TOPOLOGICAL AND UNIFORM STRUCTURES ON UNIVERSAL COVERING SPACES

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Abstract. We discuss various uniform structures and topologies on the universal covering space $\widetilde{X}$ and on the fundamental group $\pi_1(X, x_0)$. We introduce a canonical uniform structure $CU(X)$ on a topological space X and use it to relate topologies on $\widetilde{X}$ and uniform structures on $\widetilde{CU(X)}$.

Using our concept of universal Peano space we show connections between the topology introduced by Spanier [30] and a uniform structure of Berestovskii and Plaut [2]. We give a sufficient and necessary condition for Berestovskii-Plaut structure to be identical with the one generated by the uniform convergence structure on the space of paths in X. We also describe when the topology of Spanier is identical with the quotient of the compact-open topology on the space of paths.

Contents

1. Introduction

The classical way of introducing topology on the universal covering space $\widetilde{X}$ (the space of homotopy classes of based paths in X) is as the quotient space of the space of based paths $(X, x_0)^{(I,0)}$ equipped with the compact-open topology. We denote the space $\widetilde{X}$ equipped with this topology by $\widetilde{X}_{top}$. Spanier introduced a different topology on $\widetilde{X}$ (see the proof of Theorem 13 on page 82 of [30]). We call it the whisker topology and use notation $\widetilde{X}_{wh}$. The whisker topology is defined by the basis $B([\alpha], U) = \{[\alpha * \beta]\}$, where α is a path in X from x_0 to x_1, U is a neighborhood of x_1 in X, and β is a path in U originating at x_1 (we call the path β a *U-whisker*). This topology was used in 1998 by Bogley and Sieradski [3], then in 2006 by Fisher and Zastrow [14]. A new topology on $\widetilde{X}$ called the lasso topology was introduced by the authors [7] to characterize the unique path lifting property.

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The fundamental group $\pi_1(X, x_0)$ can be identified with the subset of classes of loops of the set $\widetilde{X}$, thus any topology on $\widetilde{X}$ induces a topology on the fundamental group. There have been numerous attempts to introduce topology on the fundamental group [16, 8, 25, 4]. In this paper, when we consider a topology on $\widetilde{X}$, we mainly address the question of whether the induced topology on the fundamental group makes it a topological group.

There was independent research going on in the realm of uniform structures, where Berestovskii and Plaut [2] introduced a new uniform structure on the space of chains in a uniform space. That structure can be easily generalized to $\widetilde{X}$, so we call it the Berestovskii-Plaut uniform structure on $\widetilde{X}$. In [5] the authors used ideas of [2] and Krasinkiewicz-Minc [21] to introduce generalized uniform paths on a uniform space and to define a uniform structure $GP(X, x_0)$ on the space of generalized paths originating at x_0. Generalized uniform paths lead naturally to a class of spaces called uniformly joinable and a theory of generalized uniform covering maps was developed in [5] for those spaces. See [2] for a predecessor of that theory.

Uniformly joinable spaces are connected to shape theory (see [10] or [24]) in the sense that pointed 1-movable continua are always uniformly joinable (see [5]). Recently, Fisher and Zastrow [14] found another connection between shape theory and covering maps. Namely, they showed the projection $\widetilde{X}_{wh} \to X$ is a generalized universal covering (in their sense) if the natural homomorphism $\pi_1(X) \to \check{\pi}_1(X)$ is a monomorphism and X is paracompact. The latter class of spaces includes subsets of surfaces and it is known subcontinua of surfaces are pointed 1-movable. This raises the question if one can derive results of [14] from those in [5]. Another idea is to define a new class of covering maps (called Peano covering maps) for Peano spaces as those spaces form a topological analog of uniformly joinable spaces.

The functor of universal locally-path-connected space of X, that we introduce in Section 3 and call the Peano space of X, is very useful as it shows how to derive theorems for arbitrary spaces from analogous results for locally-path-connected spaces. For example, many results of [14] follow formally from their special cases involving locally path-connected spaces only (see [7]). A good example is the main result of [14] stating that the projection $\widetilde{X} \to X$ is a generalized universal covering if X is path-connected and the fundemental group of (X, x_0) injects into its Čech fundamental group via the natural homomorphism.

It is of interest to investigate relationships between various structures on $\widetilde{X}$. One of our first results in that direction is the fact the whisker topology is philosophically very similar to the quotient compact-open topology. The difference is that, prior to applying the quotient topology, one takes the universal locally-path-connected space of $(X, x_0)^{(I,0)}$ (see 4.4). We describe when the whisker topology and the quotient compact-open topology are identical: when the space X is a small loop transfer space. The property of being a strong small loop transfer space characterizes the coinsidence of the whisker topology and the lasso topology.

If X is a uniform space, the set $\widetilde{X}$ can be equipped with a uniform structure. For basic facts on uniform spaces we refer to [18] or [19]. We consider three uniform structures on $\widetilde{X}$: the James uniformity (analogous to the quotient compact-open topology), the Berestovskii-Plaut uniformity (analogous to the whisker topology) and the lasso uniformity. For any connected and uniformly locally path-connected uniform space X the James uniformity on $\widetilde{X}$ is identical with the lasso uniformity (see 5.16). We also describe when the James uniformity and the Berestovskii-Plaut uniformity are identical: when the space X is a uniform small loop transfer space.

Given a topological space X, we show in Section 2 how to introduce a canonical uniform structure on X that induces the original topology in case X is completely regular. Then we relate various properties (such as local connectivity) of the topological space X to the corresponding uniform properties of the canonical uniform space $CU(X)$. This construction provides a relation between Sections 4 and 5: for a topological space X a topological structure on $\widetilde{X}$ can be obtained as topology induced by a uniform structure on $CU(X)$. For example, the lasso topology on $\widetilde{X}$ is identical with the topology induced by the lasso uniformity on $\widetilde{CU(X)}$.

2. Uniform vs topological structures

We will discuss exclusively symmetric subsets E of $X \times X$ (that means $(x, y) \in E$ implies $(y, x) \in E$) and the natural notation here (see [29]) is to use $f(E)$ for the set of pairs $(f(x), f(y))$, where $f\colon X \to Y$ is a function. Similarly, $f^{-1}(E)$ is the set of pairs (x, y) so that $(f(x), f(y)) \in E$ if $f\colon X \to Y$ and $E \subset Y \times Y$.

The **ball** $B(x, E)$ **at** x **of radius** E is the set of all $y \in X$ satisfying $(x, y) \in E$. A subset S of X is E**-bounded** if it is contained in $B(x, E)$ for some $x \in X$.

A **uniform structure** on X is a family $\mathcal{E}$ of symmetric subsets E of $X \times X$ (called **entourages**) that contain the diagonal of $X \times X$, form a filter (that means $E_1 \cap E_2 \in \mathcal{E}$ if $E_1, E_2 \in \mathcal{E}$ and $F_1 \in \mathcal{E}$ if $F_2 \in \mathcal{E}$ and $F_2 \subset F_1$), and every $G_1 \in \mathcal{E}$ admits $G \in \mathcal{E}$ so that $G^2 \subset G_1$ (G^2 consists of pairs $(x, z) \in X \times X$ so that there is $y \in X$ satisfying $(x, y) \in G$ and $(y, z) \in G$). A **base** $\mathcal{F}$ of a uniform structure $\mathcal{E}$ is a subfamily $\mathcal{F}$ of $\mathcal{E}$ so that for every entourage E there is a subset $F \in \mathcal{F}$ of E.

Any filter $\mathcal{F}$ of subsets of $X \times X$ that contain the diagonal serves as a base of the uniform structure $\mathcal{U}$ defined as all supersets of $\mathcal{F}$ if and only if every $G_1 \in \mathcal{F}$ admits $G \in \mathcal{F}$ so that $G^2 \subset G_1$. That is the basic case of generating a uniform structure.

Given a decomposition of a uniform space X the most pressing issue is if it induces a natural uniform structure on the decomposition space. James [19, 2.13 on p.24] has a concept of weakly compatible relation to address that issue. For the purpose of this paper we need a different approach.

Definition 2.1. Suppose $f\colon X \to Y$ is a surjective function from a uniform space X. f **generates a uniform structure on** Y if the family $f(E)$, E an entourage of X, is a base of a uniform structure on Y (that particular uniform structure on Y is said to be **generated by** f). Equivalently, for each entourage E of X there is an entourage F of X such that $f(F)^2 \subset f(E)$.

Notice $f\colon X \to Y$ is uniformly continuous if both X and Y are uniform spaces and the uniform structure on Y is generated by f. Indeed $E \subset f^{-1}(f(E))$ for any entourage E of X.

