Pointwise multipliers of Calderón-Lozanovskiĭ spaces

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Abstract

Several results concerning multipliers of symmetric Banach function spaces are presented firstly. Then the results on multipliers of Calderón-Lozanovskiĭ spaces are proved. We investigate assumptions on a Banach ideal space E and three Young functions φ_1, φ_2 and φ, generating the corresponding Calderón-Lozanovskiĭ spaces $E_{\varphi_1}, E_{\varphi_2}, E_\varphi$ so that the space of multipliers $M(E_{\varphi_1}, E_\varphi)$ of all measurable x such that $x\,y \in E_\varphi$ for any $y \in E_{\varphi_1}$ can be identified with E_{φ_2}. Sufficient conditions generalize earlier results by Ando, O'Neil, Zabreĭko-Rutickiĭ, Maligranda-Persson and Maligranda-Nakai. There are also necessary conditions on functions for the embedding $M(E_{\varphi_1}, E_\varphi) \subset E_{\varphi_2}$ to be true, which already in the case when $E = L^1$, that is, for Orlicz spaces $M(L^{\varphi_1}, L^\varphi) \subset L^{\varphi_2}$ give a solution of a problem raised in the book [26]. Some properties of a generalized complementary operation on Young functions, defined by Ando, are investigated in order to show how to construct the function φ_2 such that $M(E_{\varphi_1}, E_\varphi) = E_{\varphi_2}$. There are also several examples of independent interest.

1. Introduction and preliminaries

Pointwise multiplication and the space of pointwise multipliers between Orlicz spaces as well as between some other Banach ideal spaces were investigated by several authors. Here we try to prove such theorems for the Calderón-Lozanovskiĭ spaces E_φ generated by the Banach ideal space E and the Young function φ, which are generalizations of Orlicz spaces, Orlicz-Lorentz spaces and contain the p-convexification $E^{(p)} (1 \leq p < \infty)$ of E. The spaces E_φ were introduced by Calderón [10, p. 122] and Lozanovskiĭ [23] (see also Lozanovskiĭ [25]). Geometry of the spaces E_φ was intensively investigated during the last 20 years (see, for example, [19] and the references given there) and we should also mention here that they are, in fact, special cases of general Calderón-Lozanovskiĭ spaces $\rho(E, F)$ for $F = L^\infty$, being important in the interpolation theory (cf. [21], [26]).

Let (Ω, Σ, μ) be a complete σ-finite measure space and $L^0 = L^0(\Omega)$ be the space of all classes of μ-measurable real-valued functions defined on Ω. A Banach space $E = (E, \|\cdot\|_E)$ is said to be a *Banach ideal space* on Ω if E is a linear subspace of $L^0(\Omega)$ and satisfies the so-called ideal property, which means that if $y \in E, x \in L^0$ and $|x(t)| \leq |y(t)|$ for μ-almost all $t \in \Omega$, then $x \in E$ and $\|x\|_E \leq \|y\|_E$.

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A Banach ideal space E on Ω is *saturated* if every $A \in \Sigma$ with $\mu(A) > 0$ has a subset $B \in \Sigma$ of finite positive measure for which $\chi_B \in E$. For any such space E it is possible to construct a set $\Omega_E \in \Sigma$ such that: (i) every element of E vanishes μ-a.e. on $\Omega \setminus \Omega_E$ and (ii) every measurable $A \subset \Omega_E$ with $\mu(A) > 0$ has a measurable subset B of finite positive measure with $\chi_B \in E$. Furthermore, Ω_E is the union of an expanding sequence of sets $\{A_k\}$ such that $\mu(A_k) < \infty$ and $\chi_{A_k} \in E$ for each $k \in \mathbb{N}$. A set Ω_E is called the *support of* E and denoted by *supp*E. Note that we should say here "a support" rather than "the support" since in general there will be other sets $\tilde{\Omega}_E$ which can also satisfy (i) and (ii). However, they coincide μ-a.e with Ω_E, that is, $\mu(\tilde{\Omega}_E \setminus \Omega_E) = \mu(\Omega_E \setminus \tilde{\Omega}_E) = 0$. It is also clear that any Banach ideal space E can always be naturally identified with a saturated Banach ideal space on a possibly smaller measure space Ω_E. In such space E there exists an element x_0 which is strictly positive μ-a.e. on Ω_E, for example, $x_0 = \sum_{k=1}^{\infty} \chi_{A_k}/(2^k \|\chi_{A_k}\|_E)$. In particular, for a Banach ideal space E we have *supp*$E = \Omega$ if and only if E has a *weak unit*, i.e., a function x in E which is positive μ-a.e. on Ω (see [18] and [26]).

A point $x \in E$ is said to have *order continuous norm* if for any sequence (x_n) in E such that $0 \leq x_n \leq |x|$ and $x_n \to 0$ μ-a.e. on Ω we have $\|x_n\|_E \to 0$. By E_a we denote the subspace of all order continuous elements of E. It is known that $x \in E_a$ if and only if $\|x\chi_{A_n}\|_E \downarrow 0$ for any sequence $\{A_n\}$ satisfying $A_n \searrow \emptyset$ (that is $A_n \supset A_{n+1}$ and $\mu(\bigcap_{n=1}^{\infty} A_n) = 0$). A Banach ideal space E is called *order continuous* if every element of E has order continuous norm, that is, $E = E_a$.

We say that E has the *Fatou property* if $0 \leq x_n \uparrow x \in L^0$ with $x_n \in E$ and $\sup_{n \in \mathbb{N}} \|x_n\|_E < \infty$ imply that $x \in E$ and $\|x_n\|_E \uparrow \|x\|_E$.

If we consider the space E over a non-atomic measure μ with *supp*$E = \Omega$, then we say that E is a *Banach function space*. If we replace the measure space (Ω, Σ, μ) by the counting measure space $\left(\mathbb{N}, 2^{\mathbb{N}}, m\right)$, then we say that E is a *Banach sequence space* (denoted by e). In the last case the symbol $e_k = (0, \ldots, 0, 1, 0, \ldots)$ stands for the k-th unit vector.

The *weighted Banach function space* $E(w)$, where w is a measurable positive function (weight) on Ω, is defined by the norm $\|x\|_{E(w)} = \|xw\|_E$.

More information about Banach function spaces and Banach sequence spaces can be found, for example, in [8], [18], [21] and [22].

Let E and F be ideal Banach spaces in $L^0(\Omega)$ with their norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively. The space of pointwise multipliers $M(E, F)$ is defined by

$$M(E, F) = \{x \in L^0(\Omega) : xy \in F \text{ for all } y \in E\}$$

with the usual operator norm. This space is important, for example, in investigation of superposition operators and in factorization theorems. Some properties of superposition operators may be expressed by means of multiplicator spaces (cf. [4], [5]). They are also appearing in factorization theorems. Lozanovskiĭ proved that every function $x \in L^1$ can be factorized by $y \in E$ and $z \in E'$ in such a way that $x = yz$ and $\|y\|_E \|z\|_{E'} \leq (1+\varepsilon)\|x\|_{L^1}$, where $\varepsilon > 0$ is an arbitrary number (cf. [24]). This theorem can be generalized to the form $F = E \cdot M(E, F)$ under some assumptions on the spaces (see [35], [39]). In the case of sequence spaces (not necessarily ideal) the spaces $M(E, F)$ were investigated in [2] and used for description of different spaces of analytic functions on the disk by sequence multipliers of Taylor coefficients. More details about the space $M(E,F)$ we put in the next section.

In this paper we give improvements of the results on multipliers known for Orlicz spaces L^φ to the more general situation of Calderón-Lozanovskiĭ spaces E_φ. We need to recall some necessary definitions about Orlicz and Calderón-Lozanovskiĭ spaces.

A function $\varphi : [0,\infty) \to [0,\infty]$ is called a *Young function* (or *Orlicz function* if it is finite-valued) if φ is convex, non-decreasing with $\varphi(0) = 0$; we assume also that φ is neither identically zero nor identically infinity on $(0,\infty)$ and $\lim_{u\to b_\varphi^-} \varphi(u) = \varphi(b_\varphi)$ if $b_\varphi < \infty$, where $b_\varphi = \sup\{u > 0 : \varphi(u) < \infty\}$.

Note that from the convexity of φ and the equality $\varphi(0) = 0$ it follows that $\lim_{u\to 0+} \varphi(u) = \varphi(0) = 0$. Furthermore, from the convexity and $\varphi \not\equiv 0$ we obtain that $\lim_{u\to\infty} \varphi(u) = \infty$.

If we denote $a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\}$, then $0 \leq a_\varphi \leq b_\varphi \leq \infty$ and $a_\varphi < \infty$, $b_\varphi > 0$, since a Young function is neither identically zero nor identically infinity on $(0,\infty)$. Moreover, $a_\varphi = 0$ if φ is 0 only at 0 and $b_\varphi = \infty$ if $\varphi(u) < \infty$ for $u \in [0,\infty)$. If φ takes only two values 0 and ∞, then $0 < a_\varphi = b_\varphi < \infty$. The function φ is continuous and nondecreasing on $[0, b_\varphi)$ and is strictly increasing on $[a_\varphi, b_\varphi)$.

For a given Banach ideal space E on Ω and a Young function φ we define on $L^0(\Omega)$ a convex semimodular I_φ by

$$I_\varphi(x) := \begin{cases} \|\varphi \circ |x|\|_E & \text{if } \varphi \circ |x| \in E, \\ \infty & \text{otherwise,} \end{cases}$$

where $(\varphi \circ |x|)(t) = \varphi(|x(t)|), t \in \Omega$. By the Calderón-Lozanovskiĭ space E_φ we mean

$$E_\varphi = \{x \in L^0 : I_\varphi(cx) < \infty \text{ for some } c = c(x) > 0\},$$

which is a Banach ideal space on Ω with the so-called *Luxemburg-Nakano norm* defined by

$$\|x\|_{E_\varphi} = \inf\left\{\lambda > 0 : I_\varphi\left(x/\lambda\right) \leq 1\right\}.$$

If $E = L^1$ $(E = l^1)$, then E_φ is the Orlicz function (sequence) space L^φ (l^φ) equipped with the Luxemburg-Nakano norm (cf. [20], [26]). If E is a Lorentz function (sequence) space Λ_w (λ_w), then E_φ is the corresponding Orlicz-Lorentz function (sequence) space $\Lambda_{\varphi,w}$ $(\lambda_{\varphi,w})$, equipped with the Luxemburg-Nakano norm. On the other hand, if $\varphi(u) = u^p, 1 \leq p < \infty$, then E_φ is the p-convexification $E^{(p)}$ of E with the norm $\|x\|_{E^{(p)}} = \||x|^p\|_E^{1/p}$. If $\varphi(u) = 0$ for $0 \leq u \leq 1$ and $\varphi(u) = \infty$ for $u > 1$, then $E_\varphi = L^\infty$ with equality of the norms. If $suppE = \Omega$, then $suppE_\varphi = \Omega$, that is, E_φ has a weak unit.

For two ideal Banach spaces E and F on Ω the symbol $E \overset{C}{\hookrightarrow} F$ means that the embedding $E \subset F$ is continuous with the norm which is not bigger than C, i.e., $\|x\|_F \leq C\|x\|_E$ for all $x \in E$. In the case when the embedding $E \overset{C}{\hookrightarrow} F$ holds with some (unknown) constant $C > 0$ we simply write $E \hookrightarrow F$. Moreover, $E = F$ (and $E \equiv F$) means that the spaces are the same and the norms are equivalent (equal).

The paper is organized as follows: In Section 1 some necessary definitions and notation are collected, including the Calderón-Lozanovskiĭ spaces E_φ. In Section 2 the space of pointwise multipliers $M(E,F)$ is defined and some general results are presented. In Theorem 1, some important results in the case of symmetric spaces E,F on $[0,1]$ and $[0,\infty)$ are proved. It is important to mention here that for symmetric spaces on $[0,1]$ we have that $M(E,F)\neq\{0\}$ if and only if we have the imbedding $E\hookrightarrow F$. Also the fundamental function of $M(E,F)$ is described in terms of fundamental functions f_E and f_F. Better results appeared in two cases, when either as E we have the smallest symmetric space (the Lorentz space Λ_{f_E}) or when E is the largest Marcinkiewicz space M_{ϕ_1} and F is the smallest Lorentz space Λ_ϕ. Section 3 contains information about the Young function and its relations with its inverse. Then three relations between three Young functions are defined and some results proved for two of these relations (relations for large and small arguments).

Section 4 investigates the embedding $E_{\varphi_2}\hookrightarrow M\left(E_{\varphi_1},E_\varphi\right)$. In Theorem 2 there are sufficient conditions on the Young functions $\varphi_1,\varphi_2,\varphi$ and on the Banach ideal space E for such an inclusion. In Theorem 3 and 4 there are necessary conditions on functions under some additional assumptions on the space E. In the special case when $E=L^1$ and the corresponding spaces are Orlicz spaces, then these theorems where proved already by Ando [3] and O'Neil [32].

Section 5 deals with a more difficult reverse embedding $M\left(E_{\varphi_1},E_\varphi\right)\hookrightarrow E_{\varphi_2}$. The special case $M(L^p,L^1)\hookrightarrow L^{p'}$ is the famous Landau resonance theorem, which was extended to the case $M(L^\varphi,L^1)\hookrightarrow L^{\varphi^*}$ by several authors (see, for example, [29], where there are results for Orlicz space L^φ being even a quasi-Banach space). The first results on the embedding $M(L^{\varphi_1},L^\varphi)\hookrightarrow L^{\varphi_2}$, that is, for Orlicz spaces generated by Orlicz functions on non-atomic measure space, were given by Zabreĭko-Rutickiĭ [43] and Maligranda-Persson [28]. Using a recent result of Maligranda and Nakai [27] for Orlicz spaces on general σ-finite measure spaces and for arbitrary Young functions we were able to adopt this proof to the situation of Calderón-Lozanovskiĭ spaces E_φ (Theorem 5). Theorem 6 is interesting here since under certain monotonicity assumption it was possible to get also a necessary condition on Young functions for the embedding $M\left(E_{\varphi_1},E_\varphi\right)\hookrightarrow E_{\varphi_2}$. This result, for the special case of Orlicz spaces in which case $E=L^1$, gives an answer to the problem posed in Maligranda's book [26, Problem 4, p. 77] under additional assumption of monotonicity of ratio of the fundamental functions $f_{E_{\varphi_1}}$ and f_{E_φ}.

In Section 6 we have collected, as corollaries from some results in Sections 4 and 5, the necessary and sufficient conditions on functions so that the equality $M\left(E_{\varphi_1},E_\varphi\right)=E_{\varphi_2}$ holds provided E is a Banach ideal space with the Fatou property and $suppE=\Omega$. 2 Section 7 contains construction of a new function from two Young functions defined probably for the first time by Ando [3]. This is a complementary function to φ_1 with respect to φ given by the formula

$$\left(\varphi\ominus\varphi_1\right)(u)=\sup_{v>0}\left[\varphi\left(uv\right)-\varphi_1\left(v\right)\right].$$

The result on this construction gave possibility to improve Theorem 6 having another monotonicity condition (Theorem 7). Finally, in Example 8 we show that this last monotonicity condition cannot be dropped. This Example 8 presents construction of an Orlicz function ψ such that the non-separable Orlicz space $L^\psi[0,1]$ is a proper subspace of $L^2[0,1]$ and $M(L^\psi[0,1],L^2[0,1])=L^\infty[0,1]$. Moreover, the space $L^\psi[0,1]$ is also not $L^2[0,1]$-perfect, that is, $(L^\psi)^{L^2L^2} := M(M(L^\psi, L^2), L^2) \neq L^\psi$. This is because $(L^\psi)^{L^2L^2} = (L^\infty)^{L^2} = L^2 \neq L^\psi$.

2. On the space of pointwise multipliers $M(E,F)$

Let E and F be ideal Banach spaces in $L^0(\Omega)$ with their norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively. The space of pointwise multipliers $M(E,F)$ is defined by

$$M(E,F) = \{x \in L^0(\Omega) : xy \in F \text{ for all } y \in E\} \tag{1}$$

and the functional on it

$$\|x\|_{M(E,F)} = \sup\{\|xy\|_F,\ y \in E, \|y\|_E \leq 1\} \tag{2}$$

defines a complete semi-norm. It is a norm and $M(E,F)$ is an ideal Banach space if and only if $supp E = \Omega$, that is, E has a *weak unit*, i. e., $x_0 \in E$ such that $x_0 > 0$ μ-a.e. on Ω (in particular, $E \neq \{0\}$). In the case when $F = L^1$ we have $M(E, L^1) = E'$, where E' is the classical associated space to E or the Köthe dual space of E, and which is a Banach function space provided $supp E = \Omega$. Moreover, $supp E' \subset supp E$ and they are equal if $\|\cdot\|_E$ has the Fatou null property (if $x_n \uparrow x$ and $\|x_n\|_E = 0$ for all $n \in \mathbb{N}$, then $\|x\|_E = 0$). Always $E \overset{1}{\hookrightarrow} E''$ and $E \equiv E''$ if and only if E has the Fatou property. Note that $M(E,F)$ can be $\{0\}$ and it can be that $supp M(E,F)$ is smaller than $supp E \cap supp F$ (cf. Example 1(c) below).

The notation E' for the associated space to E is the reason why sometimes the space $M(E,F)$ is denoted as E^F. Banach ideal spaces for which $E \equiv E''$ are sometimes called perfect spaces and therefore the Banach ideal space E is called *F-perfect* if $E \equiv E^{FF}$. For example, L^∞ and F with $supp F = \Omega$ are F-perfect. Also E^F is F-perfect provided $supp F = supp E^F = \Omega$ and E is L^1-perfect if and only if E has the Fatou property.

General properties and several calculated concrete examples can be found in [5], [28] [38] (see also [2], [9], [11], [12], [26], [27], [31], [39] and [42]). Let us collect some of these properties and examples:

(i) If $E_0 \overset{C}{\hookrightarrow} E_1$, then $M(E_1, F) \overset{C}{\hookrightarrow} M(E_0, F)$.

(ii) If $F_0 \overset{C}{\hookrightarrow} F_1$, then $M(E, F_0) \overset{C}{\hookrightarrow} M(E, F_1)$.

(iii) $E \overset{1}{\hookrightarrow} E^{FF}$ and this embedding follows from the Hölder-Rogers inequality of the form
$\|xy\|_F \leq \|x\|_E \cdot \sup_{\|z\|_E \leq 1} \|yz\|_F = \|x\|_E \cdot \|y\|_{M(E,F)}$
for any $x \in E$ and $y \in M(E,F)$.

(iv) E^F is F-perfect, that is, $E^F \equiv E^{FFF}$.

(v) The embedding $L^\infty \overset{C}{\hookrightarrow} M(E,F)$ holds if and only if $E \overset{C}{\hookrightarrow} F$.

(vi) If $supp E = \Omega$, then $M(E,E) \equiv L^\infty$.

(vii) $M(E,F) \overset{1}{\hookrightarrow} M(F', E') \equiv M(E'', F'')$. If F has the Fatou property, then $M(E,F) \equiv M(F', E')$.

(viii) $M(E,F) \overset{1}{\hookrightarrow} M(F^G, E^G) \equiv M(E^{GG}, F^{GG})$. If F is G-perfect, then $M(E,F) \equiv M(E^{GG}, F^{GG})$.

(ix) For $1 < p < \infty$ we have $M(E^{(p)}, F^{(p)}) \equiv M(E, F)^{(p)}$.

