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Price manipulation in a market impact model with dark pool

Florian Klöck* Alexander Schied** Yuemeng Sun‡

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Abstract

For a market impact model, price manipulation and related notions play a role that is similar to the role of arbitrage in a derivatives pricing model. Here, we give a systematic investigation into such regularity issues when orders can be executed both at a traditional exchange and in a dark pool. To this end, we focus on a class of dark-pool models whose market impact at the exchange is described by an Almgren–Chriss model. Conditions for the absence of price manipulation for all Almgren–Chriss models include the absence of temporary cross-venue impact, the presence of full permanent cross-venue impact, and the additional penalization of orders executed in the dark pool. When a particular Almgren–Chriss model has been fixed, we show by a number of examples that the regularity of the dark-pool model hinges in a subtle way on the interplay of all model parameters and on the liquidation time constraint.

Key Words: Price manipulation, transaction-triggered price manipulation, positive expected liquidation costs, dark pool, market impact model, optimal order execution, optimal liquidation

1 Introduction

Recent years have seen a mushrooming of alternative trading platforms called *dark pools*. Orders placed in a dark pool are not visible to other market participants (hence the name) and thus do not influence the publicly quoted price of the asset. Thus, when dark-pool orders are executed against a matching order, no direct price impact is generated, although there may be certain indirect effects. Dark pools therefore promise a reduction of market impact and of the resulting liquidation costs. They are hence a popular platform for the execution of large orders.

Dark pools differ from standard limit order books in that they do not have an intrinsic price finding mechanism. Instead, the price at which orders are executed is derived from the publicly quoted prices at an exchange. Thus, by manipulating the price at the exchange through placing suitable buy or sell orders, the value of a possibly large amount of "dark liquidity" in the dark pool can be altered. For this reason, dark pools have drawn significant attention by regulators; see IOSCO (2011). We refer to Mittal (2008) for a practical overview on dark pools and some related issues of market manipulation.

In this paper, we consider a stochastic model for order execution at two possible venues: a dark pool and an exchange. This model is a continuous-time variant of the one proposed by Kratz and Schöneborn (2010). It is a natural model, because it extends the standard Almgren–Chriss market impact model for exchange prices by a dark pool, where incoming matching orders are described by a compound Poisson process. We refer to Almgren (2003) for details on the Almgren–Chriss model and also to Bertsimas and Lo (1998) for a discrete-time precursor. A different approach to modeling and analyzing dark pools was proposed by Laruelle *et al.* (2010).

*University of Mannheim, Department of Mathematics, A5, 6, 68131 Mannheim, Germany, kloeck@uni-mannheim.de
**University of Mannheim, Department of Mathematics, A5, 6, 68131 Mannheim, Germany, schied@uni-mannheim.de
‡Cornell University, ORIE, Rhodes Hall, Ithaca, NY 14850, U.S.A., ys273@cornell.edu

Kratz and Schöneborn (2010) mainly investigate optimal order execution strategies for an investor who can trade at the exchange and in the dark pool. But they are also interested in price manipulation strategies in the sense of Huberman and Stanzl (2004). Their Propositions 7.1 and 7.2 provide some first results on the existence and the absence of such strategies, and they propose the further investigation of this problem. We refer to Huberman and Stanzl (2004), Gatheral (2010), Alfonsi *et al.* (2012), and our Section 3 for discussions on the importance of the absence of price manipulation strategies. In Section 3 we will argue in particular that the absence of price manipulation and related concepts can be regarded as a regularity condition that plays a similar role for a market impact model as the absence of arbitrage for a derivatives pricing model.

Our main goal in this paper is to investigate in a systematic manner the existence and absence of price manipulation with dark pools and related topics. To this end, we modify the setup of Kratz and Schöneborn (2010) in several ways. On the one hand, we simplify their setup by using the concrete continuous-time, single-asset Almgren–Chriss model to describe market impact at the exchange and by restricting the possibilities for adjusting the sizes of orders in the dark pool[1]. On the other hand, we allow for additional possibilities of cross-impact between the two venues and for additional "slippage" in dark-pool execution.

In Section 4.1, our first main result characterizes completely those models from our class that are sufficiently regular for all underlying Almgren–Chriss models, either in the sense of the absence of price manipulation or in terms of the new condition of "positive expected liquidation costs". The critical quantities will be the size of "slippage" and the degrees of permanent and temporary cross-venue impact. In Section 4.2, we then investigate the existence of model irregularities for special model characteristics. It will turn out that generation of such irregularities hinges in a subtle way on the interplay of all model parameters and on the liquidation time constraint. In Section 4.3 we illustrate in a simplified setting that our regularity condition guarantees the existence of optimal order execution strategies, and we show how such strategies can be computed.

The paper is organized as follows. In the subsequent Section 2 we introduce the model and formulate our standing assumptions. In Section 3 we review and discuss several notions for the regularity of a market impact model, namely the absence of standard and transaction-triggered price manipulation and a new condition of positive expected liquidation costs. Our main results are stated in Section 4 and proved in Section 6. We conclude in Section 5.

2 Model setup

We will analyze a continuous-time variant of the market impact model with dark pool that was proposed in Kratz and Schöneborn (2010). This model is natural since it extends the continuous-time version of the standard Almgren–Chriss market impact model for an investor who can generate price impact by trading at an exchange; see Almgren (2003) for details on this model and also Bertsimas and Lo (1998) for a discrete-time precursor. The Almgren–Chriss model has been the basis of many academic studies pertaining to market impact and is also common in industry application.

In the Almgren–Chriss market impact model, it is assumed that the number of shares in the trader's portfolio is described by an absolutely continuous trajectory $t \mapsto X_t$, the *trading strategy.* Given this trading trajectory, the price at which transactions occur is

$$P_t = P_t^0 + \gamma(X_t - X_0) + h(\dot{X}_t). \tag{1}$$

[1]Kratz and Schöneborn (2010) allow for arbitrary adaptive adjustment of the sizes of orders in the dark pool. In our model, these orders can only be placed at the beginning of the trading period, and their remainder can be cancelled at a later time. The possibility of arbitrary adaptive adjustment of dark-pool orders influences the particular form of optimal order execution strategies, but it does not have a significant impact on the existence and absence of price manipulation in comparison to our setting, at least if we exclude so-called 'fishing' strategies (see Remark 3.5). Most dark pools operating in practice will probably have order placement and cancellation policies which lie in between these two possibilities.

Here, P_t^0 is the *unaffected stock price process*. The term $h(\dot{X}_t)$ describes the *temporary* or *instantaneous impact* of trading $\dot{X}_t\,dt$ shares at time t and only affects this current order. The term $\gamma(X_t - X_0)$ corresponds to the *permanent price impact* that has been accumulated by all transactions until time t. It is usually assumed to be linear in $X_t - X_0$ with γ denoting a positive constant, because linearity is also needed so as to exclude price manipulation; see Huberman and Stanzl (2004) or Gatheral (2010), see also Almgren *et al.* (2005) for empirical justification.

Assumption 2.1. We assume that the unaffected stock price process $(P_t^0)_{t\geq 0}$ is a càdlàg martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ for which $\mathcal{F}_0$ is $\mathbb{P}$-trivial. Trading strategies (X_t) must be adapted to the filtration $(\mathcal{F}_t)$. The permanent-impact parameter γ is assumed to be strictly positive. The temporary-impact function $h : \mathbb{R} \to \mathbb{R}$ is assumed to have the following properties: h is continuous, strictly increasing, and satisfies $h(0) = 0$ and $|h(x)| \to \infty$ for $|x| \to \infty$. Moreover, the function $f(x) := xh(x)$ is assumed to be convex.

The condition that P^0 is a martingale is a standard assumption in the market impact literature. One reason is that drift effects can be ignored due to the usually short trading horizons. In addition, we are interested here in the qualitative effects of price impact on the stability of the model. A nonzero drift would lead to the existence of profitable "round trips" that would have to be distinguished from price manipulation strategies in the sense of Definition 3.1. Our assumptions on h are satisfied for the popular choices of linear temporary impact, $h(x) = \eta x$, or more generally for power-law impact,

$$h(x) = \eta \,\mathrm{sgn}(x)|x|^{\nu} \tag{2}$$

where η and ν are positive constants and, typically, $\nu \leq 1$; see Almgren *et al.* (2005) for a discussion. An Almgren–Chriss model is thus defined in terms of the parameters

$$(\gamma, h, P^0). \tag{3}$$

The Almgren–Chriss model is a market impact model for exchange-traded orders. We will now extend this model by allowing the additional execution of orders in a *dark pool*. A dark pool is an alternative trading venue in which unexecuted orders are invisible to all other market participants. In this dark pool, buy and sell orders are matched and executed at the current price at which the asset is traded at the exchange.

In addition to a trading strategy executed at the exchange, investors can place an order of $\hat{X}$ shares into the dark pool at time $t = 0$. This order will be matched with incoming orders of the opposite side. These orders arrive at random times $0 < \tau_1 < \tau_2 < \dots$ and we denote the size of incoming matching orders by $\tilde{Y}_1, \tilde{Y}_2, \dots > 0$. We consider only those orders that are a possible match. That is, the $\tilde{Y}_i$ will describe sell orders when $\hat{X} > 0$ is a buy order and buy orders when $\hat{X} < 0$ is a sell order. These incoming orders will then be matched piece by piece with the order $\hat{X}$ until it is cancelled or completely filled. That is,

$$Y_i := \begin{cases} \mathrm{sgn}(\hat{X})\tilde{Y}_i, & \text{if } \sum_{j=1}^{i} \tilde{Y}_j \leq |\hat{X}|, \\ \hat{X} - \mathrm{sgn}(\hat{X})\sum_{j=1}^{i-1} Y_j, & \text{if } \sum_{j=1}^{i-1} \tilde{Y}_j \leq |\hat{X}| \text{ and } \sum_{j=1}^{i} \tilde{Y}_j > |\hat{X}|, \\ 0, & \text{if } \sum_{j=1}^{i-1} \tilde{Y}_j > |\hat{X}|, \end{cases}$$

is the part of the incoming order that is actually executed against the remainder of $\hat{X}$. By defining the counting process associated with the arrival times (τ_k),

$$N_t := \max\{k \in \mathbb{N} \,|\, \tau_k \leq t\}, \tag{4}$$

the amount of shares that have been executed in the dark pool until time t can be conveniently denoted by

$$Z_t := \sum_{i=1}^{N_t} Y_i.$$

By $(\mathcal{G}_t)$ we denote the rightcontinuous filtration generated by $(\mathcal{F}_t)$ and Z.