Let us demonstrate how to transfer concepts from topology to the uniform category:

Definition 2.2. A uniform space X is **uniformly locally path-connected** if for every entourage E of X there is an entourage F of X such that any pair $(x, y) \in F$ can be connected by an E-bounded path in X.

A uniform space X is **uniformly semi-locally simply-connected** if there is an entourage F of X such that any F-bounded loop in X is null-homotopic in X.

Remark 2.3. Notice our definition of uniform local path-connectedness is much simpler than [19, Definition 8.12 on p.119] and we are unsure if the definition [19, Definition 8.13 on p.119] of uniform semi-local simple connectedness is correct as it involves existence of a base of entourages rather than just one entourage.

It is well-known (see [18] or [19]) there is a forgetful functor from the uniform category to the topological category. It assigns to a uniform space X the topology defined as follows: $U \subset X$ is open if and only if for each $x \in U$ there is an entourage E_x of X satisfying $B(x, E_x) \subset U$. Notice $f\colon X \to Y$ is continuous if it is uniformly continuous.

Let us describe a functor from the topological category to the uniform category that is the inverse of the forgetful functor on the class of completely regular spaces.

Definition 2.4. Given a topological space X its **canonical uniform structure** $CU(X)$ is defined as follows: $E \subset X \times X$ is an entourage if and only if there is a partition of unity $f = \{f_s\}_{s\in S}$ on X so that E contains

$$E_f := \{(x, y) \in X \times X | \text{ there is } s \in S \text{ satisfying } f_s(x) \cdot f_s(y) > 0\}.$$

It is indeed a uniform structure:

Lemma 2.5. *If $g = \{f'_T\}_{T\subset S}$ is the derivative of $f = \{f_s\}_{s\in S}$, then $E_g^2 \subset E_f$.*

Proof. Recall (see [9]) f'_T is defined as $|T| \cdot \max(0, g_T)$, where $g_T = \min\{f_t \mid t \in T\} - \sup\{f_t \mid t \in S \setminus T\}$ and T is a finite subset of S. Suppose $(x, y) \in E_g$ and $(y, z) \in E_g$. There exist finite subsets T and Z of S such that $f'_T(x) \cdot f'_T(y) > 0$ and $f'_Z(y) \cdot f'_Z(z) > 0$. Hence $T \cap Z \neq \emptyset$ and $f_s(x) \cdot f_s(z) > 0$ for any $s \in T \cap Z$. That means $(x, y) \in E_f$. □

Notice any continuous function $f\colon X \to Y$ becomes uniformly continuous when considered as a function from $CU(X)$ to $CU(Y)$. Indeed, given a partition of unity $\{f_s\}_{s\in S}$ on X, it induces the partition of unity $\{f \circ f_s\}_{s\in S}$ on X.

Proposition 2.6. *Suppose X is a topological Hausdorff space. The topology induced on X by $CU(X)$ coincides with that of X if and only if X is completely regular.*

Proof. Notice the topology induced by $CU(X)$ is always coarser than the original one. Indeed, $B(x, E_f)$ contains the union of all $f_s^{-1}(0, 1]$ so that $f_s(x) > 0$, so every open set in $CU(X)$ belongs to the original topology of X.

If X is completely regular and U is open, then for any $x \in X$ there is a continuous function $g\colon X \to [0, 1]$ so that $g(x) = 1$ and $g(X \setminus U) \subset \{0\}$. In that case $B(x, E_f) \subset U$, where $f = \{g, 1-g\}$. That means U is open in the topology induced by $CU(X)$. □

Proposition 2.7. *Suppose X is a paracompact space.*

a. *X is locally connected if and only if $CU(X)$ is uniformly locally connected.*
b. *X is locally path-connected if and only if $CU(X)$ is uniformly locally path-connected.*
c. *If X is locally path-connected, then it is semi-locally simply-connected if and only if $CU(X)$ is uniformly semi-locally simply-connected.*

Proof. c. Suppose X is locally path-connected. Choose an open cover $\{V_x\}_{x\in X}$ of X with the property that any loop in V_x based at x is null-homotopic in X. Let W_x be the path component of V_x containing x. Observe that any loop in W_x is null-homotopic in X. Choose a partition of unity $g = \{g_x\}_{x\in X}$ on X so that $g_x^{-1}(0,1] \subset V_x$ for all $x \in X$. Let f be its derivative. Suppose α is a loop at x in $B(x, E_f)$. Observe as in 2.5 that there is $y \in X$ so that $B(x, E_f) \subset W_y$. Therefore α is null-homotopic in X. □

3. Universal spaces in the uniform category

In this section we generalize concepts from [7] to the uniform category.

Definition 3.1. A uniform space X is a **uniform Peano space** if it is uniformly locally path-connected and connected.

In analogy to the universal covering spaces we introduce the following notion:

Definition 3.2. Given a uniform space X its **universal Peano space** $P(X)$ is a uniform Peano space together with a uniformly continuous map $\pi\colon P(X) \to X$ satisfying the following universality condition:

For any map $f\colon Y \to X$ from a uniform Peano space Y there is a unique uniformly continuous lift $g\colon Y \to P(X)$ of f (that means $\pi \circ g = f$).

Notice $P(X)$ is unique if it exists.

Theorem 3.3. *Every path-connected uniform space X has a universal Peano space. It is the set X equipped with the uniform structure generated by $\{pc(E)\}_E$, where $pc(E)$ is the set of pairs (x, y) that can be connected by an E-bounded path in X, E an entourage of X.*

Proof. Notice $(pc(F))^4 \subset pc(E)$ if $F^2 \subset E$, so $\{pc(E)\}_E$ does indeed generate a uniform structure on the set X resulting in a new uniform space $P(X)$ so that the identity function $P(X) \to X$ is uniformly continuous.

Given a uniformly continuous function $f\colon Y \to X$ from a uniform Peano space Y and given an entourage E of X there is an entourage F of Y contained in $f^{-1}(E)$ with the property that any F-close points in Y can be connected by an $f^{-1}(E)$-bounded path. Therefore $F \subset f^{-1}(pc(E))$ which proves $f\colon Y \to P(X)$ is uniformly continuous. It also proves $P(X)$ is path-connected as any path in X induces a path in $P(X)$. Since any $pc(E)$-close points can be connected by E-bounded path, $P(X)$ is uniformly locally path-connected. □

Corollary 3.4. *If X is a metrizable uniform space, then so is its universal Peano space $P(X)$.*

Proof. Notice $P(X)$ is Hausdorff and has a countable base of entourages. Such spaces are metrizable by [20, Theorem 13 on p.186]. □

Theorem 3.5. *For every pointed uniform Peano space (X, x_0) there is a universal object in the class of uniformly continuous maps $f\colon (Y, y_0) \to (X, x_0)$ such that Y is a uniform Peano space and f induces the trivial homomorphism $\check{\pi}_1(f)\colon \check{\pi}_1(Y, y_0) \to \check{\pi}_1(X, x_0)$ of the uniform fundamental groups.*

Proof. It is shown in [5] that there is a uniformly continuous map $\pi_X\colon GP(X, x_0) \to (X, x_0)$ from a uniform space of the trivial uniform fundamental group with the property that for any uniformly continuous map $f\colon (Y, y_0) \to (X, x_0)$ from a pointed uniform Peano space (Y, y_0) there is a unique lift $g\colon (Y, y_0) \to GP(X, x_0)$ if and only if f induces the trivial homomorphism of uniform fundamental groups (the result in [5] is for so-called uniformly joinable spaces and it applies here as uniform Peano spaces are uniformly joinable).

Let (Z, z_0) be the universal Peano space of $GP(X, x_0)$ with the induced uniformly continuous map $\pi\colon (Z, z_0) \to (X, x_0)$. Notice any uniformly continuous function $f\colon (Y, y_0) \to (X, x_0)$ has a unique lift to (Z, z_0) provided it induces the trivial homomorphism of the uniform fundamental groups. Since $\pi\colon (Z, z_0) \to (X, x_0)$ factors through $GP(X, x_0)$ it does induce the trivial homomorphism of the uniform fundamental groups. □

In order to apply 3.5 we need to discuss simple-connectivity of the space $GP(X, x_0)$ of generalized paths in X originating from x_0.

