THE PRODUCT STRUCTURE OF THE EQUIVARIANT K-THEORY OF THE BASED LOOP GROUP OF $SU(2)$

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ABSTRACT. Let $G = SU(2)$ and let ΩG denote the space of continuous based loops in G, equipped with the pointwise conjugation action of G. It is a classical fact in topology that the ordinary cohomology $H^*(\Omega G)$ is a divided polynomial algebra $\Gamma[x]$. The algebra $\Gamma[x]$ can be described as an inverse limit as $k \to \infty$ of the symmetric subalgebra in $\Lambda(x_1, \dots, x_k)$ where $\Lambda(x_1, \dots, x_k)$ is the usual exterior algebra in the variables $x_1, \dots, x_k$. We compute the $R(G)$-algebra structure of the G-equivariant K-theory $K_G^*(\Omega G)$ which naturally generalizes the classical computation of $H^*(\Omega G)$ as $\Gamma[x]$. Specifically, we prove that $K_G^*(\Omega G)$ is an inverse limit of the symmetric (S_{2r}-invariant) subalgebra $(K_G^*((\mathbb{P}^1)^{2r}))^{S_{2r}}$ of $K_G^*((\mathbb{P}^1)^{2r})$, where the symmetric group S_{2r} acts in the natural way on the factors of the product $(\mathbb{P}^1)^{2r}$ and G acts diagonally via the standard action on each factor.

CONTENTS

1. INTRODUCTION

Let G be a compact connected Lie group and consider the conjugation action of G on itself. Let ΩG denote the space of continuous based loops in G, equipped with the pointwise conjugation action of G. The ordinary and G-equivariant cohomology rings $H^*(G)$, $H^*(\Omega G)$, $H_G^*(G)$, and $H_G^*(\Omega G)$ were computed decades ago (with contributions from many people), and these results are by now considered classical; the same is true of the computations of the ordinary K-theory rings $K^*(G)$ and $K^*(\Omega G)$. However, somewhat surprisingly, the G-equivariant K-theory computation was not addressed in the literature until quite recently. (For instance, it was only in 2000 that Brylinski and Zhang computed $K_G^*(G)$ [7].)

The main contribution of this manuscript is a concrete computation of the K_G^*-algebra structure of $K_G^*(\Omega G)$ for the specific case $G = SU(2)$. (We addressed the additive, i.e. K_G^*-module, structure of $K_G^*(\Omega G)$ in our companion paper [10].) More specifically, we describe this product structure in a concrete manner which is a straightforward and pleasant generalization of the classical fact that the ordinary cohomology ring $H^*(\Omega G)$ is a divided polynomial algebra $\Gamma[x]$. To make this more precise, let us briefly recall several descriptions (some quite familiar, and some perhaps not so) of $\Gamma[x]$. Firstly, $\Gamma[x]$ can be described as the group additively generated by elements labelled $\gamma_k(x)$ (with degree $k|x| = 2k$) satisfying the multiplicative relations

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x).$$

Alternatively, $\Gamma[x]$ is the subring of $\mathbb{Q}[x]$ generated by the elements $\{\frac{x^k}{k!}\}$ for varying k. A third description is as the Hopf algebra dual of $\mathbb{Z}[x]$. Finally, yet one more description is obtained by observing that

$$\Gamma[x]/\{\gamma_j(x) : j > k\} \cong \big(\Lambda(x_1, \ldots, x_k)\big)^{S_k} \cong \{\text{ symmetric polynomials in } \Lambda(x_1, \ldots, x_k)\}$$

where $\Lambda(x_1, \ldots, x_k)$ is the exterior algebra in the variables. The ring $\Gamma[x]$ can then be identified with the graded inverse limit as $k \to \infty$. Our computation of $K_G^*(\Omega G)$ is a suitable generalization of this last description of $\Gamma[x]$. Thus, this manuscript proves what in some sense is a "classical" theorem in topology: its statement could have been made and understood by the topologists working decades ago, and the computation itself fits naturally with the classical results. Nevertheless, since the literature appears to have been silent on this issue, we have taken this opportunity to provide the details.

The following is our main result.

Theorem 1.1. *Let $G = SU(2)$ and let ΩG be the space of (continuous) based loops in G, equipped with the natural G-action by pointwise conjugation. Then*

$$K_G(\Omega G) = \varprojlim_r \left(K_G\big((\mathbb{P}^1)^{2r}\big) \right)^{S_{2r}} = \varprojlim_r \{\text{symmetric polynomials in } K_G\big((\mathbb{P}^1)^{2r}\big)\}$$

where $K_G\big((\mathbb{P}^1)^{2r}\big) \cong R(G)[L_1, \ldots, L_{2r}]/I$, and I is the ideal generated by $\{L_j^2 - vL_j + 1\}_{j=1}^n$. Here $R(G)$ is the representation ring of G, v is the standard representation of $G = SU(2)$ on $\mathbb{C}^2$ and L_j is the pullback of either the canonical line bundle over the jth factor of $(\mathbb{P}^1)^{2r}$ or its inverse, depending on j (see Definition 3.12). The system maps in this inverse system are given by

$$i^*(s_j) = \begin{cases} s_0' & \text{if } j = 0; \\ s_1' + vs_0' & \text{if } j = 1; \\ s_j' + vs_{j-1}' + s_{j-2}' & \text{if } 1 < j \leq 2r - 2; \\ vs_{2r-2}' + s_{2r-3}' & \text{if } j = 2r - 1; \\ s_{2r-2}' & \text{if } j = 2r, \end{cases}$$

where s_j and s_j' are the jth elementary symmetric polynomials in $\{L_1, \ldots, L_{2r}\}$ and respectively $\{L_1', \ldots, L_{2r-1}'\}$. (See equation 5.15.)

We now briefly sketch the outline of our proof. From the module calculations in [10] we know that $K_G^*(\Omega G)$ is the inverse limit of $K_G^*(F_{2r})$ as $r \to \infty$ for a certain G-invariant filtration $F_0 \subseteq F_2 \subseteq \cdots \subseteq F_{2r} \cdots$ of ΩG. Thus, in the present manuscript, we focus on the computation of the K_G^*-algebra structure of $K_G^*(F_{2r})$. To accomplish this, we consider

the $G = SU(2)$-space $(\mathbb{P}^1)^{2r}$ for each $r \geq 0$, where the G acts diagonally on each factor in the usual way, induced from the standard representation of $SU(2)$ on $\mathbb{C}^2$. We define maps $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ for each $r \geq 0$ in Section 3 and then prove in Proposition 5.1 that the induced map

$$\Phi_{2r}^* : K_G^*(F_{2r}) \to K_G^*((\mathbb{P}^1)^{2r})$$

is injective for all $r > 0$. We also give an explicit and convenient presentation of the right hand side via generators and relations in Theorem 3.13. The main (and longest) technical argument in this manuscript is a computation of the image of Φ_{2r}^* in terms of the natural generators of $K_G^*((\mathbb{P}^1)^{2r})$ in a manner analogous to the description of $H^*(\Omega G)$ as a divided polynomial algebra (Theorem 5.15). Taking the inverse limit of this description yields $K_G^*(\Omega G)$. At various points in our argument, we find it necessary or useful to first prove the corresponding statements in equivariant cohomology, and then use these to deduce the analogous results in equivariant K-theory.

Notation and standard facts.

- T denotes the maximal torus of $G = SU(2)$ given by

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \,\middle|\, z \in S^1 \right\}.$$

- $W \cong S_2$ is the Weyl group of G and $w \in W$ is the nontrivial element.
- $R(T) \cong \mathbb{Z}[b, b^{-1}]$ is the representation ring of T, where b is the weight 1 (identity map) one-dimensional representation of T, and $w(b) = b^{-1}$.
- $v \in R(G)$ is the standard (two-dimensional) representation of G on $\mathbb{C}^2$. The restriction of v to T is $b \oplus b^{-1}$.
- $R(G) \cong \mathbb{Z}[b, b^{-1}]^W = \mathbb{Z}[v]$ where $v = b + b^{-1}$.
- $K_T^*(\mathrm{pt}) \cong R(T) \cong \mathbb{Z}[b, b^{-1}]$ and $K_G^*(\mathrm{pt}) \cong R(G) \cong \mathbb{Z}[v]$.
- We let $H^\pi(X)$ denote the direct product $\prod_{i=0}^\infty H^i(X)$ and $\tilde{H}^\pi(X)$ the direct product $\prod_{i=1}^\infty H^i(X)$.
- We will systematically use bars to denote elements in equivariant cohomology analogous to the corresponding letter for equivariant K-theory.

2. BACKGROUND

In this section, we assemble for the convenience of the reader brief accounts of various definitions and constructions used in later sections.

2.1. **Polynomial loops and a filtration on ΩG.** Following [27], we define the space of **polynomial based loops** $\Omega_{\mathrm{poly}} U(n)$ as the set of maps $S^1 \to U(n)$ based at the identity which can be expressed as polynomials in z (here z is the parameter on the circle S^1). More precisely, we define
(2.1)

$$\Omega_{\mathrm{poly},r} U(n) := \left\{ f : S^1 \to U(n) \,\middle|\, f(1) = \mathbf{1}_{n \times n}, \ f = \sum_{j=-r(n-1)}^{r} a_j z^j, a_j \in M(n \times n, \mathbb{C}) \right\}$$

for a positive integer r, where $\mathbf{1}_{n \times n}$ denotes the identity matrix and $M(n \times n, \mathbb{C})$ is the space of $n \times n$ complex matrices. Note that the a_j are constant matrices, and $f(z)$ is required to

be unitary (in particular invertible) for all $z \in S^1$. An element in $\Omega_{\mathrm{poly},r}U(n)$ may also be viewed as an element of $\Omega_{\mathrm{poly},r'}U(n)$ for any $r' > r$; via these natural inclusions we now define

$$\Omega_{\mathrm{poly}}U(n) := \bigcup_{r=0}^{\infty} \Omega_{\mathrm{poly},r}U(n).$$

We define $\Omega_{\mathrm{poly}}H \subseteq \Omega_{\mathrm{poly}}U(n)$ for any subgroup $H \subseteq U(n)$ by requiring the images $f(z)$ of an element $f \in \Omega_{\mathrm{poly}}U(n)$ to lie in H for all $z \in S^1$.

By definition, the space $\Omega_{\mathrm{poly}}H$ comes equipped with a filtration given by the successive subspaces $\Omega_{\mathrm{poly},r}H$. For the case $H = SU(2)$, we write F_{2r} for $\Omega_{\mathrm{poly},r}SU(2)$; the motivation for this notation comes from the fact that there exists a subspace of the Grassmannian, denoted F_{2r} in [10], which is $SU(2)$-homotopy equivalent to $\Omega_{\mathrm{poly},r}SU(2)$ [10, Thm 2.9(3), Equation (3.1)].

Remark 2.1. *The reason we index our filtration by only the* even *integers is related to the need to create a $SU(2)$-equivariant filtration, as discussed in detail in [10]. It is related to the fact that $\Omega_{\mathrm{poly},r}SU(2)$ contains Laurent polynomials whose degrees run from $-r$ to r. An analogous non-equivariant filtration indexed by* all *integers is discussed by Pressley and Segal in [27]. The distinction will not be important for the present paper, so we do not discuss this further.*

Finally, note that matrix multiplication induces a map

(2.2) $$F_{2j} \times F_{2k} \to F_{2(j+k)}.$$

This map will be crucial to our constructions below.

Remark 2.2. *In [10, Theorems 5.3-4] it is shown that there is an $SU(2)$-homotopy equivalence between $\Omega_{\mathrm{poly}}SU(2)$ and $\Omega SU(2)$.*

2.2. **The equivariant Thom space and the equivariant Thom class in K-theory.** Let H be a compact Lie group, and let ξ be an n-dimensional complex vector bundle over a base space X equipped with an action of H. Let $p : E(\xi) \to X$ denote the bundle projection and let $\tilde{p} := p(\mathbb{P}(\xi)) : \mathbb{P}(\xi) \to X$ denote the associated projective bundle with fiber $\mathbb{P}^{n-1}$. We let γ_ξ denote the canonical line bundle over $\mathbb{P}(\xi)$. We have the following [2].

Lemma 2.3. *Let H and ξ be as above. Then there exists an H-bundle β_ξ such that $\tilde{p}^*(\xi) \cong \gamma_\xi \oplus \beta_\xi$ as H-equivariant bundles.*

Proof. The bundle γ_ξ is naturally an H-subbundle of $\tilde{p}^*(\xi)$, so by using a choice of H-invariant Riemannian metric, we can define β_ξ to be the orthogonal complement of γ_ξ. This β_ξ has the required property. $\square$

We next recall a convenient description of the (equivariant) Thom space of a bundle, following Atiyah [2, p.100]. Although Atiyah does not explicitly say so, if ξ is an H-bundle, then all the maps in his description are H-equivariant.

Proposition 2.4. *Let H be a compact Lie group and let ξ be a finite-dimensional complex vector bundle over a base space X equipped with an action of H. Let ϵ denote the trivial complex line bundle over X equipped with the trivial action on the fibers. Then $\mathrm{Thom}(\xi) \cong_H \mathbb{P}(\xi \oplus \epsilon)/\mathbb{P}(\xi)$.*

Proof. By definition, $\mathrm{Thom}(\xi)$ is $D(\xi)/S(\xi)$ where $D(\xi)$ and $S(\xi)$ denote the disk and sphere bundles associated to ξ respectively. Note that we can identify the disk bundle as

$$D(\xi) \cong \{tv \mid v \in S(\xi), t \in [0,1]\}$$

while we can identify

$$\mathbb{P}(\xi \oplus \epsilon) \cong \{(u,w) \mid u \in D(\xi), w \in \mathbb{C}, |u|^2 + |w|^2 = 1\}/\sim$$

where the equivalence relation $\sim$ is given by: $(\lambda u, \lambda w) \sim (u,w)$ for $\lambda \in S^1$. With these identifications, consider the map $D(\xi) \to \mathbb{P}(\xi \oplus \epsilon)$ given by $tv \mapsto [tv, \sqrt{1-t^2}]$. This map is H-equivariant, and it restricts to an H-equivariant map $S(\xi) \to \mathbb{P}(\xi)$ by taking $t = 1$. Therefore the induced map $\mathbb{P}(\xi \oplus \epsilon) \to \mathrm{Thom}(\xi)$ in the pushout diagram

(2.3)
$$\begin{array}{ccc}
S(\xi) & \longrightarrow & \mathbb{P}(\xi) \\
\downarrow & & \downarrow{\scriptstyle j} \\
D(\xi) & \longrightarrow & \mathbb{P}(\xi \oplus \epsilon) \\
\downarrow & & \downarrow \\
\mathrm{Thom}(\xi) & = & \mathrm{Thom}(\xi)
\end{array}$$

is H-equivariant. $\qquad\square$

Under the assumption that X and BH have cells only in even degree (which will be satisfied by the spaces we study), from Proposition 2.4 it follows that we have short exact sequences

(2.4)
$$0 \to \tilde{Y}^*\big(\mathrm{Thom}(\xi)\big) \to \tilde{Y}^*\big(\mathbb{P}(\xi \oplus \epsilon)\big) \xrightarrow{\ j^*\ } \tilde{Y}^*\big(\mathbb{P}(\xi)\big) \to 0$$

where Y^* here denotes any of the following generalized (equivariant) cohomology theories: K_H^*; H_H^*; K^* and H^*. The following is immediate and we will henceforth always use this identification.