(x) We have equality $\|x\|_{M(E(w_1),F(w_2))} \equiv \|\frac{w_2}{w_1} x\|_{M(E,F)}$ for $x \in M(E(w_1), F(w_2))$. In particular, if $suppE = \Omega$, then $M(E(w_1), E(w_2)) \equiv L^\infty(w_2/w_1)$.

(xi) If F has the Fatou property, then $M(E, F)$ also has this property.

Example 1. (a) If $1 \leq q < p < \infty, 1/r = 1/q - 1/p, E$ has the Fatou property and $\operatorname{supp} E = \Omega$, then $M(E^{(p)}, E^{(q)}) \equiv E^{(r)}$. In particular, $M(E^{(p)}, E) \equiv E^{(p')}$, $M(L^p(\mu), L^q(\mu)) \equiv L^r(\mu)$ and $M(L^p_{w_1}, L^q_{w_2}) \equiv L^r_{w_2/w_1}$ for $1 \leq q \leq p \leq \infty$.

(b) Let $1 \leq p < q < \infty$. If the measure μ is non-atomic, then $M(L^p(\mu), L^q(\mu)) = \{0\}$. Moreover, $M(l^p, l^q) \equiv l^\infty$.

(c) Let $\Omega = [0, 2]$ with the Lebesgue measure m. If $E = L^1[0, 1] \oplus L^2[1, 2]$ with $\|x\|_E = \|x\|_{L^1[0,1]} + \|x\|_{L^2[1,2]}$ and $F = L^2[0, 2]$, then $M(E, F) = L^\infty[1, 2]$.

(d) Let $\Omega = [0, 2]$ with the Lebesgue measure m. If $E = L^1[0, 1] \oplus L^\infty[1, 2]$ with $\|x\|_E = \|x\|_{L^1[0,1]} + \|x\|_{L^\infty[1,2]}$, then $E_a = L^1[0, 1]$, $suppE_a = [0, 1]$, $suppE = [0, 2]$ and for any Young functions φ, φ_1 we have $E_\varphi = L^\varphi[0, 1] \oplus L^\infty[1, 2]$ and $M(E_{\varphi_1}, E_\varphi) = M(L^{\varphi_1}[0, 1], L^\varphi[0, 1]) \oplus L^\infty[1, 2]$.

We also need some results in the case of symmetric spaces. By a *symmetric function space* (symmetric Banach function space) on I, where $I = [0, 1]$ or $I = [0, \infty)$ with the Lebesgue measure m, we mean a Banach ideal space $E = (E, \|\cdot\|_E)$ with the additional property that for any two equimeasurable functions $x \sim y, x, y \in L^0(I)$ (that is, they have the same distribution functions $d_x = d_y$, where $d_x(\lambda) = m(\{t \in I : |x(t)| > \lambda\}), \lambda \geq 0$) and $x \in E$ we have that $y \in E$ and $\|x\|_E = \|y\|_E$. In particular, $\|x\|_E = \|x^*\|_E$, where $x^*(t) = \inf\{\lambda > 0 \colon\ d_x(\lambda) \leq t\},\ t \geq 0$.

The *fundamental function* f_E of a symmetric function space E on I is defined by the formula $f_E(t) = \|\chi_{[0,t]}\|_E, t \in I$. It is well-known that each fundamental function is quasi-concave on I, that is, $f_E(0) = 0, f_E(t)$ is positive, non-decreasing and $f_E(t)/t$ is non-increasing for $t \in (0, m(I))$ or, equivalently, $f_E(t) \leq \max(1, t/s) f_E(s)$ for all $s, t \in (0, m(I))$. Taking $\tilde{f}_E(t) := \inf_{s \in (0,m(I))}(1 + \frac{t}{s}) f_E(s)$ we obtain that the function $\tilde{f}_E$ is concave and $f_E(t) \leq \tilde{f}_E(t) \leq 2 f_E(t)$ for all $t \in I$. For any quasi-concave function ϕ on I the *Marcinkiewicz function space* M_ϕ is defined by the norm

$$\|x\|_{M_\phi} = \sup_{t \in I} \phi(t)\, x^{**}(t), \quad x^{**}(t) = \frac{1}{t} \int_0^t x^*(s) ds.$$

This is a symmetric Banach function space on I with the fundamental function $f_{M_\phi}(t) = \phi(t)$ and $E \overset{1}{\hookrightarrow} M_{f_E}$ since

$$x^{**}(t) \leq \frac{1}{t} \|x^*\|_E \|\chi_{[0,t]}\|_{E'} = \|x\|_E \frac{1}{f_E(t)} \text{ for any } t \in I. \tag{3}$$

The fundamental function of a symmetric function space $E = (E, \|\cdot\|_E)$ is not necessary concave but we can introduce an equivalent norm on E in such a way that the fundamental function will be concave. In fact, for the fundamental function f_E of E consider the new norm on E defined by formula

$$\|x\|_E^1 = \max(\|x\|_E, \|x\|_{M_{\tilde{f}_E}}),\ x \in E.$$

Then $\|x\|_E \le \|x\|_E^1 \le \max(\|x\|_E, 2\|x\|_{M_{f_E}}) \le 2\|x\|_E$. Moreover,

$$\|\chi_{[0,t]}\|_E^1 = \max(f_E(t), \tilde{f}_E(t)) = \tilde{f}_E(t)$$

and $(E, \|\cdot\|_E^1)$ is a symmetric Banach function space with concave fundamental function (cf. Zippin [44], Lemma 2.1).

For any symmetric function space E with concave fundamental function f_E there is also a smallest symmetric space with the same fundamental function. This space is the *Lorentz function space* given by the norm

$$\|x\|_{\Lambda_{f_E}} = \int_I x^*(t) df_E(t) = f_E(0^+)\|x\|_{L^\infty(I)} + \int_I x^*(t) f_E'(t) dt.$$

We have then embeddings

$$\Lambda_{f_E} \overset{1}{\hookrightarrow} E \overset{1}{\hookrightarrow} M_{f_E},$$

and all fundamental functions are f_E.

Any non-trivial symmetric function space E on I (E is non-trivial if $E \neq \{0\}$) is intermediate space between the spaces $L^1(I)$ and $L^\infty(I)$. More precisely,

$$L^1(I) \cap L^\infty(I) \overset{C_1}{\hookrightarrow} E \overset{C_2}{\hookrightarrow} L^1(I) + L^\infty(I),$$

where $C_1 = 2f_E(1), C_2 = 1/f_E(1)$ and $\|x\|_{L^1 \cap L^\infty} = \max(\|x\|_{L^1}, \|x\|_{L^\infty})$, $\|x\|_{L^1+L^\infty} = \inf\{\|x_0\|_{L^1} + \|x_1\|_{L^\infty} : x = x_0 + x_1, x_0 \in L^1, x_1 \in L^\infty\} = \int_0^1 x^*(s)ds$ (see [21], Theorem 4.1). In particular, supp $E = I$.

A symmetric function space E on I has the *majorant property* if for all $x \in L^0, y \in E$, the condition $\int_0^t x^*(s)\, ds \le \int_0^t y^*(s)\, ds$ for all $t \in I$ implies that $x \in E$ and $\|x\|_E \le \|y\|_E$. Every symmetric function space with the Fatou property or separable symmetric function space have the majorant property. More information about symmetric spaces on $I = [0, \infty)$ can be found in the book [21].

Theorem 1. *Let E and F be non-trivial symmetric function spaces on I.*

(i) Then the space of multipliers $M(E,F)$ is a symmetric function space on I. Moreover, if the symmetric spaces E, F are on $I = [0,1]$, then $M(E,F) \neq \{0\}$ if and only if $E \hookrightarrow F$.

(ii) If F has the majorant property, then $M(E,F)$ has also majorant property and

$$\|x\|_{M(E,F)} = \sup_{\|y\|_E \le 1} \|x^* y^*\|_F. \tag{4}$$

(iii) We have $f_{M(E,F)}(t) \ge \sup_{0<s\le t} \frac{f_F(s)}{f_E(s)}$ for all $0 < t < m(I)$, and if f_F is a concave function with $f_F(0^+) = 0$, then

$$f_{M(E,F)}(t) \le \int_0^t \frac{f_F'(s)}{f_E(s)}\, ds \text{ for all } t \in (0, m(I)). \tag{5}$$

If, in addition, $\frac{f_F(t)}{f_E(t)\,t^a}$ *is a non-decreasing function on* $(0,b)$ *for some* $a>0$ *and* $b\in(0,m(I))$*, then*

$$\frac{f_F(t)}{f_E(t)} \le f_{M(E,F)}(t) \le \frac{1}{a}\,\frac{f_F(t)}{f_E(t)} \quad \text{for all } t\in(0,b). \tag{6}$$

In the case when f_F *is only quasi-concave function, then we should multiply the right sides of inequality (5) and (6) by constant 2.*

(*iv*) *If* f_E *is a concave function on* I *with* $f_E(0^+)=0$*, then*

$$f_{M(\Lambda_{f_E},F)}(t) = \sup_{s\le t} \frac{f_F(s)}{f_E(s)} \quad \text{for all } t\in I,$$

provided the last supremum is finite.

(*v*) *Let* ϕ,ψ *be concave functions on* I*,* $\phi(0^+)=\psi(0^+)=0$ *and denote* $\phi_1(t)=\frac{t}{\psi(t)}$*. Then* $M_{\phi_1}\overset{C}{\hookrightarrow}\Lambda_\phi$ *with optimal constant* $C>0$ *if and only if* $C=\int_I \psi'(s)\phi'(s)\,ds<\infty$*. Moreover,* $M(M_{\phi_1},\Lambda_\phi)\equiv\Lambda_\eta$*, where* $\eta(t)=\int_0^t\psi'(s)\phi'(s)\,ds$ *and* $\eta(t)<\infty$ *for* $t\in I$*.*

Before the proof of Theorem 1 we give the embedding results of independent interest.

Proposition 1. *Let* E *and* F *be non-trivial symmetric function spaces on* I*.*

(*i*) *If either* $I=[0,1]$ *or* $I=[0,\infty)$ *and* $M(E,F)\neq\{0\}$*, then* $\chi_C\in M(E,F)$ *for each set* $C\subset I$ *with* $m(C)<\infty$*.*

(*ii*) *If* $I=[0,1]$*, then* $M(E,F)\neq\{0\}$ *if and only if* $E\hookrightarrow F$*.*

(*iii*) *If* $I=[0,\infty)$ *and there exists* $x\in M(E,F)$ *with* $x^*(\infty)=\lim_{t\to\infty}x^*(t)>0$*, then* $E\hookrightarrow F$*.*

(*iv*) *If* $I=[0,\infty)$ *and* $M(E,F)\neq\{0\}$*, then* $E_{\text{fin}}\subset F$*, where* E_{fin} *is the space of elements from* E *with supports having finite measure.*

Proof of Proposition 1. (i) Let $0\neq x\in M(E,F)$. Then there are a $\delta>0$ and a measurable set $A\subset I$ with $0<m(A)<\infty$ such that

$$\delta\,\chi_A \le |x|\,\chi_A.$$

Thus $\chi_A\in M(E,F)$. We will prove that $\chi_B\in M(E,F)$ for each $B\subset I$ with $m(B)=m(A)$. There is a measure preserving transformation $\omega: A\to B$ such that $\omega(A)=B$ (cf. [37], Theorem 17, p. 410). Then $\chi_A=\chi_B(\omega)$ and for any $y\in E$

$$y(\omega)\,\chi_A = y(\omega)\,\chi_B(\omega)\sim y\,\chi_B,$$

where $y(\omega)\,\chi_A$ is a function on I defined as $y(\omega(t))$ if $t\in A$ and 0 if $t\notin A$. In fact, for any $\lambda>0$ we have

$$\begin{aligned} m(\{t\in I: |y(w(t))\,\chi_A(t)|>\lambda\}) &= m(\{t\in A: |y(w(t))|>\lambda\}) \\ &= m(\omega^{-1}(\{s\in B: |y(s)|>\lambda\})) \\ &= m(\{s\in B: |y(s)|>\lambda\}) \\ &= m(\{s\in I: |y(s)\,\chi_B(s)|>\lambda\}). \end{aligned}$$

Since $\chi_A \in M(E,F)$ and E is symmetric it follows that $y(\omega)\,\chi_A \in F$. Consequently, by symmetry of F we have $y\,\chi_B \in F$. If $C \subset I$, then using the fact that measure is non-atomic we can write C as a finite sum of disjoint sets B_k such that $m(B_k) = m(A), k = 1, 2, \ldots, n-1$ and $m(B_n) \leq m(A)$. Then $\chi_C = \chi_{\cup_{k=1}^n B_k} = \sum_{k=1}^n \chi_{B_k} \in M(E,F)$.

(ii) The sufficiency follows from general properties (i) and (vi) since the embedding $E \hookrightarrow F$ gives that $M(E,F) \hookleftarrow M(F,F) = L^\infty$. We need to prove necessity. If $M(E,F) \neq \{0\}$, then by (i) $\chi_{[0,1]} \in M(E,F)$. Therefore if $x \in E$, then $x = x\,\chi_{[0,1]} \in F$.

(iii) Let $x \in M(E,F)$ be such that $x^*(\infty) > 0$. Then the set

$$A = \{t \in I : |x(t)| \geq x^*(\infty)/2\}$$

has infinite measure. Let $y \in E$ be arbitrary. Assume for the moment that there exists a measure preserving transformation $\omega : A \to I$ such that $\omega(A) = I$. Then $y(\omega)\,\chi_A \sim y$. Hence, by the symmetry of E, we obtain $y(\omega)\,\chi_A \in E$ and so $x\,y(\omega)\,\chi_A \in F$ because $x \in M(E,F)$. However,

$$\frac{x^*(\infty)}{2}\,|y(\omega)|\,\chi_A \leq |x\,y(\omega)|\,\chi_A \in F,$$

that is, $y(\omega)\,\chi_A \in F$ and also $y \in F$ by symmetry of F. The only left is to show that there exists a measure preserving transformation $\omega : A \to I$ such that $\omega(A) = I$. Since $m(A) = \infty$ and Lebesgue measure is σ-finite and non-atomic we can write A as a countable sum of disjoint subsets $(A_n)_{n=1}^\infty$ of A, each of measure 1. Using the fact that there are measure preserving transformations $\omega_n : A_n \to [n-1, n)$ such that $\omega(A_n) = [n-1, n)$ (cf. [37], Theorem 17, p. 410) we can define mapping $\omega : A \to I$ such that $\omega(A) = I$ by taking $\omega(t) := \omega_n(t)$ if $t \in A_n\,(\,n = 1, 2, \ldots)$. Observe that ω is a measure preserving transformation because for any $B \subset I$ we have

$$\begin{aligned} m(\omega^{-1}(B)) &= m\left[\omega^{-1}\left(\bigcup_{n=1}^{\infty}(B \cap [n-1, n))\right)\right] = m\left[\bigcup_{n=1}^{\infty} \omega_n^{-1}\,(B \cap [n-1, n))\right] \\ &= \bigcup_{n=1}^{\infty} m[\omega_n^{-1}(B \cap [n-1, n))] = \bigcup_{n=1}^{\infty} m(B \cap [n-1, n)) = m(B). \end{aligned}$$

(iv) If $M(E,F) \neq \{0\}$ and $y \in E_{\rm fin}$, then by Proposition 1(i) we obtain $\chi_{\operatorname{supp} y} \in M(E,F)$, which implies $y \in F$.

Remark 1. Note that E_{fin} needs not be complete therefore the inclusion $E_{fin} \subset F$ is not continuous, in general. However, for any $d > 0$ there exists a constant $c = c(d) > 0$ such that $E\mid_A \overset{c}{\hookrightarrow} F\mid_A$ for each $A \subset I$ with $m(A) = d$, that is, $\|x\chi_A\|_F \leq c\|x\chi_A\|_E$ for $x \in E$.

Of course, $E\mid_A \overset{c}{\hookrightarrow} F\mid_A$ for each A with $m(A) < \infty$, but we need to show that c depends only on d, not on A. Suppose, on the contrary, that there is sequence (A_n) of sets with $m(A_n) = d$ and $E\mid_{A_n} \overset{c_n}{\hookrightarrow} F\mid_{A_n}$, where $c_n \to \infty$ and c_n can not be taken smaller. Choose c so that $E\mid_{[0,d]} \overset{c}{\hookrightarrow} F\mid_{[0,d]}$. Moreover, since c_n were optimal, one can find a sequence $(x_n) \in \prod E\,|_{A_n}$ with $\|x_n\|_E = 1$ such that $\frac{c_n}{2}\|x_n\|_E \le \|x_n\|_F$. But for each $n \in \mathbb{N}$ one has $x_n^* \in E\,|_{[0,d]}$ by symmetry of E and finally

$$\infty \leftarrow \frac{c_n}{2}\|x_n^*\|_{E|_{[0,d]}} = \frac{c_n}{2}\|x_n\|_{E|_{A_n}} \le \|x_n\|_{F|_{A_n}} = \|x_n^*\|_{F|_{[0,d]}} \le c\|x_n^*\|_{E|_{[0,d]}} = c,$$

and this contradiction proves the claim.

Remark 2. The proof of the embedding in Proposition 1(i) can also be found in [1, Theorem 1] or [16, Lemma 5.2], where the authors showed more general results from which, in particular, it follows that the existence of a nonzero pointwise multiplier necessarily implies that $E \hookrightarrow F$. However, our proof is much simpler. Moreover, the inclusion in Proposition 1(iii) is proved in [1, Corollary 9] and in [9, Lemma 6.1] but the authors used the fact that $M(E,F)$ is already a symmetric space which we want to prove here.

Example 2. For symmetric spaces on $I = [0,\infty)$ the relation $M(E,F) \ne \{0\}$ can happen even if we don't have an embedding $E \subset F$. In fact, for $E = L^2$ and $F = L^2 \cap L^1$ on $I = [0,\infty)$ we have $E \not\subset F$ but

$$M(E,F) = M(L^2, L^2 \cap L^1) \equiv L^\infty \cap L^2.$$

Proof of Theorem 1. (i) Let $I = [0,1]$. Assume that $x \sim z$ and $0 \ne z \in M(E,F)$. By [21, Lemma 2.1, p. 60] (cf. also [6], p. 777) for any $\epsilon > 0$ there is a measure-preserving mapping $\omega : [0,1] \to [0,1]$ such that

$$\|x(\omega) - z\|_{L^\infty} \le \varepsilon.$$

Moreover, Proposition 1(ii) guarantees that $E \overset{C}{\hookrightarrow} F$. Thus, for every $y \in E, \|y\|_E \le 1$, we have

$$\begin{aligned} \|xy\|_F &= \|x(\omega)y(\omega)\|_F \le \|zy(\omega)\|_F + \|[x(\omega) - z]\,y(\omega)\|_F \\ &\le \|zy(\omega)\|_F + \varepsilon\|y(\omega)\|_F \le \|zy(\omega)\|_F + C\varepsilon\|y(\omega)\|_E \\ &\le \|z\|_{M(E,F)} + C\varepsilon. \end{aligned}$$

Taking the supremum over all such y, we obtain

$$\|x\|_{M(E,F)} \le \|z\|_{M(E,F)} + C\varepsilon,$$

or, since $\varepsilon > 0$ is arbitrary, $\|x\|_{M(E,F)} \le \|z\|_{M(E,F)}$. The reverse inequality can be proved similarly and the proof is complete in the case when $I = [0,1]$.

Let $I = [0,\infty)$. We divide the proof into two parts.