Proposition 3.6. *Suppose (X, x_0) is a pointed uniform Peano space. $GP(X, x_0)$ is simply connected if and only if the natural homomorphism $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism.*

Proof. Suppose $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism. Given a loop $\alpha\colon (S^1, 1) \to GP(X, x_0)$, its composition with $\pi_X\colon GP(X, x_0) \to (X, x_0)$ is trivial in $\check{\pi}_1(X, x_0)$ as $\check{\pi}_1(\pi_X) = 0$. Therefore $\pi_X \circ \alpha$ is trivial in $\pi_1(X, x_0)$ and can be extended over the 2-disk D^2. That map has a unique lift to $GP(X, x_0)$, hence α is null-homotopic.

Suppose $GP(X, x_0$ is simply connected and $\beta\colon (S^1, 1) \to (X, x_0)$ becomes trivial in $\check{\pi}_1(X, x_0)$. In that case β lifts to $GP(X, x_0)$ and is null-homotopic. □

Corollary 3.7. *Suppose (X, x_0) is a pointed uniform space. If the natural homomorphism $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism, then (X, x_0) has a universal simply-connected Peano space.*

Proof. From the commutativity of

$$\begin{array}{ccc} \pi_1(P(X, x_0)) & \longrightarrow & \pi_1(X, x_0) \\ \downarrow & & \downarrow \\ \check{\pi}_1(P(X, x_0)) & \longrightarrow & \check{\pi}_1(X, x_0) \end{array}$$

we get the natural homomorphism $\pi_1(P(X, x_0)) \to \check{\pi}_1(P(X, x_0))$ is a monomorphism and by 3.3 we can reduce the general case to that of (X, x_0) being a pointed uniform Peano space.

Consider $SP(X, x_0) = P(GP(X, x_0))$ with the induced map $\pi\colon SP(X, x_0) \to (X, x_0)$. Since $\check{\pi}_1(SP(X, x_0)) \to \check{\pi}_1(X, x_0)$ is trivial and $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism, $\pi_1(SP(X, x_0)) \to \pi_1(X, x_0)$ is trivial.

Given any uniformly continuous $f\colon (Y, y_0) \to (X, x_0)$ from a pointed uniform Peano space so that $\pi_1(f) = 0$ it suffices to show $\check{\pi}_1(f)\colon \check{\pi}_1(Y, y_0) \to \check{\pi}_1(P(X, x_0))$ is trivial as then it lifts uniquely to $P(GP(X, x_0))$.

Given an entourage E of $P(X)$ pick an entourage F of Y such that any two points $(x, y) \in F$ can be connected by a path contained in $B(x, f^{-1}(E))$. Given an element γ of $\check{\pi}_1(Y, y_0)$ its representative in $\mathcal{R}(Y, F)$ leads to a loop γ_E in Y at y_0 that represents γ in $\mathcal{R}(Y, f^{-1}(E))$. As $f(\gamma_E)$ is null-homotopic in $P(X)$, $\check{\pi}_1(f)(\gamma) = 1$. □

Proposition 3.8. *Suppose (X, x_0) is a pointed uniform space. If the natural homomorphism $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism, then the natural function from $\widetilde{X}_{BP}$ to $P(GP(P(X, x_0)))$ is a uniform equivalence.*

Proof. Since each path in (X, x_0) is a generalized path in $P(X, x_0)$, there is a natural function $i\colon \widetilde{X}_{BP} \to GP(P(X, x_0))$. It is uniformly continuous and $\widetilde{X}_{BP}$ is a uniform Peano space, hence $i\colon \widetilde{X}_{BP} \to P(GP(P(X, x_0)))$ is uniformly continuous. Since $P(GP(P(X, x_0)))$ is path-connected, its elements are generalized paths in $P(X, x_0)$ that are representable by genuine paths. That means i is surjective. If i is injective, then any loop in (X, x_0) □

Corollary 3.9 (Fisher-Zastrow [14])**.** *Suppose X is a path-connected paracompact space. If the natural homomorphism $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism for some $x_0 \in X$, then the projection $\widetilde{X}_{wh} \to X$ of X is a generalized universal cover.*

Proof. Consider X equipped with the canonical uniform structure $CU(X)$. As $\check{\pi}_1(CU(X), x_0) = \check{\pi}_1(X, x_0)$, there is a universal map $f\colon Y \to CU(X)$ among all uniformly continuous functions from uniform Peano spaces that induce trivial homomorphism of fundamental groups. Given a Peano space Z and a map $g\colon (Z, z_0) \to (X, x_0)$ inducing trivial homomorphism of the fundamental groups, we consider Z equipped with the canonical uniform structure $CU(Z)$. Notice in the proof of 3.7 that we only need to use $CU(Z)$ is uniformly joinable to lift g to $GP(X, x_0)$ (i.e., we do not need $CU(Z)$ to be a uniform Peano space which it certainly is if Z is paracompact by 2.7). Thus the lift exists, it lands in the path component of the trivial generalized path, and then we can lift it to the Peano space of that component as Z is a Peano space.

Consider $SP(X, x_0) = P(GP(P(X, x_0)))$ with the induced map $\pi\colon SP(X, x_0) \to (X, x_0)$. Since $\check{\pi}_1(SP(X, x_0)) \to \check{\pi}_1(X, x_0)$ is trivial and $\pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism, $\pi_1(SP(X, x_0)) \to \pi_1(X, x_0)$ is trivial.

Given any uniformly continuous $f\colon Y \to X$ from a uniform Peano space so that $\pi_1(f) = 0$ its lift $f\colon Y \to P(X)$ also satisfies $\pi_1(f) = 0$, so it suffices to show $\check{\pi}_1(f)\colon \check{\pi}_1(Y, y_0) \to \check{\pi}_1(P(X, x_0))$ is trivial as then it lifts uniquely to $P(GP(P(X, x_0)))$.

Given an entourage E of $P(X)$ pick an entourage F of Y such that any two points $(x, y) \in F$ can be connected by a path contained in $B(x, f^{-1}(E))$. Given an element γ of $\check{\pi}_1(Y, y_0)$ its representative in $\mathcal{R}(Y, F)$ leads to a loop γ_E in Y at y_0 that represents γ in $\mathcal{R}(Y, f^{-1}(E))$. As $f(\gamma_E)$ is null-homotopic in $P(X)$, $\check{\pi}_1(f)(\gamma) = 1$. □

4. Topological structures on $\widetilde{X}$

In this section we discuss topological structures on $\widetilde{X}$ and $\pi_1(X, x_0)$. Notice that neither of the two topologies on $\pi_1(X, x_0)$ discussed in this section makes $\pi_1(X, x_0)$ a topological group [7].

In case of X being metric $\pi_1(X, x_0)$ is considered as a quotient space $\pi_1^{top}(X, x_0)$ of the space $(X, x_0)^{(S^1, 1)}$ of loops equipped with the uniform convergence metric (see [12]). We generalize that concept as follows:

Definition 4.1. For any pointed topological space (X, x_0) let $\widetilde{X}_{top}$ be the quotient space of the function space $(X, x_0)^{(I, 0)}$ equipped with the compact-open topology.

$\pi_1^{top}(X,x_0)$ is the quotient space of $(X,x_0)^{(S^1,1)}$ equipped with the compact-open topology.

Spanier [30] introduced a new topology on $\pi_1(X,x_0)$ that was used by Bogley and Sieradski [3]. This topology was generalized later by Fisher and Zastrow [14] to $\widetilde{X}$ and used by the authors [7].

Definition 4.2. For any pointed topological space (X,x_0) the *whisker* topology on the set $\widetilde{X}$ is defined by the basis $B([\alpha],U)=\{[\alpha*\beta]\}$, where α is a path in X from x_0 to x_1, U is a neighborhood of x_1 in X, and β is a path in U originating at x_1 (we call the path β a U*-whisker*). We denote $\widetilde{X}$ with the whisker topology by $\widetilde{X}_{wh}$. By $\pi_1^{wh}(X,x_0)$ we mean $\pi_1(X,x_0)$ with the subspace topology inherited from $\widetilde{X}_{wh}$.

Given a neighborhood U of $\alpha(1)$ in X and a neighborhood W of x_0 in X define $B([\alpha],W,U)$ as elements of $\widetilde{X}$ of the form $\beta*\alpha*\gamma$, where β is a loop at x_0 in W and γ is a path in U originating at $\alpha(1)$.

Suppose α and β are two paths in X so that $\alpha(1)=\beta(0)$. Given a neighborhood $U_0\times U_1\times U_2$ of $(\alpha(0),\alpha(1),\beta(1))$ in X^3, we define $B([\alpha*\beta],U_0,U_1,U_2)$ as the set of homotopy classes (rel. end-points) of paths of the form $\omega*\alpha*\gamma*\beta*\lambda$, where ω is a loop in U_0 at $\alpha(0)$, γ is a loop in U_1 at $\alpha(1)$, and λ is a path in U_2 originating at $\beta(1)$.