Lemma 2.5. *Let H and ξ be as above. Then $\tilde{Y}^*\big(\mathrm{Thom}(\xi)\big) \cong \mathrm{Ker}\, j^*$ via the sequence* (2.4). $\qquad\square$

We now specialize the discussion to equivariant K-theory. In this setting the equivariant Thom class $U_\xi \in \tilde{K}_H^*(\mathrm{Thom}(\xi))$ of ξ can be described as follows [2, p.102-103]. We have

(2.5)
$$U_\xi = \lambda\big(\gamma_{\xi \oplus \epsilon}^* \otimes \bar{p}^*(\xi)\big) \in \mathrm{Ker}\, j^* \cong \tilde{K}_H^*\big(\mathrm{Thom}(\xi)\big)$$

where $\bar{p} : \mathbb{P}(\zeta \oplus \epsilon) \to X$ is the bundle projection, λ is the exterior bundle and we identify $\tilde{K}_H^*(\mathrm{Thom}(\xi)) \cong \mathrm{Ker}\, j^*$ as in the lemma above.

Let ζ be a rank n complex vector bundle with base space X, equipped with an action of H. Consider the following diagram

(2.6)

$$
\begin{array}{ccccc}
\mathbb{P}^{n-1} & \longrightarrow & \mathbb{P}(\zeta) & \xrightarrow{\tilde{p}} & X \\
\downarrow & & \downarrow{\scriptstyle j} & & \parallel \\
\mathbb{P}^n & \xrightarrow{k} & \mathbb{P}(\zeta \oplus \epsilon) & \xrightarrow{\bar{p}} & X \\
& & \downarrow & & \\
& & \mathrm{Thom}(\zeta) & &
\end{array}
$$

where $\tilde{p} : \mathbb{P}(\zeta) \to X$ and $\bar{p} : \mathbb{P}(\zeta \oplus \epsilon) \to X$ denote the bundle projections and

$$j : \mathbb{P}(\zeta) \hookrightarrow \mathbb{P}(\zeta \oplus \epsilon) \qquad \text{and} \qquad k : \mathbb{P}^n \hookrightarrow \mathbb{P}(\zeta \oplus \epsilon)$$

denote the natural inclusions. We will use this diagram repeatedly.

Remark 2.6. *In order for the inclusion $k : \mathbb{P}^1 \hookrightarrow \mathbb{P}(\zeta \oplus \epsilon)$ to be an H-equivariant map, the image $\bar{p}(k(\mathbb{P}^1)) \in X$ must be an H-fixed point. In some of our applications of the above diagram, the space X will not have an H-fixed point. In these cases, the map k is only a map of topological spaces, since the fiber has no H-action.*

Applying the argument for Lemma 2.3 to $\zeta \oplus \epsilon$ and using the fact that $\bar{p}^*(\zeta \oplus \epsilon) \cong \bar{p}^*(\zeta) \oplus \epsilon$, we obtain the following.

Lemma 2.7. *Let H and ζ be as above. Then there exists a complex line bundle $\beta_{\zeta \oplus \epsilon}$ over $\mathbb{P}(\zeta \oplus \epsilon)$ equipped with an action of H such that*

(2.7) $$\bar{p}^*(\zeta) \oplus \epsilon \cong \gamma_{\zeta \oplus \epsilon} \oplus \beta_{\zeta \oplus \epsilon}$$

as H-vector bundles. $\square$

In what follows, we will often use the above results for the special case in which ζ is a complex line bundle, i.e. $n = 1$, so we take a moment to discuss this case further. In this special case, notice that $\tilde{p}$ is the identity map, since $\mathbb{P}^{n-1} = \mathbb{P}^0$ is a point. This means the composition $\bar{p} \circ j$ can be identified with the identity map on X, in turn implying that the map j^* splits. We conclude that there exists a direct sum decomposition

$$\tilde{K}_H^*\big(\mathbb{P}(\zeta \oplus \epsilon)\big) \cong \tilde{K}_H^*\big(\mathrm{Thom}(\zeta)\big) \oplus \tilde{K}_H^*(X \cong \mathbb{P}(\zeta))$$

where again by Lemma 2.5 we identify $\tilde{K}_H^*\big(\mathrm{Thom}(\zeta)\big) \cong \mathrm{Ker}\, j^*$.

Using the description of the equivariant Thom class in (2.5), we now give a concrete computation of U_ζ for an H-equivariant line bundle ζ.

Lemma 2.8. *For H and ζ as above, the H-equivariant K-theoretic Thom class U_ζ is given by*

$$U_\zeta = 1 - \beta_{\zeta \oplus \epsilon} \in \mathrm{Ker}\, j^* \cong \tilde{K}_H^*(\mathrm{Thom}(\zeta)).$$

Proof. By (2.5) we have that

$$U_\zeta = \lambda(\gamma_{\zeta \oplus \epsilon}^* \otimes \bar{p}^*(\zeta)).$$

Recall that H-equivariant line bundles are classified by their H-equivariant first Chern class. The definition (2.7) of $\beta_{\zeta \oplus \epsilon}$ implies that

$$c_1^H(\bar{p}^*(\zeta)) = c_1^H(\gamma_{\zeta \oplus \epsilon}) + c_1^H(\beta_{\zeta \oplus \epsilon})$$

which in turn implies $\beta_{\zeta \oplus \epsilon} \cong \gamma_{\zeta \oplus \epsilon}^* \otimes \bar{p}^*(\zeta)$. Thus $U_\zeta = \lambda(\beta_{\zeta \oplus \epsilon})$, and since $\beta_{\zeta \oplus \epsilon}$ is a line bundle, its exterior power is simply $1 - \beta_{\zeta \oplus \epsilon}$, as desired. $\qquad \square$

Remark 2.9. *Note that $1 - \beta_{\zeta \oplus \epsilon}$ can be checked to lie in $\mathrm{Ker}\, j^*$ by the following direct argument. By definition of the tautological line bundle, it is straightforward to see that $j^*(\gamma_{\zeta \oplus \epsilon}) = \gamma_\zeta \cong \zeta$ where the final isomorphism follows because $\bar{p} \circ j = \tilde{p}$ can be identified with the identity map. The defining equation (2.7) then implies*

$$\begin{aligned}
(2.8) \qquad \zeta \oplus \epsilon &\cong j^* \bar{p}^*(\zeta) \oplus \epsilon \\
&\cong j^*(\gamma_{\zeta \oplus \epsilon}) \oplus j^*(\beta_{\zeta \oplus \epsilon}) \\
&\cong \zeta \oplus j^*(\beta_{\zeta \oplus \epsilon})
\end{aligned}$$

which in turn implies $j^*(1 - \beta_{\zeta \oplus \epsilon}) = 0 \in \tilde{K}_H^*(\mathbb{P}(\zeta))$.

2.3. **The bundle τ.** We now briefly recall the definition and some of the properties of a bundle τ discussed in [10]. The main reason for its appearance in this paper is Proposition 2.14.

Let $G = SU(2)$. Let γ denote the tautological bundle over $\mathbb{P}^1$ and let $\perp$ denote the orthogonal complement with respect to the standard metric in $\mathbb{C}^2$. We have the following [10, Definition 3.1].

Definition 2.10. We define τ to be the G-equivariant complex line bundle over $\mathbb{P}^1$ with total space

$$E(\tau) = \big\{(u, v) \mid u \in S^3 \subset \mathbb{C}^2, v \in (u^\perp)\big\} / \sim$$

where the equivalence relation is given by $(u, v) \sim (\zeta u, \zeta v)$ for $\zeta \in S^1$ and the bundle projection to $\mathbb{P}^1$ is given by taking the first factor.

The bundle τ can in fact be identified as the tangent bundle to $\mathbb{P}^1$ [10, Proposition 3.2].

Proposition 2.11. *The bundle τ of Definition 2.10 is G-equivariantly isomorphic to the tangent bundle $T\mathbb{P}^1$ of $\mathbb{P}^1$.* $\qquad \square$

The following is immediate.

Corollary 2.12. *The G-equivariant first Chern class of τ is given by $c_1^G(\tau) = -2c_1^G(\gamma)$. In particular, $\tau \cong_G \gamma^{-2} \cong_G (\gamma^*)^2$.*

Proof. The corresponding statement for $T\mathbb{P}^1$ is [24, Theorem 14.10], where the a in the statement of that theorem is identified in its proof as $c_1(\gamma)$. (Here we used the fact that $H_G^2(X) \cong H^2(X)$ since $G = SU(2)$ is simply connected. See the discussion below Remark 4.1.) $\qquad \square$

Remark 2.13. *The proof of [24, Theorem 14.10], to which we refer in the above proof, contains the assertion that $\tau \oplus \epsilon \cong 2\gamma^*$ as topological bundles, where γ^* denotes the dual bundle to γ. The corresponding G-equivariant statement is as follows. Consider the standard action of $G = SU(2)$ on $\mathbb{P}^1$ and $\mathbb{C}^2$ respectively. Equip $\mathbb{P}^1 \times \mathbb{C}$ with the diagonal $SU(2)$-action. Projection to the first factor $p : \mathbb{P}^1 \times \mathbb{C}^2 \to \mathbb{P}^1$ makes this a G-equivariant bundle which is topologically trivial but G-equivariantly non-trivial. We denote this rank 2 G-bundle by v, and by slight abuse of language,*

call it the standard 2-dimensional representation of G. The G-equivariant analogue of the non-equivariant statement $\tau \oplus \epsilon \cong 2\gamma^*$ above is then

$$(2.9) \qquad\qquad \tau \oplus \epsilon \cong v \otimes \gamma^*.$$

The bundle τ is an essential ingredient in our description of the filtration quotient $\frac{F_{2r}}{F_{2(r-1)}}$, where the F_{2r}'s are the subspaces of $\Omega_{\text{poly}} G$ introduced in Section 2.1. The following is [10, Proposition 3.4].

Proposition 2.14. . *Let $r \in \mathbb{Z}$ and $r \geq 0$. The quotient space F_{2r}/F_{2r-2} is G-equivariantly homeomorphic to $\text{Thom}(\tau^{2r-1})$. In particular, $F_2 \cong \text{Thom}(\tau)$.*

Remark 2.15. *It also follows from the proof of [10, Proposition 3.4] that, under the identification $F_2 \cong \text{Thom}(\tau)$, the 'point at infinity' of the Thom space gets identified with the subspace F_0 which is a single point consisting of the constant map at the identity in $F_2 = \Omega_{\text{poly},1} SU(2)$.*

2.4. **Equivariant Bott periodicity.** Here we set some notation and recall a special case of the equivariant Bott periodicity theorem, which will be used in Section 3. A reference for this section is [4].

Recall that the classical (non-equivariant) Bott periodicity theorem relates the K-theory of the Cartesian product $X \times S^2 \cong X \times \mathbb{P}^1$ with that of X. In the equivariant case, the analogous theorem relates the equivariant K-theory of $\mathbb{P}(\xi \oplus \epsilon)$ with that of X, where ξ is an equivariant bundle over X. More specifically, let H be a compact Lie group. Let ξ be a complex vector bundle over a base space X equipped with an action of H. When no confusion will arise, we denote by ξ the class $[\xi]$ in $K_H^*(X)$. In fact, we will consider the special case in which $\xi = \zeta \oplus \epsilon$, where ζ is a complex line bundle equipped with an action of H. We already saw in Section 2.2 that, in this situation,

$$\gamma^*_{\zeta \oplus \epsilon} \otimes \bar{p}^*(\zeta) \cong \beta_{\zeta \oplus \epsilon}$$

or equivalently

$$\gamma_{\zeta \oplus \epsilon} \cong \bar{p}^*(\zeta) \otimes \beta^*_{\zeta \oplus \epsilon}$$

where $\beta_{\zeta \oplus \epsilon}$ is the (line) bundle found in Lemma 2.7. Recalling that $\beta^*_{\zeta \oplus \epsilon} \cong \beta^{-1}_{\zeta \oplus \epsilon}$ for line bundles, the defining equation (2.7) becomes

$$(2.10) \qquad\qquad \bar{p}^*(\zeta) \oplus \epsilon \cong (\bar{p}^*(\zeta) \otimes \beta^{-1}_{\zeta \oplus \epsilon}) \oplus \beta_{\zeta \oplus \epsilon}.$$

This implies the equality

$$\bar{p}^*(\zeta) + 1 = \bar{p}^*(\zeta)\beta^{-1}_{\zeta \oplus \epsilon} + \beta_{\zeta \oplus \epsilon}$$

in $K_H^*(\mathbb{P}(\zeta \oplus \epsilon))$. Some simple manipulation then yields the relation

$$(\beta_{\zeta \oplus \epsilon} - \bar{p}^*(\zeta))(\beta_{\zeta \oplus \epsilon} - 1) = 0$$

in $K_H^*(\mathbb{P}(\zeta \oplus \epsilon))$, or equivalently,

$$(\gamma_{\zeta \oplus \epsilon} - 1)(\gamma_{\zeta \oplus \epsilon} - \bar{p}^*(\zeta)) = 0.$$

We have just explicitly derived a relation in $K_H^*(\mathbb{P}(\zeta \oplus \epsilon))$, but in fact, the equivariant Bott periodicity theorem (cf. [4, Theorem 2.7.1] or [8, Theorem 3.2]), applied to this case, states that this is in fact the only one. More precisely, we have the following.

Theorem 2.16. *Let H be a compact Lie group and let ζ be an H-equivariant complex line bundle over a base space X. Let $\gamma_{\zeta \oplus \epsilon}, \beta_{\zeta \oplus \epsilon}$ be as above. Then*

$$K_H^*\big(\mathbb{P}(\zeta \oplus \epsilon)\big) \cong \frac{K_H^*(X)[\gamma_{\zeta \oplus \epsilon}]}{(\gamma_{\zeta \oplus \epsilon} - 1)(\gamma_{\zeta \oplus \epsilon} - \bar{p}^*(\zeta))} \cong \frac{K_H^*(X)[\beta_{\zeta \oplus \epsilon}]}{(\beta_{\zeta \oplus \epsilon} - \bar{p}^*(\zeta))(\beta_{\zeta \oplus \epsilon} - 1)}$$

as $R(H)$ algebras.

