|_{ij}\xrightarrow{\cong}\mathcal{X}|_j|_{ij}.$$

We will similarly use the restriction notation for the pull-back functors.

The fibred category structure on $\mathcal{X}$, or more precisely the natural isomorphisms relating different pull-backs gives moreover natural isomorphisms

$$\phi_{ijk}:(can_{jk}|_{ijk})\circ(can_{ij}|_{ijk})\Longrightarrow can_{ik}|_{ijk}$$

between isomorphisms of stacks over $\mathfrak{S}_{/T_{ijk}}$, where $T_{ijk}=T_i\times_{S_0}T_j\times_{S_0}T_k$.

These natural transformations satisfy a compatibility cocycle condition (see below).

A.1.3. The situation above formalizes to the notion of 2-*descent data*: A 2-descent datum of stacks over $\mathfrak{S}$ with respect to the covering $\mathcal{U}$ is a triple $(\mathcal{X}_i,\chi_{ij},\phi_{ijk})$ of stacks $\mathcal{X}_i$ over $\mathfrak{S}_{/T_i}$, stack isomorphisms

$$\chi_{ij}:\mathcal{X}_i|_{ij}\xrightarrow{\cong}\mathcal{X}_j|_{ij},$$

and natural isomorphisms

$$\phi_{ijk}:(\chi_{jk}|_{ijk})\circ(\chi_{ij}|_{ijk})\Longrightarrow\chi_{ik}|_{ijk}$$

between functors from $\mathcal{X}_i|_{ijk}$ to $\mathcal{X}_k|_{ijk}$. These are required to satisfy the condition that for every i,j,k,l the following diagram of functors from $\mathcal{X}_i|_{ijkl}$ to $\mathcal{X}_l|_{ijkl}$ is commutative:

$$\begin{array}{ccccc}
\chi_{kl}\circ(\chi_{jk}\circ\chi_{ij}) & \xrightarrow{\chi_{kl}(\phi_{ijk})} & \chi_{kl}\circ\chi_{ik} & \xrightarrow{\phi_{ikl}} & \chi_{il}\\
\| & & & & \|\\
(\chi_{kl}\circ\chi_{jk})\circ\chi_{ij} & \xrightarrow[\phi_{jkl}\circ\chi_{ik}]{} & \chi_{jl}\circ\chi_{ij} & \xrightarrow[\phi_{ijl}]{} & \chi_{il},
\end{array}$$

where for legibility we have omitted $|_{ijkl}$ throughout.

The following is part of [Bre94, Example 1.11 (i)].

Theorem A.1. *Given a site $\mathfrak{S}$ with final object S_0 and a covering $\mathcal{U} = \{u_i : T_i \to S_0\}$, suppose we have a 2-descent datum $(\mathcal{X}_i, \chi_{ij}, \phi_{ijk})$ of stacks with respect to $\mathcal{U}$ as above. Then there is a stack $\mathcal{X}$ over $\mathfrak{S}$ along with isomorphisms*

$$\mathcal{X}|_i = u_i^{-1}\mathcal{X} \cong \mathcal{X}_i$$

of stacks over $\mathfrak{S}_{/T_i}$.

A.1.4. Now we explain how to construct twisted sheaves on 2-descended stacks. Let $(\mathcal{X}_i, \chi_{ij}, \phi_{ijk})$ be a 2-descent datum of stacks over $\mathfrak{S}$ with respect to a covering $\mathcal{U} = \{u_i : T_i \to S_0\}$ of the final object $S_0 \in \mathfrak{S}$.

Let $\mathcal{F}_i$ be a sheaf on $\mathcal{X}_i$. Suppose we are further given sheaf isomorphisms

$$\delta_{ij} : \mathcal{F}_i|_{ij} \xrightarrow{\cong} \chi_{ij}^*(\mathcal{F}_j|_{ij}),$$

where $\chi_{ij}^*(\mathcal{F}_j|_{ij}) = \mathcal{F}_i|_{ij} \circ \chi_{ij}$ as a (set-valued) functor. Then, omitting $|_{ijk}$ throughout, we have isomorphisms of sheaves on $\mathcal{X}_i|_{ijk}$:

$$\begin{array}{ccccccc} \mathcal{F}_i & \xrightarrow{\delta_{ij}} & \chi_{ij}^*(\mathcal{F}_j) & \xrightarrow{\chi_{ij}^*(\delta_{jk})} & \chi_{ij}^*\chi_{jk}^*(\mathcal{F}_k) & \xrightarrow{\chi_{ij}^*\chi_{jk}^*(\delta_{ki})} & \chi_{ij}^*\chi_{jk}^*\chi_{ki}^*(\mathcal{F}_i) \\ & & & & & & \uparrow \cong \\ & & & \xrightarrow{\eta^i_{ijk}} & & & \mathcal{F}_i, \end{array}$$

where the vertical isomorphism is induced from the natural isomorphism

$$\phi_{ijk} : \chi_{jk} \circ \chi_{ij} \Rightarrow \chi_{ik}.$$

(And the normalization $\chi_{ik} = \chi_{ki}^{-1}$.)