Lemma 4.3. *Let $\pi\colon (X,x_0)^{(I,0)}\to\widetilde{X}$ be the projection. If V is a neighborhood of the concatenation $\alpha*\beta$ of two paths in $(X,x_0)^{(I,0)}$, then there is a neighborhood $U_0\times U_1\times U_2$ of $(x_0,\alpha(1),\beta(1))$ in X^3 such that $B([\alpha*\beta],U_0,U_1,U_2)\subset\pi(V)$.*

Proof. There exist a finite sequence of pairs $\{(C_s,V_s)\}_{s\in S}$ such that $\alpha(C_s)\subset V_s$ for all $s\in S$, where C_s are compact subsets of I, V_s are open subsets of X, and any path β in X satisfying $\beta(C_s)\subset V_s$ for all $s\in S$ belongs to V. Given $t\in I$, Let U_{2t} be the intersection of $\{V_s\}_{s\in T}$, where T consists of $s\in S$ so that $(\alpha*\beta)(t)\in C_s$ (if T is empty, we put $U_{2t}=X$).

Suppose ω is a loop in U_0 at $\alpha(0)$, γ is a loop in U_1 at $\alpha(1)$, and λ is a path in U_2 originating at $\beta(1)$.

Given $s<t$ let $u_{s,t}\colon [s,t]\to[0,1]$ be the increasing linear homeomorphism. Given $t<\frac{1}{4}$, define $\mu_t\colon[0,1]\to X$ as follows:

$$\mu_t(x)=\begin{cases}\omega(x) & \text{if } x\le t,\\ \alpha(x) & \text{if } t\le x\le 1/2-t,\\ \gamma(x) & \text{if } 1/2-t\le x\le 1/2+t,\\ \beta(x) & \text{if } 1/2+t\le x\le 1-t,\\ \lambda(x) & \text{if } 1-t\le x\le 1.\end{cases}$$

Notice $\mu_t(C_s)\subset V_s$ for t sufficiently close to 0. For such t we have $\mu_t\in V$, hence $[\omega*\alpha*\gamma*\beta*\lambda]=[\mu_t]\in\pi(V)$. That proves $B([\alpha*\beta],U_0,U_1,U_2)\subset\pi(V)$. □

Theorem 4.4. *For any pointed topological space (X,x_0) its universal covering space $\widetilde{X}$ equipped with the whisker topology is identical with the quotient space of the universal Peano space of $(X,x_0)^{(I,0)}$.*

Proof. Let $\pi\colon (X,x_0)^{(I,0)} \to \widetilde{X}$ be the projection. Suppose $\beta \in \pi^{-1}(B([\alpha],U))$ for some paths $\alpha, \beta \in (X,x_0)^{(I,0)}$ such that U is an open neighborhood of the end-point of α. That means $\beta \sim \alpha * \gamma$ for some path γ in U starting from $\alpha(1)$. Consider $V = \{\omega \in (X,x_0)^{(I,0)} \mid \omega(1) \subset U\}$ and assume λ belongs to the path-component of V containing β. That implies $\lambda \sim \beta * \mu$, where μ is a path in U. Thus $\lambda \sim \alpha * (\gamma * \mu)$ and $[\lambda] \in B([\alpha],U)$. That proves $P((X,x_0)^{(I,0)}) \to \widetilde{X}_{wh}$ is continuous.

Suppose $\pi^{-1}(A)$ is open in $P((X,x_0)^{(I,0)})$ for some subset A of $\widetilde{X}$ and $\alpha \in \pi^{-1}(A)$. There is an open neighborhood V of α whose path component containing α is a subset of $\pi^{-1}(A)$. By 4.3 there is a neighborhood U of $\alpha(1)$ in X so that $B([\alpha],U) \subset A$. As $B([\alpha],U)$ is path-connected, A is open. □

Problem 4.5. *Let (X,x_0) be a pointed topological space. Is $\widetilde{X}_{wh}$ the universal Peano space of $\widetilde{X}_{top}$? Equivalently, is the identity function $P(\widetilde{X}_{top}) \to \widetilde{X}_{wh}$ continuous?*

It is of interest to determine when the compact-open topology coincides with the whisker topology on $\widetilde{X}$. It turns out (see 4.11) it is related to continuity of the induced homomorphism $h_\alpha\colon \pi_1^{wh}(X,x_1) \to \pi_1^{wh}(X,x_0)$ ($h_\alpha([\beta]) := \alpha^{-1} * \beta * \alpha$), where α is a path from x_0 to any $x_1 \in X$. Therefore it makes sense to give a necessary and sufficient condition for continuity of h_α.

Lemma 4.6. *$h_\alpha\colon \pi_1^{wh}(X,x_1) \to \pi_1^{wh}(X,x_0)$ is continuous if and only if for any neighborhood U of x_0 there is a neighborhood V of x_1 such that for any loop β in V based at x_1, the loop $\alpha * \beta * \alpha^{-1}$ is homotopic rel. x_0 to a loop contained in U.*

Proof. Suppose h_α is continuous and U is a neighborhood of x_0. There is a neighborhood V of x_1 such that $h_\alpha(B(1,V)) \subset B(1,U)$ (here 1 is used for the trivial loops at x_0 and x_1). Given any loop γ in V at x_1 one has $[\gamma] \in B(1,V)$, so $h_\alpha([\gamma]) \in B(1,U)$.

Assume $h_\alpha(B(1,V)) \subset B(1,U)$. Given any loop γ in V at x_1 and given any loop ω at x_1 one has $h_\alpha([\omega * \gamma]) = h_\alpha([\omega]) * h_\alpha([\gamma]) \in B(h_\alpha([\omega]),U)$, so h_α is continuous. □

Definition 4.7. A topological space X is a **small loop transfer space** (SLT-space for short) if for every path α in X and every neighborhood U of $x_0 = \alpha(0)$ there is a neighborhood V of $x_1 = \alpha(1)$ such that given a loop $\beta\colon (S^1,1) \to (V,x_1)$ there is a loop $\gamma\colon (S^1,1) \to (U,x_0)$ that is homotopic to $\alpha * \beta * \alpha^{-1}$ rel. x_0.

Problem 4.8. *Is there a Peano SLT-space that is not semi-locally simply connected?*

Let us explain SLT-spaces in terms of homomorphism $h_\alpha\colon \pi_1^{wh}(X,x_1) \to \pi_1^{wh}(X,x_0)$ induced by a path α in X from x_0 to x_1 ($h_\alpha([\beta]) = \alpha * \beta * \alpha^{-1}$):

Corollary 4.9. *X is an SLT-space if and only if every path α in X from x_0 to x_1 induces a homeomorphism $h_\alpha\colon \pi_1^{wh}(X,x_1) \to \pi_1^{wh}(X,x_0)$*

Proposition 4.10. *The Hawaiian Earring is not an SLT-space.*

Proof. The Hawaiian Earring HE is the union of infinitely many circles C_n tangent to each other at the point 0 so that $\mathrm{diam}(C_n) = \frac{1}{n}$. Let g_n be the loop defined by circle C_n. If HE is an SLT-space, then there is $m > 0$ such that $g_1^{-1} * g_m * g_1$ is homotopic rel. end-points to a loop α of diameter less that 1.

Consider the retraction $r\colon HE \to C_1 \cup C_m$ sending C_i homeomorphically onto C_m if $1 < i \le m$ and sending C_i to 0 if $i > m$. Applying r to $g = g_1^{-1} * g_m * g_1$ shows g is homotopic rel. end-points to a power of g_m, a contradiction. □

Theorem 4.11. *If $\widetilde{X}_{top} = \widetilde{X}_{wh}$ for all $x_0 \in X$, then X is an SLT-space.*

Proof. Consider the projections $\pi\colon (X, x_0)^{(I,0)} \to \widetilde{X}_{wh}$ from $(X, x_0)^{(I,0)}$ equipped with the compact-open topology. By 4.4 the projections π are continuous for all $x_0 \in X$.

Suppose α is a path in X from x_0 to x_1 and U is a neighborhood of x_1. Since $\pi^{-1}(B([\alpha], U))$ is open and contains α, 4.3 says there are neighborhoods V of x_0 in X and W of x_1 in X such that $B([\alpha], V, W) \subset B([\alpha], U)$. Suppose β is a loop in V at x_0. There is a loop γ in U at x_1 so that $\beta * \alpha$ is homotopic to $\alpha * \gamma$ rel. end-points. That is equivalent to X being an SLT-space. □

Theorem 4.12. *If X is a Peano SLT-space, then $\pi_1^{wh}(X, x_0) = \pi_1^{top}(X, x_0)$ and $\widetilde{X}_{top} = \widetilde{X}_{wh}$.*

Proof. Suppose α is a path from x_0 to x_1 and U is a neighborhood of x_1 in X. For each $t \in [0, 1]$ let α_t be the path from $x_t = \alpha(t)$ to x_1 determined by α. For such t choose a path-connected neighborhood V_t of x_t with the property that for any loop β in V_t at x_t there is a loop γ in U at x_1 so that $\beta * \alpha_t$ is homotopic rel. end-points to $\alpha_t * \gamma$.