We will also use the following (cf. [4, Theorem 6.1.4] or [8, Theorem 3.1]), which is a special case of the equivariant Thom isomorphism theorem.

Theorem 2.17. *Let H be a compact Lie group and let ζ be an H-equivariant complex line bundle over a base space X. Let $U = U_\zeta$ denote the equivariant Thom class associated to ζ as in* (2.5)*. Then the multiplication map $x \mapsto U \cdot \bar{p}^*(x)$ defines an isomorphism*

$$K_H(X) \cong U \cdot \bar{p}^*\big(K_H(X)\big) \cong \operatorname{Ker} j^* \cong \tilde{K}_H\big(\operatorname{Thom}(\zeta)\big).$$

3. THE EQUIVARIANT COHOMOLOGY AND EQUIVARIANT K-THEORY OF $(\mathbb{P}^1)^n$

The main purpose of this section is two-fold. First, we define a G-equivariant continuous map $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$. Second, we give convenient presentations of both $K_T^*((\mathbb{P}^1)^n)$ (respectively $K_G^*((\mathbb{P}^1)^n)$) and $H_T^*((\mathbb{P}^1)^n; \mathbb{Q})$ (respectively $H_G^*((\mathbb{P}^1)^n; \mathbb{Q})$) via generators and relations. In later sections, we show that the pullback maps

$$\Phi_{2r}^* : H_T^*(F_{2r}; \mathbb{Q}) \to H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$$

(respectively from $H_G^*(F_{2r}; \mathbb{Q})$ to $H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$) and

$$\Phi_{2r}^* : K_T^*(F_{2r}) \to K_T^*((\mathbb{P}^1)^{2r})$$

(respectively from $K_G^*(F_{2r})$ to $K_G^*((\mathbb{P}^1)^{2r})$) are injective, and use the concrete presentations of the codomains given here, in order to give explicit computations of $H_G^*(F_{2r}; \mathbb{Q})$ and, most importantly, $K_G^*(F_{2r})$. This is a key step towards our main goal, which is the computation of the algebra structure of $K_G^*(\Omega G)$.

We begin with the construction of the maps Φ_{2r}. Let τ be the $G = SU(2)$-equivariant bundle of Definition 2.10. We begin with an equivariant identification of the G-spaces $\mathbb{P}(\tau \oplus \epsilon)$ and $\mathbb{P}^1 \times \mathbb{P}^1$, where G acts on $\mathbb{P}^1 \times \mathbb{P}^1$ via the diagonal action. (The non-equivariant version of the statement below is of course standard. We choose to record the equivariant version since we did not find a reference in the literature.) We prove the lemma by constructing an explicit homeomorphism which can easily be seen to be G-equivariant.

Lemma 3.1. *There exists a G-equivariant homeomorphism $\Theta : \mathbb{P}(\tau \oplus \epsilon) \to \mathbb{P}^1 \times \mathbb{P}^1$. Under this homeomorphism, the subspace $\mathbb{P}(\tau)$ corresponds to the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$, and the bundle projection $\mathbb{P}(\tau \oplus \epsilon) \to \mathbb{P}^1$ corresponds to the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor.*

Proof. Recall that the center $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ of $SU(2)$ acts trivially on both $\mathbb{P}(\tau \oplus \epsilon)$ and $\mathbb{P}^1 \times \mathbb{P}^1$, hence the $SU(2)$ action factors through an $SO(3)$-action. It is well-known that the tangent bundle τ to $\mathbb{P}^1$ is $SO(3)$-equivariantly homeomorphic to the tangent bundle TS^2 of the unit sphere S^2 in $\mathbb{R}^3$. Here we think of $\mathbb{R}^3$ as the Lie algebra of $SU(2)$ equipped with an invariant metric and the action of $SO(3)$ on TS^2 is induced from the standard action of $SO(3)$ on $\mathbb{R}^3$. Similarly $\mathbb{P}^1 \times \mathbb{P}^1$ is equivariantly homeomorphic to $S^2 \times S^2$ equipped with the standard diagonal action of $SO(3)$.

Thus, to prove the claim it suffices to exhibit an $SO(3)$-equivariant homeomorphism between $\mathbb{P}(TS^2 \oplus \epsilon)$ and $S^2 \times S^2$. Denote by Δ the diagonal in $S^2 \times S^2$ and recall that the space $\mathbb{P}(TS^2 \oplus \epsilon) \smallsetminus \mathbb{P}(TS^2)$ can be equivariantly identified with (the total space of) TS^2 (see diagram (2.3). For $p \in S^2$ denote by $\operatorname{St}_p : S^2 \smallsetminus \{p\} \to \langle p \rangle^\perp$ the usual stereographic projection from p to the real 2-plane orthogonal to p. Consider the map

(3.1) $$(S^2 \times S^2) \smallsetminus \Delta \to TS^2, \quad (p, q) \mapsto \big(p, \operatorname{St}_p(q)\big)$$

where we think of TS^2 as pairs (x, y) in $\mathbb{R}^3$ such that $x \in S^2$ and y lies in the plane orthogonal to x. For $g \in SO(3)$, note that the usual stereographic projection satisfies $\mathrm{St}_{gp}(gq) = g\mathrm{St}_p(q)$, so the map (3.1) is $SO(3)$-equivariant, and since each St_p is a homeomorphism, it follows that (3.1) is also a homeomorphism.

The space $\mathbb{P}(TS^2 \oplus \epsilon)$ is obtained from the bundle TS^2 by taking the one-point compactification of each fiber. Denote by ∞_p the point at infinity in the fiber of $\mathbb{P}(TS^2 \oplus \epsilon)$ over p. The $SO(3)$ action fixes each ∞_p, and it is straightforward to check that the map (3.1) can be equivariantly and continuously extended to a map

$$S^2 \times S^2 \to \mathbb{P}(TS^2 \oplus \epsilon)$$

by defining $(p, p) \mapsto (p, \infty_p)$ for each $p \in S^2$. This extension is evidently a bijection. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism, so we have constructed the desired G-homeomorphism. The two properties of this homeomorphism specified in the last sentence of the statement of the lemma follow from the construction. $\square$

We now define a G-equivariant continuous map $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ as follows. First, recall that from Proposition 2.14 we know that $F_2/F_0 \cong F_2$ is G-equivariantly homeomorphic to the Thom space $\mathrm{Thom}(\tau)$. Second, we know $\mathbb{P}(\tau \oplus \epsilon)/\mathbb{P}(\tau) \cong_G \mathrm{Thom}(\tau)$ by Proposition 2.4. Putting these together with the equivariant homeomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \cong_G \mathbb{P}(\tau \oplus \epsilon)$ given in Lemma 3.1 we obtain the map Φ_{2r}. More precisely, we have the following.

Definition 3.2. Let $\Phi_{2r} : (\mathbb{P}^1)^{2r} \cong (\mathbb{P}^1 \times \mathbb{P}^1)^r \to F_{2r}$ denote the composition

$$(3.2) \qquad (\mathbb{P}^1 \times \mathbb{P}^1)^r \longrightarrow (\mathbb{P}(\tau \oplus \epsilon))^r \longrightarrow \mathrm{Thom}(\tau)^r \longrightarrow (F_2)^r \longrightarrow F_{2r}$$

where the first map is (the r-fold product of) the map Θ^{-1} from Lemma 3.1, the second is the (r-fold product of) the quotient map $\mathbb{P}(\tau \oplus \epsilon) \to \mathbb{P}(\tau \oplus \epsilon)/\mathbb{P}(\tau)$ together with the identification in Proposition 2.4, the third is the (r-fold product of the) map from Proposition 2.14, and the final arrow is induced from matrix multiplication as in (2.2).

Since all the maps in the definition are continuous, the composition Φ_{2r} is also continuous. The last arrow is G-equivariant since the action of G on $\Omega_{\mathrm{poly}}G$ is by conjugation and the map is induced by multiplication. All the other maps are already known to be equivariant, so the composition Φ_{2r} is also G-equivariant.

We now turn our attention to a computation of $K_T^*((\mathbb{P}^1)^n)$ (respectively $K_G^*((\mathbb{P}^1)^n)$) and $H_T^*((\mathbb{P}^1)^n; \mathbb{Q})$ (respectively $H_G^*((\mathbb{P}^1)^n; \mathbb{Q})$) for any positive integer n. We accomplish this by an inductive argument that computes $K_T^*(X \times \mathbb{P}^1)$ (respectively $H_T^*(X \times \mathbb{P}^1; \mathbb{Q})$) in terms of $K_T^*(X)$ (respectively $H_T^*(X; \mathbb{Q})$) for a general G-space X, using equivariant Bott periodicity. Computations of K_G^* and H_G^* then follow by taking Weyl invariants.

Let X be a G-space. By restricting the action, X is then also a T-space. Recall that $b \in R(T) \cong K_T(\mathrm{pt})$ denotes the standard 1-dimensional representation of $T = S^1$ of weight 1 and that v denotes the standard representation of $G = SU(2)$ on $\mathbb{C}^2$. As a T-representation, $v \cong b \oplus b^{-1}$. Recall also that any T-representation ρ may also be viewed as an T-equivariant bundle over a point pt. Let ρ_X denote the pullback of ρ to X via the constant map $X \to \mathrm{pt}$. This is a topologically trivial, but equivariantly non-trivial, bundle over X. (Indeed, it is this map $R(T) \to K_T^*(X), \rho \mapsto \rho_X$, which defines the $R(T)$-module structure on $K_T^*(X)$. By slight abuse of notation, when considering ρ_X as an element in $K_T^*(X)$, we will sometimes denote it simply by ρ.) In particular, the total space $E(\rho_X)$ of ρ_X is simply $X \times E(\rho)$ equipped with the diagonal T-action. Similarly, $\mathbb{P}(\rho_X) \cong_T X \times \mathbb{P}(\rho)$. In the case $\rho = b \oplus b^{-1}$,

we identify $\mathbb{P}\big((b \oplus b^{-1})_X\big)$ as a T-space with $X \times \mathbb{P}^1$ where the action of T on $\mathbb{P}^1$ is the standard one. Since $[zx : z^{-1}y] = [z^{k+1}x : z^{k-1}y] \in \mathbb{P}^1$ for any $z \in T$, we have

$$(3.3) \qquad \mathbb{P}(b^2 \oplus \epsilon) \cong \mathbb{P}(b \oplus b^{-1}) \cong \mathbb{P}^1$$

as T-spaces.

We now apply the equivariant Bott periodicity results discussed in Section 2.4 to the case where $H = T$ and ζ is the bundle $b_X^2 = b_X \otimes b_X$ over X, making use of the fact that $X \times \mathbb{P}^1 \cong_T \mathbb{P}(b_X^2 \oplus \epsilon)$. In the following, we denote by b the image of $b \in R(T)$ under the map $R(T) \to K_T^*\big(\mathbb{P}(b_{\mathbb{P}(b_X^2 \oplus \epsilon)}^2 \oplus \epsilon)\big)$. For the discussion it is useful to define the bundle

$$(3.4) \qquad \mathcal{L}_X := b^{-1} \otimes \beta_{b_X^2 \oplus \epsilon}$$

over $\mathbb{P}(b_X^2 \oplus \epsilon)$.

We begin with the following.

Lemma 3.3. *The bundle $\mathcal{L}_X$ of (3.4) has the property*

$$(3.5) \qquad \mathcal{L}_X^2 \oplus \epsilon \cong (b \otimes \mathcal{L}_X) \oplus (b^{-1} \otimes \mathcal{L}_X).$$

In $K_T^*\big(\mathbb{P}(b_X^2 \oplus \epsilon)\big)$, the (equivariant K-theory class of the) bundle $\mathcal{L}_X$ satisfies the relation

$$(3.6) \qquad (\mathcal{L}_X - b)(\mathcal{L}_X - b^{-1}) = 0$$

Proof. Applying (2.10) to our situation yields $b^2 \oplus \epsilon \cong (b^2 \otimes \beta_{b_X^2 \oplus \epsilon}^{-1}) \oplus \beta_{b_X^2 \oplus \epsilon}$. Tensoring with $b^{-2} \otimes \beta_{b_X^2 \oplus \epsilon}$ yields (3.5). Theorem 2.16 also states that

$$(\beta_{b_X^2 \oplus \epsilon} - b)(\beta_{b_X^2 \oplus \epsilon} - 1) = 0$$

in $K_T\big(\mathbb{P}(b_X^2 \oplus \epsilon)\big)$. Dividing by b^2 gives (3.6). $\qquad\square$

Remark 3.4. *Dividing (3.6) by $\mathcal{L}_X^2$ shows that $\mathcal{L}_X^{-1}$ satisfies the same relation.*

We next prove that the bundle $\mathcal{L}_X$, considered there as a T-bundle, is in fact naturally a G-bundle. In particular, the T-equivariant K-theory class of $\mathcal{L}_X$ is in the image of the forgetful map $K_G^*(\mathbb{P}(b_X^2 \oplus \epsilon)) \to K_T^*(\mathbb{P}(b_X^2 \oplus \epsilon))$. This observation will be useful when we compute $K_G^*(\mathbb{P}(b_X^2 \oplus \epsilon))$ using the results for $K_T^*(\mathbb{P}(b_X^2 \oplus \epsilon))$ (see Lemma 3.7 below).