In the first part of the paper, we make some very mild assumptions on the laws and interdependence of the random variables (τ_i), $(\tilde{Y}_i)$, and P^0:

Assumption 2.2. We assume the following conditions:

$$0 < \tau_1 < \infty \ \mathbb{P}\text{-a.s. and } \lambda_0 := \inf_{0<\delta\leq 1} \frac{1}{\delta}\mathbb{P}[\,\tau_1 \leq \delta\,] > 0; \tag{5}$$

$$\text{there exists } x_0 > 0 \text{ such that } \lambda_1 := \inf_{\delta>0} \mathbb{P}[\,\tilde{Y}_1 \geq x_0 \,|\, \tau_1 \leq \delta\,] > 0. \tag{6}$$

We furthermore assume that P^0 is a martingale also under the filtration $(\mathcal{G}_t)$ generated by $(\mathcal{F}_t)$ and Z.

Condition (5) means that the intensity for the arrival of the first matching order is bounded away from zero. Condition (6) states that there is a positive probability that the first incoming matching order has at least size x_0, conditional on the event that $\{\tau_1 \leq \delta\}$. Clearly, these assumptions are very mild. The requirement that P^0 is a $(\mathcal{G}_t)$-martingale allows (τ_i) and $(\tilde{Y}_i)$ to depend on P^0 in an arbitrary manner but, conversely, limits the dependence of P^0 on these random variables. This limitation is entirely natural since we will explicitly model the possible dependence of exchange-quoted prices on dark-pool executions via (9). Note that Assumption 2.2 is satisfied in particular when τ_1 has an exponential distribution and (τ_i), $(\tilde{Y}_i)$, and P^0 are independent random variables, as we will assume in the second part of the paper.

Now we consider an investor who must liquidate an initial asset position of $X_0 \in \mathbb{R}$ shares during the time interval $[0, T]$. The problem of how to do this in an optimal fashion is known as the optimal order execution problem; see, e.g., Gökay *et al.* (2010), Schöneborn (2008), Schied and Slynko (2011), and the references therein.

In the extended dark pool model, the investor will first place an order of $\hat{X} \in \mathbb{R}$ shares in the dark pool[2] and then choose a liquidation strategy of Almgren–Chriss-type for the execution of the remaining assets at the exchange. This latter strategy must be absolutely continuous in time. It will thus be described by a process (ξ_t) that parameterizes the speed by which shares are sold at the exchange. Moreover, until fully executed, the remaining part of the order $\hat{X}$ can be cancelled at a (possibly random) time $\rho < T$. Hence, the number of shares held by the investor at time t is

$$X_t := X_0 + \int_0^t \xi_s \, ds + Z^\rho_{t-}, \tag{7}$$

where Z^ρ_{t-} denotes the left-hand limit of $Z^\rho_t = Z_{\rho\wedge t}$.

Definition 2.3. Let an initial position $X_0 \in \mathbb{R}$ and a liquidation horizon $T > 0$ be given. An *admissible trading strategy* is a triple $\chi := (\hat{X}, \xi, \rho)$ where $\hat{X} \in \mathbb{R}$, ρ is a $(\mathcal{G}_t)$-stopping time such that

[2] If at time $t = 0$ the dark pool contains an order $\tilde{Y}_0$ of the opposite side, then the investor could fill this order immediately and then start liquidating the remaining asset position $X_0 - \tilde{Y}_0$, maybe by resizing the dark-pool order. Therefore we can assume that the dark pool does not contain a matching order at $t = 0$. Moreover, restricting the placement of dark-pool orders to $t = 0$ lets us exclude so-called 'fishing' strategies; see Remark 3.5 below.

$\rho < T$ $\mathbb{P}$-a.s., and ξ is a $(\mathcal{G}_t)$-predictable process that is $\mathbb{P}$-a.s. bounded uniformly in t and ω. In addition, the liquidation constraint

$$X_0 + \int_0^T \xi_t \, dt + Z_\rho = 0 \tag{8}$$

must be $\mathbb{P}$-a.s. satisfied. The set of all admissible strategies for given X_0 and T is denoted by $\mathcal{X}(X_0, T)$.

Due to (7) and (8), the terminal asset position of any admissible strategy is $X_T = 0$, since our requirement $\rho < T$ implies that $Z^\rho_{T-} = Z_\rho$.

Now we turn to the definition of the prices at which the orders at the exchange and in the dark pool are executed. In particular, we will specify the cross impacts of order execution in the dark pool on the exchange price and vice versa. Here, our approach is to introduce a model that is flexible enough to allow for a wide range of possible mutual influences of orders executed on both venues. Extending (1), the price at which assets can be traded at the exchange is defined as

$$P_t = P^0_t + \gamma \left(\int_0^t \xi_s \, ds + \alpha Z^\rho_{t-} \right) + h(\xi_t). \tag{9}$$

Here $\alpha \in [0, 1]$ describes the intensity of the possible permanent impact of an execution in the dark pool on the price quoted at the exchange. The existence of such a cross-venue impact can be made plausible by noting that without the dark pool the matching order would have been executed at the exchange and there would have generated permanent price impact in a favorable direction. Thus, the price impact generated by the execution of a dark-pool order can be understood in terms of a deficiency in opposite price impact.

The price at which the i^{th} incoming order is executed in the dark pool will be

$$\begin{aligned} \hat{P}_{\tau_i} &= P^0_{\tau_i} + \gamma \left(\int_0^{\tau_i} \xi_s \, ds + \alpha Z_{\tau_i -} + \beta Y_i \right) + g(\xi_{\tau_i}) \\ &= P_{\tau_i} + \beta\gamma Y_i + (g(\xi_{\tau_i}) - h(\xi_{\tau_i})). \end{aligned} \tag{10}$$

In this price, orders executed at the exchange have full permanent impact, but their possible temporary impact is described by a function $g : \mathbb{R} \to \mathbb{R}$. The parameter $\beta \geq 0$ in (10) describes additional "slippage" related to the dark-pool execution, which will result in transaction costs of the size $\beta\gamma Y_i^2$. It may also be used to account for hidden costs, which relate to dark pools but which are extremely difficult to model explicitly. For instance, one can think of costs arising from the phenomena of adverse selection or 'fishing'; see Mittal (2008) and Kratz and Schöneborn (2010). Moreover, due to the very nature of dark pools, data may be sparse so that there will be a high degree of model uncertainty. The parameter β can thus also serve as a penalization of dark-pool orders in view of adverse selection, model misspecification, fishing (see Remark 3.5), and other hidden costs that are difficult to model explicitly. In this case, $\hat{P}_{\tau_i}$ in (10) is not the actual price at which the dark-pool order is executed, but it is a virtual adjusted price that includes hidden costs and penalties.

Assumption 2.4. We assume that $\alpha \in [0, 1]$, $\beta \geq 0$, and that g either vanishes identically or satisfies the conditions on h in Assumption 2.1.

Definition 2.5. The *dark-pool extension* of a given Almgren–Chriss model is defined in terms of the new parameters

$$(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i)) \tag{11}$$

satisfying Assumptions 2.2 and 2.4.

Our main goal in this paper is to study the influence of these parameters on the stability and regularity of the model and, in particular, on the optimal execution problem. Our investigation will be based on an analysis of the revenues generated by a trading strategy. In such a strategy, $\xi_t\,dt$ shares are bought at price P_t at each time t. In addition, Y_i shares are bought at price $\hat{P}_{\tau_i}$ at each time τ_i. The revenues generated by the strategy until time T are thus given by

$$\mathcal{R}_T = -\int_0^T \xi_s P_s\,ds - \sum_{i=1}^{N_{T\wedge\rho}} Y_i \hat{P}_{\tau_i}. \tag{12}$$

To emphasize the dependence of $\mathcal{R}_T$ on the strategy $\chi = (\hat{X}, \xi, \rho)$ we will sometimes also write $\mathcal{R}_T^\chi$.

3 Price manipulation

Our main concern in this paper is to investigate the stability and regularity of the dark-pool extension in dependence on the parameters (γ, h, P^0) and $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$. This question is analogous to establishing the absence of arbitrage in a derivatives pricing model, where absence of arbitrage is a necessary condition for the existence of replicating strategies of a given contingent claim.

But there must also be a difference in the notions of regularity of a derivatives pricing model and of a market impact model. In a derivatives pricing model, one is interested in constructing strategies that almost surely replicate a given contingent claim, and this is the reason why one must exclude the existence of arbitrage opportunities defined in the usual almost-sure sense. In a market impact model, one is interested in constructing optimal order execution strategies. These strategies are not defined in terms of an almost-sure criterion but as minimizers of a cost functional of a risk averse investor. Commonly used cost functionals involve expected value as in Bertsimas and Lo (1998) and Gatheral (2010), mean-variance criteria as in Almgren and Chriss (2001), expected utility as in Schied and Schöneborn (2009) and Schöneborn (2008), or alternative risk criteria as in Forsyth *et al.* (2010) and Gatheral and Schied (2011). Therefore, also the regularity conditions to be imposed on a market impact model need to be formulated in a similar manner. To make such regularity conditions independent of particular investors preferences, it is reasonable to formulate them in a risk-neutral manner:

Definition 3.1 (Huberman and Stanzl (2004)). A *round trip* is an admissible trading strategy with $X_0 = 0$. A *price manipulation strategy* is a round trip that has strictly positive expected revenues, $\mathbb{E}[\mathcal{R}_T] > 0$.

When the revenues are a concave functional of an order execution strategy, as it is often the case, the existence of price manipulation precludes the existence of optimal execution strategies for risk-neutral investors, because one can generate arbitrarily large expected revenues by adding a multiple of a price manipulation strategy. In most cases, the same argument also works for risk-averse investors provided that risk aversion is small enough. The problem of characterizing the absence of price manipulation in a dark-pool model was formulated in Kratz and Schöneborn (2010), along with some first results in that direction. Analyses of the absence of price manipulation in various other market impact models were given, e.g., by Huberman and Stanzl (2004), Gatheral (2010), Alfonsi and Schied (2010), Alfonsi *et al.* (2012), and Gatheral *et al.* (2011).

It was observed in Alfonsi *et al.* (2012) that the absence of price manipulation may not be sufficient to guarantee the stability of the model, because optimal order execution strategies can still oscillate strongly between alternating buy and sell trades, a property one should exclude for obvious reasons. This was the reason for introducing the following notion.

Definition 3.2 (Alfonsi *et al.* (2012))**.** A market impact model admits *transaction-triggered price manipulation* if the expected execution costs of a sell (buy) program can be decreased by intermediate buy (sell) trades.

In our situation there would be transaction-triggered price manipulation if there exists $X_0 \in \mathbb{R}$ and a strategy $(\hat{X}, \xi, \rho)$ for which either $\hat{X}$ or some ξ_t have the same sign as X_0 and that has strictly higher expected revenues than all strategies $(\hat{X}', \xi', \rho')$ for which both $\hat{X}'$ and ξ'_t have always the opposite sign of X_0. We will also consider the following notion:

Definition 3.3. The model has *positive expected liquidation costs* if for all $X_0 \in \mathbb{R}$, $T > 0$, and every corresponding order execution strategy

$$\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0. \tag{13}$$

Condition (13) states that on average it is not possible to make a profit beyond the face value of a position out of the market impact generated by one's own trades. We have the following hierarchy of regularity conditions in our model.