For each $t \in I$ choose a closed subinterval I_t of I containing t in its interior (rel. I) so that $\alpha(I_t) \subset V_t$. Choose finitely many of them that cover I so that no proper subfamily covers I. Let S be the set of such chosen points t. If $I_s \cap I_t \neq \emptyset$, let $V_{s,t}$ be the path component of $V_s \cap V_t$ containing $\alpha(I_s \cap I_t)$. Otherwise, $V_{s,t} = \emptyset$. Let W be the set of all paths β in X originating at x_0 such that $\beta(I_s \cap I_t) \subset V_{s,t}$ for all $s, t \in S$. It suffices to show $\pi(W) \subset B([\alpha], U)$.

Given β in W pick points $x_{s,t} \in I_s \cap I_t$ if $s \neq t$ and $I_s \cap I_t \neq \emptyset$. Then choose paths $\gamma_{s,t}$ in X from $\alpha(x_{s,t})$ to $\beta(x_{s,t})$. Arrange points $x_{s,t}$ in an increasing sequence y_i, $1 \le i \le n$, put $y_0 = 0$, $y_{n+1} = 1$, and create loops λ_i in X at $\alpha(y_i)$ as follows: travel along α from $\alpha(y_i)$ to $\alpha(y_{i+1})$, then along γ_{i+1}, then reverse β from $\beta(y_{i+1})$ to $\beta(y_i)$, finally reverse γ_i. Notice $\alpha^{-1} * \beta$ is homotopic to $\prod \alpha_i^{-1} * \lambda_i * \alpha_i$, where α_i is the path determined by α from $\alpha(y_i)$ to x_1. Each of $\alpha_i^{-1} * \lambda_i * \alpha_i$ is homotopic rel. x_1 to a loop in U, so we are done. □

In connection to 4.8 let us introduce a subclass of SLT-spaces:

Definition 4.13. A topological space X is called a **small loops space** (or an **SL-space** for short) if for every loop α in X at x and for any neighborhood U of x there is a loop in U homotopic rel. end-points to α.

SL-spaces are completely opposite to **homotopically Hausdorff spaces** that are defined by the condition that for any non-null-homotopic loop α at any $x \in X$ there is a neighborhood U_α of x such that $[\alpha]$ does not belong to the image of $\pi_1(U_\alpha, x) \to \pi_1(X, x)$ in (see [14]).

The following problem was solved by Žiga Virk [31, 32] after the preprint of this paper was posted.

Problem 4.14. *Is there an SL-space X that is not simply-connected?*

We introduced the lasso topology on the set $\widetilde{X}$ in [7].

Definition 4.15. Let $\mathcal{U}$ be an open cover of a topological space X and x be a point in X. A path l is called *$\mathcal{U}$-lasso based at the point* x if l is equal to a finite concatenation of loops $\alpha_n * \gamma_n * \alpha_n^{-1}$, where γ_n is a loop in some $U \in \mathcal{U}$ and α_n is a path from x to $\gamma_n(0)$.

Remark 4.16. Any conjugate of a $\mathcal{U}$-lasso is a $\mathcal{U}$-lasso. Indeed,

$$\beta * \left(\prod_{i=1}^{n} \alpha_i * \gamma_i * \alpha_i^{-1}\right) * \beta^{-1} = \prod_{i=1}^{n} \left(\beta * \alpha_i * \gamma_i * \alpha_i^{-1} * \beta^{-1}\right) = \prod_{i=1}^{n} (\beta * \alpha_i) * \gamma_i * (\beta * \alpha_i)^{-1}$$

Definition 4.17. For any topological space X the *lasso* topology on the set $\widetilde{X}$ is defined by the basis $B([g], \mathcal{U}, W)$, where W is a neighborhood of the endpoint $g(1)$ and $\mathcal{U}$ is an open cover of X. A homotopy class $[h] \in \widetilde{X}$ belongs to $B([g], \mathcal{U}, W)$ if and only if this class has a representative of the form $l * g * \beta$, where l is a $\mathcal{U}$-lasso based at $g(0)$ and β is a W-whisker of g. We denote by $\widetilde{X}_l$ the set $\widetilde{X}$ equipped with the lasso topology.

Definition 4.18. A topological space X is a **strong small loop transfer space** (strong SLT-space for short) if for each point x_1, each neighborhood U of x_1, and each point x_2 there is a neighborhood V of x_2 such that for every path α in X from x_1 to x_2 given any loop $\beta\colon (S^1, 1) \to (V, x_2)$ there is a loop $\gamma\colon (S^1, 1) \to (U, x_1)$ that is homotopic to $\alpha * \beta * \alpha^{-1}$ rel. x_1.

Proposition 4.19. *A path-connected topological space X is a strong SLT-space if and only if $\widetilde{X}_l = \widetilde{X}_{wh}$ for all $x_0 \in X$.*

Proof. Suppose X is a strong SLT-space. Given any point $x_0 \in X$ and a basic neighborhood $B([\omega], U)$ in $\widetilde{X}_{wh}$, we need to find a basic neighborhood $B([\omega], \mathcal{V}, W)$ of $\widetilde{X}_l$ inside $B([\omega], U)$. Any point $x \in X$ has a neighborhood V defined by the strong SLT-property applied to the point $x_1 = \omega(1)$ and its neighborhood U. Let $\mathcal{V}$ be a cover of X by such neighborhoods V. Put $W = U$ and consider any element $[l * \omega * \beta] \in B([\omega], \mathcal{V}, W)$. Suppose l is equal to a finite concatenation of loops $l = \prod_{i=1}^{n} \alpha_i * \gamma_i * \alpha_i^{-1}$, where γ_i is a loop in some $V \in \mathcal{V}$ and α_i is a path from x_0 to $\gamma_i(0)$. Then $[l * \omega * \beta] = [\omega * \left(\prod_{i=1}^{n} \omega^{-1} * \alpha_i * \gamma_i * \alpha_i^{-1} * \omega\right) * \beta]$. By the strong SLT property each loop $\omega^{-1} * \alpha_i * \gamma_i * \alpha_i^{-1} * \omega$ is homotopic rel. x_1 to a loop inside U, thus $[l * \omega * \beta]$ belongs to $B([\omega], U)$.

Suppose now that $\widetilde{X}_l = \widetilde{X}_{wh}$ for any choice of the basepoint. Let $x_1 \in X$ be a point and U be its neighborhood. Denote by $[x_1]$ the trivial path based at x_1. Consider the spaces $\widetilde{X}_l = \widetilde{X}_{wh}$ based at x_1. The neighborhood $B([x_1], U)$ in $\widetilde{X}_{wh}$ contains a lasso neighborhood $B([x_1], \mathcal{V}, W)$, where $W \subset U$. Let x_2 be any point in X and V be any element of the cover $\mathcal{V}$ containing x_2. Then for any path α in X from x_1 to x_2 and any loop $\beta\colon (S^1, 1) \to (V, x_2)$ the class $[\alpha * \beta * \alpha^{-1}] \in B([x_1], \mathcal{V}, W) \subset B([x_1], U)$ has a representative $\gamma\colon (S^1, 1) \to (U, x_1)$. □

Since we introduce topologies on the fundamental group, it is natural to ask if these topologies make the fundamental group a topological group (i.e. if the operations of multiplication and taking inverse are continuous in the given topology). The fundamental group with the lasso topology is a topological group [7]. In general, neither the quotient of the compact-open topology nor the whisker topology make the fundamental group a topological group [13, 7]. It is interesting to note that these topologies fail to make $\pi_1(X, x_0)$ a topological group for different reasons. The inversion is always continuous in the group $\pi_1^{top}(X, x_0)$ (which makes it a quasi topological group [1]). On the other hand, the continuity of the inversion in the group $\pi_1^{wh}(X,x_0)$ would make it a topological group:

Proposition 4.20. *Let (X,x_0) be a pointed topological space. If the operation of taking inverse in $\pi_1^{wh}(X,x_0)$ is continuous, then $\pi_1^{wh}(X,x_0)$ is a topological group.*