Lemma 3.5. *Let $\pi : \mathbb{P}(b_X^2 \oplus \epsilon) \to \mathbb{P}(b^2 \oplus \epsilon) \cong \mathbb{P}^1$ denote the natural G-equivariant projection and let γ denote the tautological line bundle over $\mathbb{P}^1$. Then $\mathcal{L}_X \cong \pi^*(\gamma^{-1})$. In particular, since γ (and hence γ^{-1}) is a G-bundle, $\mathcal{L}_X$ is also a G-bundle.*

Proof. From the defining equation (2.7) of $\beta_{b_X^2 \oplus \epsilon}$ together with the definition (3.4) of $\mathcal{L}_X$, it can be deduced that

$$(3.7) \qquad b \oplus b^{-1} \cong \left(b^{-1} \otimes \gamma_{b_X^2 \oplus \epsilon}\right) \oplus \mathcal{L}_X.$$

Consider the special case $X = \mathrm{pt}$ where $\mathbb{P}(b_X^2 \oplus \epsilon) = \mathbb{P}(b^2 \oplus \epsilon) \cong \mathbb{P}(b \oplus b^{-1}) \cong \mathbb{P}^1$ as in (3.3). In addition, since the G-representation v restricts to the T-representation $b \oplus b^{-1}$, we may identify $\mathbb{P}(b^2 \oplus \epsilon)$ with $\mathbb{P}(v)$. In particular, it is a G-space. Moreover, in this setting the T-bundle isomorphism (3.7) is the restriction to T of the G-bundle isomorphism

$$v_{\mathbb{P}(v)} \cong \gamma_v \oplus \beta_v$$

where the notation is as in Lemma 2.3 and $\gamma_v \cong \gamma$ (under the identification $\mathbb{P}(v) \cong \mathbb{P}^1$). From this it follows that the bundle $\mathcal{L}_{\mathrm{pt}}$ is equivariantly isomorphic to the G-bundle β_v. A Chern class computation shows that $\beta_v \cong \gamma^{-1}$, so $\mathcal{L}_{\mathrm{pt}} \cong \gamma^{-1}$. This proves the result for the

case $X = \mathrm{pt}$. On the other hand, the bundle $\mathcal{L}_X$ for a general space X is $\mathcal{L}_X = \pi^*(\mathcal{L}_{\mathrm{pt}}) \cong \pi^*(\gamma^{-1})$, as desired. $\qquad \square$

Motivated by the above lemma, we formulate below a version of Theorem 2.16 for our situation, using the variable $\mathcal{L}_X$ instead of $\beta_{b_X^2 \oplus \epsilon}$.

Lemma 3.6. *Let X be a $G = SU(2)$-space (hence also a T-space). Let T act on $\mathbb{P}^1$ in the standard way. Then*

$$(3.8) \qquad K_T^*\big(X \times \mathbb{P}^1\big) \cong \frac{K_T^*(X)[\mathcal{L}_X]}{(\mathcal{L}_X - b)(\mathcal{L}_X - b^{-1})} \cong \frac{K_T^*(X)[\mathcal{L}_X]}{(\mathcal{L}_X^2 - (b + b^{-1})\mathcal{L}_X + 1)}.$$

Lemma 3.6 computes $K_T^*(X \times \mathbb{P}^1)$ in terms of $K_T^*(X)$, but we also need an analogous computation for $K_G^*(X \times \mathbb{P}^1)$. Recall that the forgetful functor $F : K_G^*(Y) \to K_T^*(Y)$ for any G-space Y has image contained in the Weyl-invariants $K_T^*(Y)^W$, but F need not be surjective onto $K_T^*(Y)^W$ in general (i.e., it need not be the case that $K_G^*(Y) \cong K_T^*(Y)^W$). See [12, Example 4.8]. Nevertheless, we have the following.

Lemma 3.7. *Let X be a $G = SU(2)$-space. Let G act on $\mathbb{P}^1$ in the standard way. Assume that $K_G^*(X) \cong K_T^*(X)^W$. Then $K_G^*(X \times \mathbb{P}^1) \cong K_T^*(X \times \mathbb{P}^1)$, and*

$$K_G^*(X \times \mathbb{P}^1) \cong \frac{K_G^*(X)[\mathcal{L}_X]}{(\mathcal{L}_X^2 - v\mathcal{L}_X + 1)}.$$

Proof. The equivariant Thom isomorphism (Lemma 2.17) implies that $K_G^*(X)$ injects into $K_G^*(X \times \mathbb{P}^1)$. We showed in Lemma 3.5 that $\mathcal{L}_X$ lies in $K_G^*(X \times \mathbb{P}^1 \cong \mathbb{P}(b_X^2 \oplus \epsilon))$. If we make the assumption that $K_G^*(X) \cong K_T^*(X)^W$, taking Weyl invariants in (3.8) gives

$$K_T^*(X \times \mathbb{P}^1)^W \cong \frac{K_G^*(X)[\mathcal{L}_X]}{(\mathcal{L}_X^2 - (b + b^{-1})\mathcal{L}_X + 1)}$$

since $\mathcal{L}_X$ and $(b + b^{-1}) \cong v$ are W-invariant. Hence $K_T^*(X \times \mathbb{P}^1)^W$ is generated by $K_G^*(X)$ and $\mathcal{L}_X$, i.e., the restriction map

$$K_G^*(X \times \mathbb{P}^1) \to K_T^*(X \times \mathbb{P}^1)$$

is surjective and $K_G^*(X \times \mathbb{P}^1) \cong K_T^*(X \times \mathbb{P}^1)^W$. The result follows. $\qquad \square$

Next we record the analogous computations in cohomology. Since the ideas are similar we keep exposition brief. Let $\bar{b} = c_1^T(b) \in H_T^2(\mathrm{pt}) = H^2(BT; \mathbb{Z})$ denote the equivariant Chern class of $b \in R(T)$. Then $H_T^*(\mathrm{pt}; \mathbb{Z}) = \mathbb{Z}[\bar{b}]$ and $c_1^T(b^{-1}) = -\bar{b}$. The nontrivial element w of the Weyl group $W \cong S_2 \cong \mathbb{Z}/2\mathbb{Z}$ of G acts by $w(\bar{b}) = -\bar{b}$ and

$$H_G^*(\mathrm{pt}; \mathbb{Z}) = H^*(BG; \mathbb{Z}) = \big(H^*(BT; \mathbb{Z})\big)^W = \mathbb{Z}[\bar{t}],$$

where

$$(3.9) \qquad\qquad\qquad\qquad \bar{t} := \bar{b}^2.$$

As in the case of K-theory, there is a canonical ring homomorphism

$$H_T^*(\mathrm{pt}; \mathbb{Z}) = H^*(BT; \mathbb{Z}) \to H_T^*(X; \mathbb{Z})$$

(respectively $H_G^*(\mathrm{pt}; \mathbb{Z}) \to H_G^*(X; \mathbb{Z})$) for any T-space (respectively G-space) X, and by slight abuse of notation we will use the same symbol to denote an element in $H_T^*(\mathrm{pt}; \mathbb{Z})$ (respectively $H_G^*(\mathrm{pt}; \mathbb{Z})$) to denote its image in $H_T^*(X; \mathbb{Z})$ (respectively $H_G^*(X; \mathbb{Z})$).

Let $\bar{\mathcal{L}}_X$ denote the T-equivariant first Chern class of $\mathcal{L}_X$, i.e., $\bar{\mathcal{L}}_X := c_1^T(b_{\mathbb{P}(b_X^2 \oplus \epsilon)}^{-1} \otimes \beta_{b_X^2 \oplus \epsilon})$ in $H_T^2\big(\mathbb{P}(b_X^2 \oplus \epsilon); \mathbb{Q}\big)$.

Lemma 3.8. *Let $\bar{\mathcal{L}}_X$ and $\bar{t}$ be as above. Then $\bar{\mathcal{L}}_X^2 = \bar{t}$.*

Proof. The isomorphism (3.5) implies that $\mathcal{L}_X^2 \oplus \epsilon$ is a sum of a line bundle with a trivial bundle. Hence its 2nd equivariant Chern class is 0 and we have

$$
\begin{aligned}
0 &= c_2^T\big((b_{\mathbb{P}(b_X^2 \oplus \epsilon)} \oplus b_{\mathbb{P}(b_X^2 \oplus \epsilon)}^{-1}) \otimes \mathcal{L}_X\big) \\
&= c_1^T(b_{\mathbb{P}(b_X^2 \oplus \epsilon)} \otimes \mathcal{L}_X)c_1^T(b_{\mathbb{P}(b_X^2 \oplus \epsilon)}^{-1} \otimes \mathcal{L}_X) \\
&= (\bar{\mathcal{L}}_X + \bar{b})(\bar{\mathcal{L}}_X - \bar{b}) \\
&= \bar{\mathcal{L}}_X^2 - \bar{b}^2 \\
&= \bar{\mathcal{L}}_X^2 - \bar{t}
\end{aligned}
$$

(3.10)

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.9. *Lemma 3.8 is the cohomology analogue of* (3.6).

The equivariant Thom isomorphism in cohomology implies

$$
\tilde{H}_T^*\big(\mathbb{P}(b_X^2 \oplus \epsilon); \mathbb{Q}\big) \cong \bar{p}^*\big(H_T^*(X; \mathbb{Q})\big) \oplus \bar{U} \cdot \bar{p}^*\big(H_T^*(X; \mathbb{Q})\big) \tag{3.11}
$$

as $H_T^*(\mathrm{pt}; \mathbb{Q})$-modules, where $\bar{U} := \bar{U}_{b_X^2} := c_1^T(U_{b_X^2}) \in H_T^2\big(\mathbb{P}(\tilde{b}^2 \oplus \epsilon); \mathbb{Q}\big)$ is the equivariant Thom class in cohomology. Recalling from Lemma 2.8 that $U_{b_X^2} = 1 - \beta_{b_X^2 \oplus \epsilon}$, we obtain $\bar{U} = -\bar{\mathcal{L}}_X - \bar{b}$. Thus (3.11) can be rewritten as

$$
\tilde{H}_T^*\big(\mathbb{P}(b_X^2 \oplus \epsilon); \mathbb{Q}\big) \cong \bar{p}^*\big(\tilde{H}_T^*(X; \mathbb{Q})\big) \oplus \bar{\mathcal{L}}_X \cdot \bar{p}^*\big(H_T^*(X; \mathbb{Q})\big)
$$

as $H_T^*(\mathrm{pt}; \mathbb{Q})$-modules. Together with $\bar{L}^2 = \bar{t}$, this determines the $H_T^*(\mathrm{pt}; \mathbb{Q})$-algebra structure as follows.

Lemma 3.10. *With notation as above,*

$$
H_T^*\big(\mathbb{P}(\tilde{b}^2 \oplus \epsilon); \mathbb{Q}\big) \cong H_T^*(X; \mathbb{Q})[\bar{\mathcal{L}}_X]/\big(\bar{\mathcal{L}}_X^2 - \bar{t}\big).
$$

Taking Weyl invariants and using the fact that $H_G^*(X; \mathbb{Q}) \cong H_T^*(X; \mathbb{Q})^W$ for any G-space X, we also conclude the following.[1]

Lemma 3.11. *Let X be a G-space and let G act on $\mathbb{P}^1$ in the standard way. Then*

$$
H_G^*\big(X \times \mathbb{P}^1 \cong \mathbb{P}(b_X^2 \oplus \epsilon); \mathbb{Q}\big) \cong \Big(H_T^*\big(X \times \mathbb{P}^1\big); \mathbb{Q}\Big)^W \cong H_G^*(X; \mathbb{Q})[\bar{\mathcal{L}}_X]/\big(\bar{\mathcal{L}}_X^2 - \bar{t}\big)
$$

as $H_G^*(\mathrm{pt}; \mathbb{Q})$-algebras.

With the preceding lemmas in place, it is now straightforward to compute $K_T^*((\mathbb{P}^1)^n)$, $K_G^*((\mathbb{P}^1)^n)$, $H_T^*((\mathbb{P}^1)^n; \mathbb{Q})$, and $H_G^*((\mathbb{P}^1)^n; \mathbb{Q})$ by a simple inductive argument starting with $X = \mathrm{pt}$. Let $\pi_j : (\mathbb{P}^1)^n \to \mathbb{P}^1$ denote the projection to the j-th coordinate. Motivated by Lemma 3.5, we define the following collection of line bundles over $(\mathbb{P}^1)^n$.

Definition 3.12. *For any j with $1 \le j \le n$, define*

$$
L_j := \begin{cases} \pi_j^*(\gamma^{-1}) \text{ if } j \text{ is odd;} \\ \pi_j^*(\gamma) \text{ if } j \text{ is even.} \end{cases}
$$

We denote by $\bar{L}_j$ the first Chern class $c_1(L_j)$ of L_j in $H_T^2\big((\mathbb{P}^1)^n; \mathbb{Q}\big)$.

---

[1]We warn the reader that for cohomology with $\mathbb{Z}$ coefficients it is not always the case that $H_G^*(X; \mathbb{Z}) \cong H_T^*(X; \mathbb{Z})^W$. See [14].

We remark that all the bundles L_j in the above definition are equipped with a natural G-action (and hence also a T-action). In particular they may be viewed as elements of $K_T^*\big((\mathbb{P}^1)^n\big)$ or $K_G^*\big((\mathbb{P}^1)^n\big)$.

Using Lemmas 3.6, 3.7, 3.10 and 3.11 and and a simple induction argument beginning with $X = \mathrm{pt}$ we obtain the following.

Theorem 3.13. *Let $G = SU(2)$ act on $\mathbb{P}^1$ in the standard way. Let L_j be the bundles defined above, viewed as elements of $K_T^*((\mathbb{P}^1)^n)$ or $K_G^*((\mathbb{P}^1)^n)$ as appropriate, and let $\bar{L}_j$ denote the first Chern classes of L_j, viewed as elements of $H_T^2((\mathbb{P}^1)^n; \mathbb{Q})$ or $H_G^2((\mathbb{P}^1)^n; \mathbb{Q})$ as appropriate. Then*

$$(3.12) \qquad K_T^*\big((\mathbb{P}^1)^n\big) \cong R(T)[L_1, \ldots, L_n]/\langle\{L_j^2 - (b + b^{-1})L_j + 1, \ 1 \le j \le n\}\rangle$$

and

$$(3.13) \qquad K_G^*\big((\mathbb{P}^1)^n\big) \cong R(G)[L_1, \ldots, L_n]/\langle\{L_j^2 - vL_j + 1, \ 1 \le j \le n\}\rangle.$$

We also have

$$(3.14) \qquad H_T^*\big((\mathbb{P}^1)^n; \mathbb{Q}\big) \cong \mathbb{Z}[\bar{b}][\bar{L}_1, \ldots, \bar{L}_n]/\langle\{\bar{L}_j^2 - \bar{b}^2, \ 1 \le j \le n\}\rangle$$

and

$$(3.15) \qquad H_G^*\big((\mathbb{P}^1)^n; \mathbb{Q}\big) \cong \mathbb{Z}[\bar{t}][\bar{L}_1, \ldots, \bar{L}_n]/\langle\{\bar{L}_j^2 - \bar{t}, \ 1 \le j \le n\}\rangle.$$

Remark 3.14. *The relation $L_j^2 - vL_j + 1 = 0$ appearing in the theorem above is the pullback to the jth factor of the relation in Remark 2.13.*

4. COMPUTATION OF $\Phi_{2r}^* : H_G^*(F_{2r}; \mathbb{Q}) \to H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$

The main goal of this section is to give explicit descriptions of $H_G^*(F_{2r}; \mathbb{Q})$ by using the map $\Phi_{2r}^* : H_G^*(F_{2r}; \mathbb{Q}) \to H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ and using the presentation of the codomain given in Theorem 3.13. Specifically, we show in Theorem 4.7 that

$$\Phi_{2r}^* : H_T^*(F_{2r}; \mathbb{Q}) \to H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$$

is injective and concretely describe the subring $\Phi_{2r}^*(H_T^*(F_{2r}; \mathbb{Q}))$. In Theorem 4.8 we take Weyl invariants to obtain the corresponding result in H_G^*. We will need the cohomology results in the present section in order to complete the analogous computation for K-theory in Section 5.