A. Suppose $x \in M(E,F)$ and $x^*(\infty) > 0$. Set

$$x_1(t) := \max(|x(t)|, x^*(\infty)),\ t \in I.$$

Then, for any $y \in E$, we obtain $x\,y \in F$ and $x^*(\infty)\,y \in F$ since by Proposition 1(iii) we have imbedding $E \subset F$. Thus $x_1\,y \in F$ for any $y \in E$ and whence $x_1 \in M(E,F)$. We will prove that

$$\|x\|_{M(E,F)} = \|x_1\|_{M(E,F)}. \tag{7}$$

Clearly, it is enough to show that $\|x\|_{M(E,F)} \geq \|x_1\|_{M(E,F)}$. Let $\varepsilon > 0$ be arbitrary. We can find $y \in E$ with $\|y\|_E \leq 1$ such that

$$(1-\varepsilon)\|x_1\|_{M(E,F)} \leq \|x_1\, y\|_F.$$

Consider two sets

$$A := \{t \in I : (1-\varepsilon)\, x^*(\infty) \leq |x(t)| \leq (1+\varepsilon)\, x^*(\infty)\},$$

and

$$B := \{t \in I : |x(t)| > (1+\varepsilon)\, x^*(\infty)\}.$$

Then

$$x_1\, \chi_B = |x|\, \chi_B,\ m(A) = \infty,\ \text{and } x_1\, \chi_A \geq |x|\, \chi_A \geq (1-\varepsilon)\, x_1\, \chi_A.$$

Similarly as in the proof of the case (iii) of Proposition 1 we can find measure preserving transformation $\omega_0 : A \to I\backslash B$ such that $\omega_0(A) = I\backslash B$. Define $\omega : A \cup B \to I$ by

$$\omega(t) := \begin{cases} \omega_0(t), & \text{if } t \in A, \\ t, & \text{if } t \in B. \end{cases}$$

Then ω is measure preserving transformation and $\omega(A \cup B) = I$. Moreover, $y(\omega)\, \chi_{A\cup B} \sim y$ and

$$\begin{aligned} |x\, y(\omega)|\, \chi_{A\cup B} &= |x\, y(\omega)|\, \chi_A + |x\, y(\omega)|\, \chi_B \\ &\geq (1-\varepsilon)\, x_1\, |y(\omega)|\, \chi_A + x_1\, |y(\omega)|\, \chi_B \\ &\geq (1-\varepsilon)\, x_1\, |y(\omega)|\, \chi_{A\cup B}. \end{aligned}$$

On the other hand, since $y(\omega_0)\, \chi_A \sim y\, \chi_{I\backslash B}$ it follows that

$$\begin{aligned} d_{x_1 y} &= d_{x_1 y \chi_B} + d_{x_1 y \chi_{I\backslash B}} \leq d_{x_1 y \chi_B} + d_{(1+\varepsilon) x^*(\infty)\, y \chi_{I\backslash B}} \\ &= d_{x_1 y \chi_B} + d_{(1+\varepsilon) x^*(\infty)\, y(\omega_0) \chi_A} \\ &\leq d_{(1+\varepsilon) x_1 y(\omega) \chi_B} + d_{(1+\varepsilon) x_1 y(\omega_0) \chi_A} \\ &= d_{(1+\varepsilon) x_1 y(\omega) \chi_{A\cup B}}. \end{aligned}$$

Thus, $(x_1\, y)^*(t) \leq [(1+\varepsilon)\, x_1\, y(\omega)\, \chi_{A\cup B}]^*(t)$ and

$$\begin{aligned} (1-\varepsilon)\|x_1\|_{M(E,F)} &\leq \|x_1\, y\|_F \leq (1+\varepsilon)\|x_1\, y(\omega)\, \chi_{A\cup B}\|_F \\ &\leq \frac{1+\varepsilon}{1-\varepsilon}\, \|x\, y(\omega)\, \chi_{A\cup B}\|_F \leq \frac{1+\varepsilon}{1-\varepsilon}\, \|x\|_{M(E,F)}, \end{aligned}$$

which means that

$$\|x_1\|_{M(E,F)} \leq \frac{1+\varepsilon}{(1-\varepsilon)^2}\, \|x\|_{M(E,F)},$$

or, since $\varepsilon > 0$ is arbitrary, $\|x_1\|_{M(E,F)} \leq \|x\|_{M(E,F)}$.

Let $z \sim x$. Then $z_1 \sim x_1$, where $x_1 \in M(E,F)$ and $z_1(t) = \max(|z(t)|, z^*(\infty)), t \in I$. We may follow the proof of (i), applying Lemma 2.1, p. 60 in [21], to conclude that

$z_1 \in M(E,F)$ and $\|x_1\|_{M(E,F)} = \|z_1\|_{M(E,F)}$. Clearly, $z_1 \geq |z|$ and so $z \in M(E,F)$. Using then equality (7) we obtain

$$\|x\|_{M(E,F)} = \|x_1\|_{M(E,F)} = \|z_1\|_{M(E,F)} = \|z\|_{M(E,F)},$$

and symmetry of $M(E,F)$ is proved.

B. Assume $x^*(\infty) = 0$ and $0 \neq x \in M(E,F)$. Take any $z \sim x$. Then, by [21, Lemma 2.1, p. 60], for any $\epsilon > 0$ there is a measure-preserving mapping $\omega : I \to I$ such that

$$\|x - z(\omega)\|_{L^1 \cap L^\infty} < \varepsilon,$$

and, for any $y \in E$, $\|y\|_E \leq 1$, we have

$$\begin{aligned} \|zy\|_F &= \|z(\omega)y(\omega)\|_F \leq \|xy(\omega)\|_F + \|[z(\omega) - x]\, y(\omega)\|_F \\ &\leq \|x\|_{M(E,F)} + \|[z(\omega) - x]\, y(\omega)\|_F. \end{aligned}$$

Since $\|y(\omega)\|_E \leq 1$, then, by using Lemma 1 proved below, we can find a decomposition $y(\omega) = u + v$ such that $u \in E, m(\operatorname{supp} u) \leq 1$ and $v \in E \cap L^\infty$ with $\|v\|_{L^\infty} \leq \frac{1}{f_E(1)}$. Therefore, applying Proposition 1(iv) on inclusion $E_{\text{fin}} \subset F$, we obtain

$$\begin{aligned} \|zy\|_F &\leq \|x\|_{M(E,F)} + \|[z(\omega) - x]\,[u + v]\|_F \\ &\leq \|x\|_{M(E,F)} + \|[z(\omega) - x]\, u\|_F + \|[z(\omega) - x]\, v\|_F \\ &\leq \|x\|_{M(E,F)} + \|z(\omega) - x\|_{L^1 \cap L^\infty} \|u\|_F + \|[z(\omega) - x]\, v\|_F \\ &\leq \|x\|_{M(E,F)} + \varepsilon\, \|u\|_F + \|v\|_{L^\infty} \|z(\omega) - x\|_F \\ &\leq \|x\|_{M(E,F)} + \varepsilon\, \|u\|_F + \frac{1}{f_E(1)} \|z(\omega) - x\|_{L^1 \cap L^\infty} \cdot 2 f_F(1) \\ &= \|x\|_{M(E,F)} + \varepsilon\, \|u\|_F + 2\varepsilon \frac{f_F(1)}{f_E(1)}. \end{aligned}$$

Using Remark 1 we have $\|u\|_F \leq c\|u\|_E$ since $m(\operatorname{supp} u) \leq 1$ and, hence,

$$\|zy\|_F \leq \|x\|_{M(E,F)} + \varepsilon\, c\|u\|_E + 2\varepsilon \frac{f_F(1)}{f_E(1)} \leq \|x\|_{M(E,F)} + \varepsilon\, c + 2\varepsilon \frac{f_F(1)}{f_E(1)}.$$

Taking the supremum over all $y \in E$, $\|y\|_E \leq 1$, we obtain

$$\|z\|_{M(E,F)} \leq \|x\|_{M(E,F)} + \varepsilon\, c + 2\varepsilon \frac{f_F(1)}{f_E(1)},$$

and since $\varepsilon > 0$ is arbitrary $\|z\|_{M(E,F)} \leq \|x\|_{M(E,F)}$. The reversed inequality can be proved similarly turning the roles of both functions and the proof is complete.

Lemma 1. *Let E be a symmetric space on $[0, \infty)$. If $y \in E$ and $\|y\|_E \leq 1$, then we can find a decomposition $y = u + v$ such that $u \in E, \|u\|_E \leq 1$ with $m(\operatorname{supp} u) \leq 1$ and $v \in E \cap L^\infty$ with $\|v\|_{L^\infty} \leq 1/f_E(1)$.*

Proof of Lemma 1. If either $y^*(1) > y^*(\infty) \geq 0$ or $y^*(1) = y^*(\infty) > 0$, then we can take $y = u + v$, where $u = y\chi_A$ and $v = y\chi_{I\setminus A}$ with $A = \{s > 0 : |y(s)| > y^*(1)\}$. In fact, $m(A) \leq 1$ and

$$\|v\|_{L^\infty} \leq y^*(1) \leq \int_0^1 y^*(t)dt \leq \|y\|_E \|\chi_{[0,1]}\|_{E'} \leq 1/f_E(1).$$

If $y^*(1) = y^*(\infty) = 0$ and $A\operatorname{supp} y$, then $m(A) = t$ for some $0 < t \leq 1$. Then $y = y\chi_A + 0$ is such a decomposition.

Proof of Theorem 1. (ii) Let $x \in M(E,F), y \in E$. By Theorem 1(i) and symmetry of E we have $x^* \in M(E,F), y^* \in E$ and thus $x^*y^* \in F$. Moreover,

$$\begin{aligned} \sup_{\|y\|_E \leq 1} \|x^*y^*\|_F &= \sup_{\|y^*\|_E \leq 1} \|x^*y^*\|_F \\ &\leq \sup_{\|z\|_E \leq 1} \|x^*z\|_F = \|x^*\|_{M(E,F)}. \end{aligned}$$

On the other hand, since a symmetric space F has the majorant property and

$$\int_0^t (xy)^*(s)ds = \sup_{m(A)=t} \int_A |x(s)y(s)|ds \leq \int_0^t x^*(s)y^*(s)ds \quad \text{for all} \quad t \in I,$$

it follows that $\|xy\|_F \leq \|x^*y^*\|_F$ and, hence,

$$\|x\|_{M(E,F)} = \sup_{\|y\|_E \leq 1} \|xy\|_F \leq \sup_{\|y\|_E \leq 1} \|x^*y^*\|_F \leq \|x^*\|_{M(E,F)} = \|x\|_{M(E,F)},$$

where the last equality follows from (i), that is, from symmetry of $M(E,F)$ and equality (4) is proved.

We show that if F has the majorant property, then $M(E,F)$ has it as well. Let $x \in M(E,F), z \in L^0$ and

$$\int_0^t z^*(s)\, ds \leq \int_0^t x^*(s)\, ds$$

for all $t \in I$. By Hardy lemma

$$\int_0^t z^*(s)\, y^*(s)\, ds \leq \int_0^t x^*(s)\, y^*(s)\, ds \tag{8}$$

for all $t \in I$ and each $y \in E$. Since, by (i), $M(E,F)$ is symmetric it follows that $x^* \in M(E,F)$. Then $x^*y^* \in F$ for each $y \in E$. The majorant property of F and inequality (8) give $z^*y^* \in F$. Following analogously as above and applying majorant property of F we obtain $zy \in F$. Thus, $z \in M(E,F)$ and by (8) $\|z^*y^*\|_F \leq \|x^*y^*\|_F$. Taking the supremum over all $y \in E$ with $\|y\|_E \leq 1$ we have $\|z\|_{M(E,F)} \leq \|x\|_{M(E,F)}$, which shows the majorant property of $M(E,F)$.

(iii) For any $t \in I$ we have

$$f_{M(E,F)}(t) = \sup_{\|y\|_E \leq 1} \|y\, \chi_{[0,t]}\|_F \geq \sup_{s \in I} \|\frac{\chi_{[0,s]}}{f_E(s)}\, \chi_{[0,t]}\|_F = \sup_{s \leq t} \frac{f_F(s)}{f_E(s)}.$$

On the other hand, if f_F is a concave function and $f_F(0^+) = 0$, then by the general properties (i) and (ii) we have $M(M_{f_E}, \Lambda_{f_F}) \overset{1}{\hookrightarrow} M(E, F)$ and, hence,

$$\begin{aligned} f_{M(E,F)}(t) &\leq f_{M(M_{f_E},\Lambda_{f_F})}(t) = \sup_{\|y\|_{M_{f_E}} \leq 1} \|y\chi_{[0,t]}\|_{\Lambda_{f_F}} \\ &\leq \sup_{y^* \leq 1/f_E} \int_0^t y^*(s) df_F(s) \leq \int_0^t \frac{f_F'(s)}{f_E(s)} ds. \end{aligned}$$

If, in addition, we have monotonicity assumption on $\frac{f_F(t)}{f_E(t)\, t^a}$, then, by using the fact that $f_F'(s) \leq f_F(s)/s$ for almost all $s \in I$, we obtain for $t \in (0, b)$

$$\begin{aligned} \int_0^t \frac{f_F'(s)}{f_E(s)} ds &\leq \int_0^t \frac{f_F(s)}{s\, f_E(s)} ds = \int_0^t \frac{f_F(s)}{f_E(s)\, s^a} s^{a-1} ds \\ &\leq \frac{f_F(t)}{f_E(t)\, t^a} \int_0^t s^{a-1} ds = \frac{1}{a} \frac{f_F(t)}{f_E(t)}. \end{aligned}$$

(iv) The estimation follows from the equality $\|x^*\|_{M(\Lambda_{f_E},F)} = \sup_{s\in I} \frac{\|x^*\chi_{[0,s]}\|_F}{f_E(s)}$ proved in [28, Theorem 3] since then

$$f_{M(\Lambda_{f_E},F)}(t) = \sup_{s\in I} \frac{\|\chi_{[0,t]}\chi_{[0,s]}\|_F}{f_E(s)} = \sup_{s\leq t} \frac{f_F(s)}{f_E(s)}.$$

(v) First we show the equivalence. Suppose $\int_I \psi'(s)\phi'(s)\, ds = C < \infty$. If $\|x\|_{M_{\phi_1}} \leq 1$, then for all $0 < t < m(I)$

$$\frac{1}{\psi(t)} \int_0^t x^*(s)\, ds = \frac{\phi_1(t)}{t} \int_0^t x^*(s)\, ds \leq 1,$$

whence

$$\int_0^t x^*(s) ds \leq \psi(t) = \int_0^t \psi'(s) ds$$

for each $t \in I$. Thus, by Hardy lemma (see Proposition 3.6, p. 56 in [21]),

$$\int_I x^*(s)\phi'(s)\, ds \leq \int_I \psi'(s)\phi'(s)\, ds = C \tag{9}$$

and so $\|x\|_{\Lambda_\phi} \leq C$, which means that $M_{\phi_1} \overset{C}{\hookrightarrow} \Lambda_\phi$. Moreover,

$$\|\psi'\|_{M_{\phi_1}} = \sup_{t\in I} \phi_1(t) \left(\psi'\right)^{**}(t) = \sup_{t\in I} \frac{1}{\psi(t)} \int_0^t \psi'(s)\, ds = 1. \tag{10}$$

Note that for $x = x^* = \psi'$ we have equality in estimate (9), that is, C is optimal.

Suppose, conversely, $M_{\phi_1} \overset{C}{\hookrightarrow} \Lambda_\phi$ with optimal constant C. Then

$$\int_I \psi'(s)\phi'(s)\, ds \leq c\, \|\psi'\|_{M_{\phi_1}} = C. \tag{11}$$

Moreover, if $x = x^*$ and $\|x\|_{M_{\phi_1}} = 1$, then $1 = \sup_{t\in I} \frac{\phi_1(t)}{t} \int_0^t x(s)\, ds$, whence $\int_0^t x(s)\, ds \leq \psi(t) = \int_0^t \psi'(s)\, ds$ for any $t \in I$. By Hardy lemma, $\int_0^t x(s)\phi'(s)\, ds \leq \int_0^t \psi'(s)\phi'(s)\, ds$ for each $t \in I$. Consequently

$$\|x\|_{\Lambda_\phi} = \int_I x(s)\phi'(s)\, ds \leq \int_0^t \psi'(s)\phi'(s)\, ds$$

for any $x \in M_{\phi_1}$ with $\|x\|_{M_{\phi_1}} = 1$. Thus

$$\|x\|_{\Lambda_\phi} \leq \int_0^t \psi'(s)\phi'(s)\, ds\, \|x\|_{M_{\phi_1}},$$

for all $x \in M_{\phi_1}$. But $C \leq \int_I \psi'(s)\phi'(s)\, ds$, because C is optimal, which together with (11) gives the equality $C = \int_0^t \psi'(s)\phi'(s)\, ds$.

The equality of spaces follows from the following facts: if $x^* \in M(M_{\phi_1}, \Lambda_\phi)$ and since $\|\psi'\|_{M_{\phi_1}} = 1$ we obtain $x^*\psi' \in \Lambda_\phi$ or

$$\int_I (x^*\psi')^*(s)\phi'(s)\, ds = \int_I x^*(s)\, \psi'(s)\phi'(s)\, ds \leq \|x^*\|_{M(M_{\phi_1},\Lambda_\phi)},$$

and, thus, $\|x\|_{\Lambda_\eta} \leq \|x^*\|_{M(M_{\phi_1},\Lambda_\phi)}$. On the other hand, if $x \in \Lambda_\eta$ and $\|y\|_{M_{\phi_1}} \leq 1$ is arbitrary or, equivalently, $\int_0^t y^*(s)\, ds \leq \psi(t) = \int_0^t \psi'(s)\, ds$ for all $t \in I$, then, by the Hardy inequality,

$$\int_I x^*(s)y^*(s)\phi'(s)\, ds \leq \int_I x^*(s)\psi'(s)\phi'(s)\, ds = \|x\|_{\Lambda_\eta}.$$

Thus $\|xy\|_{\Lambda_\phi} \leq \|x\|_{\Lambda_\eta}$ for any $\|y\|_{M_{\phi_1}} \leq 1$ and so $\|x^*\|_{M(M_{\phi_1},\Lambda_\phi)} \leq \|x\|_{\Lambda_\eta}$. The proof is complete. ☐

Example 3. A special case of symmetric spaces for which we can calculate the fundamental function of their space of multipliers was given in [15, Example 4.2] and we give a short proof of this result. Let $E \hookrightarrow F$ be two ultrasymmetric spaces on $[0, 1]$ with the same parameter $\tilde{G}$, that is,

$$\|x\|_E = \|f_E(t)x^*(t)\|_{\tilde{G}},\ \|x\|_F = \|f_F(t)x^*(t)\|_{\tilde{G}},$$

where $\tilde{G}$ is a symmetric space on $(0, 1)$ with respect to the measure dt/t (see Pustylnik [33] and Astashkin-Maligranda [7]). Then

$$f_{M(E,F)}(t) = \sup_{0<s\leq t} \frac{f_F(s)}{f_E(s)} \quad \text{for all } t \in (0, 1]. \tag{12}$$

In fact, for any $\|x\|_E \leq 1$ we have

$$\begin{aligned} \|x\chi_{[0,t]}\|_F &= \|f_F(x\chi_{[0,t]})^*\|_{\tilde{G}} \leq \|f_F x^*\chi_{[0,t]}\|_{\tilde{G}} = \|\frac{f_F}{f_E} f_E x^*\chi_{[0,t]}\|_{\tilde{G}} \\ &\leq \sup_{0<s\leq t} \frac{f_F(s)}{f_E(s)} \|f_E x^*\chi_{[0,t]}\|_{\tilde{G}} \leq \sup_{0<s\leq t} \frac{f_F(s)}{f_E(s)}, \end{aligned}$$

and the reverse estimate is always true by Theorem 1(iii). Thus we obtain equality (12). Another example of spaces with equality (12) will be given in Example 9.