Proposition 3.4. (a) *If there is no transaction-triggered price manipulation, we have positive expected liquidation costs.*

(b) *If we have positive expected liquidation costs, then there is no price manipulation.*

Implication (a) holds for every market impact model in which buy orders increase the price and sell orders decrease the price. This implication is particularly useful in models where the condition of positive expected liquidation costs is violated, since it immediately yields the existence of transaction-triggered price manipulation. Implication (b) clearly holds for every market impact model.

Remark 3.5. A common price manipulation strategy is the so-called 'fishing' strategy in dark pools; see Mittal (2008). In a fishing strategy, agents first send small orders to dark pools so as to detect dark liquidity. Once a dark-pool order is detected, the visible price at the exchange is manipulated for a short period in a direction that is unfavorable for that order. Finally, a large order is sent to the dark pool so as to be executed against the dark liquidity at the manipulated price.

Here, we are not interested in the profitability of such predatory fishing strategies but primarily in the stability and regularity of optimal order execution algorithms in dark pool and exchange. We therefore exclude fishing strategies by allowing the placement of orders in the dark pool only at time $t = 0$. Allowing for the placement of dark-pool orders at times $t > 0$ will increase the class of admissible strategies. In such an extended setting, the conditions of no-price manipulation or of positive expected liquidation costs will be violated as soon as they are violated in our present setting.

4 Results

An Almgren–Chriss model is specified by the parameters (γ, h, P^0) satisfying Assumption 2.1. Its extension incorporating a dark pool is based on the additional parameter set $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$, which will always be assumed to satisfy Assumptions 2.2 and 2.4. We are interested in the conditions we need to impose on these parameters such that the extended market model is regular. Here, regularity refers to the absence of price manipulation and related notions as explained in the preceding section.

4.1 General regularity results

Our first result characterizes completely those parameters $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$ for which the dark-pool extension of *every* Almgren–Chriss model is sufficiently regular for all time horizons.

Theorem 4.1. *For given $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$, the following conditions are equivalent.*

(a) *For any Almgren–Chriss model, the dark-pool extension has positive expected liquidation costs.*

(b) *For any Almgren–Chriss model, the dark-pool extension does not admit price manipulation for every time horizon $T > 0$.*

(c) *We have $\alpha = 1, \beta \geq \frac{1}{2}$ and $g = 0$.*

Remark 4.2. Let us comment on the three conditions in part (d) of the preceding theorem.

(i) The requirement $\alpha = 1$ means that an execution of a dark-pool order must generate the same permanent impact on the exchange-quoted price as a similar order that is executed at the exchange. At first sight, this requirement might seem artificial. At second thought, however, one realizes that price impact generated by the execution of a dark-pool order can be understood in terms of a deficiency in opposite-price impact; see the discussion following (9).

(ii) The condition $\beta \geq \frac{1}{2}$ means that the execution of a dark-pool order of size Y_i needs to generate "slippage" of at least $\frac{\gamma}{2} Y_i^2$. This latter amount is just equal to the costs from permanent impact one would have incurred by executing the order at the exchange. With this amount of slippage, the savings by executing an order not at the exchange but at a dark pool would thus be equal to the costs generated by permanent impact. It seems that dark pools that are currently operative do not charge transaction costs or taxes of this magnitude. Nevertheless, our theorem states that a penalization with a factor $\beta \geq \frac{1}{2}$ is needed for a robust stabilization of the model against irregularities.

(iii) The requirement $g = 0$ means that temporary impact from trades executed at the exchange must not affect the price at which dark-pool orders are executed. This requirement may not be surprising, although the $(\mathcal{G}_t)$-predictability of the exchange-traded part (ξ_t) of an admissible strategy excludes short-term manipulation in immediate response of the arrival of a matching order in the dark pool.

In Theorem 4.1, it is crucial that we may vary at least the parameter h of the underlying Almgren-Chriss model. If all parameters are fixed, we can only obtain the following implication instead of an equivalent characterization of regular models.

Theorem 4.3. *Suppose an Almgren–Chriss model with parameters (γ, h, P^0) has been fixed. When a dark-pool extension $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$ of this model does not admit price manipulation for all $T > 0$, then*

$$\beta \geq \alpha - \frac{1}{2}. \tag{14}$$

If, in addition, there is equality in (14) *and $g(x) = \kappa h(x)$ for some constant $\kappa \geq 0$, then $\kappa = 0$ and $\alpha = 1$.*

In the next section, we will analyze several concrete situations in which some of the model parameters are chosen in a particular way. Our corresponding results will first illustrate that (14) cannot be improved in general. For instance, it will follow from Corollary 4.11 that even in the case $\alpha = \beta = 0$ it may happen that there is no price manipulation for all $T > 0$, but this situation is then characterized in terms of relations between γ, h, and the law of τ_1.

4.2 Regularity and irregularity for special model characteristics

In this section, we will investigate in more detail the regularity and irregularity of a dark-pool extension of a *fixed* Almgren–Chriss model. To this end, we will assume throughout this section that slippage is zero, $\beta = 0$, which is the natural (naive) first guess in setting up a dark-pool extension of an Almgren–Chriss model. We know from Theorem 4.1, though, that there must be some Almgren–Chriss model such that there is price manipulation for sufficiently large time horizon T.

First, we will look into the role played by T in the existence of price manipulation. We show there exists a critical threshold $T^* \geq 0$ such that there is no price manipulation for $T < T^*$ but price manipulation does exist for $T > T^*$. We will show that all three cases $T^* = \infty$, $0 < T^* < \infty$, and $T^* = 0$ can occur. Second, we will analyze the stronger requirements of absence of transaction-triggered price manipulation and of positive expected liquidation costs. We will find situations in which there is no price manipulation for all $T > 0$ but where the condition of positive expected liquidation costs fails and where there is transaction-triggered price manipulation for sufficiently large T.

We will make the following simplifying but natural assumption on the dark-pool extension defined through $(\alpha, \beta, g, (\tau_i), (\tilde{Y}_i))$.

Assumption 4.4. We assume the following conditions throughout Section 4.2.

(a) Slippage is zero: $\beta = 0$.

(b) The process (N_t), as defined in (4), is a standard Poisson process with parameter $\theta > 0$ and $(\tilde{Y}_i)$ are i.i.d. random variables with common distribution μ on $(0, \infty]$. We also assume that the stochastic processes (P_t^0), (N_t), and $(\tilde{Y}_i)$ are independent.

Note that Assumption 4.4 (b) implies that

$$\lim_{t \uparrow \infty} \sum_{i=1}^{N_t} Y_i = \hat{X} \qquad \mathbb{P}\text{-a.s.} \tag{15}$$

Note also that we do not exclude the possibility that $\tilde{Y}_i$ takes the value $+\infty$ with positive probability. The particular case $\tilde{Y}_i = +\infty$ $\mathbb{P}$-a.s., corresponding to $\mu = \delta_\infty$, can be regarded as the limiting case of infinite liquidity in the dark pool. It results in $Y_1 = \hat{X}$ and hence in an immediate execution of the entire order $\hat{X}$. In fact, many dark pools allow the specification of lower limits on the size of matching orders, for instance to avoid the effects of 'fishing'. So, in principle, it should be possible to set this lower limit equal to $\hat{X}$. Unless $\mu = \delta_\infty$, setting such a limit will however lower the arrival rate θ of matching orders.

In Propositions 4.5 and 4.7, we will consider the situation in which the execution of a dark-pool order has full permanent impact on the price at the exchange, i.e., $\alpha = 1$. In view of our assumption $\beta = 0$, Theorem 4.3 implies that there will be price manipulation for sufficiently large T. The following proposition shows that one then can also generate arbitrarily large expected revenues. In contrast to the situation in many other market impact models, this conclusion is not obvious in our case, because the expected revenues are typically not a concave functional of admissible strategies.

Proposition 4.5. *Suppose that an Almgren–Chriss model has been fixed and that $\alpha = 1$. Then, for any $X_0 \in \mathbb{R}$,*

$$\lim_{T \uparrow \infty} \sup_{\chi \in \mathcal{X}(X_0, T)} \mathbb{E}[\mathcal{R}_T^\chi] = +\infty.$$

In particular, the condition of positive expected liquidation costs is violated.

Now we examine in more detail the role played by T in the existence of price manipulation. First, we show that for a certain class of models there is no price manipulation for small T.

Proposition 4.6. *Let $g = 0$ and $h(x) = \eta x$. If $T \leq \frac{2\eta}{\gamma}$, then there is no price manipulation.*

Since the class $\mathcal{X}(X_0, T)$ of admissible strategies increases with T, the existence of price manipulation for one T implies the existence of price manipulation for any $T' \geq T$. Hence there exists a critical value T^* such that there is no price manipulation for $T < T^*$ but price manipulation does exist for $T > T^*$. For $\alpha = 1$ and linear temporary impact, the next proposition shows, that $T^* = \frac{2\eta}{\gamma}$. Furthermore, we show that $T^* = 0$ for sublinear temporary impact.

Proposition 4.7. *Suppose that an Almgren–Chriss model has been fixed and that $\alpha = 1$.*

(a) *If $g = 0$ and temporary impact in the Almgren–Chriss model is linear, i.e., $h(x) = \eta x$, then there is no price manipulation if and only if*

$$T \leq \frac{2\eta}{\gamma}. \tag{16}$$

(b) *If $\mathbb{P}[\tilde{Y}_1 > x] > 0$ for all x and h has sublinear growth, i.e.,*

$$\lim_{|x|\to\infty} \frac{h(x)}{x} = 0,$$

then there is price manipulation for every $T > 0$.

For the next set of results, we will assume that

$$\alpha = 0, \qquad g = 0 \qquad \text{and} \qquad h(x) = \eta x, \tag{17}$$

in addition to Assumption 4.4. By Theorem 4.1, we know that there exists an Almgren–Chriss model for which there is price manipulation and for which the condition of positive expected liquidation costs is violated for sufficiently large T. Our aim is to give a refined analysis for the class of Almgren–Chriss models with linear temporary price impact. We first take a look at the condition of positive expected liquidation costs.

Proposition 4.8. *Consider a fixed Almgren–Chriss model and suppose that condition (17) holds.*

(a) *If $\frac{\gamma}{\eta} < 2\theta$, we have for any $X_0 \in \mathbb{R} \setminus \{0\}$,*

$$\lim_{T\uparrow\infty} \sup_{\mathcal{X}\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^{\mathcal{X}}] \geq X_0 P_0 + \frac{1}{2}\gamma^2 X_0^2 \frac{1}{2\eta\theta - \gamma} > X_0 P_0. \tag{18}$$

(b) *If either $\frac{\gamma}{\eta} = 2\theta$ and $X_0 \neq 0$ or $\frac{\gamma}{\eta} > 2\theta$, then*

$$\lim_{T\uparrow\infty} \sup_{\mathcal{X}\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^{\mathcal{X}}] = +\infty.$$

In particular, the condition of positive expected liquidation costs is violated in both cases.