Consider the commutative diagram

$$(4.1)$$

$$
\begin{array}{ccc}
\mathrm{pt} & \xrightarrow{i_1 \circ i_1} & \mathbb{P}^1 \times \mathbb{P}^1 \\
\Big\| & & \Big\downarrow{\scriptstyle \Phi_2} \\
\mathrm{pt} & \longrightarrow & F_2
\end{array}
$$

where $i_1 \circ i_1(\mathrm{pt})$ is by definition the point $([T], [T]) \in \mathbb{P}^1 \times \mathbb{P}^1 \cong G/T \times G/T$, and the bottom horizontal arrow has image the 'point at infinity' in the Thom space $F_2 \cong \mathrm{Thom}(\tau)$.

Remark 4.1. *Note that $i_1 \circ i_1$ is not a G-equivariant map, since the image point*

$$i_1 \circ i_1(\mathrm{pt}) = ([T], [T])$$

is not a G-fixed point. All other maps in the diagram are G-equivariant.

Recall that since $G = SU(2)$ is simply-connected and hence

$$H^1(BG) = H^2(BG) = 0,$$

the Serre spectral sequence for H_G^* implies that $H_G^2(X) \cong H^2(X)$ for any G-space X. Consider the diagram 2.6 for the special case $\zeta = \tau$ and $X = \mathbb{P}^1$. In this setting, the map $\mathbb{P}(\tau \oplus \epsilon) \to \mathrm{Thom}(\tau) \cong F_2$ is the composition $\Phi_2 \circ \Theta$, where $\Theta : \mathbb{P}(\tau \oplus \epsilon) \to \mathbb{P}^1 \times \mathbb{P}^1$ is the equivariant homeomorphism constructed in Lemma 3.1. (To be completely precise, we should include in the notation the homeomorphism $\mathrm{Thom}(\tau) \cong F_2$ mentioned in Proposition 2.14 and which is part of the definition of the map Φ_2, but since this map plays no role in the argument we will ignore this ambiguity.) The fact that j splits in diagram 2.6 implies that $(\Phi_2 \circ \Theta)^* : H_G^2(F_2; \mathbb{Z}) \cong H^2(F_2; \mathbb{Z}) \to H_G^2(\mathbb{P}(\tau \oplus \epsilon); \mathbb{Z}) \cong H^2(\mathbb{P}(\tau \oplus \epsilon); \mathbb{Z})$ is an injection. Let $\bar{U}_\tau \in \mathrm{Ker}\, j^* \subset H_G^2(\mathbb{P}(\tau \oplus \epsilon); \mathbb{Z}) \cong H^2(\mathbb{P}(\tau \oplus \epsilon); \mathbb{Z})$ denote the (equivariant) cohomology Thom class of the bundle τ over $\mathbb{P}^1$. With these preliminaries in place we may now define the following.

Definition 4.2. We define L as the unique line bundle over F_2 such that

$$c_1((\Phi_2 \circ \Theta)^*(L)) = (\Phi_2 \circ \Theta)^*(c_1(L)) = \bar{U}_\tau.$$

The next lemma is fundamental; it shows that the pullback of L is symmetric in L_1 and L_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ (where the L_i are as in Definition 3.12).

Lemma 4.3. *Let $\Phi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to F_2$ be the map given in Definition 3.2 and L, L_1, L_2 be the line bundles given in Definitions 4.2 and 3.12. Then*

$$\Phi_2^*(L) \cong L_1 \otimes L_2.$$

Remark 4.4. *The definition of the L_j given in Definition 3.12, which treated differently the cases when j is odd and j is even, is motivated by Lemma 4.3. If we define L_j by the same formula for all j, then we lose the symmetry in the statement of Lemma 4.3.*

Proof. As noted above, $H_G^2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z}) \cong H^2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$ since G is simply-connected, and $H^2(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by $c_1(L_1)$ and $c_1(L_2)$ by definition of the L_i. Hence $c_1(\Phi_2^*(L)) = m \cdot c_1(L_1) + n \cdot c_1(L_2)$ for some integers m and n. Equivalently,

$$\Phi_2^*(L) \cong L_1^{\otimes m} \otimes L_2^{\otimes n}.$$

Moreover, since Θ is a G-equivariant homeomorphism, this is equivalent to

$$c_1((\Phi_2 \circ \Theta)^*(L)) = m \cdot c_1(\Theta^*(L_1)) + n \cdot c_1(\Theta^*(L_2)).$$

We first give a different description of the left hand side of this equation. Since τ is a line bundle, Lemma 2.8 implies that $U_\tau = 1 - \beta_{\tau \oplus \epsilon} \in \mathrm{Ker}\, j^* \subset K_G^*(\mathbb{P}(\tau \oplus \epsilon))$ which in turn means $\bar{U}_\tau = -c_1(\beta_{\tau \oplus \epsilon}) \in H_G^2(\mathbb{P}(\tau \oplus \epsilon)) \cong H^2(\mathbb{P}(\tau \oplus \epsilon))$. Thus, by definition

$$c_1((\Phi_2 \circ \Theta)^*(L)) = -c_1(\beta_{\tau \oplus \epsilon}),$$

so in fact it suffices to compute the coefficients $a, b \in \mathbb{Z}$ in the equality

$$c_1(\beta_{\tau \oplus \epsilon}) = a \cdot c_1(\Theta^*(L_1)) + b \cdot c_1(\Theta^*(L_2)).$$

By Lemma 3.1, the composition $\Theta \circ k : \mathbb{P}^1 \to \mathbb{P}(\tau \oplus \epsilon) \to \mathbb{P}^1 \times \mathbb{P}^1$ (where k is the inclusion of the fiber in (2.6)) is the inclusion of $\mathbb{P}^1$ as the second factor. This implies $k^*\Theta^*L_1$ is the trivial bundle, so $k^*c_1(\Theta^*L_1) = 0$ and $k^*c_1(\beta_{\tau \oplus \epsilon}) = bk^*c_1(\Theta^*L_2) = bc_1(L_2)$. Next recall that the defining equation of $\beta_{\tau \oplus \epsilon}$ is

$$\gamma_{\tau \oplus \epsilon} \oplus \beta_{\tau \oplus \epsilon} = \bar{p}^*(\tau \oplus \epsilon)$$

so in particular $k^*(\gamma_{\tau\oplus\epsilon} \oplus \beta_{\tau\oplus\epsilon})$, being the restriction of $\bar{p}^*(\tau \oplus \epsilon)$ to a fiber of $\bar{p}$, is the trivial bundle. Hence

$$c_1(k^*\gamma_{\tau\oplus\epsilon}) = k^*c_1(\gamma_{\tau\oplus\epsilon}) = -k^*c_1(\beta_{\tau\oplus\epsilon}).$$

On the other hand, $k^*(\gamma_{\tau\oplus\epsilon}) \cong \gamma$ by definition of the tautological bundle, so $c_1(k^*\gamma_{\tau\oplus\epsilon}) = c_1(L_2)$ by definition of L_2. Hence $k^*c_1(\beta_{\tau\oplus\epsilon}) = -c_1(L_2)$ and we conclude $b = -1$ and hence $n = 1$.

It remains to compute the coefficient m. Again by Lemma 3.1, the composition $\Theta \circ j : \mathbb{P}^1 \cong \mathbb{P}(\tau) \to \mathbb{P}(\tau \oplus \epsilon) \to \mathbb{P}^1 \times \mathbb{P}^1$ is the inclusion of $\mathbb{P}^1$ as the diagonal in the direct product. This implies that $(\Theta \circ j)^*(L_1) = \gamma^{-1}$ and $(\Theta \circ j)^*(L_2) = \gamma$, so $(\Theta \circ j)^*(L_1^m \otimes L_2^n) = 1$ if and only if $n = m$. Thus $n = m = 1$, as desired.

$\square$

The following is immediate from Lemma 4.3. Recall that we also denote by $\bar{L}_i$ the Chern class $c_1^T(L_i)$ of the L_i.

Lemma 4.5. *Under $\Phi_2^* : H_T^*(F_2; \mathbb{Z}) \to H_T^*(\mathbb{P}^1 \times \mathbb{P}^1)$, the Chern class $\bar{L} := c_1^T(L) \in H_T^2(F_2; \mathbb{Z})$ maps to $\bar{L}_1 + \bar{L}_2$.*

From [10] it follows that the inclusion $F_{2r-2} \to F_{2r}$ induces isomorphisms on

$$H_G^2(F_{2r}; \mathbb{Z}) \cong \mathbb{Z}$$

for all $r > 0$. By slight abuse of notation we define the line bundle L on F_{2r} to be the unique bundle such that the composite inclusion $F_2 \to F_4 \to \cdots \to F_{2r}$ pulls L on F_{2r} back to the bundle L over F_2 given in Definition 4.2. Since the inclusion is a T-equivariant map and the bundle L of Definition 4.2 is a T-bundle, the bundle L over F_{2r} is also a T-bundle.

A straightforward generalization of the argument given for Lemma 4.3 yields the following.

Lemma 4.6. *Let $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ be the map given in Definition 3.2 and let L be the line bundle over F_{2r} defined above. Let $L_i, 1 \le i \le 2r$, be the line bundles given in Definition 3.12. Then*

$$\Phi_{2r}^*(L) \cong L_1 \otimes \cdots \otimes L_{2r}.$$

In particular,

$$\Phi_{2r}^*(c_1^T(L)) = \bar{L}_1 + \cdots + \bar{L}_{2r} \in H_T^2((\mathbb{P}^1)^{2r}; \mathbb{Q}).$$

We are now ready to prove the main results of this section. Let $s_j(y_1, \ldots, y_n)$ denote the jth elementary symmetric polynomial in the variables $y_1, \ldots, y_n$, where by standard convention $s_0 := 1$. We let $\bar{s}_j$ denote the element

$$(4.2) \qquad \bar{s}_j(\bar{L}_1, \ldots, \bar{L}_{2r}) \in H_G^*((\mathbb{P}^1)^{2r}) \subset H_T^*((\mathbb{P}^1)^{2r}).$$

The next result proves that the image of $\Phi_{2r}^* : H_T^*(F_{2r}; \mathbb{Q}) \to H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ is generated precisely by the $\bar{s}_j$, i.e., it is the subalgebra of symmetric polynomials in the elements $\bar{L}_i$ with respect to the presentation of $H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ given in Theorem 3.13.

Theorem 4.7. *Let $G = SU(2)$ and $T \subset G$ be the standard maximal torus. Let $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ be the map given in Definition 3.2. Then:*

(1) *The pullback $\Phi_{2r}^* : H_T^*(F_{2r}; \mathbb{Q}) \to H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ is injective.*

(2) *The ring $H_T^*(F_{2r}; \mathbb{Q})$, or equivalently the image of $\Phi_{2r}^* : H_T^*(F_{2r}; \mathbb{Q}) \to H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ is the symmetric subalgebra of $H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$. More specifically,*

$$H_T^*(F_{2r}; \mathbb{Q}) \cong \Phi_{2r}^* \big(H_T^*(F_{2r}; \mathbb{Q}) \big)$$
$$= \big(H_T^*(\mathbb{P}^1)^{2r}; \mathbb{Q} \big)^{S_{2r}}$$
$$= \{ \text{symmetric polynomials in } H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q}) \}$$
$$= \text{the } H_T^*(\text{pt}; \mathbb{Q})\text{-subalgebra of } H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q}) \text{ generated by } \bar{s}_1, \ldots, \bar{s}_{2r}$$
$$= \text{the } H_T^*(\text{pt}; \mathbb{Q})\text{-submodule of } H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q}) \text{ generated by } \bar{s}_0, \ldots, \bar{s}_{2r}.$$

For the proof of Theorem 4.7 we need the following notation. Let $\mathbb{Q}[\bar{t}]$ be a polynomial ring in one variable $\bar{t}$ and let $A = \mathbb{Q}[\bar{t}, \bar{L}_1, \ldots, \bar{L}_N] = \mathbb{Q}[t][\bar{L}_1, \ldots, \bar{L}_N]$ denote a polynomial ring in the variable $\bar{t}$ of degree 2 and the variables $\bar{L}_1, \ldots, \bar{L}_{2r}$, each of degree 1. Then A is also a $\mathbb{Q}[t]$-module. (The cohomology degree of the corresponding cohomology classes are obtained by multiplying the polynomial degree by 2.) Let I denote the ideal in A generated by the homogeneous polynomials $\{\bar{L}_j^2 - \bar{t}, 1 \leq j \leq N\}$. Then by Theorem 3.13, we know the ring $H_T^*((\mathbb{P}^1)^N; \mathbb{Q})$ can be presented as the quotient ring A/I. We will use this identification in the proof below.

Further, for a monomial $\bar{t}^b \bar{L}^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_{2r})$, we define its *signature* by

$$\text{sig}\,(\bar{t}^b \bar{L}^\alpha) := \text{sig}\,(\alpha) := \#j \text{ such that } \alpha_j \text{ is odd}.$$

By definition, the signature takes values in $\{0, 1, \ldots, 2r\}$ and induces a $\mathbb{Q}[\bar{t}]$-module decomposition $\oplus_{s=0}^{2r} A_{\text{sig}\,(\alpha)=s}$ of A, where $A_{\text{sig}\,(\alpha)=s}$ consists of the $\mathbb{Q}[\bar{t}]$-submodule of A generated by the monomials of signature s. Notice that I is homogeneous with respect to this grading in the sense that $I = \oplus_{s=0}^{2r}(I \cap A_{\text{sig}\,(\alpha)=s})$ since each monomial of the generators $\bar{L}_j^2 - \bar{t}$ are of signature 0. Finally, for a fixed degree $M > 0$ and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_{2r})$, we set

$$b(\alpha) := \frac{M - (\alpha_1 + \ldots + \alpha_{2r})}{2}$$

so that the monomial $\bar{t}^{b(\alpha)} \bar{L}^\alpha$ has total degree M. Similarly we set

$$b'(\alpha) = \frac{M - 2 - (\alpha_1 + \ldots + \alpha_{2r})}{2} = b(\alpha) - 1$$

so that the monomial $\bar{t}^{b'(\alpha)} \bar{L}^\alpha$ has total degree $M - 2$.