3. Some properties of Young functions

To state and prove our main results we will need to define some subclasses of Young functions, an inverse of Young function and their properties. We write $\varphi > 0$ when $a_\varphi = 0$ and $\varphi < \infty$ if $b_\varphi = \infty$. Define the sets of Young functions $\mathcal{Y}^{(i)}$, for $i = 1, 2, 3$, as

$$\begin{aligned} \mathcal{Y}^{(1)} &= \{\varphi : b_\varphi = \infty\}, \\ \mathcal{Y}^{(2)} &= \{\varphi : b_\varphi < \infty \text{ and } \varphi(b_\varphi) = \infty\}, \\ \mathcal{Y}^{(3)} &= \{\varphi : b_\varphi < \infty \text{ and } \varphi(b_\varphi) < \infty\}. \end{aligned}$$

For an Young function φ we define its right-continuous inverse in a generalized sense by the formula (cf. O'Neil [32]):

$$\varphi^{-1}(v) = \inf\{u \geq 0 : \varphi(u) > v\} \text{ for } v \in [0, \infty) \text{ and } \varphi^{-1}(\infty) = \lim_{v \to \infty} \varphi^{-1}(v). \tag{13}$$

Note that $\{u \geq 0 : \varphi(u) > v\} \neq \emptyset$ for each $v \in [0, \infty)$.

We will often use properties of an Young function φ and its generalized inverse φ^{-1}. Therefore let us collect these properties here.

Lemma 2. *We have*

(i) $\varphi(\varphi^{-1}(u)) \leq u$ *for all* $u \in [0, \infty)$ *and* $u \leq \varphi^{-1}(\varphi(u))$ *if* $\varphi(u) < \infty$.

(ii) $\varphi^{-1}(\varphi(u)) = u$ *for* $a_\varphi \leq u \leq b_\varphi$ *if* $b_\varphi < \infty$ *and* $\varphi(b_\varphi) < \infty$.

(iii) $\varphi^{-1}(\varphi(u)) = u$ *for* $a_\varphi \leq u < b_\varphi$ *if either* $b_\varphi = \infty$ *or* $b_\varphi < \infty$ *and* $\varphi(b_\varphi) = \infty$.

(iv) $\varphi^{-1}(\varphi(u)) > u$ *for* $0 \leq u < a_\varphi$.

(v) $\varphi^{-1}(\varphi(u)) < u$ *for* $u > b_\varphi$.

(vi) $\varphi(\varphi^{-1}(u)) = u$ *if* $u \in [0, \infty)$ *and* $\varphi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$.

(vii) $\varphi(\varphi^{-1}(u)) = u$ *if* $u \in [0, u_0]$ *and* $\varphi \in \mathcal{Y}^{(3)}$, *where* $u_0 = \inf\{u > 0 : \varphi^{-1}(u) = b_\varphi\}$.

(viii) $\varphi(\varphi^{-1}(u)) < u$ *if* $u > u_0$ *and* $\varphi \in \mathcal{Y}^{(3)}$, *where* $u_0 = \inf\{u > 0 : \varphi^{-1}(u) = b_\varphi\}$.

Note that (i) follows from (vi)–(viii) and (ii)–(iv).

We will use also the following notations: the symbol $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for all arguments [for large arguments] (for small arguments) means that there is a constant $C > 0$ [there are constants $C, u_0 > 0$] (there are constants $C, u_0 > 0$) such that the inequality

$$C\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq \varphi^{-1}(u) \tag{14}$$

holds for all $u > 0$ [for all $u \geq u_0$] (for all $0 < u \leq u_0$), respectively.

The symbol $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for all arguments [for large arguments] (for small arguments) means that there is a constant $D > 0$ [there are constants $D, u_0 > 0$] (there are constants $D, u_0 > 0$) such that the inequality

$$\varphi^{-1}(u) \leq D\varphi_1^{-1}(u)\varphi_2^{-1}(u) \tag{15}$$

holds for all $u > 0$ [for all $u \geq u_0$] (for all $0 < u \leq u_0$), respectively.

The symbol $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments [for large arguments] (for small arguments) means that $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ and $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$, that is provided there are constants $C, D > 0$ [there are constants $C, D, u_0 > 0$] (there are constants $C, D, u_0 > 0$) such that the inequalities

$$C\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq \varphi^{-1}(u) \leq D\varphi_1^{-1}(u)\varphi_2^{-1}(u)$$

hold for all $u > 0$ [for all $u \geq u_0$] (for all $0 < u \leq u_0$), respectively.

Lemma 3. *(i) If $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for large arguments, then*

(a) for any $0 < u_1 < u_0$ there are contants $C_1 \leq C, D_1 \geq D$ such that

$$C_1\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq \varphi^{-1}(u) \leq D_1\varphi_1^{-1}(u)\varphi_2^{-1}(u) \text{ for any } u \geq u_1. \tag{16}$$

(b) $b_\varphi < \infty$ if and only if $b_{\varphi_1} < \infty$ and $b_{\varphi_2} < \infty$.

(ii) If $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for small arguments, then

(a) for any $u_1 > u_0$ there are contants $C_1 \leq C, D_1 \geq D$ such that

$$C_1\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq \varphi^{-1}(u) \leq D_1\varphi_1^{-1}(u)\varphi_2^{-1}(u) \text{ for any } 0 < u \leq u_1. \tag{17}$$

(b) $a_\varphi = 0$ if and only if $a_{\varphi_1} = 0$ or $a_{\varphi_2} = 0$.

Proof. (i). In order to prove (a) it is enough to take

$$C_1 = \min\left\{C, \inf_{u_1 \leq u \leq u_0} \frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)\varphi_2^{-1}(u)}\right\} \text{ and } D_1 = \max\left\{D, \sup_{u_1 \leq u \leq u_0} \frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)\varphi_2^{-1}(u)}\right\}.$$

We prove (ii),(b). Necessity. Suppose $a_\varphi = 0$ and $a_{\varphi_1} > 0$ or $a_{\varphi_2} > 0$. Taking $u_n \to 0$ we get that $\varphi_1^{-1}(u_n)\varphi_2^{-1}(u_n) \to a_{\varphi_1}a_{\varphi_2} > 0$ and $\varphi^{-1}(u_n) \to 0$, a contradiction with inequality (14). Sufficiency. If $a_{\varphi_1} = 0$ and $a_\varphi > 0$, then $\varphi_1^{-1}(u_n)\varphi_2^{-1}(u_n) \to 0$ and $\varphi^{-1}(u_n) \to a_\varphi$, a contradiction with inequality (15). The case $a_{\varphi_2} = 0$ and $a_\varphi > 0$ can be proved in an analogous way.

The proofs of (i),(b) and (ii),(a) are similar. □

4. On the inclusion $E_{\varphi_2} \hookrightarrow M(E_{\varphi_1}, E_\varphi)$

We will consider the question when the product $xy \in E_\varphi$ provided $x \in E_{\varphi_1}$ and $y \in E_{\varphi_2}$.

THEOREM 2. *Suppose E is a Banach ideal space with the Fatou property and φ, φ_1 and φ_2 are Young functions. Assume also that at least one of the following conditions holds:*

(i) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for all arguments.

(ii) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for large arguments and $L^\infty \hookrightarrow E$.

(iii) $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for small arguments and $E \hookrightarrow L^\infty$.

Then, for every $x \in E_{\varphi_1}$ and $y \in E_{\varphi_2}$ the product $xy \in E_\varphi$, which means that $E_{\varphi_2} \hookrightarrow M(E_{\varphi_1}, E_\varphi)$.

Proof. (i). It is well-known that inequality (14) implies

$$\varphi\left(Cuv\right) \leq \varphi_1\left(u\right) + \varphi_2\left(v\right) \tag{18}$$

for each $u, v > 0$ with $\varphi_1(u), \varphi_2(v) < \infty$ (cf. O'Neil [32]). In fact, taking $w = \max[\varphi_1(u), \varphi_2(v)]$, we obtain

$$uv \leq \varphi_1^{-1}(\varphi_1(u))\varphi_2^{-1}(\varphi_2(v)) \leq \varphi_1^{-1}(w)\varphi_2^{-1}(w) \leq \frac{1}{C}\varphi^{-1}(w),$$

so that $\varphi(Cuv) \leq \varphi(\varphi^{-1}(w)) \leq w \leq \varphi_1(u) + \varphi_2(v)$. Then, taking any $x \in E_{\varphi_1}$ and $y \in E_{\varphi_2}$ with $\|x\|_{\varphi_1} = \|y\|_{\varphi_2} = 1$, we obtain

$$I_\varphi\left(\frac{Cxy}{2}\right) \leq \frac{1}{2}\, I_\varphi\left(Cxy\right) \leq \frac{1}{2}\,\left[I_{\varphi_1}(x) + I_{\varphi_2}(y)\right] \leq 1.$$

This means that $\|xy\|_{E_\varphi} \leq \frac{2}{C}\,\|x\|_{E_{\varphi_1}}\,\|y\|_{E_{\varphi_2}}$ for any $x \in E_{\varphi_1}$ and $y \in E_{\varphi_2}$, and so $E_{\varphi_2} \overset{2/C}{\hookrightarrow} M\left(E_{\varphi_1}, E_\varphi\right)$.

(ii) Set $u_1 = \frac{1}{\|\chi_\Omega\|_E}$. Let $C_1 = C_1(u_1)$ be the corresponding number from (16). Then, analogously as in (i), we conclude

$$\varphi\left(C_1 uv\right) \leq \varphi_1(u) + \varphi_2(v)$$

for each $u, v > 0$ with $\varphi_1(u), \varphi_2(v) < \infty$ and $\max\{\varphi_1(u), \varphi_2(v)\} \geq u_1$. Let $x \in E_{\varphi_1}, y \in E_{\varphi_2}$ with $\|x\|_{\varphi_1} = \|y\|_{\varphi_2} = 1$ and

$$A = \{t \in \Omega : \max[\varphi_1(|x(t)|), \varphi_2(|y(t)|)] \geq u_1\}, \text{ and } \ B = \Omega \backslash A.$$

Then

$$I_\varphi\left(\frac{C_1 xy}{3}\chi_A\right) \leq \frac{1}{3}\,\left[I_{\varphi_1}(x\chi_A) + I_{\varphi_2}(y\chi_A)\right] \leq \frac{2}{3}.$$

Since $I_{\varphi_1}(x) \leq 1$ it follows that $\varphi_1(|x(t)|) < \infty$ for μ-a.e. $t \in \Omega$ and, consequently,

$$|x(t)| \leq \varphi_1^{-1}(\varphi_1(|x(t)|)) \leq \varphi_1^{-1}(u_1) \text{ for each } t \in B.$$

Analogously, $|y(t)| \leq \varphi_2^{-1}(u_1)$. Then, by (16), we obtain

$$I_\varphi(C_1 xy\chi_B) \leq I_\varphi(C_1\varphi_1^{-1}(u_1)\varphi_2^{-1}(u_1)\chi_B) \leq \varphi(\varphi^{-1}(u_1))\|\chi_\Omega\|_E \leq u_1\|\chi_\Omega\|_E = 1.$$

Finally,

$$I_\varphi\left(\frac{C_1 xy}{3}\right) \leq I_\varphi\left(\frac{C_1 xy}{3}\,\chi_A\right) + \frac{1}{3}\, I_\varphi\left(C_1 xy\,\chi_B\right) \leq \frac{2}{3} + \frac{1}{3} = 1,$$

and, thus, $\|xy\|_{E_\varphi} \leq \frac{3}{C_1}$. Consequently, $\|xy\|_{E_\varphi} \leq \frac{3}{C_1}\|x\|_{E_{\varphi_1}}\|y\|_{E_{\varphi_2}}$ for any $x \in E_{\varphi_1}$ and $y \in E_{\varphi_2}$. Thus $E_{\varphi_2} \overset{3/C_1}{\hookrightarrow} M\left(E_{\varphi_1}, E_\varphi\right)$.

(iii) Assume that $E \overset{A}{\hookrightarrow} L^\infty$. We then follow analogously as above case (i) showing $E_{\varphi_2} \overset{2/C_1}{\hookrightarrow} M\left(E_{\varphi_1}, E_\varphi\right)$, where $C_1 = C_1(A)$ is from (17) for $u_1 = A$ and A is such that $\operatorname{ess\,sup}_{t\in\Omega} |\varphi(u(t))| \leq A$ for any $u \in E_\varphi$ with $\|u\|_{E_\varphi} \leq 1$. □

The next result shows the necessity of the estimate $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ in Theorem 2.

THEOREM 3. *Let E be a Banach function space with the Fatou property and let $\varphi, \varphi_1, \varphi_2$ be Young functions. Suppose*

$$E_{\varphi_2} \hookrightarrow M\left(E_{\varphi_1}, E_{\varphi}\right). \tag{19}$$

(i) If $E_a \neq \{0\}$, then $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for large arguments.

(ii) If $suppE_a = suppE$ and $L^\infty \not\hookrightarrow E$, then $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for all arguments.

Proof. (i) Suppose the condition $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ is not satisfied for large arguments. This means that there exists a sequence (u_n) with $u_n \nearrow \infty$ such that, for any $n \in \mathbb{N}$,

$$\varphi_1^{-1}(u_n)\varphi_2^{-1}(u_n) \geq 2^n \varphi^{-1}(u_n). \tag{20}$$

We want to construct a sequence $\{x_n\} \subset E_{\varphi_2}$ such that $\|x_n\|_{E_{\varphi_2}} \leq 1$ but $\|x_n\|_{M\left(E_{\varphi_1}, E_\varphi\right)} \to \infty$, which is equivalent to the fact that $E_{\varphi_2} \not\hookrightarrow M\left(E_{\varphi_1}, E_\varphi\right)$.
First of all, note that for almost all $n \in \mathbb{N}$ we can find measurable sets A_n satisfying

$$\|u_n \chi_{A_n}\|_E = 1. \tag{21}$$

In fact, if $E_a \neq \{0\}$, then there is a nonzero $0 \leq x \in E_a$ and therefore there is also a set A of positive measure such that $\chi_A \in E_a$. Of course, for large enough n one has $\|u_n\chi_A\|_E \geq 1$. Applying Dobrakov result from [14] we conclude that the submeasure $\omega(B) = \|\chi_B\|_E$ for $B \in \Sigma, B \subset A$, has the Darboux property. Consequently, for each such n there exists a set A_n satisfying (21). Define

$$x_n = \varphi_2^{-1}(u_n)\chi_{A_n}, \quad y_n = \varphi_1^{-1}(u_n)\chi_{A_n}.$$

Then

$$I_{\varphi_1}(y_n) = \|\varphi_1(\varphi_1^{-1}(u_n)\chi_{A_n})\|_E \leq \|u_n\chi_{A_n}\|_E = 1$$

and thus $\|y_n\|_{E_{\varphi_1}} \leq 1$. Similarly, we can show that $\|x_n\|_{E_{\varphi_2}} \leq 1$. However, for large enough n, one has by (20)

$$\begin{aligned} I_\varphi\left(\frac{x_n y_n}{\lambda}\right) &= \|\varphi\left(\frac{\varphi_1^{-1}(u_n)\,\varphi_2^{-1}(u_n)}{\lambda}\right)\chi_{A_n}\|_E \\ &\geq \|\varphi\left(\frac{2^n\varphi^{-1}(u_n)}{\lambda}\right)\chi_{A_n}\|_E. \end{aligned}$$

If $\varphi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ and $\lambda \leq 2^n$, then

$$\begin{aligned} \|\varphi\left(\frac{2^n\varphi^{-1}(u_n)}{\lambda}\right)\chi_{A_n}\|_E &\geq \|\frac{2^n}{\lambda}\varphi\left(\varphi^{-1}(u_n)\right)\chi_{A_n}\|_E \\ &= \|\frac{2^n}{\lambda}\,u_n\,\chi_{A_n}\|_E = \frac{2^n}{\lambda} \geq 1. \end{aligned}$$

If $\varphi \in \mathcal{Y}^{(3)}$, then for sufficiently large n and $\lambda < 2^n$ we obtain that

$$I_\varphi\left(\frac{x_n y_n}{\lambda}\right) = \|\varphi\left(\frac{2^n\,b_\varphi}{\lambda}\right)\chi_{A_n}\|_E = \infty,$$

which implies $\|x_n y_n\|_\varphi \geq 2^n$. Finally,

$$\|x_n\|_{M(E_{\varphi_1}, E_\varphi)} = \sup_{\|y\|_{E_{\varphi_1}} \leq 1} \|x_n y\|_{E_\varphi} \geq \|x_n y_n\|_{E_\varphi} \geq 2^n,$$

whereas $\|x_n\|_{E_{\varphi_2}} \leq 1$ and this is the required sequence.

(ii) Of course, the assumption $suppE_a = suppE$ implies that $E_a \neq \{0\}$. Therefore we need only to prove that $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for small arguments. Note that in this case the proof is almost the same as in (i). One only has to prove that there are sets like in (21). Since $suppE_a = suppE$ we see that there is $x \in E_a$ with $x > 0$ a.e. Define $B_k = \{t \in \Omega : x(t) > \frac{1}{k}\}$. Of course, B_k have positive measure for sufficently large k, $\Omega = \bigcup_{k=1}^{\infty} B_k$ and $B_1 \subset B_2 \subset B_3 \subset \ldots$. We have $\|\chi_{B_k}\|_E \to \infty$ because $L^\infty \not\hookrightarrow E$ and E has the Fatou property. Moreover, $\chi_{B_k} \in E_a$ for any $n \in \mathbb{N}$. Therefore, for each u_n one can find $k\,(n)$ such that $\|u_n \chi_{B_{k(n)}}\|_E > 1$ and the argument is the same as before. □

The following example explains why the conditions concerning E_a in Theorem 3 are reasonable but not necessary.

Example 4. Note that Theorem 3 is not true without any additional assumption on the space E. In fact, for $E = L^\infty$ and for any non-trivial Young function φ we have that $E_\varphi = L^\infty$, which gives

$$M(E_{\varphi_1}, E_\varphi) = M(L^\infty, L^\infty) = L^\infty = E_{\varphi_2},$$

and no relation between the functions $\varphi_1, \varphi_2, \varphi$ is necessary. On the other hand, for the weighted space $E = L_t^\infty[0,1]$ with its norm $\|x\|_E = \operatorname{ess\,sup}_{t\in[0,1]} |t\,x(t)|$ we have $E_a = \{0\}$ but the condition $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for large arguments is necessary for inclusion $E_{\varphi_2} \hookrightarrow M\,(E_{\varphi_1}, E_\varphi)$. To see this it is enough to proceed like in Theorem 3(i) because the function $\eta : [0,1] \to [0,1]$ given by $\eta(t) = \|\chi_{[0,t]}\|_E = t$ is continuous and therefore has the Darboux property.

Remark 3. The condition $E_a \neq \{0\}$ from Theorem 3(i) may be changed by the following weaker one: there is $a > 0$ such that for any $0 < t < a$ we can find $A \in \Sigma$ with $\|\chi_A\|_E = t$.

Now we investigate necessity condition on the Young functions in the case of Banach sequence space.