Proof. One only needs to show continuity of the concatenation operation. Let α and β be two loops and $B([\alpha*\beta],U)$ be a neighborhood of their concatenation in $\pi_1^{wh}(X,x_0)$. The continuity of taking inverse (applied at the element $[\beta]^{-1}\in\pi_1^{wh}(X,x_0)$) implies the existence of a neighborhood V of x_0 such that for any loop γ in V the inverse of $[\beta^{-1}*\gamma]$ belongs to $B([\beta],U)$, i.e. $[\beta^{-1}*\gamma]^{-1}=[\gamma^{-1}*\beta]=[\beta*\mu]$ for some loop μ in U. Then for any $[\alpha*\gamma]\in B([\alpha],V)$ and any $[\beta*\delta]\in B([\beta],U)$ the concatenation $[\alpha*\gamma*\beta*\delta]=[\alpha*\beta*\mu*\delta]$ belongs to $B([\alpha*\beta],U)$. □

Proposition 4.21. *$\pi_1^{wh}(X,x_0)$ is discrete if and only if X is semi-locally simple-connected at x_0.*

Proof. If there is a neighborhood U of x_0 such that every loop in U at x_0 is null-homotopic in X, then $B([\alpha],U)$ contains only $[\alpha]$ among all homotopy classes of loops at x_0. Conversely, if $B([\alpha],U)$ contains only $[\alpha]$ among all homotopy classes of loops at x_0, then $[\alpha]=[\alpha*\gamma]$ for every loop γ in U at x_0, i.e. $[\gamma]=1$.

Theorem 4.22. *Suppose X is a path-connected space. If $\pi_1^{top}(X,x_0)$ contains an isolated point, then X is semi-locally simply-connected. If X is semi-locally simply-connected Peano space, then $\pi_1^{top}(X,x_0)$ is discrete.*

Proof. Choose an open set V in $\widetilde{X}_{top}$ whose intersection with $\pi_1^{top}(X,x_0)$ is exactly $[\alpha]$. Given $x_1\in X$ choose a path λ from x_0 to x_1. Since $[(\alpha*\lambda)*\lambda^{-1}]\in V$, by 4.3 there is a neighborhood U of x_1 such that $[(\alpha*\lambda)*\gamma*\lambda^{-1}]\in V$ for any loop γ in U at x_1. Since V contains only the homotopy class of the loop α, $[\alpha]=[(\alpha*\lambda)*\gamma*\lambda^{-1}]$ and that implies γ is null-homotopic in X.

If X is semi-locally simply-connected, then it is an SLT-space. By 4.12, we have $\pi_1^{top}(X,x)=\pi_1^{wh}(X,x)$ for every $x\in X$. Application of 4.21 completes the proof. □

Corollary 4.23 (Fabel [12])**.** *Suppose X is a connected locally path-connected metrizable space. The group $\pi_1^{top}(X,x_0)$ is discrete if and only if X is semi-locally simply-connected.*

5. Uniform structures on $\widetilde{X}$

In this section we introduce uniform structures on $\widetilde{X}$ and $\pi_1(X,x_0)$.

Given a uniform space X and $x_0\in X$ we consider the space $(X,x_0)^{(I,0)}$ of continuous functions with the **uniform convergence structure** denoted by $(X,x_0)^{(I,0)}_{UC}$. The base of that structure consists of elements $uc(E):=\{(\alpha,\beta)|(\alpha(t),\beta(t))\in E \text{ for all } t\in I\}$, where E is an entourage of X.

The projection $\pi\colon (X,x_0)^{(I,0)}\to\widetilde{X}$ is defined by $\pi(\alpha)=[\alpha]$.

Our first example of a uniform structure on $\widetilde{X}$ is the **James uniform structure** (see [19, p.120]):

Definition 5.1. The **James uniform structure** on $\widetilde{X}$ is generated by sets E^* consisting of pairs $([\alpha],[\beta])$ such that (α,β) is a path in $E\subset X\times X$ originating at (x_0,x_0).

$\widetilde{X}_J$ is the space $\widetilde{X}$ equipped with the James uniform structure.

Since it is not entirely obvious $\{E^*\}$ form a base of a uniform structure, let us provide some details that will be also useful later on.

Definition 5.2. By a **punctured square** we mean a subset of $I \times I$ whose complement in $I \times I$ is a finite union of interiors of disks D_i such that $D_i \cap D_j = \emptyset$ if $i \neq j$ and $\bigcup D_i$ is contained in the interior of $I \times I$.

By an **E-punctured-homotopy** from a path α to a path β originating from the same point $x_0 \in X$ we mean a map H from a punctured square to X satisfying the following conditions:

(1) $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$ for all $t \in I$.
(2) $H(0,s) = x_0$ for all $s \in I$.
(3) $H(\{1\} \times I)$ is E-bounded.
(4) $H(\partial D_i)$ is E-bounded for all disks D_i.

Lemma 5.3. *Let X be a uniform space with two given entourages E and F so that any pair $(x,y) \in F$ can be connected by an E-bounded path in X. Suppose $\alpha, \beta \colon (I,0) \to (X,x_0)$ are two paths in X. If the corresponding elements $[\alpha],[\beta]$ of $\widetilde{X}$ satisfy $([\alpha],[\beta]) \in \pi(uc(F))$, then there is an E^2-punctured-homotopy from α to β.*

Proof. If $([\alpha],[\beta]) \in \pi(uc(F))$, then there is $(\alpha',\beta') \in F$ so that α' is homotopic to α rel. end-points and β' is homotopic to β rel. end-points. Find $\epsilon > 0$ with the property that $\alpha'(J)$ and $\beta'(J)$ are E-bounded for any subinterval J of I of diameter at most ϵ. Subdivide I into an increasing sequence of points $t_0 = 0, t_1, \ldots, t_n = 1$ so that $t_{i+1} - t_i < \epsilon$ for all $0 \leq i < n$ and choose an E-bounded path γ_i in X from $\alpha'(t_i)$ to $\beta'(t_i)$. Use it to construct an E^2-punctured-homotopy from α' to β'. Combine it with homotopies from α to α' and from β to β' to create an E^2-punctured-homotopy from α to β. $\square$

Lemma 5.4. *Given two paths $\alpha, \beta \colon (I,0) \to (X,x_0)$, the corresponding elements $[\alpha],[\beta]$ of $\widetilde{X}$ satisfy $([\alpha],[\beta]) \in \pi(uc(E^2))$ if there is an E-punctured-homotopy H from α to β.*

Proof. Without loss of generality we may assume the domain of H is obtained from $I \times I$ by removing mutually disjoint disks D_i, $1 \leq i \leq n$, centered at $(x_i, \frac{1}{2})$ so that $x_i > x_j$ if $i < j$. Let α' be the path in X obtained from H by starting at $(0, \frac{1}{2})$ and traversing upper semicircles of boundries of disks D_i, reaching $(1, \frac{1}{2})$, and then going up to $(1,1)$. Let β' be the path in X obtained from H by starting at $(0, \frac{1}{2})$ and traversing lower semicircles of boundries of disks D_i, reaching $(1, \frac{1}{2})$, and then going down to $(1,0)$. Notice α' is homotopic to α rel. end-points, β' is homotopic to β rel. end-points, and $(\alpha',\beta') \in uc(E^2)$. $\square$

Corollary 5.5. *If X is uniformly locally path-connected, then the projection $\pi \colon (X,x_0)^{(I,0)} \to \widetilde{X}$ $(\pi(\alpha) = [\alpha])$ generates the James uniform structure on $\widetilde{X}$.*

Proof. First observe $E^* = \pi(uc(E))$, so if π generates a uniform structure, it must be identical with that of James.

Given an entourage E of X find an entourage F such that $F^4 \subset E$. If $([\alpha],[\beta]) \in \pi(uc(F))$ and $([\beta],[\gamma]) \in \pi(uc(F))$ there exist E^2-punctured homotopies H from α to β and G from β to γ. The concatenation $H * G$ is an E^2-punctured-homotopy from α to γ hence $([\alpha],[\gamma]) \in \pi(uc(F^4)) \subset \pi(uc(E))$. $\square$

Berestovskii and Plaut [2] introduced a uniform structure on the space of E-chains in X. It easily generalizes to a uniform structure on $\widetilde{X}$:

Definition 5.6. The **Berestovskii-Plaut uniform structure** on $\widetilde{X}$ (denoted by $\widetilde{X}_{BP}$) is generated by sets $bp(E)$ consisting of pairs $([\alpha],[\beta])$ such that $\alpha^{-1}*\beta$ is homotopic in X rel. end-points to an E-bounded path.