Proposition 3.4 immediately yields that there is transaction-triggered price manipulation in the situations considered in Proposition 4.8. We are interested in the form of these manipulations.

Proposition 4.9. *Consider a fixed Almgren–Chriss model and suppose that condition (17) holds. The violation of positive expected liquidation costs in Proposition 4.8 can only be obtained by intermediate buy (sell) trades at the exchange during an overall sell (buy) program.*

Therefore, if T is large enough, only a strategy that manipulates the exchange-quoted price can be more profitable than other strategies.

Some of the preceding results can be strengthened in the infinite-liquidity limit $\mu = \delta_\infty$. We refer to the paragraph after Assumption 4.4 for a discussion of this condition. We first show that (18) actually becomes an equality.

Proposition 4.10. *Consider a fixed Almgren–Chriss model. Suppose moreover that condition* (17) *holds and that $\mu = \delta_\infty$. Then, for $X_0 \in \mathbb{R}$ and $\frac{\gamma}{\eta} < 2\theta$,*

$$\lim_{T\uparrow\infty} \sup_{\chi\in\mathcal{X}(X_0,T)} \mathbb{E}[\mathcal{R}_T^\chi] = X_0P_0 + \frac{1}{2}\gamma^2 X_0^2 \frac{1}{2\eta\theta - \gamma}. \tag{19}$$

Equation (19) is remarkable, because it implies on the one hand that the condition of positive expected liquidation costs is violated. By taking $X_0 = 0$ we see, on the other hand, that there is no price manipulation and $T^* = \infty$. In fact, we have the following result.

Corollary 4.11. *Consider a fixed Almgren–Chriss model. Suppose moreover that condition* (17) *holds and that $\mu = \delta_\infty$. Then there is no price manipulation for every $T > 0$ if and only if $\frac{\gamma}{\eta} \le 2\theta$.*

By comparing the preceding result with Propositions 3.4 and 4.10, we arrive the following statement.

Corollary 4.12. *Under the assumptions of Corollary 4.11 there is always transaction-triggered price manipulation for sufficiently large T. Standard price manipulation, however, exists only for $\frac{\gamma}{\eta} > 2\theta$.*

4.3 Optimal order execution strategies

In this section, we illustrate some of our results by determining an optimal strategy for selling $X_0 > 0$ shares. To this end, we will make a number of simplifying assumptions, because our main goal is to analyze the regularity of the model. In particular, for us, optimality of a strategies refers to the maximization of the expected revenues. For a detailed analysis of optimal order execution strategies in a discrete-time model with dark pool we refer to Kratz and Schöneborn (2010).

We fix an Almgren–Chriss model and assume that Assumption 4.4 (b) holds and that

$$\alpha = 1,\ \beta = \frac{1}{2},\ g = 0. \tag{20}$$

Then Theorem 4.1 guarantees that there is no price manipulation. For simplicity, we will also assume that there is infinite liquidity in the dark pool in the sense that

$$\mu = \delta_\infty. \tag{21}$$

Then the entirety of the dark-pool order $\hat{X}$ will either be filled when $\tau_1 \le \rho$ or it will be cancelled when $\tau_1 > \rho$. In this setting, an admissible strategy $(\hat{X}, \xi, \rho)$ will be called a *single-update strategy* if ρ is a deterministic time in $[0, T)$ and ξ is predictable with respect to the filtration generated by the stochastic process $\mathbb{1}_{\{\tau_1 \le t\}}$, $t \ge 0$.

Note that the process ξ of a single-update strategy evolves deterministically until there is an execution in the dark pool, i.e., until time τ_1. At that time, ξ can be updated. But the update will only depend on the time τ_1 and not on any other random quantities. In particular, ξ can be written as

$$\xi_t = \begin{cases} \xi_t^0, & \text{if } t \le \tau_1 \text{ or } \tau_1 > \rho, \\ \xi_t^1, & \text{if } t > \tau_1 \text{ and } \tau_1 \le \rho, \end{cases} \tag{22}$$

where ξ^0 is deterministic and ξ^1 depends on τ_1.

Proposition 4.13. *Suppose that that Assumption 4.4* (b), (20)*, and* (21) *hold. For any* $X_0 \in \mathbb{R}$ *and* $T > 0$ *there exists a single-update strategy that maximizes the expected revenues* $\mathbb{E}[\mathcal{R}_T]$ *in the class of all admissible strategies.*

Now we show how an optimal single-update strategy can be computed. To this end, we make the additional simplifying assumption that temporary impact is linear, $h(x) = \eta x$. It will follow from Equation (40) in the proof of Proposition 4.13 that the expected revenues of a single-update strategy are given by

$$\begin{aligned}\mathbb{E}[\mathcal{R}_T] = X_0 P_0 - \frac{1}{2}\gamma X_0^2 - \int_0^\rho \eta(\xi_s^0)^2 e^{-\theta s}\, ds - \eta e^{-\theta\rho}\frac{(X_0 - \int_0^\rho \xi_s^0\, ds)^2}{T-\rho} \\ - \int_0^\rho \eta\theta e^{-\theta t}\frac{(X_0 - \int_0^t \xi_s^0\, ds - \hat{X})^2}{T-t}\, dt.\end{aligned} \tag{23}$$

A standard calculation shows that the strategy $X_t^0 := X_0 - \int_0^t \xi_s^0\, ds$, $0 \le t \le \rho$, minimizing this expression is the solution of the Euler–Lagrange equation

$$-\ddot{X}_t^0 + \theta \dot{X}_t^0 + \theta\frac{X_t^0 - \hat{X}}{T-t} = 0 \tag{24}$$

with initial condition $X_0^0 = X_0$ and a terminal condition X_ρ^0 that will be determined later. By using the computer algebra software Mathematica, we found the analytic solution

$$\begin{aligned}X_t^0 = \hat{X} + \\ \Big(\theta T e^{\theta T}(T-\rho)(\mathrm{Ei}(-T\theta) - \mathrm{Ei}(\theta(\rho - T))) - \rho + T(1 - e^{\theta\rho})\Big)^{-1}\Big\{ - e^{\theta t}\rho X_0 \\ +(t-T)\big(X_0 e^{\theta\rho} - X_\rho^0 + \hat{X}\big) + \theta(T-t)e^{\theta T}\Big[\mathrm{Ei}((t-T)\theta)(T(X_0 - X_\rho^0 + \hat{X}) \\ -\rho X_0) + X_0(\rho - T)\mathrm{Ei}(\theta(\rho - T)) + T(X_\rho^0 - \hat{X})\mathrm{Ei}(-T\theta)\Big] + e^{\theta t}T(X_0 - X_\rho^0 + \hat{X})\Big\},\end{aligned}$$

where $\mathrm{Ei}(t) = \int_{-\infty}^t s^{-1}e^s\, ds$ is the exponential integral function. The constants ρ, $\hat{X}$ and X_ρ^0 can then be determined by optimizing the expression (23) numerically. Finally, the part ξ^1 of the strategy, which describes the trades to be executed at the exchange after time $\rho \wedge \tau_1$ is given by

$$\xi_t^1 = \begin{cases} \dfrac{X_{\tau_1} - \hat{X}}{T - \tau_1} & \text{on } \{\tau_1 \le \rho\}, \\[2ex] \dfrac{X_\rho}{T-\rho} & \text{on } \{\rho < \tau_1\}; \end{cases}$$

see (39) in the proof of Proposition 4.13.

5 Conclusion

We have analyzed the regularity of a class of dark-pool extensions of an Almgren–Chriss model and found that such models admit price manipulation strategies unless the model parameters satisfy certain restrictions. These restrictions are satisfied for every Almgren–Chriss model when the penalty parameter β is equal to $\frac{1}{2}$, the cross-venue impact parameter α is 1, and there is no temporary price impact from the exchange on dark-pool prices. With these choices, the dark-pool extension of any

Almgren–Chriss model is free of price manipulation, has positive expected liquidation costs, and hence admits reasonable optimal order execution strategies. In this sense, the model is then regular.

It should be noted, however, that the parameter values $\alpha = 1$ and $\beta = \frac{1}{2}$ will typically not correspond to values found in empirical analysis or calibration of real-world dark pools. Our results can therefore provide some indication that dark pools may create market inefficiencies and disturb the price finding mechanism of markets, although further empirical analysis will be needed to support this conjecture.

6 Proofs

Recall from Assumption 2.2 that the martingale property of (P_t^0) is retained by passing to the enlarged filtration $(\mathcal{G}_t)$. Next, for an admissible strategy, the asset position process X defined in (7) is an admissible integrand for P^0 since it is leftcontinuous, $(\mathcal{G}_t)$-adapted, and hence $(\mathcal{G}_t)$-predictable. Recall also that $f(x) = x\,h(x)$.

Lemma 6.1. *The terminal revenues of an admissible strategy for given X_0 and T are given by*

$$\begin{aligned}\mathcal{R}_T &= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\Big(X_0 + \sum_{i=1}^{N_\rho} Y_i\Big)^2 - \int_0^T f(\xi_t)\,dt \\ &\quad - \sum_{i=1}^{N_\rho} Y_i\left(\gamma\int_0^{\tau_i}\xi_s\,ds - \alpha\gamma X_{\tau_i+} + \beta\gamma Y_i + g(\xi_{\tau_i})\right).\end{aligned}$$

Proof. First we prove that

$$-\int_0^T \xi_t P_t^0\,dt - \sum_{i=1}^{N_\rho} Y_i P_{\tau_i}^0 = X_0P_0^0 + \int_0^T X_t\,dP_t^0.$$

To this end, we first define $\tilde{X}_t := \int_0^t \xi_s\,ds$ and note that $\mathbb{P}$-a.s. $\tilde{X}_T = -X_0 - Z_{T-}^\rho = -X_0 - Z_T^\rho$. Moreover, the independence of N and P^0 implies that $\mathbb{P}$-a.s. $P_{\tau_i-}^0 = P_{\tau_i}^0$. In particular, the quadratic co-variations $[P^0, N]$ and $[P^0, Z]$ vanish $\mathbb{P}$-a.s. It follows that $\mathbb{P}$-a.s.

$$\begin{aligned}X_0P_0^0 + \int_0^T X_t\,dP_t^0 &= X_0P_0^0 + \int_0^T\left(X_0 + \int_0^t \xi_s\,ds + Z_{t-}^\rho\right)dP_t^0 \\ &= X_0P_0^0 + X_0(P_T^0 - P_0^0) + \int_0^T \tilde{X}_t\,dP_t^0 + \int_0^T Z_{t-}^\rho\,dP_t^0 \\ &= -P_T^0Z_{T-}^\rho - \int_0^T \xi_t P_t^0\,dt + Z_T^\rho P_T^0 - \int_0^T P_{t-}^0\,dZ_t^\rho \\ &= -\int_0^T \xi_t P_t^0\,dt - \sum_{i=1}^{N_\rho} Y_i P_{\tau_i-}^0 \\ &= -\int_0^T \xi_t P_t^0\,dt - \sum_{i=1}^{N_\rho} Y_i P_{\tau_i}^0.\end{aligned}$$