THEOREM 4. *Let e be a Banach sequence space with the Fatou property and let φ, φ_1, φ_2 be Young functions. Suppose*

$$e_{\varphi_2} \hookrightarrow M\,(e_{\varphi_1}, e_\varphi)\,. \tag{22}$$

- (i) *If $l^\infty \not\hookrightarrow e$ and $\sup_{i\in\mathbb{N}} \|e_i\|_e < \infty$, then $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for small arguments.*
- (ii) *If $e \not\hookrightarrow l^\infty$ and for each $a > 1$ there is a set B_a with $\frac{1}{2} \leq a\|\chi_{B_a}\| \leq 1$, then $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for large arguments.*

Proof. (i) Suppose that the condition $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ is not satisfied for small arguments. Then there exists a sequence (u_n) with $u_n \to 0$ such that for any $n \in \mathbb{N}$ we have

$$\varphi_1^{-1}(u_n)\varphi_2^{-1}(u_n) \geq 2^n \varphi^{-1}(u_n). \tag{23}$$

Since $l^\infty \not\hookrightarrow e$ and e has the Fatou property, it follows that $\|\chi_{\{1,2,...,n\}}\|_e \to \infty$ as $n \to \infty$. From the assumption $\sup_{i\in\mathbb{N}} \|e_i\|_e < \infty$ one can find N large enough such that $\|u_n\, e_i\|_e \leq 1/2$ for each $n > N$ and each $i \in \mathbb{N}$. Furthermore, for each $n > N$ we can find k_n satisfying

$$\left\|u_n\chi_{\{1,2,...,k_n\}}\right\|_e \leq 1 \text{ and } \left\|u_n\chi_{\{1,2,...,k_n,k_n+1\}}\right\|_e > 1.$$

Because $\|u_n e_{k_n+1}\|_e \leq 1/2$ one has also $1/2 \leq \|u_n\chi_{\{1,2,...,k_n\}}\|_e$. For n large enough we put $A_n = \{1, 2, ..., k_n\}$, and

$$x_n = \varphi_2^{-1}(u_n)\chi_{A_n},\ y_n = \varphi_1^{-1}(u_n)\chi_{A_n}.$$

Then $\|y_n\|_{e_{\varphi_1}} \leq 1$ and $\|x_n\|_{e_{\varphi_2}} \leq 1$. Moreover, by (23), one has

$$\begin{aligned} I_\varphi\left(\frac{x_n y_n}{\lambda}\right) &= \|\varphi\left(\frac{\varphi_1^{-1}(u_n)\,\varphi_2^{-1}(u_n)}{\lambda}\right)\chi_{A_n}\|_e \\ &\geq \|\varphi\left(\frac{2^n\varphi^{-1}(u_n)}{\lambda}\right)\chi_{A_n}\|_e\,. \end{aligned}$$

If $\lambda \leq 2^{n-1}$, then, by applying Lemma 2(vi) and (vii), we obtain

$$\|\varphi\left(\frac{2^n\varphi^{-1}(u_n)}{\lambda}\right)\chi_{A_n}\|_e \geq \|\frac{2^n u_n}{\lambda}\chi_{A_n}\|_e \geq \frac{2^{n-1}}{\lambda} \geq 1,$$

for sufficiently large n, which implies that $\|x_n y_n\|_{e_\varphi} \geq 2^{n-1}$ and, consequently,

$$\|x_n\|_{M\left(e_{\varphi_1},e_\varphi\right)} = \sup_{\|y\|_{e_{\varphi_1}}\leq 1} \|x_n y\|_{e_\varphi} \geq \|x_n y_n\|_{e_\varphi} \geq 2^{n-1},$$

whereas $\|x_n\|_{e_{\varphi_2}} \leq 1$. Therefore (22) is not satisfied.

(ii) We proceed as in (i). Note that the condition is satisfied in many non-symmetric spaces, for example, in $e = l^1(\{\frac{1}{i}\})$ for which $l^\infty \not\hookrightarrow e$ or $e = l^1(\{\frac{1}{i^2}\})$ in which $l^\infty \hookrightarrow e$. $\square$

Putting Theorems 2 and 3 together we obtain:

Corollary 1. *Let E be a Banach function space with the Fatou property and let φ, φ_1, φ_2 be Young functions.*

- (i) *Suppose $L^\infty \hookrightarrow E$ and $E_a \neq \{0\}$. Then $E_{\varphi_2} \hookrightarrow M\left(E_{\varphi_1}, E_\varphi\right)$ if and only if $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for large arguments.*
- (ii) *Assume $L^\infty \not\hookrightarrow E$ and $suppE_a = suppE$. Then $E_{\varphi_2} \hookrightarrow M\left(E_{\varphi_1}, E_\varphi\right)$ if and only if $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for all arguments.*

Before giving a similar characterization for the sequence case note that the equality $e = l^\infty$ implies then $e_{\varphi_2} = e_{\varphi_1} = e_\varphi = M(e_{\varphi_1}, e_\varphi) = l^\infty$ for any Orlicz functions. Consequently, looking for a neccesary and sufficient condition for the inclusion $e_{\varphi_2} \hookrightarrow M(e_{\varphi_1}, e_\varphi)$ we need to consider the following three cases: (i) $e \hookrightarrow l^\infty$ and $l^\infty \not\hookrightarrow e$, (ii) $l^\infty \hookrightarrow e$ and $e \not\hookrightarrow l^\infty$, (iii) $l^\infty \not\hookrightarrow e$ and $e \not\hookrightarrow l^\infty$.
Taking into account Theorems 2 and 4, we then get immediately

Corollary 2. *Let e be a Banach sequence space with the Fatou property and let $\varphi, \varphi_1, \varphi_2$ be Young functions.*

(i) Suppose $e \hookrightarrow l^\infty, l^\infty \not\hookrightarrow e$ and $\sup_{i\in\mathbb{N}} \|e_i\|_e < \infty$. Then $e_{\varphi_2} \hookrightarrow M(e_{\varphi_1}, e_\varphi)$ if and only if $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for small arguments.

Suppose additionally that for each $a > 1$ there is a set B_a with $\frac{1}{2} \le a\|\chi_{B_a}\| \le 1$.

(ii) Assume that $l^\infty \hookrightarrow e$ and $e \not\hookrightarrow l^\infty$. Then $e_{\varphi_2} \hookrightarrow M(e_{\varphi_1}, e_\varphi)$ if and only if $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for large arguments.

(iii) Let $l^\infty \not\hookrightarrow e$, $e \not\hookrightarrow l^\infty$ and $\sup_{i\in\mathbb{N}} \|e_i\|_e < \infty$. Then $e_{\varphi_2} \hookrightarrow M(e_{\varphi_1}, e_\varphi)$ if and only if $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ for all arguments.

Note that conditions (14) and (18) are equivalent. The implication (14) $\Rightarrow$ (18) was shown in the proof of Theorem 2. We prove that $\varphi(Cuv) \le \varphi_1(u) + \varphi_2(v)$ for all $u, v > 0$ implies $\varphi_1^{-1}(w)\varphi_2^{-1}(w) \le \frac{2}{C}\varphi^{-1}(w)$ for all $w > 0$. In fact, for $w > 0$ let $u = \varphi_1^{-1}(w)$ and $v = \varphi_2^{-1}(w)$. Then, by assumption and Lemma 2, we have

$$\varphi(Cuv) \le \varphi_1(u) + \varphi_2(v) = \varphi_1(\varphi_1^{-1}(w)) + \varphi_2(\varphi_2^{-1}(w)) \le 2w,$$

and again, by Lemma 2 and the concavity of φ^{-1}, we obtain

$$Cuv \le \varphi^{-1}(\varphi(Cuv)) \le \varphi^{-1}(2w) \le 2\varphi^{-1}(w),$$

which gives $\varphi_1^{-1}(w)\varphi_2^{-1}(w) \le \frac{2}{C}\varphi^{-1}(w)$. Now we consider the respective case for large and small arguments. Discussing the case for large arguments we prove that the following conditions are equivalent:

(a) for any $u_0 > 0$ there is $C_0 > 0$ such that $C_0\varphi_1^{-1}(u)\varphi_2^{-1}(u) \le \varphi^{-1}(u)$ for all $u \ge u_0$.

(b) for any $u_1 > 0$ there is $C_1 > 0$ such that $\varphi(C_1uv) \le \varphi_1(u) + \varphi_2(v)$ for all u, v with $u_1 \le \max\{\varphi_1(u), \varphi_2(v)\} < \infty$.

The implication $(a) \Rightarrow (b)$ has been shown in the proof of Theorem 2(ii). We prove $(b) \Rightarrow (a)$. For any Orlicz function $\varphi \in \mathcal{Y}^{(3)}$ denote

$$\alpha_\varphi = \inf\left\{u > 0 : \varphi^{-1}(u) = b_\varphi\right\}$$

and for $\varphi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ we set $\alpha_\varphi = \infty$. Let

$$u_1 = \min\{u_0, \alpha_{\varphi_1}, \alpha_{\varphi_2}\}.$$

Take $w \ge u_0$ and $u = \varphi_1^{-1}(w), v = \varphi_2^{-1}(w)$. Then $\varphi_1(u) = \varphi_1(\varphi_1^{-1}(w)) \in [u_1, \infty)$. Similarly $\varphi_2(v) \in [u_1, \infty)$ and we finish as above. In the case of nondegenerate Orlicz functions we simply get the following equivalence:

(a) there are $u_0 > 0$ and $C_0 > 0$ such that $C_0\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq \varphi^{-1}(u)$ for all $u \geq u_0$.
(b) there are $u_1 > 0$ and $C_1 > 0$ such that $\varphi(C_1uv) \leq \varphi_1(u) + \varphi_2(v)$ for all u, v with $u, v \geq u_1$.

Finally, for all Orlicz functions, it is easy to show the following equivalence:
(a) for any $u_0 > 0$ there is $C_0 > 0$ such that $C_0\varphi_1^{-1}(u)\varphi_2^{-1}(u) \leq \varphi^{-1}(u)$ for all $u \leq u_0$.
(b) for any $u_1 > 0$ there is $C_1 > 0$ such that $\varphi(C_1uv) \leq \varphi_1(u) + \varphi_2(v)$ for all u, v with $\max\{\varphi_1(u), \varphi_2(v)\} \leq u_1$.

In the case $E = L^1$ the space E_φ is an Orlicz space L^φ and our Theorems 2–4 together with the equivalence of conditions (14) and (18) give the following results of Ando [3] and O'Neil [32] (see also [26], Theorems 10.2-10.4).

Ando theorem ([3], Theorem 1): *Let μ be a non-atomic measure and $0 < \mu(\Omega) < \infty$. For any $x \in L^{\varphi_1}$ and any $y \in L^{\varphi_2}$ the product $xy \in L^\varphi$ if and only if there exist $C, u_0 > 0$ such that $\varphi(Cuv) \leq \varphi_1(u) + \varphi_2(v)$ for all $u, v \geq u_0$.*

O'Neil [32, Theorem 6.5] observed that the last condition on Young functions is equivalent to the condition $\limsup_{u\to\infty} \frac{\varphi_1^{-1}(u)\,\varphi_2^{-1}(u)}{\varphi^{-1}(u)} < \infty$. Krasnoselskiĭ and Rutickiĭ [20] noted that relation on embedding of sets is equivalent to estimations of the norms, that is, there is a number $A > 0$ such that $\|xy\|_\varphi \leq A\|x\|_{\varphi_1}\|y\|_{\varphi_2}$ for all $x \in L^{\varphi_1}$ and $y \in L^{\varphi_2}$.

O'Neil theorems ([32], Theorems 6.6 and 6.7): *(i) Let μ be non-atomic measure with $\mu(\Omega) = \infty$. Then the following conditions are equivalent: for any $x \in L^{\varphi_1}$ and any $y \in L^{\varphi_2}$ the product $xy \in L^\varphi \iff$ there exist $C > 0$ such that $\varphi(Cuv) \leq \varphi_1(u) + \varphi_2(v)$ for all $u, v > 0 \iff \sup_{u>0} \frac{\varphi_1^{-1}(u)\,\varphi_2^{-1}(u)}{\varphi^{-1}(u)} < \infty \iff$ there is a number $B > 0$ such that $\|xy\|_\varphi \leq B\|x\|_{\varphi_1}\|y\|_{\varphi_2}$ for all $x \in L^{\varphi_1}$ and $y \in L^{\varphi_2}$.*

(ii) Let $I = \mathbb{N}$ with the counting measure. The following conditions are equivalent: for any $x \in l^{\varphi_1}$ and any $y \in l^{\varphi_2}$ the product $x\,y \in l^\varphi \iff$ there exist $C, u_0 > 0$ such that $\varphi(Cuv) \leq \varphi_1(u) + \varphi_2(v)$ for all $0 < u, v \leq u_0 \iff \limsup_{u\to 0^+} \frac{\varphi_1^{-1}(u)\,\varphi_2^{-1}(u)}{\varphi^{-1}(u)} < \infty \iff$ there is a number $D > 0$ such that $\|x\,y\|_\varphi \leq D\|x\|_{\varphi_1}\|y\|_{\varphi_2}$ for all $x \in L^{\varphi_1}$ and $y \in L^{\varphi_2}$.

5. On the inclusion $M(E_{\varphi_1}, E_\varphi) \hookrightarrow E_{\varphi_2}$

We start by stating a crucial lemma, which in the case $E = L^1$ was proved in [27].

Lemma 4. *If $\varphi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$ and $x = \sum_{k=1}^N c_k\chi_{A_k}, x \neq 0$ is a simple function, then $I_\varphi\left(\frac{x}{\|x\|_{E_\varphi}}\right) = 1$.*

Proof. We follow arguments as it was done in the proof of Lemma 3 in [27]. It is enough to show that the function

$$h(\lambda) = I_\varphi\left(\frac{x}{\lambda}\right) = \|\sum_{k=1}^N \varphi\left(\frac{c_k}{\lambda}\right)\chi_{A_k}\|_E$$

is continuous, non-increasing and $h : (0, c_N/a_\varphi) \to (0, \infty)$. Suppose that $\varphi \in \mathcal{Y}^{(1)}$. If $\lambda_m \to \lambda_0$, then

$$
\begin{aligned}
|h(\lambda_m) - h(\lambda_0)| &\leq \| \sum_{k=1}^{N} \left| \varphi\left(\frac{c_k}{\lambda_m}\right) - \varphi\left(\frac{c_k}{\lambda_0}\right) \right| \chi_{A_k} \|_E \\
&\leq \sum_{k=1}^{N} \left| \varphi\left(\frac{c_k}{\lambda_m}\right) - \varphi\left(\frac{c_k}{\lambda_0}\right) \right| \|\chi_{A_k}\|_E \to 0
\end{aligned}
$$

as $m \to \infty$. Clearly, h is non-increasing and

$$
\lim_{\lambda \to 0^+} h(\lambda) \geq \lim_{\lambda \to 0^+} \varphi(\frac{c_1}{\lambda}) \|\chi_{A_1}\|_E = \infty, \quad \lim_{\lambda \to c_N/a_\varphi} h(\lambda) = 0,
$$

since for $\lambda > \frac{c_{N-1}}{a_\varphi}$ we have that $h(\lambda) = \varphi\left(\frac{c_N}{\lambda}\right) \|\chi_{A_N}\|_E$. Consequently, there is a number $\lambda_0 \in (0, c_N/a_\varphi)$ with $I_\varphi\left(\frac{x}{\lambda_0}\right) = 1$.

If $\varphi \in \mathcal{Y}^{(2)}$ the proof is the same as in [27] and Lemma 4 is proved. $\square$

The following result is a generalization of Theorem 1 from [27].

THEOREM 5. *Suppose E is a Banach ideal space with the Fatou property and $suppE = \Omega$. Let $\varphi, \varphi_1, \varphi_2$ be Young functions. Assume that the condition $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ holds:*

(i) for all arguments.

(ii) for large arguments and $L^\infty \hookrightarrow E$.

(iii) for small arguments and $E \hookrightarrow L^\infty$.

Then $M(E_{\varphi_1}, E_\varphi) \hookrightarrow E_{\varphi_2}$.

Proof. We apply the technique from the proof of Theorem 1 in [27]. The case (i) follows in the same way as in [27] with one restriction. Namely, if $x \in M(E_{\varphi_1}, E_\varphi)$ is a simple function, then x need not belong to E_{φ_2}. Consider the case $x = \sum_{i=1}^n a_i \chi_{A_i}$ and $\chi_{A_i} \notin E$ for some i. Then there is an increasing sequence $\left(A_i^k\right)_{k=1}^\infty$ of subsets of A_i satisfying $\bigcup_{k=1}^\infty A_i^k = A_i$ and $\chi_{A_i^k} \in E$ for each k. Taking $x_k = \sum_{i=1}^n a_i \chi_{A_i^k}$ we get $x_k \in E_{\varphi_2}$ for each k. Then we follow just the proof of Theorem 1 in [27] and get $\|x_k\|_{M(E_{\varphi_1},E_\varphi)} \geq \frac{1}{D}\|x_k\|_{\varphi_2}$ for each k. But $0 \leq x_k \leq x$ and $x_k \uparrow x$. Since E_{φ_2} has the Fatou property it follows that $\|x\|_{M\left(E_{\varphi_1},E_\varphi\right)} \geq \frac{1}{D}\|x\|_{\varphi_2}$ for each simple function $x \in M(E_{\varphi_1}, E_\varphi)$.

(ii) Assume that $L^\infty \hookrightarrow E$. Take $\alpha > a_{\varphi_2}$ with $\varphi_2(\alpha)\|\chi_\Omega\|_E < \frac{1}{2}$. Applying Lemma 3 we find a contant $D_1 \geq D$ such that

$$
\varphi^{-1}(u) \leq D_1 \varphi_1^{-1}(u) \varphi_2^{-1}(u) \text{ for any } u \geq \varphi_2(\alpha).
$$

Observe that

$$
I_{\varphi_2}(z\chi_B) \geq 1/2 \tag{24}
$$

for any $z = z(t)$ with $I_{\varphi_2}(z) = 1$, where $B = \{t \in \operatorname{supp} z : |z(t)| \geq \alpha\}$. Really, otherwise

$$
1 = I_{\varphi_2}(z) \leq I_{\varphi_2}(z\chi_B) + I_{\varphi_2}(z\chi_{\Omega \setminus B}) < \frac{1}{2} + \varphi_2(\alpha)\|\chi_\Omega\|_E < 1,
$$

and we get a contradiction. Although some steps are similar as in the proof of Theorem 1 from [27] we present the whole proof for the sake of completeness. Assume that φ and φ_2 are in $\mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$. Let $x \in M(E_{\varphi_1}, E_\varphi)$. First suppose that x is a simple function. Then $x \in E_{\varphi_2}$, because $\varphi_2 \circ (\lambda |x|)$ is a simple function in E for some λ and $\chi_A \in L^\infty \hookrightarrow E$ for each $A \in \Sigma$. Consequently,

$$y(t) = \varphi_2 \left(\frac{|x(t)|}{\|x\|_{E_{\varphi_2}}} \right) < \infty \quad \text{for} \quad \mu - \text{a.e. } t \in \Omega.$$

Set

$$z(t) = \begin{cases} \varphi_1^{-1}(y(t)) & \text{if} \quad 0 < y(t) < \infty, \\ 0 & \text{if} \quad y(t) = 0. \end{cases}$$

Then $I_{\varphi_1}(z) \le I_{\varphi_2}(\frac{x}{\|x\|_{E_{\varphi_2}}}) \le 1$ implies $\|z\|_{E_{\varphi_1}} \le 1$ and, by the assumption, we have $zx \in E_\varphi$. Denote

$$A = \{t \in \operatorname{supp} y : \frac{|x(t)|}{\|x\|_{E_{\varphi_2}}} < \alpha\} \text{ and } B = \{t \in \operatorname{supp} y : \frac{|x(t)|}{\|x\|_{E_{\varphi_2}}} \ge \alpha\}.$$

Then, for μ-a.e. $t \in B$,

$$z(t) \frac{|x(t)|}{\|x\|_{E_{\varphi_2}}} = \varphi_1^{-1}(y(t)) \varphi_2^{-1}(y(t)) \ge \frac{1}{D_1} \varphi^{-1}(y(t)),$$

so that

$$\varphi \left(D_1 z(t) \frac{2|x(t)|}{\|x\|_{E_{\varphi_2}}} \right) \ge \varphi(2\varphi^{-1}(y(t))) \ge 2\varphi(\varphi^{-1}(y(t))) = 2y(t),$$

where the last equality follows from the fact that $\varphi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$. By Lemma 4, we have $I_{\varphi_2}\left(\frac{x}{\|x\|_{E_{\varphi_2}}}\right) = 1$ and, consequently, by (24),

$$I_\varphi \left(D_1 z \frac{2x}{\|x\|_{E_{\varphi_2}}} \chi_B \right) \ge 2 I_{\varphi_2} \left(\frac{x}{\|x\|_{E_{\varphi_2}}} \chi_B \right) \ge 1.$$

Thus $\|zx\|_{E_\varphi} \ge \frac{1}{2D_1} \|x\|_{E_{\varphi_2}}$ and so $\|x\|_{M\left(E_{\varphi_1}, E_\varphi\right)} \ge \frac{1}{2D_1} \|x\|_{E_{\varphi_2}}$. For arbitrary function $x \in M\left(E_{\varphi_1}, E_\varphi\right)$ we can follow the proof of Theorem 1 from [27], because the space E_{φ_2} has the Fatou property provided that E has it.