Proof. First we claim that the $H_T^*(\text{pt})$-subalgebra generated by $\bar{s}_1, \ldots, \bar{s}_{2r}$ is contained in $\text{Im}\Phi_{2r}^*$. Recall that, working over $\mathbb{Q}$, the monomials in the power functions $\bar{\psi}^k := \bar{L}_1^k + \ldots + \bar{L}_{2r}^k$ form a basis for all symmetric polynomials in $\bar{L}_j$. Using the relations $\bar{L}_i^2 - \bar{t}$, this means we can express the (equivalence class of) any symmetric polynomial in the $\bar{L}_j$ as a polynomial in $\bar{s}_1$. We saw in Lemma 4.6 above that $\bar{s}_1$ is in the image of Φ_{2r}^*, so the claim follows.

Note that we have the natural inclusions
(4.3)
$$\mathbb{Q}[t]\text{-module generated by } \bar{s}_0, \ldots, \bar{s}_{2r} \subseteq \mathbb{Q}[t]\text{-algebra generated by } \bar{s}_1, \ldots, \bar{s}_{2r} \subseteq \text{Im}\,\Phi_{2r}^*.$$

We show next that the $\mathbb{Q}[t]$-submodule generated by $\bar{s}_0, \ldots, \bar{s}_{2r}$ is a free $\mathbb{Q}[t]$-module of rank $2r + 1$.

Since I is a homogeneous ideal in the usual grading on A, in order to prove the claim, it suffices to prove that if (the equivalence class of) a linear combination $y = \sum_{j=0}^{2r} n_j t^{\frac{M-j}{2}} \bar{s}_j$ of homogeneous degree M is equal to 0 in the quotient ring $A/I \cong H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$, where

each $n_j \in \mathbb{Q}$, then each n_j must be equal to 0. (Here we use the convention that if $\frac{M-j}{2}$ is not integral then n_j is automatically equal to 0.) Suppose we have such a y. The equivalence class of y is 0 in A/I if and only if $y \in I$, so there exists a relation

$$
(4.4) \qquad y = \sum_{j=1}^{N} a_j (\bar{L}_j^2 - \bar{t})
$$

in A, where $a_j = a_j(\bar{t}, \bar{L}_1, \ldots, \bar{L}_N) \in A$.

Fix an integer k with $0 \le k \le 2r$. Assume $Q := \frac{M-k}{2}$ is integral. We wish to show that $n_k = 0$. Taking the component of (4.4) lying in $A_{s(\alpha)=k}$ yields the equality

$$
(4.5) \qquad n_k \bar{t}^Q \bar{s}_k = \sum_{j=1}^{2r} a_{j,k} (\bar{L}_j^2 - \bar{t}),
$$

where $a_{j,k}$ denotes the component of a_j lying in $A_{s(\alpha)=k}$. Let $c_{j,b,\alpha} \in \mathbb{Q}$ denote the coefficient of $\bar{t}^b \bar{L}_1^{\alpha_1} \ldots \bar{L}_{2r}^{\alpha_{2r}}$ in the polynomial $a_{j,k}$. By convention we set $c_{j,b,\alpha} = 0$ if any entry of $\alpha \in \mathbb{Z}^{2r}$ is negative or if $b'(\alpha)$ is not integral. Note also that b must satisfy $b = b'(\alpha)$ for $c_{j,b,\alpha}$ to be non-zero since $a_{j,k}$ is homogeneous of degree $M - 2$.

Let $m_{b,\alpha}$ denote the coefficient of the monomial $\bar{t}^b \bar{L}_1^{\alpha_1} \ldots \bar{L}_{2r}^{\alpha_{2r}}$ after collecting terms on the right hand side of (4.4). It follows that

$$
(4.6) \qquad m_{b,\alpha} = \sum_{j} c_{j,b,(\alpha_1,\ldots,\alpha_j-2,\ldots,\alpha_{2r})} + \sum_{j} c_{j,b-1,\alpha}.
$$

Now define for any $b \in \mathbb{Z}$ the expression

$$
C_b := \sum_{\{\alpha \ s.t. \ b'(\alpha)=b\}} \sum_{j} c_{j,b,\alpha}.
$$

Note that a given coefficient of the form $c_{j,b,\gamma}$ occurs in the first summation of (4.6) for $2r$ different values of the multi-index γ. Thus

$$
(4.7) \qquad \sum_{\{\alpha \ s.t. \ b(\alpha)=b\}} m_{b,\alpha} = (2r) \cdot C_b + C_{b-1}.
$$

Equating this to the corresponding sum on the left hand side of (4.5) yields

$$
(2r) \cdot C_b + C_{b-1} = \begin{cases} 0 & \text{if } b \neq Q; \\ n_k \binom{2r}{k} & \text{if } b = Q. \end{cases}
$$

Since $C_{-1} = 0$ we inductively conclude that $C_b = 0$ for $b \le Q - 1$, and in particular, $C_{Q-1} = 0$.

We next claim that $C_Q = 0$, To see this, recall $a_{j,k}$ is of degree $M - 2$ and each monomial $t^b L^\alpha$ appearing with non-zero coefficient in $a_{j,k}$ must also have signature k. This implies that the degree of the L^α must be $\ge k$ (and hence b must be $\le Q - 1$), so any coefficient of the form $c_{j,Q,\alpha}$ is 0. Hence $C_Q = 0$ and we conclude

$$
C_{Q-1} = n_k \binom{2r}{k} = 0
$$

from which it also immediately follows that for any k with $0 \le k \le 2r$, we also have $n_k = 0$.

We conclude that the $\{\bar{s}_j\}_{j=0}^{2r}$ generate a free $\mathbb{Q}[t]$-module in $H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$. Denote this free module by $\mathcal{M}$. Let $\mathcal{M}^{2\ell}$ denote the (cohomology) degree-2ℓ piece of $\mathcal{M}$. Since the (cohomology) degree of $\bar{s}_j$ is $2j$, the dimension of $\mathcal{M}^{2\ell}$ as a $\mathbb{Q}$-vector space is $\ell + 1$ for

$0 \leq \ell \leq 2r$ and $2r+1$ for $\ell > 2r$. This agrees with the dimensions of $H_T^{2\ell}(F_{2r}; \mathbb{Q})$ for all ℓ [10]. Thus the containments in (4.3) must be equalities. In particular, $\operatorname{Im} \Phi_{2r}^* \cong H_T^*(F_{2r}; \mathbb{Q})$ and we conclude Φ_{2r}^* is injective. The rest of the theorem also follows. $\qquad \square$

Taking Weyl invariants gives the following.

Theorem 4.8. *Let* $G = SU(2)$ *and* $T \subset G$ *be the standard maximal torus. Let* $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ *be the map given in Definition 3.2. Then:*

(1) *The pullback* $\Phi_{2r}^* : H_G^*(F_{2r}; \mathbb{Q}) \to H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ *is injective.*
(2) *The ring* $H_G^*(F_{2r}; \mathbb{Q})$, *or equivalently the image of* $\Phi_{2r}^* : H_G^*(F_{2r}; \mathbb{Q}) \to H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ *is the symmetric subalgebra of* $H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$. *More specifically,*

$$
\begin{aligned}
H_G^*(F_{2r}; \mathbb{Q}) &\cong \Phi_{2r}^* \big(H_G^*(F_{2r}; \mathbb{Q}) \big) \\
&= \big(H_G^*(\mathbb{P}^1)^{2r}; \mathbb{Q} \big)^{S_{2r}} \\
&= \{ \text{symmetric polynomials in } H_G^*\big((\mathbb{P}^1)^{2r}; \mathbb{Q} \big) \} \\
&= \text{the } H_G^*(\mathrm{pt}; \mathbb{Q})\text{-subalgebra of } H_G^*\big((\mathbb{P}^1)^{2r}; \mathbb{Q} \big) \text{ generated by } \bar{s}_1, \ldots, \bar{s}_{2r} \\
&= \text{the } H_G^*(\mathrm{pt}; \mathbb{Q})\text{-submodule of } H_G^*\big((\mathbb{P}^1)^{2r}; \mathbb{Q} \big) \text{ generated by } \bar{s}_0, \ldots, \bar{s}_{2r}.
\end{aligned}
$$

Proof. The fact that $\Phi_{2r}^* : H_G^*(F_{2r}; \mathbb{Q}) \to H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q})$ is injective follows from the commutative diagram

$$
\begin{array}{ccc}
H_G^*(F_{2r}; \mathbb{Q}) & \xrightarrow{\;\Phi_{2r}^*\;} & H_G^*((\mathbb{P}^1)^{2r}; \mathbb{Q}) \\
\downarrow & & \downarrow \\
H_T^*(F_{2r}; \mathbb{Q}) & \xrightarrow{\;\Phi_{2r}^*\;} & H_T^*((\mathbb{P}^1)^{2r}; \mathbb{Q})
\end{array}
$$

and the fact that both vertical arrows and the bottom horizontal arrow are injective. For part (2), recall from Theorem 4.7 that

$$
H_T^*(F_{2r}; \mathbb{Q}) \cong \text{the } H_T^*(\mathrm{pt}; \mathbb{Q})\text{-submodule of } H_T^*\big((\mathbb{P}^1)^{2r}; \mathbb{Q} \big) \text{ generated by } \bar{s}_1, \ldots, \bar{s}_{2r}.
$$

It is clear that an $H_T^*(\mathrm{pt}; \mathbb{Q})$-linear combination of $\bar{s}_0, \ldots, \bar{s}_{2r}$ is Weyl invariant if and only if its coefficients lie in $\big(H_T^*(\mathrm{pt}); \mathbb{Q} \big)^W = H_G^*(\mathrm{pt}; \mathbb{Q})$. Thus we have

$$
\begin{aligned}
\operatorname{Im} \Phi_{2r} \subset \big(H_T^*(F_{2r}); \mathbb{Q} \big)^W &\cong \big(H_T^*(\mathrm{pt}; \mathbb{Q})\text{-submodule generated by } \bar{s}_1, \ldots, \bar{s}_{2r} \big)^W \\
&\cong H_G^*(\mathrm{pt}; \mathbb{Q})\text{-submodule generated by } \bar{s}_0, \ldots, \bar{s}_{2r}.
\end{aligned}
$$

The result follows. $\qquad \square$

5. COMPUTATION OF $\Phi_{2r}^* : K_G^*(F_{2r}) \to K_G^*((\mathbb{P}^1)^{2r})$

We now come to the technical heart of this manuscript, which is the explicit computation of $K_T^*(F_{2r})$ and $K_G^*(F_{2r})$. By results in [10] we know that $K_G^*(\Omega G)$ is the inverse limit of $K_G^*(F_{2r})$ as $r \to \infty$, so knowledge of $K_G^*(F_{2r})$ is a key step in the computation of $K_G^*(\Omega G)$. This section is long, so we have divided the exposition into pieces.

5.1. **Preliminaries and general setup.** We first prove that, similar to the case of cohomology in the previous section, the map $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to (F_2)^r \to F_{2r}$ induces an injection in equivariant K-theory.

Proposition 5.1. *Let* $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ *be the map given in Definition 3.2. Then*

(5.1) $$\Phi_{2r}^* : K_G^*(F_{2r}) \to K_G^*\big((\mathbb{P}^1)^{2r}\big)$$

is an injective ring homomorphism, and similarly

(5.2) $$\Phi_{2r}^* : K_T^*(F_{2r}) \to K_T^*\big((\mathbb{P}^1)^{2r}\big)$$

is an injective ring homomorphism.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
K_G^*(F_{2r}) & \xrightarrow{\Phi_{2r}^*} & K_G^*\Big((\mathbb{P}^1)^{2r}\Big) = K_G^*(\mathrm{pt})[L_1, \ldots, L_{2r}]/\sim \\
\Big\downarrow{\scriptstyle \mathrm{ch}_G} & & \Big\downarrow{\scriptstyle \mathrm{ch}_G} \\
H_G^\pi(F_{2r}; \mathbb{Q}) & \xrightarrow{\Phi_{2r}^*} & H_G^\pi\Big((\mathbb{P}^1)^{2r}; \mathbb{Q}\Big) = H^\pi(\mathrm{pt}; \mathbb{Q})[\bar{L}_1, \ldots, \bar{L}_{2r}]/\sim
\end{array}
$$

By Theorem 4.8 we know that the bottom horizontal map is an injection. The vertical maps are Chern character maps and so are injective since all the groups are torsion free. This implies that the upper horizontal map must also be injective. The proof for K_T^* is identical. $\qquad\square$

By the above proposition, in order to compute $K_T^*(F_{2r})$ and $K_G^*(F_{2r})$, it therefore remains to compute their images in $K_T^*((\mathbb{P}^1)^{2r})$ and $K_G^*((\mathbb{P}^1)^{2r})$ respectively. We will accomplish this by an induction argument on the variable r. As a first step, we note the following.

Lemma 5.2. *The diagram*

$$
\begin{array}{ccc}
(\mathbb{P}^1)^{2r-2} & \xrightarrow{\Phi_{2(r-1)}} & F_{2(r-1)} \\
\Big\downarrow & & \Big\downarrow \\
(\mathbb{P}^1)^{2r} & \xrightarrow{\Phi_{2r}} & F_{2r}
\end{array}
$$

commutes, where the right vertical arrow is the canonical inclusion and the left vertical arrow is the inclusion $(x_1, x_2, \ldots, x_{2r-2}) \mapsto (x_1, \ldots, x_{2r-2}, [T], [T]) \in (\mathbb{P}^1)^{2r} \cong (G/T)^{2r}$.

Proof. This follows from the definition of the maps Φ_{2r} and Remark 2.15. $\qquad\square$

Remark 5.3. *The left vertical map is not a G-equivariant map since $([T], [T])$ is not a G-fixed point, as in Remark 4.1. All other maps in* (5.2) *are G-equivariant.*

Let $\kappa : F_{2r} \to \mathrm{Thom}(\tau^{2r-1})$ be the composition of the projection $F_{2r} \to F_{2r}/F_{2r-2}$ with the G-equivariant homeomorphism $F_{2r}/F_{2r-2} \cong \mathrm{Thom}(\tau^{2r-1}$ discussed in Proposition 2.14. By Lemma 5.2 we may consider the following commutative diagram, which

provides the framework for all the arguments in this section.

(5.3)

$$0 \longrightarrow \tilde{K}_T^*\big(\mathrm{Thom}(\tau^{2r-1})\big) \xrightarrow{\kappa^*} K_T^*(F_{2r}) \xrightarrow{i^*} K_T^*(F_{2r-2}) \longrightarrow 0$$

$$\Big\downarrow{\scriptstyle\Phi_{2r}^*} \qquad\qquad \Big\downarrow{\scriptstyle\Phi_{2r-2}^*}$$

$$K_T^*\big((\mathbb{P}^1)^{2r}\big) \xrightarrow{i^*} K_T^*\big((\mathbb{P}^1)^{2r-2}\big)$$

Here it is important that we use T-equivariant K-theory instead of G-equivariant K-theory, since by Remark 5.3 the diagram (5.2) is not a diagram of G-equivariant maps.