Thus, $\mathbb{P}$-a.s.,

$$
\begin{aligned}
\mathcal{R}_T &= -\int_0^T \xi_t P_t \, dt - \sum_{i=1}^{N_\rho} Y_i \hat{P}_{\tau_i} \\
&= -\int_0^T \xi_t \left(P_t^0 + \gamma \left(\int_0^t \xi_s \, ds + \alpha Z_{t-}^\rho \right) + h(\xi_t) \right) dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(P_{\tau_i}^0 + \gamma \left(\int_0^{\tau_i} \xi_s \, ds + \alpha \sum_{j=1}^{i-1} Y_j \right) + \beta\gamma Y_i + g(\xi_{\tau_i}) \right) \\
&= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \gamma \int_0^T \int_0^t \xi_s \, ds \, \xi_t \, dt - \int_0^T \xi_t \gamma\alpha Z_{t-}^\rho \, dt - \int_0^T f(\xi_t) \, dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(\gamma \int_0^{\tau_i} \xi_s \, ds + \alpha\gamma \sum_{j=i+1}^{N_\rho} Y_j + \beta\gamma Y_i + g(\xi_{\tau_i}) \right) \\
&= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma \left(\int_0^T \xi_t \, dt \right)^2 - \gamma\alpha \sum_{i=1}^{N_\rho} Y_i \int_{\tau_i}^T \xi_t \, dt - \int_0^T f(\xi_t) \, dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(\gamma \int_0^{\tau_i} \xi_s \, ds + \alpha\gamma \sum_{j=i+1}^{N_\rho} Y_j + \beta\gamma Y_i + g(\xi_{\tau_i}) \right) \\
&= X_0 P_0^0 + \int_0^T X_t \, dP_t^0 - \frac{1}{2}\gamma \left(X_0 + \sum_{i=1}^{N_\rho} Y_i \right)^2 - \int_0^T f(\xi_t) \, dt \\
&\quad - \sum_{i=1}^{N_\rho} Y_i \left(\gamma \int_0^{\tau_i} \xi_s \, ds - \alpha\gamma X_{\tau_i+} + \beta\gamma Y_i + g(\xi_{\tau_i}) \right).
\end{aligned}
$$

In the last step, we have again used the fact that $X_T = X_{T+} = 0$ $\mathbb{P}$-a.s. □

Proof of Proposition 3.4. (a): Assume $X_0 \geq 0$, and let the trading strategy be selling only, i.e. $\xi_t \leq 0$ for all t and $Y_i \leq 0$ for all i. Then $P_t \leq P_t^0$ for all t and $\hat{P}_{\tau_i} \leq P_{\tau_i}^0$ for all i. Using integration by parts, we find that

$$\mathcal{R}_T \leq -\int_0^T \xi_s P_s^0 \, ds - \sum_{i=1}^{N_{T\wedge\rho}} Y_i P_{\tau_i}^0 = X_0 P_0^0 + \int_0^T X_t \, dP_t^0.$$

Since (P^0) is a martingale, $\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0^0$ for such a trading strategy. Absence of transaction-triggered price manipulation implies that the expected revenues cannot be increased by intermediate sell trades and therefore, we have $\mathbb{E}[\mathcal{R}_T] \leq X_0 P_0^0$ for all trading strategies. The case $X_0 \leq 0$ works analogously.

(b): By setting $X_0 = 0$ in (13) we find that $\mathbb{E}[\mathcal{R}_T] \leq 0$ for round trips. □

In the following, we will consider round trips which cancel the order in the dark pool after the first execution, i.e. $X_0 = 0$ and $\rho = \tau_1 \wedge r$ with some $r < T$. With Lemma 6.1 we find that the revenues of such a round trip are

$$
\begin{aligned}
\mathcal{R}_T &= \int_0^T X_t \, dP_t^0 - \frac{\gamma}{2} \mathbb{1}_{\{\tau_1 \leq r\}} Y_1^2 - \int_0^T f(\xi_t) \, dt \\
&\quad - \mathbb{1}_{\{\tau_1 \leq r\}} Y_1 \left(\gamma X_{\tau_1 -} - \alpha\gamma (X_{\tau_1 -} + Y_1) + \beta\gamma Y_1 + g(\xi_{\tau_i}) \right).
\end{aligned}
$$

Furthermore, we will consider strategies that do not depend on P^0, i.e. they only depend on τ_1 and Y_1. In particular, these strategies can be written as

$$\xi_t = \begin{cases} \xi_t^0, & \text{if } t \le \tau_1 \text{ or } \tau_1 > r, \\ \xi_t^1, & \text{if } t > \tau_1 \text{ and } \tau_1 \le r, \end{cases} \tag{25}$$

where ξ^0 is deterministic and ξ^1 depends on τ_1 and Y_1. As in Section 4.3, we will call these strategies *single-update round trips*. We define further $X_t^0 = \int_0^t \xi_s^0 \, ds$.

The revenues of a single-update round trip are

$$\begin{aligned} \mathbb{E}[\mathcal{R}_T] &= -\int_0^r f(\xi_t^0)\mathbb{P}[\, t \le \tau_1 \,]\, dt - \mathbb{P}[\, \tau_1 > r \,] \int_r^T f(\xi_t^0)\, dt \\ &\quad -\mathbb{E}\left[\int_{\tau_1}^T f(\xi_t^1)\, dt; \tau_1 \le r\right] - \mathbb{E}\Big[\gamma\Delta Y_1^2 + \gamma(1-\alpha)X_{\tau_1}^0 Y_1 + g(\xi_{\tau_1}^0)Y_1\,; \tau_1 \le r\Big] \end{aligned} \tag{26}$$

where

$$\Delta := -\alpha + \frac{1}{2} + \beta. \tag{27}$$

We will next prove Theorem 4.3. The proof of Theorem 4.1 will be based on Theorem 4.3.

Proof of Theorem 4.3. We first show that we must have that $\Delta \ge 0$ when there is no price manipulation. To this end, we assume by way of contradiction that $\Delta < 0$ but that there is no price manipulation for all T. Consider the single-update round trip with $r = T/2$, $\hat{X} > 0$, and

$$\xi_t = \begin{cases} \frac{-2Y_1}{T} & \text{if } t > r \text{ and } \tau_1 \le r \\ 0 & \text{otherwise.} \end{cases}$$

The expected revenues of this strategy satisfy

$$\mathbb{E}[\mathcal{R}_T] \;=\; -\mathbb{E}\left[\frac{T}{2} f\left(-\frac{2Y_1}{T}\right) + \gamma\Delta Y_1^2\,; \tau_1 \le \frac{T}{2}\right] = \mathbb{E}\left[Y_1 h\left(-\frac{2Y_1}{T}\right) - \gamma\Delta Y_1^2\,; \tau_1 \le \frac{T}{2}\right].$$

The continuity of $h(x)$ at $x = 0$ yields that $h(-2Y_1/T) \nearrow 0$ for $T \uparrow \infty$. Dominated convergence and our assumption $\Delta < 0$ hence imply that

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = -\gamma\Delta\mathbb{E}\,[\, Y_1^2 \,] > 0.$$

It follows that for sufficiently large T the expected revenues are strictly positive, and so there is price manipulation. But this contradicts our assumption.

We now consider the special case in which $\Delta = 0$ and $g(x) = \kappa h(x)$ for some $\kappa \ge 0$ and deduce $\kappa = 0$. By way of contradiction, we will show that there is price manipulation for sufficiently large T when $\kappa > 0$. Consider any single-update round trip. When holding r fixed and taking T arbitrarily large, we can liquidate the asset position $X_{\tau_1 \wedge r}$ arbitrarily slowly during $[r, T]$ and thus achieve that both $\xi_t^0 \searrow 0$ and $\xi_t^1 \searrow 0$ for $t \ge r$ as $T \uparrow \infty$. By sending T to infinity in (26), it follows that we can achieve via monotone convergence that

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = -\mathbb{E}\Big[\int_0^r f(\xi_t^0)\mathbb{1}_{\{t\le\tau_1\}}\, dt + \Big(\gamma\Delta Y_1^2 + \gamma(1-\alpha)X_{\tau_1}^0 Y_1 + g(\xi_{\tau_1}^0)Y_1\Big)\mathbb{1}_{\{\tau_1\le r\}}\Big], \tag{28}$$

where we keep the term with Δ for the moment, although $\Delta = 0$ here, because this and the subsequent formulas will also be used later on. Now we take some $\delta \in (0, 1)$, which will be specified later, and let

$r = \delta$ and $\xi^0_t = -\delta$ for $0 \le t \le \delta$. We also suppose that $\hat{X} > 0$. Then

$$\begin{aligned}
&\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] \\
&= -\mathbb{E}\Big[f(-\delta)\int_0^\delta \mathbb{1}_{\{t\le\tau_1\}}\,dt + \Big(\gamma\Delta Y_1^2 + Y_1\big(\gamma(1-\alpha)X^0_{\tau_1} + g(-\delta)\big)\Big)\mathbb{1}_{\{\tau_1\le\delta\}}\Big] \qquad (29)\\
&\ge -\delta f(-\delta) - \mathbb{E}\Big[\Big(\gamma\Delta Y_1^2 + Y_1 g(-\delta)\Big)\mathbb{1}_{\{\tau_1\le\delta\}}\Big] \qquad (30)\\
&= -\kappa h(-\delta)\Big(\frac{\delta f(-\delta)}{\kappa h(-\delta)} + \mathbb{E}[\,Y_1\,|\,\tau_1\le\delta\,]\mathbb{P}[\,\tau_1\le\delta\,]\Big)\\
&\ge -\kappa h(-\delta)\delta\Big(-\frac{\delta}{\kappa} + \lambda_0\lambda_1 x_0\Big),
\end{aligned}$$

where we have used (5) and (6) in the last step. Due to the assumption $\lambda_0\lambda_1 x_0 > 0$, this expression is strictly positive as soon as $\delta > 0$ is small enough. This implies the desired existence of price manipulation for sufficiently large T.

We now show that we must have $\alpha = 1$ when $\Delta = 0$ and $g = 0$. To this end, we assume by way of contradiction that $\alpha < 1$. As before, we may assume that (28) holds. When taking $r = 1$ and $\xi^0_t := -\delta\mathbb{1}_{[0,1]}$ for $\delta \in (0,1)$, we have $X^0_{\tau_1} = -\delta\tau_1$ on $\{\tau_1 \le r\}$, and (28) yields that

$$\begin{aligned}
\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\Big[\int_0^1 f(-\delta)\mathbb{1}_{\{t\le\tau_1\}}\,dt + \gamma(1-\alpha)X^0_{\tau_1}Y_1\mathbb{1}_{\{\tau_1\le r\}}\Big]\\
&\ge \delta\Big[h(-\delta) + \gamma(1-\alpha)\mathbb{E}\big[\,\tau_1 Y_1\mathbb{1}_{\{\tau_1\le r\}}\,\big]\Big].
\end{aligned}$$

But the latter expression is strictly positive as soon as δ is small enough, because $\mathbb{E}[\,\tau_1 Y_1 \mathbb{1}_{\{\tau_1\le r\}}\,]$ is strictly positive by (5) and (6). Hence, $\alpha < 1$ implies the existence of price manipulation for sufficiently large T. □

Proof of Theorem 4.1. The implication (a)⇒(b) follows immediately by taking $X_0 = 0$.