If φ or φ_2 is in $\mathcal{Y}^{(3)}$ we follow again the same way as in case 2 of the proof of Theorem 1 in [27] to show that $\|x\|_{M\left(E_{\varphi_1}, E_\varphi\right)} \ge \frac{1}{2D_1} \|x\|_{E_{\varphi_2}}$.

(iii) Let $E \overset{A}{\hookrightarrow} L^\infty$. First observe that if $\|u\|_{E_\varphi} \le 1$, then $\operatorname{ess\,sup}_{t \in \Omega} |\varphi(u(t))| \le A$. Furthermore, by assumption *(iii)* and Lemma 3, there exists a contant D_2 such that

$$\varphi^{-1}(u) \le D_2 \varphi_1^{-1}(u) \varphi_2^{-1}(u) \text{ for any } 0 < u \le A.$$

The rest of the proof goes as in the case *(i)* (see also the proof of Theorem 1 in [27]). The proof is complete. ☐

To find cases when the condition $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ is necessary for the imbedding $M\left(E_{\varphi_1}, E_\varphi\right) \hookrightarrow E_{\varphi_2}$ we will take as E a symmetric function space on I. Then E_φ is also a symmetric function space and an easy calculation gives that the fundamental function f_{E_φ} of E_φ is equal to $f_{E_\varphi}(t) = \frac{1}{\varphi^{-1}(1/f_E(t))}$ for $t \in (0, m(I))$.

THEOREM 6. *Let E be a symmetric function space on I and let $\varphi, \varphi_1, \varphi_2$ be Young functions. Suppose*

$$M\left(E_{\varphi_1}, E_\varphi\right) \hookrightarrow E_{\varphi_2}. \tag{25}$$

(i) If there are numbers $a, b > 0$ such that $\frac{f_{E_\varphi}(t)}{f_{E_{\varphi_1}}(t)\, t^a} = \frac{\varphi_1^{-1}(1/f_E(t))}{\varphi^{-1}(1/f_E(t))\, t^a}$ is a non-decreasing function of t on an interval $(0, b)$ and $E_a \neq \{0\}$, then $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for large arguments.

(ii) Let $b_\varphi = \infty$. If there is a number $a > 0$ such that $\frac{f_{E_\varphi}(t)}{f_{E_{\varphi_1}}(t)\, t^a} = \frac{\varphi_1^{-1}(1/f_E(t))}{\varphi^{-1}(1/f_E(t))\, t^a}$ is a non-decreasing function of t on $(0, \infty), L^\infty \not\hookrightarrow E$ and $suppE_a = suppE$, then $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for all arguments.

Proof. (i) Assume $f_{E_\varphi}(0^+) = 0$ and suppose that the condition $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ is not satisfied for large arguments, i.e., there is a sequence (u_n) tending to infinity such that for any $n \in \mathbb{N}$

$$2^n \varphi_1^{-1}(u_n)\varphi_2^{-1}(u_n) \le \varphi^{-1}(u_n).$$

It is enough to find a sequence (x_n) both in $M(E_{\varphi_1}, E_\varphi)$ and E_{φ_2} such that

$$\frac{\|x_n\|_{E_{\varphi_2}}}{\|x_n\|_{M(E_{\varphi_1}, E_\varphi)}} \longrightarrow \infty.$$

Analogously as in Theorem 3(i) for each u_n one can find measurable set A_n satisfying $\|u_n \chi_{A_n}\|_E = 1$. Define

$$x_n = \varphi_2^{-1}(u_n)\chi_{A_n}.$$

Then $\|x_n\|_{E_{\varphi_2}} = 1$. In fact, if $\varphi_2 \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)}$, then

$$I_{\varphi_2}(x_n) = \left\|\varphi_2\left(\varphi_2^{-1}(u_n)\right)\chi_{A_n}\right\|_E = u_n \left\|\chi_{A_n}\right\|_E = 1.$$

If $\varphi_2 \in \mathcal{Y}^{(3)}$, then there is N_0 with $\varphi_2^{-1}(u_n) = b_{\varphi_2}$ for $n \ge N_0$, whence $I_{\varphi_2}(x_n) \le 1$ and $I_{\varphi_2}(x_n/\lambda) = \infty$ for $0 < \lambda < 1$. Thus, $\|x_n\|_{E_{\varphi_2}} = 1$ for sufficiently large n.

Putting $t_n = m(A_n)$ we obtain by symmetry of E that $f_E(t_n) = \|\chi_{[0,t_n]}\|_E = \|\chi_{A_n}\|_E = \frac{1}{u_n} \to 0$ as $n \to \infty$ and so $t_n \to 0$ as $n \to \infty$. Therefore, according to Theorem 1(iii), for $t_n \in (0, b)$, we obtain

$$\begin{aligned}
\|x_n\|_{M(E_{\varphi_1}, E_\varphi)} &= \varphi_2^{-1}(u_n)\, \|\chi_{A_n}\|_{M(E_{\varphi_1}, E_\varphi)} = \varphi_2^{-1}(u_n) f_{M(E_{\varphi_1}, E_\varphi)}(t_n) \\
&\le 2\,\frac{1}{a}\,\varphi_2^{-1}(u_n)\frac{f_{E_\varphi}(t_n)}{f_{E_{\varphi_1}}(t_n)} \le 2\,\frac{\varphi^{-1}(u_n)}{a\,2^n\,\varphi_1^{-1}(u_n)}\frac{f_{E_\varphi}(t_n)}{f_{E_{\varphi_1}}(t_n)} \\
&= \frac{2}{a\,2^n}\,\frac{\varphi^{-1}(u_n)}{\varphi_1^{-1}(u_n)}\,\frac{\varphi_1^{-1}(1/f_E(t_n))}{\varphi^{-1}(1/f_E(t_n))} = \frac{2}{a\,2^n} \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which finishes the proof.

In the case when $f_{E_\varphi}(0^+) > 0$ we have $f_E(0^+) > 0$ which implies $b_\varphi < \infty$ and estimate on Young functions is automatically satisfied.

(ii) This part is analogous to the above and Theorem 3(ii). $\square$

Combining Theorems 5 and 6, we obtain the following result:

Corollary 3. *Let E be a symmetric function space on I with the Fatou property. Let $\varphi, \varphi_1, \varphi_2$ be Young functions.*

(i) *Suppose $L^\infty \hookrightarrow E, E_a \neq \{0\}$ and that there are numbers $a, b > 0$ such that $\frac{\varphi_1^{-1}(1/f_E(t))}{\varphi^{-1}(1/f_E(t))\, t^a}$ is a non-decreasing function of t on the interval $(0, b)$. Then $M(E_{\varphi_1}, E_\varphi) \hookrightarrow E_{\varphi_2}$ if and only if $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for large arguments.*

(ii) *Assume $b_\varphi = \infty$, $L^\infty \not\hookrightarrow E$, $suppE_a = I$ and that there is a number $a > 0$ such that $\frac{\varphi_1^{-1}(1/f_E(t))}{\varphi^{-1}(1/f_E(t))\, t^a}$ is a non-decreasing function on $(0, \infty)$. Then $M(E_{\varphi_1}, E_\varphi) \hookrightarrow E_{\varphi_2}$ if and only if $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for all arguments.*

Now, if we take in Corollary 3 as $E = L^1$ we obtain the results, which give an answer for the problem posed in the book [26] (Problem 4, p. 77) in the case of Orlicz spaces (under an additional assumption):

(i) Let $I = [0, 1]$ and let $\frac{\varphi_1^{-1}(u)\, u^a}{\varphi^{-1}(u)}$ be a non-increasing function for some $a > 0$ and suffieciently large u. Then $M\left(L^{\varphi_1}, L^\varphi\right) \hookrightarrow L^{\varphi_2}$ if and only if $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for large arguments.

(ii) Let $I = [0, \infty)$ and let $\frac{\varphi_1^{-1}(u)\, u^a}{\varphi^{-1}(u)}$ be a non-increasing function for some $a > 0$ and all $u > 0$. Then $M\left(L^{\varphi_1}, L^\varphi\right) \hookrightarrow L^{\varphi_2}$ if and only if $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for all arguments.

The monotonicity assumption in (i) is essential for the equivalence (see Example 9 (g) below).

6. On the equality $M\left(E_{\varphi_1}, E_\varphi\right) = E_{\varphi_2}$

Putting together Theorems 2 and 5 we obtain sufficient conditions for coincidence of the space of pointwise multipliers $M\left(E_{\varphi_1}, E_\varphi\right)$ with E_{φ_2}.

Corollary 4. *Let φ, φ_1 and φ_2 be Young functions. Suppose E is a Banach ideal space with the Fatou property and $suppE = \Omega$. Assume also that at least one of the following conditions holds:*

(i) *$\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments.*

(ii) *$\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for large arguments and $L^\infty \hookrightarrow E$.*

(iii) *$\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for small arguments and $E \hookrightarrow L^\infty$.*

Then $M\left(E_{\varphi_1}, E_\varphi\right) = E_{\varphi_2}$ with equivalent norms.

Taking into account Corollary 1 and 3, we obtain

Corollary 5. *Let E be a symmetric function space with the Fatou property. Let $\varphi, \varphi_1, \varphi_2$ be Young functions.*

(i) *Suppose $L^\infty \hookrightarrow E, E_a \neq \{0\}$ and that there are numbers $a, b > 0$ such that $\frac{\varphi_1^{-1}(1/f_E(t)}{\varphi^{-1}(1/f_E(t))\, t^a}$ is a non-decreasing function of t on the interval $(0, b)$. Then $M(E_{\varphi_1}, E_\varphi)$ $= E_{\varphi_2}$ if and only if $\varphi^{-1} \approx \varphi_1^{-1}\varphi_2^{-1}$ for large arguments.*

(ii) Assume $b_\varphi = \infty$, $L^\infty \not\hookrightarrow E$, *supp*$E_a =$ *supp*E *and that there is a number* $a > 0$ *such that* $\frac{\varphi_1^{-1}(1/f_E(t))}{\varphi^{-1}(1/f_E(t))\,t^a}$ *is a non-decreasing function on* $(0,\infty)$. *Then* $M(E_{\varphi_1}, E_\varphi) = E_{\varphi_2}$ *if and only if* $\varphi^{-1} \approx \varphi_1^{-1}\varphi_2^{-1}$ *for all arguments.*

7. On the construction of a Young function generating the space $M(E_{\varphi_1}, E_\varphi)$

The following questions arises: having two Young functions φ_1, φ how can one find a Young function φ_2 satisfying $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$? Does such a function always exist?

It appears that such a function may not exist. The following example describes such possibility.

Example 5. Let $\varphi(u) = u^2, \varphi_1(u) = u$ and $E = L_t^\infty[0,1]$ with the norm $\|x\|_E = \operatorname{ess\,sup}_{t\in[0,1]}|t\,x(t)|$. The equivalence $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ means that $u\,\varphi_2^{-1} \approx \sqrt{u}$, i.e., $\varphi_2^{-1} \approx 1/\sqrt{u}$, which is not possible for any Young function φ_2. Moreover, $M(E_{\varphi_1}, E_\varphi)$ is not a Calderón-Lozanovskiĭ space of the form E_{φ_3} for any Young function φ_3. In fact, $E_\varphi = L^\infty_{\sqrt{t}}[0,1]$ and

$$M(E_{\varphi_1}, E_\varphi) = M(L_t^\infty[0,1], L^\infty_{\sqrt{t}}[0,1]) = L^\infty_{1/\sqrt{t}}[0,1].$$

This space cannot be of the form E_{φ_3} since $\chi_{[0,1]} \in E_{\varphi_3}$ and $\|\chi_{[0,1]}\|_{E_{\varphi_3}} = 1/\varphi_3^{-1}(1)$, but $\chi_{[0,1]} \notin L^\infty_{1/\sqrt{t}}[0,1]$.

We have seen in the proof of Theorem 2 that the inequality $\varphi_1^{-1}(u)\,\varphi_2^{-1}(u) \le \varphi^{-1}(u)$ for all $u > 0$ gives that $\varphi(uv) \le \varphi_1(v) + \varphi_2(u)$ for all $u, v > 0$ and that the last estimate suggests to consider an operation on two Young functions φ_1, φ and compare it with φ_2. Define a new function $\varphi \ominus \varphi_1 : [0,\infty) \to [0,\infty]$ by the formula

$$(\varphi \ominus \varphi_1)(u) = \sup_{v\ge 0}\left[\varphi(uv) - \varphi_1(v)\right].$$

We may say that $\varphi \ominus \varphi_1$ is the conjugate (complementary) function (in the sense of Young) to φ_1 with respect to φ. In particular, if $\varphi(u) = u$, then $\varphi \ominus \varphi_1 = \varphi_1^*$ is the usual conjugate (complementary) function (in sense of Young) to φ_1. This operation on the class of N-functions was defined by Ando [3, p. 180] and on the class of extended Young functions (by word "extended Young" functions we mean nondecreasing convex functions $\varphi : [0,\infty) \to [0,\infty]$ with $\varphi(0) = 0$ and they can be trivial) by O'Neil [32, p. 325] and he referred to Ando.

Note that it can happen that the function $\varphi \ominus \varphi_1(u) = \infty$ for $u > 0$, and then we have that the corresponding Orlicz space is the zero space. To avoid a confusion when $\max\{b_\varphi, b_{\varphi_1}\} < \infty$ since then we will have symbol $\infty - \infty$ we better skip this case.

Moreover, in the case when we work with sequence spaces (or in case $E \hookrightarrow L^\infty$) it is reasonable to define $\varphi \ominus \varphi_1$ in a different way, namely,

$$(\varphi \ominus \varphi_1)_0(u) = \sup_{0\le v\le 1}\left[\varphi(uv) - \varphi_1(v)\right],$$

since then only the behaviour of the functions in a neighbourhood of zero is important. Djakov and Ramanujan [13] proved that in the case of Orlicz sequence spaces we have that $M(l^{\varphi_1}, l^\varphi) = l^{\varphi_2}$, where $\varphi_2 = (\varphi \ominus \varphi_1)_0$. It is easy to see that the function $(\varphi \ominus \varphi_1)_0$ is smaller than $\varphi \ominus \varphi_1$ and it can be different from $\varphi \ominus \varphi_1$.

Example 6. Let $\varphi(u) = u^p/p, \varphi_1(u) = u^{p_1}/p_1$ with $1 \le p, p_1 < \infty$. If $p > p_1$, then $(\varphi \ominus \varphi_1)(u) = \infty$ for $u > 0$ and

$$(\varphi \ominus \varphi_1)_0(u) = \begin{cases} 0 & \text{if } 0 \le u \le (p/p_1)^{1/p}, \\ \frac{u^p}{p} - \frac{1}{p_1} & \text{if } u \ge (p/p_1)^{1/p}. \end{cases}$$

If $p = p_1$, then

$$(\varphi \ominus \varphi_1)(u) = \begin{cases} 0 & \text{if } 0 \le u \le 1, \\ \infty & \text{if } u > 1, \end{cases}$$

and

$$(\varphi \ominus \varphi_1)_0(u) = \begin{cases} 0 & \text{if } 0 \le u \le 1, \\ \frac{u^p - 1}{p} & \text{if } u \ge 1. \end{cases}$$

If $p < p_1$, then $(\varphi \ominus \varphi_1)(u) = \frac{u^{p_2}}{p_2}$, where $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$ and

$$(\varphi \ominus \varphi_1)_0(u) = \begin{cases} \frac{u^{p_2}}{p_2} & \text{if } 0 \le u \le 1, \\ \frac{u^p}{p} - \frac{1}{p_1} & \text{if } u \ge 1. \end{cases}$$

Example 7. Let

$$\varphi(u) = \begin{cases} 0 & \text{if } 0 \le u \le 1, \\ u - 1 & \text{if } u \ge 1, \end{cases}$$

and $\varphi_1(u) = u^2$. Then

$$\varphi_2(u) = (\varphi \ominus \varphi_1)(u) = \begin{cases} 0 & \text{if } 0 \le u \le 2, \\ \frac{u^2}{4} - 1 & \text{if } u \ge 2, \end{cases}$$

$$\varphi_3(u) = (\varphi \ominus \varphi_2)(u) = \begin{cases} 0 & \text{if } 0 \le u \le 1/2, \\ 2u - 1 & \text{if } 1/2 \le u \le 1, \\ u^2 & \text{if } u \ge 1, \end{cases}$$

and $\varphi_4(u) = (\varphi \ominus \varphi_3)(u) = \varphi_2(u)$ for all $u \ge 0$. The last equality was proved by O'Neil [32, p. 325]. For Orlicz spaces considered on $I = [0, \infty)$ we have

$$M(L^{\varphi_1}, L^{\varphi}) = M(L^2, L^1 + L^\infty) = L^2 + L^\infty = L^{\varphi_2} \tag{26}$$

and

$$M(L^{\varphi_2}, L^{\varphi}) = M(L^2 + L^\infty, L^1 + L^\infty) = L^{\varphi_3} = L^2 + L^\infty. \tag{27}$$

The second equality in (26) we can get in the following way: if $x \in L^2 + L^\infty$ and $y \in L^2$, then

$$\begin{aligned} \|xy\|_{L^1+L^\infty} &= \int_0^1 (xy)^*(t)dt \le (\int_0^1 x^*(t/2)^2 dt)^{1/2} (\int_0^1 y^*(t/2)^2 dt)^{1/2} \\ &\le 2\|x\|_{L^2+L^\infty} \|y\|_{L^2}, \end{aligned}$$

and, hence, $L^2+L^\infty \overset{2}{\hookrightarrow} M(L^2, L^1+L^\infty)$. This embedding also follows from Theorem 2(i) since $\varphi_1^{-1}(u)\varphi_2^{-1}(u) \le 2\varphi^{-1}(u)$ for all $u>0$.