Here η^i_{ijk} is an automorphism of the sheaf $\mathcal{F}_i|_{ijk}$ on $\mathcal{X}_i|_{ijk}$. There is a natural compatibility relation between η^i_{ijk} and η^j_{ijk} in terms of the isomorphism δ_{ij}. If for every i, j, k the automorphism η^i_{ijk} is the identify automorphism, then the datum

$$(\mathcal{F}_i, \delta_{ij})$$

defines a sheaf on the stack $\mathcal{X}$. In general, η^i_{ijk} needs not be the identity, and we get a twisted sheaf on $\mathcal{X}$.

A.2. **2-cocycle on an abelian category.**

A.2.1. Now let A be an abelian tensor category, and let $\pi : \mathcal{B} \to \mathfrak{S}$ be a category fibred in abelian tensor categories satisfying the conditions of Theorem 2.4. Notice that for every i the restriction

$$\pi_i : \mathcal{B}|_i := \mathcal{B}_{/T_i} \to \mathfrak{S}_{/T_i}$$

also satisfies the conditions of Theorem 2.4, and therefore we have a stack $\underline{A}(\mathcal{B}|_i)$ over $\mathfrak{S}_{/T_i}$ for every i which is isomorphic to $\underline{A}(\mathcal{B})|_i = u_i^{-1}\underline{A}(\mathcal{B})$.

A.2.2. With notations as above, we would like to glue the stacks $\underline{A}(\mathcal{B}|_i)$ using 2-descent to produce new stacks over $\mathfrak{S}$.

To this end, let $\mathrm{Aut}^{\otimes}(A)$ denote the set of autoequivalences of A which are tensor functors. Choose $\lambda_{ij} \in \mathrm{Aut}^{\otimes}(A)$ for every i, j and natural isomorphisms

$$\mu_{ijk} : \lambda_{ij} \circ \lambda_{jk} \Longrightarrow \lambda_{ik},$$

satisfying the "tetrahedron" condition that the following diagram of functors is commutative for every i, j, k, l:

$$\begin{array}{ccccc} (\lambda_{ij} \circ \lambda_{jk}) \circ \lambda_{kl} & \xRightarrow{\mu_{ijk} \circ \lambda_{kl}} & \lambda_{ik} \circ \lambda_{kl} & \xRightarrow{\mu_{ikl}} & \lambda_{il} \\ \| & & & & \| \\ \lambda_{ij} \circ (\lambda_{jk} \circ \lambda_{kl}) & \xRightarrow[\lambda_{ij} \circ \mu_{jkl}]{} & \lambda_{ij} \circ \lambda_{jl} & \xRightarrow[\mu_{ijl}]{} & \lambda_{il}. \end{array}$$

These conditions imply that the triple

$$(\underline{A}(\mathcal{B}|_i), \lambda_{ij}^*, \mu_{ijk})$$

is a 2-descent datum of stacks over $\mathfrak{S}$ with respect to the covering $\mathcal{U}$. Therefore by Theorem A.1 there is a stack $\mathcal{X}$ over $\mathfrak{S}$ satisfying

$$\mathcal{X}|_i = u_i^{-1}\mathcal{X} \cong \underline{A}(\mathcal{B}).$$

Here λ_{ij}^* denotes the isomorphism

$$\lambda_{ij}^* : \underline{A}(\mathcal{B}|_i)|_{ij} \longrightarrow \underline{A}(\mathcal{B}|_j)|_{ij}$$

between stacks over $\mathfrak{S}_{/T_{ij}}$ defined by

$$(f, (S, F)) \mapsto (f, (S, F \circ \lambda_{ij}))$$

for every $f : S \to T_{ij}$; here $F : A \to \mathcal{B}|_{i,S} \cong \mathcal{B}_S$ is a tensor functor.

A.2.3. *Remark.* It may be interesting to study the set of such 2-cocycles for given A. It should form some sort of cohomology set in degree 2.

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