(b)⇒(c): We already know from Theorem 4.3 that we must have $\Delta \ge 0$, where Δ is as in (27). Thus, it remains to show that $g = 0$ and $\alpha = 1$.

We start by showing that $g = 0$. To this end, we assume by way of contradiction that there is no price manipulation but $g \ne 0$. Then g must satisfy the conditions on a temporary-impact function in Assumption 2.1. When $\Delta = 0$, we can take $h := g$ and get the desired contradiction from the second part of Theorem 4.3. So let us now consider the case $\Delta > 0$. To this end, we consider again the situation in the proof of Theorem 4.3 in which (28) holds and where $r = \delta$, $\delta \in (0,1)$, and $\xi^0_t = -\delta$ for $0 \le t \le \delta$. Then we have from (30) that

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] \ge -\delta f(-\delta) - \gamma\Delta\mathbb{E}\big[\,Y_1^2\mathbb{1}_{\{\tau_1\le\delta\}}\,\big] - g(-\delta)\mathbb{E}\big[\,Y_1\mathbb{1}_{\{\tau_1\le\delta\}}\,\big].$$

On the one hand, we have $0 < Y_1 = \tilde{Y}_1 \wedge \hat{X} \le \hat{X}$ and hence

$$\mathbb{E}\big[\,Y_1^2\mathbb{1}_{\{\tau_1\le\delta\}}\,\big] \le \hat{X}^2\mathbb{P}[\,\tau_1\le\delta\,].$$

On the other hand, our assumption (6) implies that for all $\hat{X}$ such that $0 < \hat{X} \le x_0$ we have $\mathbb{E}[\,Y_1\,|\,\tau_1\le\delta\,] \ge \lambda_1\hat{X}$. Thus, for $0 < \hat{X} \le x_0$ we have

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] \ge -\delta f(-\delta) - \big(\gamma\Delta\hat{X}^2 + g(-\delta)\lambda_1\hat{X}\big)\mathbb{P}[\,\tau_1\le\delta\,].$$

Choosing $\hat{X} = -g(-\delta)\lambda_1/(2\gamma\Delta)$ yields

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] \geq -\delta f(-\delta) + \frac{g(-\delta)^2\lambda_1^2}{4\Delta\gamma}\mathbb{P}[\,\tau_1 \leq \delta\,] \geq \delta g(-\delta)^2\Big(\frac{\delta h(-\delta)}{g(-\delta)^2} + \frac{\lambda_1^2}{4\Delta\gamma}\lambda_0\Big), \tag{31}$$

where we have used (5) in the last step. One can check that the function $h(x) := xg(x)^2$ satisfies Assumption 2.1. But with this choice, the right-hand side of (31) becomes strictly positive for sufficiently small $\delta > 0$, and we obtain price manipulation for sufficiently large T. This completes the proof of $g = 0$.

Now we show that we must have $\alpha = 1$. To this end, we assume by way of contradiction that $\alpha < 1$ and start from the identity (29), which holds for $r = \delta$, $\hat{X} > 0$, $\xi^0_t = -\delta$ for $0 \leq t \leq \delta$, and a suitable choice for ξ^1 and ξ^0_t $(t > \delta)$, depending on T. We take $\delta = 1$ and hence have $X^0_{\tau_1} = -\tau_1$ on $\{\tau_1 \leq 1\}$. Since $g = 0$, (29) implies that

$$\begin{aligned}
\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\Big[\,f(-1)\int_0^1 \mathbb{1}_{\{t\leq\tau_1\}}\,dt + \Big(\gamma\Delta Y_1^2 - \gamma(1-\alpha)Y_1\tau_1\Big)\mathbb{1}_{\{\tau_1\leq 1\}}\Big]\,dt \\
&\geq -f(-1) + \mathbb{E}\Big[\Big(-\gamma\Delta Y_1^2 + \gamma(1-\alpha)Y_1\tau_1\Big)\mathbb{1}_{\{\tau_1\leq 1\}}\Big].
\end{aligned} \tag{32}$$

Next we consider Almgren–Chriss models with fixed permanent-impact parameter $\gamma > 0$ and with temporary impact function εh, where h is fixed and $\varepsilon > 0$. Suppose first that $\Delta = 0$. Then we get in the limit $\varepsilon \downarrow 0$,

$$\lim_{\varepsilon\downarrow 0}\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] \geq \gamma(1-\alpha)\mathbb{E}\big[\,Y_1\tau_1\mathbb{1}_{\{\tau_1\leq 1\}}\,\big] > 0,$$

which implies that there is price manipulation for small enough ε and large enough T.

For $\Delta > 0$, we get

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] &\geq \mathbb{E}\Big[\Big(-\gamma\Delta Y_1^2 + \gamma(1-\alpha)Y_1\tau_1\Big)\mathbb{1}_{\{\tau_1\leq 1\}}\Big] \\
&= \gamma\Delta\hat{X}\mathbb{E}\Big[\Big(-\frac{(\tilde{Y}_1\wedge\hat{X})^2}{\hat{X}} + \frac{1-\alpha}{\Delta}\cdot\frac{Y_1\wedge\hat{X}}{\hat{X}}\tau_1\Big)\mathbb{1}_{\{\tau_1\leq 1\}}\Big].
\end{aligned}$$

But it is easy to see that the expectation on the right will be strictly positive as soon as $\hat{X}$ is sufficiently small, since

$$\frac{(Y_1\wedge\hat{X})^2}{\hat{X}} \longrightarrow 0 \qquad \text{and} \qquad \frac{Y_1\wedge\hat{X}}{\hat{X}} \longrightarrow 1 \qquad \text{as } \hat{X}\downarrow 0.$$

This shows that there is price manipulation for small enough ε and large enough T when $\alpha < 1$.

(c)⇒(a): Assume that $\alpha = 1$, $\beta \geq \frac{1}{2}$, $g \equiv 0$. Note that

$$\int_0^{\tau_i} \xi_s\,ds - X_{\tau_i+} = -\sum_{j=1}^{i} Y_i - X_0. \tag{33}$$

With Lemma 6.1 we get for the revenues of an admissible strategy $(\hat{X},\rho,\xi)$

$$\begin{aligned}
\mathcal{R}_T &= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(X_0 + \sum_{i=1}^{N_\rho} Y_i\right)^2 - \int_0^T f(\xi_t)\,dt \\
&\quad + \sum_{i=1}^{N_\rho} Y_i\left(\gamma\sum_{j=1}^{i} Y_i + \gamma X_0 - \beta\gamma Y_i\right) \\
&= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma X_0^2 - \int_0^T f(\xi_t)\,dt - \big(\beta - \frac{1}{2}\big)\gamma\sum_{i=1}^{N_\rho} Y_i^2.
\end{aligned}$$

Therefore,

$$\mathbb{E}[\mathcal{R}_T] = X_0P_0^0 - \frac{1}{2}\gamma X_0^2 - \mathbb{E}\left[\int_0^T f(\xi_t)\,dt\right] - \big(\beta - \frac{1}{2}\big)\gamma\mathbb{E}\left[\sum_{i=1}^{N_\rho} Y_i^2\right] \le X_0P_0^0.$$

This establishes (a). □

Proof of Proposition 4.5. Let $X_0 \in \mathbb{R}$ and $\alpha = 1, \beta = 0$. The revenues for a strategy are given by

$$\begin{aligned}\mathcal{R}_T &= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma X_0^2 - \int_0^T f(\xi_t)\,dt \\ &\quad + \frac{1}{2}\gamma\sum_{i=1}^{N_\rho} Y_i^2 - \sum_{i=1}^{N_\rho} Y_i g(\xi_{\tau_i}).\end{aligned}$$

Consider the following trading strategy with $\rho = \frac{T}{2}$ and given $\hat{X} \neq 0$,

$$\xi_t = \begin{cases} 0, & \text{if } 0 \le t \le \rho, \\ -2\frac{X_0+Z_\rho}{T}, & \text{if } \rho < t \le T. \end{cases}$$

The expected revenues of this strategy are

$$\begin{aligned}\mathbb{E}[\mathcal{R}_T] &= X_0P_0^0 - \frac{1}{2}\gamma X_0^2 - \mathbb{E}\left[\frac{T}{2}f\left(-2\frac{X_0 + Z_{T/2}}{T}\right)\right] + \frac{1}{2}\gamma\mathbb{E}\left[\sum_{i=1}^{N_{T/2}} Y_i^2\right] \\ &= X_0P_0^0 - \frac{1}{2}\gamma X_0^2 + \mathbb{E}\left[\left(X_0 + Z_{T/2}\right)h\left(-2\frac{X_0 + Z_{T/2}}{T}\right)\right] + \frac{1}{2}\gamma\mathbb{E}\left[\sum_{i=1}^{N_{T/2}} Y_i^2\right].\end{aligned}$$

Recall that $|Z_{T/2}|$ is bounded by $|\hat{X}|$ for all T and that Y_i is nonzero only as long as $|\sum_{j=1}^{i-1}\tilde{Y}_j| < |\hat{X}|$. Hence, with probability one, only finitely many Y_i are nonzero. Therefore, and by dominated convergence,

$$\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] = X_0P_0^0 - \frac{1}{2}\gamma X_0^2 + \frac{1}{2}\gamma\mathbb{E}\left[\sum_{i=1}^{\infty} Y_i^2\right].$$

When sending $|\hat{X}|$ to infinity, $\sum_{i=1}^{\infty} Y_i^2$ tends to infinity with probability one. Hence, we can make $\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T]$ arbitrarily large. □

Proof of Proposition 4.6. Lemma 6.1 and (33) yield

$$\begin{aligned}\mathcal{R}_T &= \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(\sum_{i=1}^{N_\rho} Y_i\right)^2 - \eta\int_0^T \xi_t^2\,dt - \gamma\sum_{i=1}^{N_\rho} Y_i\left((1-\alpha)\int_0^{\tau_i}\xi_s\,ds - \alpha\sum_{j=1}^{i} Y_j\right) \\ &= \int_0^T X_t\,dP_t^0 + \alpha\left(\frac{1}{2}\gamma\sum_{i=1}^{N_\rho} Y_i^2 - \eta\int_0^T \xi_t^2\,dt\right) \\ &\quad + (1-\alpha)\left(-\eta\int_0^T \xi_t^2\,dt - \gamma\sum_{i=1}^{N_\rho} Y_i\int_0^{\tau_i}\xi_s\,ds - \frac{1}{2}\gamma\left(\sum_{i=1}^{N_\rho} Y_i\right)^2\right).\end{aligned}$$