Remark 5.4. *In our computation of $K_G^*(F_{2r})$ below, we occasionally use expressions such as* $\mathrm{ch}_G(y)$, *when the element $y \in K_T^*(F_{2r})$ happens to be Weyl-invariant. This is justified by the fact that our results in [10] show that $K_G^*(F_{2r})$ is isomorphic to $K_T^*(F_{2r})^W$; in this situation there is no harm in using $\mathrm{ch}_G(y)$, since $\mathrm{ch}_G(y)$ determines $\mathrm{ch}_T(y)$.*

In order to describe the image of $\Phi_{2r}^* : K_T^*(F_{2r}) \to K_T^*((\mathbb{P}^1)^{2r})$ (respectively for K_G^*), we note first that for any $r > 0$ there is a natural S_{2r}-action on $(\mathbb{P}^1)^{2r}$, commuting with the given (diagonal) G-action on $(\mathbb{P}^1)^{2r}$, obtained by interchanging the factors. This geometric action induces an action on $K_T^*((\mathbb{P}^1)^{2r})$ (and $K_G^*((\mathbb{P}^1)^{2r})$).

Definition 5.5. Fix $r \in \mathbb{Z}^+$. Let $\Big(K_T^*\big((\mathbb{P}^1)^{2r}\big)\Big)^{S_{2r}}$ denote the subring which is invariant under the S_{2r}-action above. We call this the **symmetric subring of** $K_T^*((\mathbb{P}^1)^{2r})$. We will use similar notation and terminology for statements with K_G^* replacing K_T^*.

Motivated by Definition 5.5 and following our notation (4.2) in cohomology, let s_j denote the element

(5.4)
$$s_j(L_1, \ldots, L_{2r}) \in K_G^*((\mathbb{P}^1)^{2r}) \subseteq K_T^*((\mathbb{P}^1)^{2r}),$$

where $s_j(L_1, \ldots, L_{2r})$ is the j-th elementary symmetric polynomial in $L_1, \ldots, L_{2r}$. From the relations $L_j^2 = vL_j - 1$ in Theorem 3.13, it follows that any element in $\Big(K_T^*\big((\mathbb{P}^1)^{2r}\big)\Big)^{S_{2r}}$ can be written as an $R(T)$-linear combination of $\{s_0, \ldots, s_{2r}\}$. Thus we can also refer to $\Big(K_T^*\big((\mathbb{P}^1)^{2r}\big)\Big)^{S_{2r}}$ as the ring of **symmetric polynomials** in $L_1, \ldots, L_{2r}$.

We can now state the main result of this section, which is that the image $\Phi_{2r}^*\big(K_T^*(F_{2r})\big)$ is precisely equal to the symmetric subring $\Big(K_T^*\big((\mathbb{P}^1)^{2r}\big)\Big)^{S_{2r}}$, in analogy with the result in cohomology in Section 4.

Theorem 5.6. *Let $G = SU(2)$ and $T \subset SU(2)$ its maximal torus. Let $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ be the map given in Definition 3.2. Then*

$$K_T^*(F_{2r}) \cong \Phi_{2r}^*\big(K_T^*(F_{2r})\big)$$
$$= \Big(K_T^*((\mathbb{P}^1)^{2r})\Big)^{S_{2r}}$$
$$= \{\text{symmetric polynomials in } L_1, \ldots, L_{2r} \text{ in } K_T^*\big((\mathbb{P}^1)^{2r}\big)\}$$
$$= \text{the } K_T^*(\mathrm{pt})\text{-subalgebra of } K_T^*\big((\mathbb{P}^1)^{2r}\big) \text{ generated by } s_1, \ldots, s_{2r}$$
$$= \text{the } K_T^*(\mathrm{pt})\text{-submodule of } K_T^*\big((\mathbb{P}^1)^{2r}\big) \text{ generated by } s_0, \ldots, s_{2r}.$$

As a first step towards the proof of Theorem 5.6, we prove containment in one direction.

Lemma 5.7. *The image $\Phi_{2r}^*(K_T^*(F_{2r}))$ is contained in the symmetric subring $\left(K_T^*((\mathbb{P}^1)^{2r})\right)^{S_{2r}}$ of $K_T^*((\mathbb{P}^1)^{2r})$.*

Proof. Let $x \in \Phi_{2r}^*(K_T^*(F_{2r})) \subset K_T^*((\mathbb{P}^1)^{2r})$. To show $\sigma(x) = x$ for all $\sigma \in S_{2r}$, it suffices to consider the case where σ is a transposition. The map σ is induced by a G-map of spaces which permutes factors within $(\mathbb{P}^1)^{2r}$. Thus the question reduces to showing that the diagram

$$
\begin{array}{ccc}
(\mathbb{P}^1)^{2r} & \xrightarrow{\;\sigma\;} & (\mathbb{P}^1)^{2r} \\[2pt]
\Big\downarrow{\scriptstyle \Phi_{2r}} & & \Big\downarrow{\scriptstyle \Phi_{2r}} \\[6pt]
F_{2r} & =\!=\!=\!=\!= & F_{2r} \\[2pt]
\Big\downarrow & & \Big\downarrow \\[10pt]
\Omega G & =\!=\!=\!=\!= & \Omega G
\end{array}
$$

G-homotopy commutes. The standard proof that ΩG is homotopy-abelian goes through G-equivariantly to prove this. $\qquad\square$

The remainder of this section is devoted to proving the reverse inclusion

$$(5.5) \qquad\qquad \left(K_T^*((\mathbb{P}^1)^{2r})\right)^{S_{2r}} \subseteq \Phi_{2r}^*(K_T^*(F_{2r}))$$

by an inductive argument. The following is straightforward, giving us a sufficient condition for proving (5.5) inductively.

Lemma 5.8. *Fix $r \in \mathbb{Z}^+$. Suppose that $K_T^*((\mathbb{P}^1)^{2r-2})^{S_{2r-2}}$ is contained in $\Phi_{2r-2}^*(K_T^*(F_{2r-2}))$. Let i^* be the bottom horizontal arrow in diagram (5.3). If*

$$(5.6) \qquad\qquad \mathrm{Ker}\, i^* \cap \left(K_T^*((\mathbb{P}^1)^{2r})\right)^{S_{2r}} \subseteq \Phi_{2r}^*(K_T^*(F_{2r}))$$

then

$$\left(K_T^*((\mathbb{P}^1)^{2r})\right)^{S_{2r}} \subseteq \Phi_{2r}^*(K_T^*(F_{2r})).$$

Proof. The inclusion map $i : (\mathbb{P}^1)^{2r-2} \hookrightarrow (\mathbb{P}^1)^{2r}$ is S_{2r-2}-equivariant, where $S_{2r-2} \subseteq S_{2r}$ is the subgroup of S_{2r} which only moves the first $2r - 2$ factors. Thus the induced map i^* on equivariant K-theory appearing has the property that the image under i^* of the symmetric subring of $K_T^*((\mathbb{P}^1)^{2r})$ is contained in $\left(K_T^*((\mathbb{P}^1)^{2r-2})\right)^{S_{2r-2}}$. The claim now follows from a straightforward diagram chase using the inductive hypothesis. $\qquad\square$

In fact, we will prove a stronger result than (5.6), recorded in Proposition 5.9 below, for the statement of which we need some notation. In particular, it will be useful for the remainder of the discussion to choose specific generators of $\tilde{K}_T(\mathrm{Thom}(\tau^{2r-1}))$ as follows. Consider the diagram (2.6) and the corresponding maps $\tilde{p}, \bar{p}, j$, and k for the case $X = \mathbb{P}^1$ and $\zeta = \tau^{2r-1}$. Let γ be the canonical line bundle over $\mathbb{P}^1$. Define

$$(5.7) \qquad\qquad x := \bar{p}^*(\gamma) - 1 \in K_G^*\left(\mathbb{P}(\tau^{2r-1} \oplus \epsilon)\right) \subset K_T^*\left(\mathbb{P}(\tau^{2r-1} \oplus \epsilon)\right).$$

Let $U := U_{\tau^{2r-1}} = \lambda\big(\gamma^*_{\tau^{2r-1}\oplus\epsilon} \otimes \bar{p}^*(\tau^{2r-1})\big)$ denote the Thom class of τ^{2r-1} as in (2.5). The equivariant Thom isomorphism says in this setting that

$$\tilde{K}^*_G(F_{2r}/F_{2r-2}) = \tilde{K}^*_G\big(\mathrm{Thom}(\tau^{2r-1})\big)$$

is freely generated as an $R(G)$-module by U and xU. We will use these generators in our arguments below.

We can now state the main technical proposition of this section.

Proposition 5.9. *The map* $\Phi^*_{2r}\circ\kappa^* : \tilde{K}^*_T\big(\mathrm{Thom}(\tau^{2r-1})\big) \to K_T\big((\mathbb{P}^1)^{2r}\big)$ *induces an isomorphism of* $\tilde{K}^*_T(\mathrm{Thom}(\tau^{2r-1}))$ *onto the subspace*

$$\mathrm{Ker}\, i^* \cap \Big(K^*_T\big((\mathbb{P}^1)^{2r}\big) \Big)^{S_{2r}} \subseteq K^*_T\big((\mathbb{P}^1)^{2r}\big).$$

Specifically, the images of U and xU under $\Phi^*_{2r} \circ \kappa^*$ form an $R(T)$-module basis for $\mathrm{Ker}\, i^* \cap \Big(K^*_T\big((\mathbb{P}^1)^{2r}\big) \Big)^{S_{2r}}$. In particular,

$$\mathrm{Ker}\, i^* \cap K^*_T\big((\mathbb{P}^1)^{2r}\big)^{S_{2r}} \subset \mathrm{Im}\, \Phi^*_{2r}.$$

The proof of Proposition 5.9 is both long and technical, and occupies Sections 5.2 to 5.4. Thus, before embarking on the details, we briefly sketch the main ideas of the proof. Our strategy is to relate the map $\Phi^*_{2r} \circ \kappa^*$ to its analogue in H^π_T via the homomorphism ch_T. This method allows us to take advantage of the presence of a $\mathbb{Z}$-grading in cohomology. The following commutative diagram

(5.8)
$$
\begin{array}{ccc}
\tilde{K}^*_T\big(\mathrm{Thom}(\tau^{2r-1})\big) & \xrightarrow{\;\Phi^*_{2r}\circ\kappa^*\;} & \mathrm{Ker}\, i^* \cap \Big(\tilde{K}^*_T\big((\mathbb{P}^1)^{2r}\big) \Big)^{S_{2r}} \\
\Big\downarrow{\scriptstyle\mathrm{ch}_T} & & \Big\downarrow{\scriptstyle\mathrm{ch}_T} \\
\tilde{H}^\pi_T\big(\mathrm{Thom}(\tau^{2r-1})\big);\mathbb{Q}) & \xrightarrow{\;\Phi_{2r}{}^*\circ\kappa^*\;} & \mathrm{Ker}\, i^* \cap \Big(\tilde{H}^\pi_T\big((\mathbb{P}^1)^{2r};\mathbb{Q}\big) \Big)^{S_{2r}}.
\end{array}
$$

will be central in our analysis. Note that the diagram is well-defined since, by naturality, ch_T takes S_{2r}-invariant elements to S_{2r}-invariant elements, and also takes $\mathrm{Ker}\, i^*$ to $\mathrm{Ker}\, i^*$. Moreover, the cohomology version of (5.3) shows that the image of $\Phi^*_{2r}\kappa^*$ on $\tilde{H}^\pi_T(\mathrm{Thom}(\tau^{2r-1});\mathbb{Q})$ lie in $\mathrm{Ker}\, i^*$, and Theorem 4.7 shows that they are symmetric. Also note that the top horizontal arrow in (5.8) is a morphism of $R(T)$-modules, while the bottom horizontal arrow is a morphism of $H^\pi_T(\mathrm{pt})$-modules. The vertical arrows satisfy the relation

$$\mathrm{ch}_T(\rho m) = \mathrm{ch}_T(\rho)\mathrm{ch}_T(m)$$

for all $\rho \in R(T)$ and any m in the domain. Our goal, stated in terms of (5.8), is to prove that the top horizontal arrow is an isomorphism.

We will accomplish this goal by concrete linear algebra. Recall that $\tilde{K}^*_T\big(\mathrm{Thom}(\tau^{2r-1})\big)$ is a free $R(T)$-module of rank 2, where we have fixed a choice of basis $\{U, xU\}$. Letting $\bar{U}$ denote the (equivariant) cohomology Thom class of τ^{2r-1} and

(5.9)
$$\bar{x} := c^G_1(x) = c^G_1\big(\bar{p}^*(\gamma)\big) \in H^2_G\big(\mathbb{P}(\tau^{2r-1}\oplus\epsilon)\big) \subset \tilde{H}^2_T\big(\mathbb{P}(\tau^{2r-1}\oplus\epsilon)\big)$$

it is also clear from the (equivariant) cohomology Thom isomorphism that $\{\bar{U}, \bar{x}\bar{U}\}$ form a basis for $\tilde{H}^\pi_T(\mathrm{Thom}(\tau^{2r-1});\mathbb{Q})$ as a free $\tilde{H}^\pi_T(\mathrm{pt};\mathbb{Q})$-module. We show in Section 5.2 that

both $\operatorname{Ker} i^* \cap (K_T^*((\mathbb{P}^1)^{2r}))^{S_{2r}}$ and $\operatorname{Ker} i^* \cap (H_T^\pi((\mathbb{P}^1)^{2r}; \mathbb{Q}))^{S_{2r}}$ are also free rank-2 modules over $R(T)$ and $H_T^\pi(\mathrm{pt}; \mathbb{Q})$ respectively, and find explicit bases $\{K_1, K_2\}$ and $\{\bar{K}_1, \bar{K}_2\}$ respectively.

Given these choices of bases, we can construct a 2×2 matrix determining any of the four maps in the diagram (5.8). For example, for the right vertical arrow, we may write

(5.10) $$\operatorname{ch}_T(K_1) = n_{11}\bar{K}_1 + n_{21}\bar{K}_2, \quad \operatorname{ch}_T(K_2) = n_{12}\bar{K}_1 + n_{22}\bar{K}_2$$

where $n_{ij} \in H_T^\pi(\mathrm{pt}; \mathbb{Q})$. This defines a 2×2 matrix $\mathbf{N} = (n_{ij})$. Similarly we may define matrices $\mathbf{M}$ corresponding to the left vertical arrow, Q for the top horizontal arrow, and $\bar{Q}$ for the bottom horizontal arrow. Note that the entries of $\mathbf{N}, \mathbf{M}, \bar{Q}$ are all in $H_T^\pi(\mathrm{pt}; \mathbb{Q})$, whereas the entries of Q are in $R(T)$. We record these definitions schematically in the following diagram.