On the other hand, the function $y_t(s) = \chi_{[0,t]}(s)/\max(1,t) \in L^1 \cap L^\infty$ and $\|y_t\|_{L^1\cap L^\infty} = 1$. Thus, by the general property in (vii) and Theorem 1(i),

$$\begin{aligned}\|x\|_{M(L^2,L^1+L^\infty)} &= \|x\|_{M(L^1\cap L^\infty,L^2)} = \|x^*\|_{M(L^1\cap L^\infty,L^2)}\\ &\ge \|x^* y_t\|_{L^2} = \frac{1}{\max(1,t)}\|x^*\chi_{[0,t]}\|_{L^2} \ \text{ for any } t>0.\end{aligned}$$

Hence,

$$\begin{aligned}\|x\|_{M(L^2,L^1+L^\infty)} &\ge \sup_{t>0}\frac{1}{\max(1,t)}\|x^*\chi_{[0,t]}\|_{L^2} = \Big(\int_0^1 x^*(s)^2 ds\Big)^{1/2}\\ &\ge \frac{1}{\sqrt{2}}\|x\|_{L^2+L^\infty},\end{aligned}$$

and we have the reverse embedding $M(L^2, L^1+L^\infty) \overset{\sqrt{2}}{\hookrightarrow} L^2+L^\infty$. This embedding does not follow from Theorem 5(i) or Corollary 4(i) since $\lim_{u\to 0^+}\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)\varphi_2^{-1}(u)} = \lim_{u\to 0^+}\frac{u+1}{\sqrt{u}\cdot 2\sqrt{u+1}} = \infty$. This is also not a contradiction with Theorem 6(ii), Corollary 3(ii) and Corollary 5(ii) since the function $\varphi_1^{-1}(1/t)/[\varphi^{-1}(1/t)t^a]$ is not non-decreasing for any $a>0$.

The second equality in (27) follows from Corollary 4(i) since $\varphi^{-1}(u) \le \varphi_2^{-1}(u)\varphi_3^{-1}(u) \le 2\varphi^{-1}(u)$ for all $u>0$.

Some properties of operation $\varphi \ominus \varphi_1$ are collected in the next lemma (part (iii) in the Lemma 5 below was proved in [43, Theorem 3] with some additional assumptions; cf. also [26] and [28]).

Lemma 5. *Let φ, φ_1 be two Young functions with* $\max\{b_\varphi, b_{\varphi_1}\} = \infty$ *and* $\varphi_2 = \varphi \ominus \varphi_1$.

(i) The function φ_2 is non-decreasing, convex, left-continuous on $[0,\infty)$ *with* $\varphi_2(0)=0$ *and it can be ∞ on* $(0,\infty)$.

(ii) We have

$$\varphi_1^{-1}(u)\,\varphi_2^{-1}(u) \le 2\,\varphi^{-1}(u) \quad \textit{for all } u \ge 0.$$

(iii) If $b_\varphi = b_{\varphi_1} = \infty$ (which means that φ, φ_1 are, in fact, Orlicz functions) and for any $v>0$ the function $\frac{\varphi_1(u)}{\varphi(uv)}$ is equivalent to a non-decreasing function, then $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments.

Remark 4. This lemma gives a constructive way to define the function φ_2 such that $M(E_{\varphi_1}, E_\varphi) \hookrightarrow E_{\varphi_2}$ (see Theorem 5) and $E_{\varphi_2} \hookrightarrow M(E_{\varphi_1}, E_\varphi)$ (see Theorem 2).

Proof of Lemma 5. (i) Of course, $\varphi_2(0)=0$ and φ_2 is a non-decreasing function together with φ. Moreover, φ_2 is a convex function since φ is convex. We only need to show that φ_2 is left-continuous at $u_0>0$. We consider two cases.

1^0. Let $0 < \varphi_2(u_0) < \infty$. Suppose, on the contrary, that φ_2 is not left-continuous at u_0. Then, since φ_2 is non-decreasing, we can find a $\delta>0$ such that for all $u<u_0$ we have $\varphi_2(u) \le \varphi_2(u_0)-\delta$. Also, by the definition of φ_2, there is $v>0$ such that

$\varphi_2(u_0) \leq \varphi(u_0 v) - \varphi_1(v) + \frac{\delta}{3}$ and, by the left-continuity of φ, there is $t < u_0$ such that $0 \leq \varphi(u_0 v) - \varphi(tv) \leq \frac{\delta}{3}$. Thus

$$\varphi_2(t) \geq \varphi(tv) - \varphi_1(v) \geq \varphi(u_0 v) - \varphi_1(v) - \frac{\delta}{3} \geq \varphi_2(u_0) - \frac{2\delta}{3},$$

which is a contradiction. This contradiction shows that φ_2 is left-continuous at $u_0 > 0$.

2^0. Let $\varphi_2(u_0) = \infty$. Suppose again that φ_2 is not left-continuous at u_0. Then, since φ_2 is non-decreasing, we can find $M > 0$ such that for all $u < u_0$ we have that $\varphi_2(u) \leq M$. Moreover, by the definition of φ_2, there is $v > 0$ such that $\varphi(u_0 v) - \varphi_1(v) \geq 3M$ and, by the left-continuity of φ, there is $t < u_0$ such that $\varphi(tv) = \infty$ (in the case $u_0 v > b_\varphi$) or $\varphi(tv) \geq \varphi(u_0 v) - M$ (in the case $u_0 v \leq b_\varphi$). Then, in the case $u_o v \leq b_\varphi$, we have

$$\varphi_2(t) \geq \varphi(tv) - \varphi_1(v) \geq \varphi(u_0 v) - \varphi_1(v) - M \geq 2M,$$

or in the case $u_0 v > b_\varphi$ we obtain

$$\varphi_2(t) \geq \varphi(tv) - \varphi_1(v) = \infty \geq 3M,$$

which give contradictions. Thus, our claim is proved.

(ii) By the definition of φ_2 we have $\varphi(uv) \leq \varphi_1(v) + \varphi_2(u)$ for all $u, v > 0$. Then (ii) follows from remarks after Corollary 2.

(iii) The equivalence of the function $\frac{\varphi_1(u)}{\varphi(uv)}$ to a non-decreasing function means that there is a number $K > 0$ such that for each $v > 0$ there is a non-decreasing function ψ_v with estimates $\frac{1}{K}\psi_v(u) \leq \frac{\varphi_1(u)}{\varphi(uv)} \leq K\psi_v(u)$ for all $u > 0$.

Let $u > 0$ be fixed and suppose $0 < \varphi_1^{-1}(u) < v$. Then, by the monotonicity of ψ_w, one has for $w = \frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}$

$$\begin{aligned} \frac{\varphi_1(v)}{\varphi(vw)} &\geq \frac{1}{K}\psi_w(v) \geq \frac{1}{K}\psi_w(\varphi_1^{-1}(u)) \geq \frac{1}{K^2}\frac{\varphi_1(\varphi_1^{-1}(u))}{\varphi(\varphi_1^{-1}(u)w)} \\ &= \frac{1}{K^2}\frac{u}{\varphi(\varphi^{-1}(u))} = \frac{1}{K^2}, \end{aligned}$$

which gives $\varphi(vw) \leq K^2\varphi_1(v)$. If $\varphi_1^{-1}(u) \geq v$, then by monotonicity of φ, we obtain $\varphi(vw) \leq \varphi(\varphi_1^{-1}(u)w) = \varphi(\varphi^{-1}(u)) = u$. Consequently, by convexity of φ_1, for any $v > 0$ we have that $\varphi(vw) \leq K^2\varphi_1(v) + u \leq \varphi_1(K^2 v) + u$ and, therefore, $\varphi_2(\frac{w}{K^2}) \leq u$. Thus $\varphi^{-1}(u) \leq K^2\varphi_1^{-1}(u)\varphi_2^{-1}(u)$ for all $u > 0$ and the proof is complete. □

Using Lemma 5(iii), we obtain the following other version of Theorem 6.

THEOREM 7. *Let E be a Banach function space with the Fatou property and let $\varphi, \varphi_1, \varphi_2$ be Orlicz functions. Suppose*

$$M(E_{\varphi_1}, E_\varphi) \hookrightarrow E_{\varphi_2}.$$

Assume that for any $v > 0$ the function $\frac{\varphi_1(u)}{\varphi(uv)}$ is equivalent to a non-decreasing function of $u > 0$. If $L^\infty \hookrightarrow E$ and $E_a \neq \{0\}$, then $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for large arguments.

Proof. By Lemma 5(iii), we know that there is $\varphi_3 = \varphi \ominus \varphi_1$ satisfying $\varphi_1^{-1}\varphi_3^{-1} \approx \varphi^{-1}$. Therefore, according to Corollary 4, we have

$$M(E_{\varphi_1}, E_\varphi) = E_{\varphi_3} \hookrightarrow E_{\varphi_2}.$$

Moreover, it is known (see [17], Theorem 2.4) that if $E_a \neq \{0\}$ and $E_{\varphi_3} \hookrightarrow E_{\varphi_2}$ then there is $k > 0$ such that $\limsup_{u\to\infty} \frac{\varphi_2(k\,u)}{\varphi_3(u)} < \infty$. Therefore, we have $\varphi_2(k\,u) \leq C\,\varphi_3(u)$ for some $C > 1$ and large u. Consequently, for $u = \varphi_3^{-1}(v)$ from Lemma 2 we obtain

$$\varphi_2(k\varphi_3^{-1}(v)) \leq C\,\varphi_3(\varphi_3^{-1}(v)) \leq Cv$$

and

$$k\varphi_3^{-1}(v) \leq \varphi_2^{-1}(\varphi_2(k\varphi_3^{-1}(v))) \leq \varphi_2^{-1}(Cv) \leq C\varphi_2^{-1}(v) \text{ for large } v.$$

Finally, we have $\varphi^{-1} \approx \varphi_1^{-1}\varphi_3^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for large arguments and the theorem is proved. □

It is worth to notice that there are Orlicz spaces L^φ, L^{φ_1} such that for any $v > 0$ the function $\frac{\varphi_1(u)}{\varphi(uv)}$ is non-decreasing in u, but there is no $a > 0$ such that $\frac{f_{L^\varphi}(t)}{f_{L^{\varphi_1}}(t)\,t^a}$ is non-decreasing in t.

Example 8. Consider the Orlicz functions $\varphi(u) = u^2$ and $\varphi_1(u) = u^2\ln(u+1)$. Then $L^{\varphi_1}[0,1] \hookrightarrow L^\varphi[0,1]$ and the function $\frac{\varphi_1(u)}{\varphi(uv)} = \frac{\ln(u+1)}{v^2}$ is non-decreasing in $u > 0$ for any $v > 0$. On the other hand, if in the quotient $\frac{f_{L^\varphi}(t)}{f_{L^{\varphi_1}}(t)\,t^a} = \frac{\varphi_1^{-1}(\frac{1}{t})}{\varphi^{-1}(\frac{1}{t})\,t^a}$ after substitution $t = \frac{1}{\varphi_1(u)}$ we obtain

$$\frac{\varphi_1(u)^a\,u}{\varphi^{-1}(\varphi_1(u))} = \frac{u^{2a+1}\ln^a(u+1)}{\sqrt{u^2\ln(u+1)}} = u^{2a}\ln^{a-1/2}(u+1) \to \infty,$$

as $u \to \infty$ for any $a > 0$. Consequently, $\frac{f_{L^\varphi}(t)}{f_{L^{\varphi_1}}(t)\,t^a} \to \infty$ as $t \to 0^+$ and therefore it cannot be non-decreasing for small $t > 0$.

If we drop the assumption that $\frac{\varphi_1(u)}{\varphi(uv)}$ is non-decreasing in Theorem 7, then the result may not be true.

Example 9. Let $\varphi(u) = \frac{u^2}{2}$ and we will construct a new function ψ which does not satisfy the Δ_2-condition for large arguments, i.e, $\limsup_{u\to\infty} \frac{\psi(2u)}{\psi(u)} = \infty$ and such that

$$\psi(u) \geq \varphi(u) \text{ for all } u > 0 \text{ and } \psi(u_n) = \varphi(u_n)$$

for some sequence (u_n) tending to infinity with $\frac{\psi(2u_n)}{\psi(u_n)} \nearrow \infty$.

Take any sequence (a_n) of positive real numbers satisfying two conditions

$$\frac{a_{n+1}}{a_n} \nearrow \infty \text{ and } 2\sum_{k=1}^{n}(-1)^{n-k}\,a_k < \sum_{k=1}^{n+1}(-1)^{n+1-k}\,a_k \text{ for all } n \in \mathbb{N}. \tag{28}$$

It is easy to see that, for example, the sequence $a_n = (n+2)!$ satisfies those conditions. Define the required sequence as $u_n = 2\sum_{k=1}^{n}(-1)^{n-k}\,a_k$, $u_0 = 0$ and consider the sequence of pairwise disjoint subintervals of $[0,\infty)$ defined by $I_n = [u_{n-1}, u_n), n = 1, 2, \ldots$. The numbers a_n are the centers of I_n, since $\frac{u_n+u_{n-1}}{2} = a_n$. Now define the following Orlicz function

$$\psi(u) = \int_0^u \sum_{n=1}^{\infty} a_n \, \chi_{I_n}(s) ds. \tag{29}$$

For any $n \in \mathbb{N}$ we have

$$\int_{I_n} a_n ds = a_n(u_n - u_{n-1}) = \frac{1}{2}(u_n + u_{n-1})(u_n - u_{n-1}) = \frac{u_n^2 - u_{n-1}^2}{2} = \int_{I_n} s ds$$

and, thus,

$$\begin{aligned}
\psi(u_n) &= \int_0^{u_n} \sum_{k=1}^{\infty} a_k \, \chi_{I_k}(s) ds = \sum_{k=1}^{n} \int_{I_k} a_k ds \\
&= \sum_{k=1}^{n} \int_{I_k} s ds = \int_0^{u_n} s ds = \frac{u_n^2}{2} = \varphi(u_n).
\end{aligned}$$

We must now check that the function ψ is bigger than the function φ. For $u \in [0, u_1] = [0, 2a_1]$ we have $\psi(u) = a_1 u \geq u^2/2$ and, for $u \in [u_{n-1}, u_n], n = 2, 3, \ldots$, it yields

$$\begin{aligned}
\psi(u) &= \int_0^{u} \sum_{k=1}^{\infty} a_k \, \chi_{I_k}(s) ds = \sum_{k=1}^{n-1} a_k(u_k - u_{k-1}) + a_n \, (u - u_{n-1}) \\
&= \frac{1}{2} \sum_{k=1}^{n-1} (u_k^2 - u_{k-1}^2) + \frac{u_n + u_{n-1}}{2}(u - u_{n-1}) \\
&= \frac{1}{2} \, u_{n-1}^2 + \frac{u_n + u_{n-1}}{2} \, u - \frac{u_n + u_{n-1}}{2} u_{n-1} \\
&= \frac{u_n + u_{n-1}}{2} u - \frac{u_n u_{n-1}}{2} = \frac{h(u)}{2} + \frac{u^2}{2},
\end{aligned}$$

where

$$h(u) = -u^2 + (u_n + u_{n-1}) \, u - u_n \, u_{n-1}.$$

Since for $u \in [u_{n-1}, u_n]$ one has $h(u) \geq \max\left[h(u_{n-1}), h(u_n)\right] = 0$ it follows that $\psi(u) \geq \frac{u^2}{2}$ for any $u \in [u_{n-1}, u_n]$, and consequently $\psi(u) \geq \frac{u^2}{2}$ for any $u \geq 0$. Moreover, by assumptions (28) on a_n, we see that $2u_n \in I_{n+1} = [u_n, u_{n+1})$ and one has

$$\begin{aligned}
\frac{\psi(2u_n)}{\psi(u_n)} &= \frac{(u_{n+1} + u_n) \, u_n - \frac{u_{n+1} \, u_n}{2}}{u_n^2/2} \\
&= \frac{2u_{n+1} \, u_n + 2u_n^2 - u_{n+1} u_n}{u_n^2} = 2 + \frac{u_{n+1}}{u_n} \\
&= 2 + \frac{2a_{n+1} - u_n}{u_n} = 1 + \frac{2a_{n+1}}{u_n} \\
&= 1 + \frac{2a_{n+1}}{2a_n - u_{n-1}} > 1 + \frac{a_{n+1}}{a_n} \to \infty,
\end{aligned}$$

as $n \to \infty$.

Of course, $L^\psi[0,1] \subset L^\varphi[0,1] = L^2[0,1]$ because $\psi(u) \geq \varphi(u)$ for all $u > 0$ and thus $M(L^\psi, L^\varphi)$ is non-trivial. Moreover, $L^\psi \neq L^\varphi = L^2$ since ψ does not satisfy the Δ_2-condition for large u. Let us calculate $\varphi_2 = \varphi \ominus \psi$. For $u > 1$

$$\begin{aligned}\varphi_2(u) &= \sup_{v>0}\,[\varphi(uv) - \psi(v)] \geq \limsup_{n\to\infty}\,[\varphi(uu_n) - \psi(u_n)] \\ &= \limsup_{n\to\infty} \frac{1}{2}u_n^2(u^2-1) = \infty,\end{aligned}$$

and for $0 < u \leq 1$ one has $\varphi(u\,v) - \psi(v) \leq \varphi(v) - \psi(v) \leq 0$ for each $v > 0$ and so $\varphi_2(u) = 0$. Therefore,

$$\varphi_2(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ \infty & \text{if } u > 1, \end{cases}$$

and $\psi^{-1}(u)\varphi_2^{-1}(u) = \psi^{-1}(u) \leq \varphi^{-1}(u)$ for all $u > 0$.

Let us now collect some properties of functions and spaces that arise in Example 9.

(a) We don't have the relation $\varphi^{-1} \prec \psi^{-1}\varphi_2^{-1}$ for large u, since

$$\liminf_{u\to\infty} \frac{\psi^{-1}(u)}{\varphi^{-1}(u)} = \liminf_{v\to\infty} \frac{v}{\varphi^{-1}(\psi(v))}$$

$$\begin{aligned}&\leq \lim_{n\to\infty} \frac{2u_n}{\varphi^{-1}(\psi(2u_n))} = \lim_{n\to\infty} \frac{\sqrt{2}u_n}{\sqrt{\psi(2u_n)}} = \sqrt{2}\lim_{n\to\infty} \frac{u_n}{\sqrt{(u_{n+1}+u_n)u_n - u_{n+1}u_n/2}} \\ &= \sqrt{2}\lim_{n\to\infty} \frac{u_n}{\sqrt{2a_{n+1}u_n - (2a_{n+1}-u_n)u_n/2}} = \sqrt{2}\lim_{n\to\infty} \frac{u_n}{\sqrt{a_{n+1}u_n + u_n^2/2}} \\ &= \sqrt{2}\lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{2} + \frac{a_{n+1}}{u_n}}} \leq \sqrt{2}\lim_{n\to\infty} \frac{1}{\sqrt{\frac{1}{2} + \frac{a_{n+1}}{2a_n}}} = 0.\end{aligned}$$

(b) The function $\frac{\psi(u)}{\varphi(u)}$ cannot be monotone since

$$\frac{\psi(u_n)}{\varphi(u_n)} = 1 \text{ and } \frac{\psi(2u_n)}{\varphi(2u_n)} = \frac{\psi(2u_n)}{2u_n^2} = \frac{\psi(2u_n)}{4\psi(u_n)} \to \infty \text{ as } n \to \infty.$$

In consequence, from (a) and (b), we obtain

(c) we cannot drop the assumption of monotonicity of $\frac{\psi(u)}{\varphi(uv)}$ in Lemma 5(iii), in general.

Moreover,

(d) there is no $a > 0$ such that $\frac{f_{L^\varphi}(t)}{f_{L^\psi}(t)t^a}$ is non-decreasing near zero, because

$$\limsup_{t\to 0^+} \frac{f_{L^\varphi}(t)}{f_{L^\psi}(t)} = \limsup_{u\to\infty} \frac{\psi^{-1}(u)}{\varphi^{-1}(u)} \geq \limsup_{n\to\infty} \frac{\psi^{-1}(u_n)}{\varphi^{-1}(u_n)} = 1 \tag{30}$$

and for each $a > 0$ and every sequence $t_n \to 0^+$ we have $t_n^{-a} \to +\infty$.