Note that

$$\eta \int_0^T \xi_t^2\, dt \ge \frac{\eta}{T}\Big(\int_0^T \xi_t\, dt\Big)^2 = \frac{\eta}{T}\Big(\sum_{i=1}^{N_\rho} Y_i\Big)^2 \ge \frac{\eta}{T}\sum_{i=1}^{N_\rho} Y_i^2,$$

where we have used Jensen's inequality in the first step. Thus,

$$\frac{1}{2}\gamma \sum_{i=1}^{N_\rho} Y_i^2 - \eta \int_0^T \xi_t^2\, dt \le \Big(\frac{\gamma}{2} - \frac{\eta}{T}\Big)\sum_{i=1}^{N_\rho} Y_i^2. \tag{34}$$

Furthermore, let $\Xi := \sup_{t\in[0,T]} | \int_0^t \xi_s\, ds|$. Then, by Jensen's inequality,

$$\int_0^T \xi_t^2\, dt \ge T\Big(\frac{1}{T}\int_0^T |\xi_t|\, dt\Big)^2 \ge \frac{\Xi^2}{T}.$$

We can estimate

$$-\eta \int_0^T \xi_t^2\, dt - \gamma \sum_{i=1}^{N_\rho} Y_i \int_0^{\tau_i} \xi_s\, ds \le -\eta\frac{\Xi^2}{T} + \sum_{i=1}^{N_\rho} |Y_i|\gamma\Xi.$$

The right-hand side is maximized by

$$\Xi = \frac{\gamma T}{2\eta}\sum_{i=1}^{N_\rho} |Y_i|.$$

Therefore,

$$-\eta \int_0^T \xi_t^2\, dt - \gamma \sum_{i=1}^{N_\rho} Y_i \int_0^{\tau_i} \xi_s\, ds - \frac{1}{2}\gamma\left(\sum_{i=1}^{N_\rho} Y_i\right)^2 \le \frac{\gamma}{2}\left(\sum_{i=1}^{N_\rho} |Y_i|\right)^2 \left(\frac{\gamma T}{2\eta} - 1\right). \tag{35}$$

Combining (34) and (35) yields

$$\begin{aligned}\mathcal{R}_T &\le \int_0^T X_t\, dP_t^0 + \alpha\left(\frac{\gamma}{2} - \frac{\eta}{T}\right)\sum_{i=1}^{N_\rho} Y_i^2 + (1-\alpha)\frac{\gamma}{2}\left(\sum_{i=1}^{N_\rho} |Y_i|\right)^2\left(\frac{\gamma T}{2\eta} - 1\right)\\ &\le \int_0^T X_t\, dP_t^0\end{aligned}$$

for $T \le 2\eta/\gamma$ and we find $\mathbb{E}[\mathcal{R}_T] \le 0$. □

Proof of Proposition 4.7. Part (a): Necessity follows from Proposition 4.6. For the proof of sufficiency, let us assume that $T > 2\eta/\gamma$. Then there exists $\varepsilon \in (0, T)$ such that

$$\frac{1}{2}\gamma - \frac{\eta}{T-\varepsilon} > 0.$$

Consider the round trip with $\rho = \tau_1 \wedge \varepsilon$, arbitrary $\hat{X} \neq 0$, and

$$\xi_t = \begin{cases} -\frac{Y_1}{T-\varepsilon}, & \text{if } t > \varepsilon \text{ and } \tau_1 \le \varepsilon,\\ 0, & \text{otherwise.}\end{cases}$$

The expected revenues of this strategy are

$$\mathbb{E}[\mathcal{R}_T] = \left(\frac{1}{2}\gamma - \frac{\eta}{T-\varepsilon}\right)\mathbb{E}[Y_1^2; \tau_1 \le \varepsilon] > 0.$$

Hence, there is price manipulation for $T > 2\eta/\gamma$.

Part (b): Let $T > 0$ and fix $\hat{X}$ such that

$$\frac{\gamma}{2}\hat{X}^2 - \frac{T}{2}f\Big(2\frac{\hat{X}}{T}\Big) = \frac{\hat{X}^2}{2}\Big(\gamma - \frac{h(2\hat{X}/T)}{\hat{X}}\Big) > 0,$$

which is possible due to the sublinearity of h. Now we impose $\hat{X}$ as lower limit on matching orders in the dark pool so that $Y_1 = \hat{X}$ $\mathbb{P}$-a.s, which is possible to our assumption that $\tilde{Y}_1$ is unbounded. The arrival of this first matching order will again be exponentially distributed with a parameter $\tilde{\theta} < \theta$.

Now we take $\rho = T/2$ and

$$\xi_t := \begin{cases} 0, & t \le \rho, \\ 0, & t > \rho \text{ and } \tau > \rho, \\ -2\hat{X}/T, & t > \rho \text{ and } \tau \le \rho. \end{cases}$$

The expected revenues of this strategy are

$$\begin{aligned} \mathbb{E}[\mathcal{R}_T] &= -\mathbb{E}\left[\int_0^T f(\xi_t)\,dt\right] + \int_0^\rho \tilde{\theta}e^{-\tilde{\theta}t}\left(\frac{1}{2}\gamma\hat{X}^2 + \hat{X}g(0)\right)\,dt \\ &= (1 - e^{-\tilde{\theta}\rho})\left(\frac{1}{2}\gamma\hat{X}^2 - \frac{T}{2}f\left(2\frac{\hat{X}}{T}\right)\right) \\ &> 0. \end{aligned}$$

So there is price manipulation. □

Now we prove the results pertaining to the assumptions that $\alpha = \beta = 0$, $g \equiv 0$, and $h(\xi) = \eta\xi$. Under this conditions, Lemma 6.1 yields

$$\mathcal{R}_T = X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(X_0 + \sum_{i=1}^{N_\rho} Y_i\right)^2 - \eta\int_0^T \xi_t^2\,dt - \sum_{i=1}^{N_\rho} Y_i\left(\gamma\int_0^{\tau_i}\xi_s\,ds\right). \tag{36}$$

Proof of Proposition 4.8. Proof of (a): Take $\rho = \frac{T}{2}$ and

$$\xi_t = \begin{cases} -\frac{\gamma}{2\eta}\hat{X}, & \text{if } t \le \tau_1, t \le \rho, \\ 0, & \text{if } t > \tau_1, t \le \rho \\ -\frac{X_{\rho+}}{\rho}, & \text{if } t > \rho, \end{cases}$$

where $\hat{X}$ will be specified later. By (36), we find that

$$\mathcal{R}_T = X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(X_0 + \sum_{i=1}^{N_\rho} Y_i\right)^2 - \eta(\rho\wedge\tau_1)\frac{\gamma^2}{4\eta^2}\hat{X}^2 - \eta\frac{X_{\rho+}^2}{\rho} + \sum_{i=1}^{N_\rho} Y_i\tau_1\frac{\gamma^2}{2\eta}\hat{X}.$$

In the limit $T \uparrow \infty$ we will have

$$\sum_{i=1}^{N_\rho} Y_i = \sum_{i=1}^{N_{T/2}} Y_i \longrightarrow \hat{X}.$$

Hence, using the fact that $\mathbb{E}[\tau_1] = \frac{1}{\theta}$,

$$\lim_{T\uparrow\infty}\mathbb{E}[\mathcal{R}_T] = X_0P_0^0 - \frac{1}{2}\gamma(X_0 + \hat{X})^2 + \frac{1}{\theta}\frac{\gamma^2}{4\eta}\hat{X}^2. \tag{37}$$

Choosing

$$\hat{X} = -\frac{2X_0\eta\theta}{\gamma - 2\eta\theta}$$

yields

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0P_0^0 + \frac{1}{2}\gamma^2 X_0^2 \frac{1}{2\eta\theta - \gamma} > X_0P_0.$$

This concludes the proof of part (a).

Proof of (b): We first consider the case in which $\frac{\gamma}{\eta\theta} = 2$ and $X_0 \neq 0$. With the same strategy as in part (a) we find with (37) that

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0P_0^0 - \frac{1}{2}\gamma X_0^2 - \gamma X_0\hat{X}.$$

For $X_0 \neq 0$, the right-hand side can be made arbitrarily large by taking $\hat{X}$ with the opposite sign of X_0 and making $|\hat{X}|$ large.

Now we consider the case in which $\frac{\gamma}{\eta\theta} > 2$. With (37) we find

$$\lim_{T\uparrow\infty} \mathbb{E}[\mathcal{R}_T] = X_0P_0^0 - \frac{1}{2}\gamma X_0^2 - \gamma X_0\hat{X} + \varepsilon\hat{X}^2,$$

where $\varepsilon > 0$. Again, the right-hand side can be made arbitrarily large by sending $\hat{X}$ to infinity. $\square$

Proof of Proposition 4.9. In view of Proposition 4.8, the assertion will be implied by the following claim: If, for $0 \leq t < \rho$, we have $\xi_t \leq 0$ when $X_0 > 0$ or $\xi_t \geq 0$ when $X_0 < 0$, then

$$\mathbb{E}[\mathcal{R}_T] \leq X_0P_0^0.$$

In proving this claim, we will consider the case $X_0 > 0$. The case $X_0 < 0$ is analogous. With Lemma 6.1 we find that

$$\mathcal{R}_T = X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(X_0 + \sum_{i=1}^{N_\rho} Y_i\right)^2 - \int_0^T f(\xi_t)\,dt - \sum_{i=1}^{N_\rho} Y_i\gamma\int_0^{\tau_i}\xi_s\,ds.$$

Consider first the case $\hat{X} \leq 0$. Then

$$-\sum_{i=1}^{N_\rho} Y_i\gamma\int_0^{\tau_i}\xi_s\,ds \leq 0$$

and $\mathbb{E}[\mathcal{R}_T] \leq X_0P_0^0$ follows.