It is indeed a uniform structure: if $F^4\subset E$, then $(bp(F))^2\subset bp(E)$. Note how much easier it is to introduce it in comparison to the compact-open topology or James structure.

Notice that the projection $\pi_X\colon \widetilde{X}_{BP}\to X$ is uniformly continuous. Also, it is surjective if and only if X is path-connected.

Proposition 5.7. *If X is completely regular, then the whisker topology on $\widetilde{X}$ is identical with the topology induced on $\widetilde{X}$ by the Berestovskii-Plaut uniform structure corresponding to the canonical uniform structure on X.*

Proof. Given a path α from x_0 to x_1 in X and given a neighborhood U of x_1 in X, the set $B([\alpha],U)$ contains $B([\alpha],bp(E))$ for any entourage E so that $B(x_1,E^2)\subset U$. Also, $B([\alpha],bp(E))\supset B([\alpha],U)$ if $U\subset B(x_1,E)$. $\square$

Proposition 5.8. *The map $id\colon \widetilde{X}_{BP}\to\widetilde{X}_J$ is uniformly continuous for any uniform space X.*

Proof. If two paths α and β have their homotopy classes close in the Berestovskii-Plaut uniform structure (i.e. $([\alpha],[\beta])\in bp(E)$ for some entourage E), then $\alpha^{-1}*\beta$ is homotopic in X rel. end-points to some E-bounded path γ. Thus β is homotopic rel. end-points to $\alpha*\gamma$. Since γ is E-bounded, the path $(\alpha*\gamma,\alpha*const)\subset X\times X$ is contained in E, therefore $[\alpha*\gamma]$ and $[\alpha*const]$ are E^*-close in the James uniform structure. Notice that $[\alpha*\gamma]=[\beta]$ and $[\alpha*const]=[\alpha]$. $\square$

It is of interest to characterize spaces X so that $id\colon\widetilde{X}_J\to\widetilde{X}_{BP}$ is uniformly continuous. Here is the corresponding concept to 4.7 in the uniform category:

Definition 5.9. A uniform space X is a **uniform small loop transfer space** (a uniform SLT-space for short) if for every entourage E of X there is an entourage F of X such that given two loops $\alpha\colon(S^1,1)\to(B(y,F),y)$ and $\beta\colon(S^1,1)\to(X,x)$ that are freely homotopic, there is a loop $\gamma\colon(S^1,1)\to(B(x,E),x)$ that is homotopic to β rel. base point.

Theorem 5.10. *If X is path-connected and uniformly locally path-connected, then $id\colon\widetilde{X}_J\to\widetilde{X}_{BP}$ is uniformly continuous if and only if X is a uniform SLT-space.*

Proof. Suppose X is an SLT-space and E is an entourage of X. Find an entourage F of X such that given two loops $\alpha\colon(S^1,1)\to(B(y,F^2),y)$ and $\beta\colon(S^1,1)\to(X,x)$ that are freely homotopic, there is a loop $\gamma\colon(S^1,1)\to(B(x,E),x)$ that is homotopic to β rel. base point. It suffices to show $\pi(uc(F))\subset bp(E^3)$, so assume $([\alpha],[\beta])\in\pi(uc(F))$. By 5.3 there is an F^2-punctured-homotopy H from α to β. Without loss of generality we may assume the domain of H is obtained from $I\times I$ by removing mutually disjoint disks D_i, $1\le i\le n$, centered at $(x_i,\frac12)$ so that $x_i>x_j$ if $i<j$. Let α_1 be the path in X obtained from re-parametrizing H on the interval $[r_1,1]\times\{\frac12\}$ traversed to the left (in the decreasing direction of the first coordinate), where $(r_1,\frac12)$ belongs to ∂D_1. More generally, α_i, $n\ge i>1$, is the path in X obtained from H by traveling from the left-most point L_{i-1} of

$\partial D_{i-1} \cap I \times \{\frac{1}{2}\}$ to the right-most point R_i of $\partial D_i \cap I \times \{\frac{1}{2}\}$. Paths β_i, $1 \le i \le n$, are obtained from H by traveling from R_i to L_i along the lower part of ∂D_i. Paths γ_i, $1 \le i \le n$, are obtained from H by traveling from R_i to L_i along the upper part of ∂D_i.

Define l_k as $\prod_{i=1}^{k} \alpha_i * \beta_i$ and define u_k as $\prod_{i=1}^{k} \alpha_i * \gamma_i$ for $k \le n$. We will show by induction on k that $l_k * u_k^{-1}$ is homotopic rel. end-points to a loop ω_k in $B(x_1, E)$, where $x_1 = H(1, \frac{1}{2})$. That will complete the proof of the first part of 5.10 as $\alpha^{-1} * \beta$ is homotopic rel. end-points to $\delta * \omega_n * \mu$, where δ corresponds to $H|\{1\} \times [0, \frac{1}{2}]$ and μ corresponds to $H|\{1\} \times [\frac{1}{2}, 1]$.

Notice $l_1 * u_1^{-1}$ is freely homotopic to $\beta_1 * \gamma_1^{-1}$ that is contained in $B(R_1, F^2)$, so ω_1 exists. To do inductive step observe $l_k * u_k^{-1} \sim l_{k-1} * \beta_k * \gamma_k^{-1} * u_{k-1}^{-1} \sim (l_{k-1} * \beta_k * \gamma_k^{-1} * l_{k-1}^{-1}) * (l_{k-1} * u_{n-1}$ (here $\sim$ stands for homotopy rel. end-points). Since $l_{k-1} * \beta_k * \gamma_k^{-1} * l_{k-1}^{-1}$ is freely homotopic to a small loop $\beta_k * \gamma_k^{-1}$, it is homotopic to a loop λ contained in $B(x_1, E)$. Thus $l_k * u_k^{-1} \sim \lambda * \omega_{k-1}$ and $\lambda * \omega_{k-1}$ is contained in $B(x_1, E)$.

Suppose $id \colon \widetilde{X}_{UC} \to \widetilde{X}_J$ is uniformly continuous and E is an entourage of X. Pick an entourage F of X such that $uc(F^2) \subset bp(E)$. Given a loop α in $B(x_1, F)$ and a path λ from x_1 to x_0 represent α as $\beta * \gamma$ and notice $([\beta^{-1} * \lambda], [\gamma * \lambda]) \in uc(F^2)$. Hence $([\beta^{-1} * \lambda], [\gamma * \lambda]) \in bp(E)$ and there is a loop μ in $B(x_0, E)$ such that μ is homotopic rel. end-points to $(\beta^{-1} * \lambda)^{-1} * (\gamma * \lambda) \sim \lambda^{-1} * \alpha * \lambda$. That proves any loop freely homotopic to α is homotopic rel. end-points to a small loop. $\square$

Problem 5.11. *Is there a uniform SLT-space X that is not a uniform Poincare space?*

We introduce the lasso uniformity on the set $\widetilde{X}$ for any uniform space X.

Definition 5.12. Let E be an entourage of a uniform space X and x be a point in X. A path l is called *E-lasso based at the point x* if l is equal to a finite concatenation of loops $\alpha_n * \gamma_n * \alpha_n^{-1}$, where each loop γ_n is E-bounded and α_n is a path from x to $\gamma_n(0)$.

Definition 5.13. The **lasso uniform structure** on $\widetilde{X}$ (denoted by $\widetilde{X}_l$) is generated by sets $l(E)$ consisting of pairs $([\alpha], [\beta])$ such that for some E-lasso l the path $\alpha^{-1} * l * \beta$ is homotopic in X rel. end-points to an E-bounded path.

Lemma 5.14. *Let X be a uniform space, E be an entourage, and α and β be two paths in X. The classes $[\alpha]$ and $[\beta]$ are E-close in the lasso uniformity on $\widetilde{X}$ if and only if there is an E-punctured homotopy from α to β.*

Proof. We may assume that the domain of an E-punctured homotopy H from α to β is obtained from $I \times I$ by removing mutually disjoint congruent disks D_i, $1 \le i \le n$, centered at $(x_i, \frac{1}{2})$ so that $x_i > x_j$ if $i < j$. Let α_i be the straight path from the point $(0,0)$ to the point p_i of the boundary of the disk D_i with the smallest y-coordinate (so that any path α_i is disjoint from other paths α_j and disks D_j). Let the loop γ_i be based at the point p_i and traversing the boundary of the disk D_i once. Then H makes the path $\prod_{i=1}^{n} \alpha_i * \gamma_i * \alpha_i^{-1}$ an E-lasso l in X and thus H can be interpreted as a homotopy between $\alpha^{-1} * l * \beta$ and an E-bounded path. Conversely, a homotopy between $\alpha^{-1} * l * \beta$ and an E-bounded path can be interpreted as an E-punctured homotopy from α to β. □

The universality condition of the universal Peano space $P(X)$ (see 3.2) allows us to identify the sets $\widetilde{X}$ and $\widetilde{P(X)}$ since any path in $P(X)$ projects to a path in X and any path in X lifts uniquely to a path in $P(X)$.