(e) The function $\frac{f_{L^\varphi}(t)}{f_{L^\psi}(t)}$ cannot be equivalent at 0 to any pseudo-concave function, because of (30) and

$$\liminf_{t\to 0^+} \frac{f_{L^\varphi}(t)}{f_{L^\psi}(t)} = \liminf_{t\to 0^+} \frac{\psi^{-1}(\frac{1}{t})}{\varphi^{-1}(\frac{1}{t})} = \liminf_{u\to\infty} \frac{\psi^{-1}(u)}{\varphi^{-1}(u)} = 0.$$

Thus, in particular, formula (5.21) $f_{M(E,F)}(t) = \frac{f_F(t)}{f_E(t)}$ in the book [5] is false, in general (even up to equivalence). Note that in this example we have $f_{M(E,F)}(t) = \sup_{0<s\leq t} \frac{f_F(s)}{f_E(s)} = 1$.

(f) We have $M(L^\psi, L^\varphi) = L^\infty = L^{\varphi_2}$.

In fact, as we have seen in the proof of Theorem 1(iii), we always have that $f_{M(E,F)}(t) \geq \frac{f_F(t)}{f_E(t)}$. Therefore, $\limsup_{t\to 0^+} f_{M(L^\psi,L^\varphi)}(t) \geq \limsup_{t\to 0^+} \frac{f_{L^\varphi}(t)}{f_{L^\psi}(t)} \geq 1$ and $M(L^\psi, L^\varphi)$ as a symmetric space on $[0,1]$ with the Fatou property such that $\lim_{t\to 0^+} f_{M(L^\psi,L^\varphi)}(t) > 0$ must be $L^\infty[0,1]$ (cf. [22], p. 118).

(g) The assumption of Theorem 6(i) is not satisfied since we have (d) and also the assumption of Theorem 7 is not satisfied since we have (b) but one can see that $M(L^\psi, L^\varphi) \subset L^{\varphi_2}$ while $\varphi^{-1} \not\prec \psi^{-1}\varphi_2^{-1}$ as it was shown in (a).

(h) There is no Young function φ_3 satisfying the equivalence $\varphi^{-1} \approx \psi^{-1}\varphi_3^{-1}$. If such a function should exists, then, by Corollary 4, we have that $M(L^\psi, L^\varphi) = L^{\varphi_3}$. On the other hand, from (f) we have $M(L^\psi, L^\varphi) = L^\infty$, which will mean that $\varphi_3 \approx \varphi_2$ for large arguments, but this is not possible because of (a).

As we have seen earlier we have equality $M(E,E) = L^\infty$ and we can ask if $M(E,F) = L^\infty$ implies that $E = F$? From (f) we see that this is not always the case.

Assume here that $suppE = suppF = \Omega$. Note that $M(E,F) = L^\infty$ if and only if $E^{FF} = F$. Really, if $E^F = L^\infty$, then $E^{FF} = (L^\infty)^F = F$. On the other hand, if $E^{FF} = F$, then $E^F = E^{FFF} = F^F = L^\infty$. We can also have equality $M(E,F) = L^\infty$ if E is a proper subspace of F and the norms of $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent on E. For example, if $E = F_a$ with $F \neq F_a$ and $suppF_a = suppF = \Omega$. Thus

(i) L^ψ *is not* L^φ*-perfect*. In fact, from (f) and $L^\psi \neq L^\varphi$ we obtain

$$(L^\psi)^{L^\varphi L^\varphi} = M(M(L^\psi, L^\varphi), L^\varphi) = M(L^\infty, L^\varphi) = L^\varphi \neq L^\psi.$$

Using Lemma 5, Corollary 4(ii) and the operation $\varphi \ominus \varphi_1$ we are able to prove that the multiplier space between two Orlicz spaces $M(L^{\varphi_1}, L^\varphi)$ on $[0,1]$ is an Orlicz space L^{φ_2} with $\varphi_2 = \varphi \ominus \varphi_1$ under some additional assumptions on the Orlicz functions φ, φ_1, which is a certain similarity to the case of sequence Orlicz spaces. Our proof is presented even for the Calderón-Lozanovskiĭ spaces.

Theorem 8. *Let φ, φ_1 be increasing Orlicz functions and let E be a symmetric space on $[0,1]$ with the Fatou property.*

(i) If $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} = 0$ for any $v > 0$ and additionally at least one of the following three conditions holds: either the function $f_v(u) := \frac{\varphi(uv)}{\varphi_1(u)}$ is non-increasing on $(0,\infty)$ for any $v > 0$ or $\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}$ is a non-decreasing function for large u or the function

$$\varphi_2(u) = (\varphi \ominus \varphi_1)(u) = \sup_{v>0}[\varphi(uv) - \varphi_1(v)] \tag{31}$$

satisfies the Δ_2-condition for large arguments, then $M(E_{\varphi_1}, E_\varphi) = E_{\varphi_2}$.

(ii) If $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} < \infty$ for some $v > 0$ and $\limsup_{u\to\infty} \frac{\varphi(uw)}{\varphi_1(u)} > 0$ for some $w > 0$, then $M(E_{\varphi_1}, E_\varphi) = L^\infty$.

(iii) If $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} = \infty$ for all $v > 0$, then $M(E_{\varphi_1}, E_\varphi) = \{0\}$.

Proof. (i) We only need to prove that in all these three cases we have $\varphi^{-1} \prec \varphi_1^{-1}\varphi_2^{-1}$ for large arguments, since by Lemma 5(ii), we have $\varphi_1^{-1}\varphi_2^{-1} \prec \varphi^{-1}$ even for all arguments, which means that $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for large arguments and then Corollary 4(ii)l implies $M(E_{\varphi_1}, E_\varphi) = E_{\varphi_2}$. Therefore, in each of these three cases we will proceed as follows:

1^0. If for any $v > 0$, $f_v(u)$ is a non-increasing function on $(0,\infty)$, then, by Lemma 5(iii), we obtain that $\varphi_1^{-1}\varphi_2^{-1} \approx \varphi^{-1}$ for all arguments.

2^0. Let $\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}$ be non-decreasing for $u > u_0 \geq 0$. Since $\limsup_{v\to\infty} \frac{\varphi(vw)}{\varphi_1(v)} = 0$ for any $w > 0$ it follows that the supremum in the definition of φ_2 is attained at some $v_0 = v_0(w) > 0$. Can we say something more about this v_0? If $w = \frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}$ and we have $v_0 \geq \varphi_1^{-1}(u)$, then, by the monotonicity assumption, we get

$$\varphi_2[\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}] = \varphi[\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}\, v_0] - \varphi_1(v_0) \leq \varphi[\frac{\varphi^{-1}(\varphi_1(v_0))}{v_0}\, v_0] - \varphi_1(v_0) = 0$$

and this case is not important since $\varphi_2 \geq 0$. Therefore it must be $v_0 \leq \varphi_1^{-1}(u)$ with $u > u_0 \geq 0$, which, in its turn gives,

$$\varphi_2[\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}] = \varphi[\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}\, v_0] - \varphi_1(v_0) \leq \varphi[\frac{\varphi^{-1}(u)}{\varphi_1^{-1}(u)}\, v_0] \leq \varphi[\varphi^{-1}(u)] = u,$$

i.e., $\varphi^{-1}(u) \leq \varphi_1^{-1}(u)\, \varphi_2^{-1}(u)$ for $u > u_0 \geq 0$.

3^0. Let φ_2 satisfy the Δ_2-condition for large arguments, that is, there exist constants $C \geq 1, u_0 \geq 0$ such that $\varphi_2(2u) \leq C\varphi_2(u)$ for all $u > u_0$. Similarly as in 2^0 we find that for any $w > 0$ there exists a $v_0 = v_0(w) > 0$ such that $\varphi_2(w) = \varphi(wv_0) - \varphi_1(v_0)$. For $w = \varphi_2^{-1}(u)$ we have

$$u = \varphi_2[\varphi_2^{-1}(u)] = \varphi[\varphi_2^{-1}(u)\, v_0] - \varphi_1(v_0) > 0,$$

that is, $\varphi_2^{-1}(u) \geq \frac{\varphi^{-1}[\varphi_1(v_0)]}{v_0}$. Hence, by using Lemma 5(ii), we obtain

$$1 \geq \frac{\varphi^{-1}[\varphi_1(v_0)]}{v_0\, \varphi_2^{-1}(u)} \geq \frac{\varphi_1^{-1}[\varphi_1(v_0)]\, \varphi_2^{-1}[\varphi_1[v_0)]}{2v_0\, \varphi_2^{-1}(u)} = \frac{\varphi_2^{-1}[\varphi_1(v_0)]}{2\, \varphi_2^{-1}(u)},$$

and, by the Δ_2-condition of φ_2 for $u > u_1 = \varphi_2(u_0) \geq 0$, we get

$$v_0 \leq \varphi_1^{-1}[\varphi_2(2\,\varphi_2^{-1}(u))] \leq \varphi_1^{-1}(Cu).$$

Since $\varphi^{-1}[u + \varphi_1(v_0)] = \varphi_2^{-1}(u)\, v_0$ it follows that

$$\begin{aligned}\varphi^{-1}(u) &\leq \frac{\varphi_1^{-1}(Cu)}{v_0}\varphi^{-1}(u) \leq \frac{\varphi_1^{-1}(Cu)}{v_0}\varphi^{-1}[u + \varphi_1(v_0)] \\ &\leq \frac{\varphi_1^{-1}(Cu)}{v_0}\varphi_2^{-1}(u)v_0 = \varphi_1^{-1}(Cu)\,\varphi_2^{-1}(u) \leq C\varphi_1^{-1}(u)\,\varphi_2^{-1}(u)\end{aligned}$$

for $u > u_1 = \varphi_2(u_0) \geq 0$. Therefore, all three cases are proved.

(ii) Suppose $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} < \infty$ for some $v > 0$. Then there is $K > 0$ such that $\varphi(uv) \leq K\varphi_1(u)$ for large u and, by [17, Theorem 2.3], we obtain $E_{\varphi_1} \hookrightarrow E_\varphi$. Thus $L^\infty \hookrightarrow M(E_{\varphi_1}, E_\varphi)$. On the other hand, suppose on the contrary that $\limsup_{u\to\infty} \frac{\varphi(uw)}{\varphi_1(u)} = \eta > 0$ for some $w > 0$ and $M(E_{\varphi_1}, E_\varphi) \neq L^\infty$. Define the new function $\psi(u) = \frac{2}{\eta}\varphi(uw)$. Then, again, by [17, Theorem 2.3] we have that $E_\varphi = E_\psi$ and so $M(E_{\varphi_1}, E_\varphi) = M(E_{\varphi_1}, E_\psi)$. The fundamental function f_M of the symmetric space $M = M(E_{\varphi_1}, E_\psi)$ satisfies the condition $\lim_{t\to 0^+} f_M(t) = 0$ because $M(E_{\varphi_1}, E_\psi) \neq L^\infty$. Since

$$1 = \left\| \frac{\chi_{[0,t]}}{f_M(t)} \right\|_{M(E_{\varphi_1}, E_\psi)} \geq \left\| \frac{\chi_{[0,t]}}{f_M(t)} \frac{\chi_{[0,t]}}{f_{E_{\varphi_1}}(t)} \right\|_\psi = \frac{1}{f_M(t)} \frac{f_{E_\psi}(t)}{f_{E_{\varphi_1}}(t)}$$

and $\lim_{t\to 0^+} f_M(t) = 0$ it follows that $\lim_{t\to 0^+} \frac{f_{E_\psi}(t)}{f_{E_{\varphi_1}}(t)} = 0$. This means

$$0 = \lim_{t\to 0^+} \frac{f_{E_\psi}(t)}{f_{E_{\varphi_1}}(t)} = \lim_{t\to 0^+} \frac{\varphi_1^{-1}(1/f_E(t))}{\varphi^{-1}(1/f_E(t))} = \lim_{u\to\infty} \frac{\varphi_1^{-1}(u)}{\psi^{-1}(u)}.$$

But

$$\eta < \limsup_{u\to\infty} \frac{\varphi(uw)}{\varphi_1(u)} = \limsup_{u\to\infty} \frac{\frac{\eta}{2}\psi(u)}{\varphi_1(u)},$$

and thus we can find a sequence $u_n \to \infty$ such that $\psi(u_n) \geq \varphi_1(u_n)$. Putting $v_n = \psi(u_n)$ we see that $\frac{\varphi_1^{-1}(v_n)}{\psi^{-1}(v_n)} \geq 1$, which is a contradiction with the just mentioned equality $\lim_{u\to\infty} \frac{\varphi_1^{-1}(u)}{\psi^{-1}(u)} = 0$.

(iii) The condition: there are $K, u_0, M > 0$ such that $\varphi(Ku) \leq M\varphi_1(u)$ for all $u > u_0$ is necessary for the inclusion $E_{\varphi_1} \hookrightarrow E_\varphi$ (see [17], Theorem 2.4) and this inclusion is necessary for $M(E_{\varphi_1}, E_\varphi) \neq \{0\}$ by Proposition 1(i). But $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} = \infty$ means that the just mentioned condition on the function φ is not satisfied.

Example 10. Let φ_1 be an increasing Orlicz function and $\varphi(u) = 2\varphi_1(\sqrt{u})$. Then $\varphi_2(u) = \sup_{v>0}[\varphi(uv) - \varphi_1(v)] = \varphi_1(u)$ since, by convexity of φ_1, we have

$$\varphi(uv) - \varphi_1(v) = 2\varphi_1(\sqrt{uv}) - \varphi_1(v) \leq 2\varphi_1(\frac{u+v}{2}) - \varphi_1(v) \leq \varphi_1(u)$$

with equality for $v = u$ (see also [43, p. 269] and [26, p. 79]). If φ is a convex function, then

$$\varphi^{-1}(u) = \varphi_1^{-1}(u/2)^2 \leq \varphi_1^{-1}(u)^2 = \varphi^{-1}(2u) \leq 2\varphi^{-1}(u)$$

for any $u > 0$ and from Corollary 4 we obtain that $M(E_{\varphi_1}, E_\varphi) = E_{\varphi_1}$ for any Banach ideal space E with the Fatou property. Note that for the concrete $\varphi_1(u) = \exp(u^2) - 1$ we have that $\varphi(u) = 2(e^u - 1)$ and $f_4(u) = \frac{\varphi(4u)}{\varphi_1(u)}$ is not decreasing on $(0, \infty)$ since $f_4'(1) > 0$, φ_1 does not satisfy the Δ_2-condition for large arguments, but the function $\varphi^{-1}(u)/\varphi_1^{-1}(u)$ is increasing on $(0, \infty)$.

Theorem 8 with $E = L^1[0,1]$, that is, for Orlicz spaces L^φ, L^{φ_1} on $[0,1]$ and the space of multipiers $M(L^{\varphi_1}, L^\varphi)$ has the following form:

Corollary 6. *Let φ, φ_1 be increasing Orlicz functions generating the corresponding Orlicz spaces L^φ and L^{φ_1} on $[0,1]$.*

(i) If $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} = 0$ for any $v > 0$ and additionally at least one of three conditions on $\varphi, \varphi_1, \varphi_2$ from Theorem 8(i) hold, then $M(L^{\varphi_1}, L^\varphi) = L^{\varphi_2}$.

(ii) If $\lim_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} < \infty$ for some $v > 0$ and $\limsup_{u\to\infty} \frac{\varphi(uw)}{\varphi_1(u)} > 0$ for some $w > 0$, then $M(L^{\varphi_1}, L^\varphi) = L^\infty$.

(iii) If $\limsup_{u\to\infty} \frac{\varphi(uv)}{\varphi_1(u)} = \infty$ for all $v > 0$, then $M(L^{\varphi_1}, L^\varphi) = \{0\}$.

Corollary 6(i) without any proof was written in the papers by Wang [41, Lemma 2], Zabreĭko [42, p. 109] and in the book by Appell and Zabrejko [5, p. 123]. In these mentioned sources the authors formulate Corollary 6(i) without additional assumptions on φ, φ_1, but we were able to prove only the result with these three additional assumptions. Of course, it will be nice to give the proof without these additional conditions. The proof of the first case in Corollary 6(i) was already given in the book [26, pp. 77-78]. Note that Ando [3, Theorem 5] for given Orlicz functions φ_1, φ defined the function φ_2 by the formula (31) and proved that L^{φ_2} is a largest Orlicz space on $[0,1]$ such that $L^{\varphi_2} \subset M(L^{\varphi_1}, L^\varphi) \neq \{0\}$.

Parts (ii) and (iii) appeared without any proof in Zabreĭko [42, pp. 108-109] and with different proofs than our in the book [5, pp. 132, 148-149] (the same proof of part (iii) appeared also earlier in [4, pp. 309-310]).

Already in 1957 Shragin [Sh57] proved that $x \in M(L^{\varphi_1}, L^\varphi)$ if and only if there are $c > 0$ and $\lambda > 0$ such that $\int_0^1 \varphi(\lambda|x(t)|\, f_c(\lambda|x(t)|)\, dt < \infty$, where $f_c(u) = \sup\{v \geq 0 : \varphi(uv) \geq c\varphi_1(v)\}$. It seems that the last condition, in general, cannot be discribed in terms of the function φ_2.

B. Maurey in the paper [29] on pages 128-138 is proving that if φ, φ_1 are two Orlicz functions which additionally are N-functions at infinity, that is, $\lim_{u\to\infty} \frac{\varphi(u)}{u} = \lim_{u\to\infty} \frac{\varphi_1(u)}{u} = \infty$ and $\lim_{u\to\infty} \frac{\varphi(vu)}{\varphi_1(u)} = 0$ for any $v > 0$, then for any measure space (Ω, μ) we have (cf. [29], Proposition 107)

$$M\left(L^{\varphi_1}(\Omega, \mu), L^\varphi(\Omega, \mu)\right) = L^\theta(\Omega, \mu),$$

where $\theta = \varphi \ominus \varphi_1$. His proof of this result is using one important property of operation $\ominus$, namely that $\varphi \ominus [\varphi \ominus \varphi_1](u) = \varphi_1(u)$ for all $u > 0$ (cf. [29], Proposition 104(b), p. 130).

Unfortunately, the last equality is not true for all $u > 0$, as we can see on the example below.

Example 11. Let $\varphi(u) = u^2$ and let φ_p for $1 \le p \le 4$ be defined by

$$\varphi_p(u) = \begin{cases} u^p & \text{if } 0 \le u \le 1, \\ u^4 & \text{if } u \ge 1. \end{cases}$$

If $1 \le p \le 2$, then

$$\theta(u) = \varphi \ominus \varphi_p(u) = \begin{cases} 0 & \text{if } 0 \le u \le 1, \\ u^2 - 1 & \text{if } 1 \le u \le \sqrt{2}, \\ u^4/4 & \text{if } u \ge \sqrt{2}, \end{cases}$$

and

$$\varphi \ominus \theta(u) = \varphi \ominus [\varphi \ominus \varphi_p](u) = \varphi_2(u) = \begin{cases} u^2 & \text{if } 0 \le u \le 1, \\ u^4 & \text{if } u \ge 1. \end{cases}$$

Therefore, for $1 \le p < 2, \varphi \ominus [\varphi \ominus \varphi_p]$ is equal to φ_p only on interval $[1, \infty)$ but not on the interval $(0, 1)$ where it is φ_2.

We finish our considerations with a conjecture motivated by the above Example 9, Theorem 8 and Example 10.

Conjecture. We have equality $M(E_{\varphi_1}, E_\varphi) = E_{\varphi \ominus \varphi_1}$ for any Banach ideal space E.

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