Consider next the case $\hat{X} > 0$. Since $\xi_t \leq 0$ this implies $X_t \geq 0$ for all t. Especially, $X_{\tau_i-} \geq 0$, or equivalently

$$\int_0^{\tau_i}\xi_s\,ds \geq -X_0 - \sum_{j=1}^{i-1} Y_i.$$

Therefore, we find that

$$\mathcal{R}_T \leq X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma X_0^2 - \frac{1}{2}\gamma\sum_{i=1}^{N_\rho} Y_i^2 - \int_0^T f(\xi_t)\,dt$$

and $\mathbb{E}[\mathcal{R}_T] \leq X_0P_0^0$ follows. $\square$

Proof of Proposition 4.10. The revenues in this case are

$$\begin{aligned}\mathcal{R}_T &= X_0 P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma(X_0 + \mathbb{1}_{\{\tau_1\le\rho\}}\hat{X})^2 - \eta\int_0^T \xi_t^2\,dt - \mathbb{1}_{\{\tau_1\le\rho\}}\gamma\hat{X}(X_{\tau_1-} - X_0)\\ &\le X_0 P_0^0 + \int_0^T X_t\,dP_t^0\\ &\quad + \mathbb{1}_{\{\tau_1\le\rho\}}\left(-\frac{1}{2}\gamma(X_0+\hat{X})^2 - \eta\frac{(X_{\tau_1-}-X_0)^2}{\tau_1} - \gamma\hat{X}(X_{\tau_1-}-X_0)\right).\end{aligned}$$

The rightmost expression is maximized by

$$X_{\tau_1-} = X_0 - \frac{\gamma}{2\eta}\tau_1\hat{X}$$

and we find

$$\mathcal{R}_T \le X_0 P_0^0 + \int_0^T X_t\,dP_t^0 + \mathbb{1}_{\{\tau_1<\rho\}}\left(-\frac{1}{2}\gamma(X_0+\hat{X})^2 + \frac{\gamma^2}{4\eta}\tau_1\hat{X}^2\right) \tag{38}$$

and thus

$$\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0 + \mathbb{E}[f(\rho,\hat{X})],$$

where

$$f(\rho,\hat{X}) := \int_0^\rho \theta e^{-\theta t}\left(-\frac{1}{2}\gamma(X_0+\hat{X})^2 + \frac{\gamma^2}{4\eta}t\hat{X}^2\right)dt.$$

We see that $f(0,\hat{X}) = 0$ and the term in parenthesis is increasing in t. Therefore, if $\hat{X}$ is such that $f(\infty,\hat{X}) > 0$, then we have $f(\rho,\hat{X}) \le f(\infty,\hat{X})$ for all $\rho < \infty$. For $\hat{X}$ with $f(\infty,\hat{X}) \le 0$ we have $f(\rho,\hat{X}) \le 0$ for all $\rho < \infty$. Thus,

$$\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0 + 0 \vee f(\infty,\hat{X}) = X_0 P_0^0 + \Big(-\frac{1}{2}\gamma(X_0+\hat{X})^2 + \frac{\gamma^2}{4\eta\theta}\hat{X}^2\Big)^+.$$

The right-hand side is maximized by taking

$$\hat{X} = 2X_0\frac{\eta\theta}{\gamma - 2\eta\theta},$$

and so

$$\mathbb{E}[\mathcal{R}_T] \le X_0 P_0^0 + \frac{1}{2}\gamma^2 X_0^2\frac{1}{2\eta\theta-\gamma}.$$

The statement now follows with Proposition 4.8. □

Proof of Corollary 4.11. We already know from Proposition 4.8 (b) that there is price manipulation for $\frac{\gamma}{\eta} > 2\theta$. On the other hand, Proposition 4.10 implies that is no price manipulation for $\frac{\gamma}{\eta} < 2\theta$. Hence, it remains to analyze the case $\frac{\gamma}{\eta} = 2\theta$. For a round trip with $X_0 = 0$, our estimate (38) yields that in this case

$$\mathcal{R}_T \le \int_0^T X_t\,dP_t^0 + \mathbb{1}_{\{\tau_1<\rho\}}\gamma\left(\frac{\gamma}{4\eta}\tau_1 - \frac{1}{2}\right)\hat{X}^2.$$

Hence,

$$\mathbb{E}[\mathcal{R}_T] \le \gamma\hat{X}^2\mathbb{E}[g(\rho)]$$

where

$$g(\rho) := \int_0^\rho \theta e^{-\theta t}\left(\frac{\gamma}{4\eta}t - \frac{1}{2}\right)dt = \frac{\gamma}{4\eta\theta}\big(1 - e^{-\theta\rho}(1+\theta\rho)\big) - \frac{1}{2}(1-e^{-\theta\rho}) = -\frac{1}{2}\theta\rho e^{-\theta\rho} \le 0.$$

This gives $\mathbb{E}[\mathcal{R}_T] \le 0$. □

Proof of Proposition 4.13. Under the assumptions $\alpha = 1$, $\beta = \frac{1}{2}$, and $g = 0$, the revenues of an admissible strategy are given by

$$\begin{aligned}\mathcal{R}_T &= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma\left(X_0 + \mathbb{1}_{\{\tau_1\le\rho\}}\hat{X}\right)^2 - \int_0^T f(\xi_t)\,dt + \gamma\hat{X}\left(\frac{1}{2}\hat{X} + X_0\right)\mathbb{1}_{\{\tau_1\le\rho\}} \\ &= X_0P_0^0 + \int_0^T X_t\,dP_t^0 - \frac{1}{2}\gamma X_0^2 - \int_0^T f(\xi_t)\,dt.\end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_{\tau_1\wedge\rho}$ and using optional sampling yields

$$\mathbb{E}[\,\mathcal{R}_T\,|\,\mathcal{F}_{\tau_1\wedge\rho}\,] = X_0P_0^0 + \int_0^{\tau_1\wedge\rho} X_t\,dP_t^0 - \frac{\gamma}{2}X_0^2 - \int_0^{\tau_1\wedge\rho} f(\xi_t)\,dt - \mathbb{E}\Big[\int_{\tau_1\wedge\rho}^T f(\xi_t)\,dt\,\Big|\,\mathcal{F}_{\tau_1\wedge\rho}\Big].$$

Due to the liquidation constraint, we must have $\int_{\tau_1\wedge\rho}^T \xi_t\,dt = X_{\tau_1\wedge\rho} - \mathbb{1}_{\{\tau_1<\rho\}}\hat{X}$, and so the convexity of f and Jensen's inequality yield that

$$\int_{\tau_1\wedge\rho}^T f(\xi_t)\,dt \ge (T-\tau_1\wedge\rho)f\Big(\frac{X_{\tau_1\wedge\rho} - \mathbb{1}_{\{\tau_1<\rho\}}\hat{X}}{T-\tau_1\wedge\rho}\Big)$$

with equality if, for $\tau_1\wedge\rho\le t\le T$,

$$\xi_t = \begin{cases}\dfrac{X_{\tau_1}-\hat{X}}{T-\tau_1} & \text{on } \{\tau_1\le\rho\}\\[2ex] \dfrac{X_\rho}{T-\rho} & \text{on } \{\rho<\tau_1\}.\end{cases} \tag{39}$$

These two possibilities will correspond to the single update of $\bar{\xi}$ at τ_1.

Note next that, due to the $(\mathcal{G}_t)$-predictability of the processes (ξ_t) and $(\rho\wedge t)_{t\ge0}$, $(\xi_s)_{s\le t}$ and $\rho\wedge t$ are independent of τ_1, conditional on $\{t\le\tau_1\}$. It follows that

$$\begin{aligned}\mathbb{E}[\,\mathcal{R}_T\,] &= \mathbb{E}[\,\mathbb{E}[\,\mathcal{R}_T\,|\,\mathcal{F}_{\tau_1\wedge\rho}\,]\,]\\ &\le X_0P_0^0 - \frac{\gamma}{2}X_0^2 - \mathbb{E}\Big[\int_0^{\tau_1\wedge\rho} f(\xi_t)\,dt + (T-\tau_1\wedge\rho)f\Big(\frac{X_{\tau_1\wedge\rho} - \mathbb{1}_{\{\tau_1\le\rho\}}\hat{X}}{T-\tau_1\wedge\rho}\Big)\Big]\\ &= X_0P_0^0 - \frac{\gamma}{2}X_0^2 - \mathbb{E}[\,F(\hat{X},\xi,\rho)\,],\end{aligned} \tag{40}$$

where the functional F maps $\hat{X}\in\mathbb{R}$, $\xi\in L^1[0,T]$, and $r\in[0,T]$ to

$$F(\hat{X},\xi,r) = \int_0^\infty du\,\theta e^{-\theta u}\Big\{\int_0^{u\wedge r} f(\xi_t)\,dt + (T-u\wedge r)f\Big(\frac{X_0 + \int_0^{u\wedge r}\xi_t\,dt - \mathbb{1}_{\{u\le r\}}\hat{X}}{T-u\wedge r}\Big)\Big\}.$$

When F admits a minimizer $(\hat{X}^*,\xi^*,r^*)$, then concatenating ξ^* with (39) in $r^*\wedge\tau_1$ yields an optimal strategy that is a single-update strategy.

To show the existence of a minimizer of F, take any triple $(\tilde{X},\tilde{\xi},\tilde{r})$ for which $C := F(\tilde{X},\tilde{\xi},\tilde{r}) < \infty$. We then only need to look into those triples $(\hat{X},\xi,r)$ for which $F(\hat{X},\xi,r)\le C$. Without loss of generality, we can pick the component ξ from the set

$$K_C := \Big\{\xi\in L^1[0,T]\,\Big|\,\int_0^T f(\xi_t)\,dt \le Ce^{\theta T}\Big\},$$

because we clearly have

$$F(\hat{X}, \xi, r) \geq \int_T^\infty du\, \theta e^{-\theta u} \int_0^{u\wedge r} f(\xi_t)\, dt = e^{-\theta T} \int_0^r f(\xi_t)\, dt$$

and we can set $\xi_t := 0$ for $t > r$.

The set K_C is a closed convex subset of $L^1[0,T]$. Hence it is also weakly closed in $L^1[0,T]$. It is also uniformly integrable according to the criterion of de la Vallée Poussin and our assumption that f has superlinear growth. Hence, the Dunford–Pettis theorem (Dunford and Schwartz, 1958, Corollary IV.8.11) implies that K_C is weakly sequentially compact in $L^1[0,T]$. From now on we will endow K_C with the weak topology.

It follows in particular that

$$\sup_{\xi\in K_C} \int_0^T |\xi_t|\, dt < \infty. \tag{41}$$

Since

$$F(\hat{X}, \xi, r) \geq \int_0^r du\, \theta e^{-\theta u}(T-u) f\Big(\frac{X_0 + \int_0^u \xi_t\, dt - \hat{X}}{T-u}\Big),$$

the superlinear growth of f and (41) imply that there is a constant $C_1 \geq 0$ such that $|\hat{X}| \leq C_1$ when $F(\hat{X}, \xi, r) \leq C$. Hence we can restrict our search of a minimizer to the sequentially compact set

$$\mathcal{K} := [-C_1, C_1] \times K_C \times [0,T].$$

Next,

$$[0,T] \times K_C \ni (r, \xi) \longrightarrow \int_0^r \xi_t\, dt = \int_0^T \xi_t \mathbb{1}_{[0,r]}(t)\, dt$$

is a continuous map. Moreover, denoting by f^* the Fenchel-Legendre transform of the convex function f, we have $f^{**} = f$ due to the biduality theorem, and so

$$[0,T] \times K_C \ni (r, \xi) \longmapsto \int_0^r f(\xi_t)\, dt = \sup_{\varphi\in L^\infty} \Big[\int_0^T \mathbb{1}_{[0,r]}(t)\xi_t\varphi_t\, dt - \int_0^r f^*(\varphi_t)\, dt \Big];$$

see, e.g., Theorem 2 in Rockafellar (1968). It follows that this map is lower semicontinuous as the supremum of continuous maps.

Altogether, it follows that F is lower semicontinuous on the sequentially compact set $\mathcal{K}$ and so admits a minimizer. □

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