Proposition 5.15. *Let X be a uniform space. The natural identification of the sets $\widetilde{X}$ and $\widetilde{P(X)}$ generates a uniform homeomorphism of $\widetilde{X}_l$ and $\widetilde{P(X)}_J$*

Proof. Suppose that two paths α and β in X are E-close in the lasso uniformity. By 5.14, there is an E-punctured homotopy from α to β. By 5.4, the classes $[\alpha]$ and $[\beta]$ have uniformly E^2-close representatives (notice that the representatives constructed in the proof of 5.4 are also uniformly E^2-close in $P(X)$). By 5.5, the classes $[\alpha]$ and $[\beta]$ are close in the James uniform structure on $\widetilde{P(X)}$.

Suppose that two paths α and β in $P(X)$ are $pc(E)$-close in the James uniform structure on $\widetilde{P(X)}$. By 5.3, there is an E^2-punctured-homotopy from α to β. By 5.14, the classes $[\alpha]$ and $[\beta]$ are E^2-close in the lasso uniformity on $\widetilde{X}$. □

Corollary 5.16. *For any connected and uniformly locally path-connected uniform space X the James uniformity on $\widetilde{X}$ is identical with the lasso uniformity.*

Proof. For any connected and uniformly locally path-connected uniform space X we have $P(X) = X$. □

A uniform structure on $\widetilde{X}$ induces a uniform structure on the fundamental group $\pi_1(X, x_0)$. It is natural to ask if this uniform structure makes the group a topological group.

Recall that the quotient of the compact-open topology does not make $\pi_1(X, x_0)$ a topological group for the single reason of the product of quotient maps not being a quotient map [13]. If the space X is a uniform space, the situation is better: each product of uniform quotient mappings is a uniform quotient mapping [17]. Thus the fundamental group equipped with the James uniform structure is a topological group. The Berestovskii-Plaut uniform structure is unlikely to make the fundamental group a topological group for the same reason as the whisker topology does not make $\pi_1(X, x_0)$ a topological group (recall 5.7).

Although the lasso uniformity on $\pi_1(X, x_0)$ can be related to the James uniformity, we give a direct simple proof of the following:

Proposition 5.17. *The lasso uniform structure makes the fundamental group a topological group.*

Proof. By definition, two elements of $[\alpha], [\beta] \in \pi_1(X, x_0)$ are E-close if for some E-lasso l the loop $\alpha^{-1} * l * \beta$ is homotopic to an E-bounded loop γ. That means for the E-lasso $L = l * \beta * \gamma * \beta^{-1}$ the loop $\alpha^{-1} * L * \beta$ is homotopically trivial. Thus two elements of $[\alpha], [\beta] \in \pi_1(X, x_0)$ are E-close if the loop $\alpha * \beta^{-1}$ is homotopic to an E-lasso.

The inverse operation in $\pi_1(X, x_0)$ is continuous in the lasso uniformity because if the loop $\alpha * \beta^{-1}$ is homotopic to an E-lasso L, then the loop $(\alpha^{-1}) * (\beta^{-1})^{-1} = \alpha^{-1} * \beta$ is homotopic to an E-lasso $\alpha^{-1} * L^{-1} * \alpha$ (use 4.16).

The product operation in $\pi_1(X, x_0)$ is continuous in the lasso uniformity because if the loop $\alpha * \beta^{-1}$ is homotopic to an E-lasso L and the loop $\gamma * \delta^{-1}$ is homotopic to an E-lasso M, then the loop $(\alpha*\gamma)*(\beta*\delta)^{-1} \sim \alpha*M*\beta^{-1} \sim \alpha*\beta^{-1}*\beta*M*\beta^{-1} \sim L*(\beta*M*\beta^{-1})$ is homotopic to an E-lasso (use 4.16). □

References

[1] A. Arhangelskii, M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, 1. Atlantis Press, Paris, 2008.

[2] V. Berestovskii, C. Plaut, *Uniform universal covers of uniform spaces*, Topology Appl. 154 (2007), 1748–1777.

[3] W.A. Bogley, A.J. Sieradski, *Universal path spaces*, http://oregonstate.edu/~bogleyw/#research

[4] J. Brazas, *The fundamental group as a topological group*, Preprint arXiv:1009.3972v5.

[5] N. Brodskiy, J. Dydak, B. Labuz, A. Mitra, *Rips complexes and covers in the uniform category*, Preprint math.MG/0706.3937, accepted by the Houston Journal of Math.

[6] N. Brodskiy, J. Dydak, B. Labuz, A. Mitra, *Group actions and covering maps in the uniform category*, Topology Appl., 157 (2010), 2593–2603. doi:10.1016/j.topol.2010.07.011.

[7] N. Brodskiy, J. Dydak, B. Labuz, A. Mitra, *Covering maps for locally path-connected spaces*, Preprint arXiv:0801.4967.

[8] J. Dugundji, *A topologized fundamental group*, Proc. Nat. Acad. Sci. U. S. A. 36, (1950). 141–143.

[9] J. Dydak, *Partitions of unity*, Topology Proceedings 27 (2003), 125–171.

[10] J. Dydak, J. Segal, *Shape theory: An introduction*, Lecture Notes in Math. 688, 1–150, Springer Verlag 1978.

[11] P. Fabel, *The Hawaiian earring group and metrizability*, ArXiv:math.GT/0603252.

[12] P. Fabel, *Metric spaces with discrete topological fundamental group*, Topology and its Applications 154 (2007), 635–638.

[13] P. Fabel, *Multiplication is Discontinuous in the Hawaiian Earring Group (with the Quotient Topology)*, Bull. Pol. Acad. Sci. Math. 59 (2011), no. 1, 7783. DOI: 10.4064/ba59-1-09

[14] H. Fischer, A. Zastrow, *Generalized universal coverings and the shape group*, Fundamenta Mathematicae 197 (2007), 167–196.

[15] E. Ghys and E. de la Harpe (eds), *Sur les groupes hyperboliques d'apres de Mikhael Gromov*, Birkhäuser, 1990.

[16] W. Hurewicz, *Homotopie, Homologie und lokaler Zusammenhang*, Fundamenta Mathematicae 25 (1935), 467–485.

[17] M. Hušek, M. Rice, *Productivity of coreflective subcategories of uniform spaces*, General Topology Appl. 9 (1978), no. 3, 295–306.

[18] J. Isbell, *Uniform Spaces*, Mathematical Surveys, vol. 12, American Mathematical Society, Providence, RI, 1964.

[19] I.M. James, *Introduction to Uniform Spaces*, London Math. Soc. Lecture Notes Series 144, Cambridge University Press, 1990.

[20] J.L. Kelley, *General topology*, D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[21] J. Krasinkiewicz, P. Minc, *Generalized paths and pointed 1-movability*, Fund. Math. 104 (1979), no. 2, 141–153.

[22] B. Labuz, *Inverse limits and uniform covers*, in preparation.

[23] E.L. Lima, *Fundamental groups and covering spaces*, AK Peters, Natick, Massachusetts, 2003.

[24] S. Mardešić, J. Segal, *Shape theory*, North-Holland Publ.Co., Amsterdam 1982.

[25] S.A. Melikhov, *Steenrod homotopy*, (Russian) Uspekhi Mat. Nauk 64 (2009), no. 3(387), 73–166; translation in Russian Math. Surveys 64 (2009), no. 3, 469–551.

[26] J.R. Munkres, *Topology*, Prentice Hall, Upper Saddle River, NJ 2000.

[27] H. Nakano, *Uniform spaces and transformation groups*. Wayne State University Press, Detroit, Mich. 1968 xv+253 pp.

[28] J. Pawlikowski, *The fundamental group of a compact metric space*, Proceedings of the American Mathematical Society, 126 (1998), 3083–3087.

[29] C. Plaut, *Quotients of uniform spaces*, Topology Appl. 153 (2006), 2430–2444.

[30] E. Spanier, *Algebraic topology*, McGraw-Hill, New York 1966.

[31] Ž. Virk, *Small loop spaces*, Topology Appl. 157 (2010), 451–455.

[32] Ž. Virk, *Homotopical smallness and closeness*, Topology Appl. 158 (2011), 360–378.

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