(5.11)
$$
\begin{array}{ccc}
\langle U, xU \rangle & \xrightarrow{\quad Q \quad} & \langle K_1, K_2 \rangle \\
\downarrow{\scriptstyle \mathbf{M}} & & \downarrow{\scriptstyle \mathbf{N}} \\
\langle \bar{U}, \bar{x}\bar{U} \rangle & \xrightarrow{\quad \bar{Q} \quad} & \langle \bar{K}_1, \bar{K}_2 \rangle
\end{array}
$$

The commutativity of (5.8) implies that these matrices satisfy

(5.12) $$\mathbf{N} \operatorname{ch}_T(Q) = \bar{Q}\,\mathbf{M}$$

where the notation $\operatorname{ch}_T(Q)$ denotes the 2×2 matrix obtained by applying ch_T to each entry of Q.

With this notation in place it is immediate that the following is sufficient to prove Proposition 5.9:

(5.13)   the matrix Q in (5.11) has determinant 1 (and is hence invertible).

We will prove the claim in (5.13) via the indirect route of computing $\mathbf{M}, \mathbf{N}$, and $\bar{Q}$, and then using the relation (5.12) to deduce that the determinant of $\operatorname{ch}_T(Q)$ is 1. This implies $\det(Q) = 1$ since ch_T is injective. More specifically, in Section 5.3 we explicitly compute both $\mathbf{M}$ and $\mathbf{N}$. We compute $\bar{Q}$ and some determinants to finish the proof of Proposition 5.9 (and hence Theorem 5.6) in Section 5.4.

5.2. **Module bases for** $\operatorname{Ker} i^* \cap (K_T^*((\mathbb{P}^1)^{2r}))^{S_{2r}}$ **and** $\operatorname{Ker} i^* \cap (H_T^\pi((\mathbb{P}^1)^{2r}; \mathbb{Q}))^{S_{2r}}$. As above, we must prove that both $\operatorname{Ker} i^* \cap (K_T^*((\mathbb{P}^1)^{2r}))^{S_{2r}}$ and $\operatorname{Ker} i^* \cap (H_T^\pi((\mathbb{P}^1)^{2r}; \mathbb{Q}))^{S_{2r}}$ are free rank-2 modules over appropriate rings, and then to fix particular choices of module bases for each. (In fact, we will not specify the bases completely, since for our later arguments only the 'highest-order terms' are needed.)

We begin with an explicit description of the map i^* in terms of the presentations of $K_T^*((\mathbb{P}^1)^{2r})$ and $K_T^*((\mathbb{P}^1)^{2r-2})$ given in Theorem 3.13. Let $\{L_1, \ldots, L_{2r}\}$ denote the generators of $K_T^*((\mathbb{P}^1)^{2r})$ as before, and let $\{L_1', \ldots, L_{2r-2}'\}$ denote the generators of $K_T^*((\mathbb{P}^1)^{2r-2})$. With respect to these variables the map $i^* : K_T^*((\mathbb{P}^1)^{2r}) \to K_T^*((\mathbb{P}^1)^{2r-2})$ is given by

(5.14) $$i^*(L_j) = \begin{cases} L_j' & \text{if } j \leq 2r - 2; \\ b^{-1} & \text{if } j = 2r - 1; \\ b & \text{if } j = 2r. \end{cases}$$

Also note that the quadratic relations $L_j^2 = vL_j - 1$ in Theorem 3.13 imply that any symmetric polynomial in the L_j in $K_T^*((\mathbb{P}^1)^{2r})$ can be expressed as an $R(T)$-linear combination

of $s_0, \dots, s_{2r}$. Since we are interested in the kernel of i^* restricted to the symmetric polynomials, it is useful to compute i^* on the s_k. Let $s'_0, \dots, s'_{2r-2}$ denote the analogous elements in $K_T^*((\mathbb{P}^1)^{2r-2})$. Using the expression

$$
\begin{aligned}
s_j(L_1, \dots, L_{2r}) &= s_{j-2}(L_1, \dots, L_{2r-2})L_{2r-1}L_{2r} + s_{j-1}(L_1, \dots, L_{2r-2})L_{2r-1} \\
&\quad + s_{j-1}(L_1, \dots, L_{2r-2})L_{2r} + s_j(L_1, \dots, L_{2r-2})
\end{aligned}
$$

it follows from a straightforward computation that

$$
\text{(5.15)} \qquad i^*(s_j) = \begin{cases}
s'_0 & \text{if } j = 0; \\
s'_1 + vs'_0 & \text{if } j = 1; \\
s'_j + vs'_{j-1} + s'_{j-2} & \text{if } 1 < j \le 2r - 2; \\
vs'_{2r-2} + s'_{2r-3} & \text{if } j = 2r - 1; \\
s'_{2r-2} & \text{if } j = 2r.
\end{cases}
$$

The corresponding matrix is

$$
\text{(5.16)} \qquad
\begin{pmatrix}
1 & v & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & v & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & v & 1 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 1 & v & 1 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 1 & v & 1
\end{pmatrix}
$$

From (5.15) and (5.16) it follows that $\operatorname{Ker}(i^*) \cap \left(K_T^*((\mathbb{P}^1)^{2r})\right)^{S_{2r}}$ is a free rank-2-module and that there exists a basis K_1, K_2 of the form

$$
K_1 = s_{2r-1} + \text{lower-order terms in } s_0, \dots, s_{2r-2}
$$

and

$$
K_2 = -s_{2r} + \text{lower-order terms in } s_0, \dots, s_{2r-2}.
$$

(Only the leading terms s_{2r-1} and s_{2r} of K_1 and K_2 are of concern to us, so we do not record details about the lower-order terms.)

We now make analogous computations for $i^* : H_T^\pi\left((\mathbb{P}^1)^{2r}; \mathbb{Q}\right) \to H_T^\pi\left((\mathbb{P}^1)^{2r-2}; \mathbb{Q}\right)$. Noting that $v = b + b^{-1}$ and that $\bar{b}(-\bar{b}) = -\bar{t}$, we have the following equations for $i^*(\bar{L}_j)$ analogous to (5.14) with b and b^{-1} replaced by $\bar{b}$ and $-\bar{b}$ respectively.

$$
\text{(5.17)} \qquad i^*(\bar{s}_j) = \begin{cases}
\bar{s}'_0 & \text{if } j = 0; \\
\bar{s}'_1 & \text{if } j = 1; \\
\bar{s}'_j - \bar{t}\bar{s}'_{j-2} & \text{if } 1 < j \le 2r - 2; \\
-\bar{t}\bar{s}'_{2r-3} & \text{if } j = 2r - 1; \\
-\bar{t}\bar{s}'_{2r-2} & \text{if } j = 2r.
\end{cases}
$$

The corresponding matrix for i^* is

(5.18)
$$
\begin{pmatrix}
1 & 0 & -\bar{t} & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 & -\bar{t} & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 1 & 0 & -\bar{t} & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & -\bar{t}
\end{pmatrix}
$$

Again it follows that $\mathrm{Ker}(i^*) \cap \left(H_T^\pi((\mathbb{P}^1)^{2r}; \mathbb{Q}) \right)^{S_{2r}}$ is a free rank-2 module and that there exists a basis of the form

$$\bar{K}_1 = \bar{s}_{2r-1} + \text{lower-order terms in } \bar{s}_0, \dots, \bar{s}_{2r-2}$$

and

$$\bar{K}_2 = -\bar{s}_{2r} + \text{lower-order terms in } \bar{s}_0, \dots, \bar{s}_{2r-1}$$

where both $\bar{K}_1$ and $\bar{K}_2$ are homogeneous.

5.3. **Computation of the matrices N and M.** We next turn to a computation of the matrix N in (5.11), the entries of which are determined by (5.10). Hence our task is to compute $\mathrm{ch}_T(K_1)$ and $\mathrm{ch}_T(K_2)$ in terms of $\{\bar{K}_1, \bar{K}_2\}$, with respect to the bases chosen in Section 5.2. Since the K_i are written in terms of the s_k, we first compute $\mathrm{ch}_T(s_k)$.

Note that in general if two variables t and y satisfy the relation $y^2 = t$, then the formal series $e^y = \sum_k \frac{y^k}{k!}$ can be expressed as

(5.19)
$$e^y = yp(t) + q(t)$$

where

(5.20)
$$p(t) := \sum_{k=0}^\infty \frac{t^k}{(2k+1)!} = \sinh(\sqrt{t})/\sqrt{t}$$

and

(5.21)
$$q(t) := \sum_{k=0}^\infty \frac{t^k}{(2k)!} = \cosh(\sqrt{t}).$$

In our setting, recall that $\bar{L}_k$ satisfies $\bar{L}_k^2 = \bar{t}$. Using (5.19) we obtain, by definition of the Chern character,

$$\mathrm{ch}_T(L_k) = e^{\bar{L}_k} = \bar{L}_k p(\bar{t}) + q(\bar{t})$$

so $\mathrm{ch}_T(L_k)$ is an expression in $H^\pi((\mathbb{P}^1)^{2r}; \mathbb{Q})$ which is linear in $\bar{L}_k$ with coefficients in $H_T^\pi(\mathrm{pt}; \mathbb{Q})$. Therefore we conclude

$$
\begin{aligned}
\mathrm{ch}_T(s_k) &= s_k\big(\mathrm{ch}_T(L_1), \dots, \mathrm{ch}_T(L_{2r-2}) \big) \\
&= s_k\big(\bar{L}_1 p(\bar{t}) + q(\bar{t}), \dots, \bar{L}_{2r} p(\bar{t}) + q(\bar{t}) \big) \\
&= p(\bar{t})^k \bar{s}_k + \text{lower-order terms in } \bar{s}_0, \dots, \bar{s}_{k-1}.
\end{aligned}
$$

It then follows that

$$\mathrm{ch}_T(K_1) = p(\bar{t})^{2r-1}\bar{s}_{2r-1} + \text{lower-order terms in } \bar{s}_0, \ldots, \bar{s}_{2r-2}$$

and

$$\mathrm{ch}_T(K_2) = -p(\bar{t})^{2r}\bar{s}_{2r} + \text{lower-order terms in } \bar{s}_0, \ldots, \bar{s}_{2r-1}.$$

Comparing the coefficients of $\bar{s}_{2r-1}$ and $\bar{s}_{2r}$ in $\mathrm{ch}_T(K_1)$ and $\mathrm{ch}_T(K_2)$ with those in $\bar{K}_1$ and $\bar{K}_2$, we conclude that

$$\mathrm{ch}_T(K_1) = p(\bar{t})^{2r-1}\bar{K}_1$$

and

$$\mathrm{ch}_T(K_2) = p(\bar{t})^{2r}\bar{K}_2 + A(\bar{t})\bar{K}_1$$

for some $A(\bar{t}) \in H_T^\pi(\mathrm{pt}; \mathbb{Q})$. Hence we have

(5.22)
$$\mathbf{N} = \begin{pmatrix} \left(p(\bar{t})\right)^{2r-1} & A(\bar{t}) \\ 0 & \left(p(\bar{t})\right)^{2r} \end{pmatrix}.$$

We now turn to a computation of $\mathbf{M}$, for which we must first compute the Chern character of U and xU. We will in fact compute $\mathrm{ch}_G(U)$ and $\mathrm{ch}_G(xU)$, from which we can deduce $\mathrm{ch}_T(U)$ and $\mathrm{ch}_T(xU)$. For this computation it is useful to introduce the bundle

$$a := \gamma_{\tau^{2r-1}\oplus\epsilon}^* \otimes \bar{p}^*(\tau)$$

on $\mathrm{Thom}(\tau^{2r-1})$. Then it follows from (2.5) that $U = \lambda(a^{2r-1})$. Letting $\bar{a}$ denote the first Chern class $c_1(a)$, it also follows from the definition of a, Corollary 2.12, and the definition (5.9) of $\bar{x}$ that

(5.23)
$$c_1^G(\gamma_{\tau^{2r-1}\oplus\epsilon}) = -\bar{a} - 2\bar{x}.$$

We are now in a position to compute $\mathrm{ch}_G(U)$ in terms of $\bar{a}$. Recall that the definition of the Chern character implies that if $\xi = \xi_1 + \ldots + \xi_n$ is a sum of equivariant line bundles then

$$\mathrm{ch}_G\left(\lambda(\xi)\right) = \sum_{k=0}^n (-1)^k s_k(e^{\overline{\xi_1}}, \ldots, e^{\overline{\xi_n}})$$

where $\overline{\xi_j} := c_1^G(\xi_j)$. Applying this to $U = a^{2r-1}$ yields

$$\mathrm{ch}_G(U) = \sum_{k=0}^{2r-1} (-1)^k s_k(e^{\bar{a}}, e^{\bar{a}}, \ldots, e^{\bar{a}})$$

$$= 1 - (2r-1)e^{\bar{a}} + \binom{2r-1}{2}e^{2\bar{a}} + \ldots = (1 - e^{\bar{a}})^{2r-1}$$

$$= (1 - e^{\bar{a}})^{2r-1}.$$

In order to compute the matrix $\mathbf{M}$ with respect to the chosen bases $\{U, xU\}$ and $\{\bar{U}, \bar{x}\bar{U}\}$, we must now relate $\bar{a}$ to $\bar{U}$ and $\bar{x}\bar{U}$. The next two lemmas serve this purpose.

Lemma 5.10. $\bar{a}^{2r} + 2\bar{x}\bar{a}^{2r-1} = 0$.

Proof. For an equivariant bundle ξ, we write $c^G(\xi) = c_0^G(\xi) + c_1^G(\xi) + \ldots$ for its total (equivariant) Chern class. Then the Whitney sum formula and the defining equation (2.7) for the bundle $\beta_{\tau^{2r-1}\oplus\epsilon}$ together imply

$$c^G(\bar{p}^*\tau^{2r-1}) = c^G(\gamma_{\tau^{2r-1}\oplus\epsilon})c^G(\beta).$$

Since $c^G(\bar{p}^*(\tau^{2r-1})) = c^G(\bar{p}^*\tau)^{2r-1} = (1-2\bar{x})^{2r-1}$ and $c^G(\gamma_{\tau^{2r-1}\oplus\epsilon}) = 1 - (\bar{a}+2\bar{x})$ by (5.23), we conclude
(5.24)
$$c^G(\beta) = c^G(\bar{p}^*(\tau^{2r-1}))\big(c^G(\gamma_{\tau^{2r-1}\oplus\epsilon})\big)^{-1} = \frac{(1-2\bar{x})^{2r-1}}{1-(\bar{a}+2\bar{x})} = (1-2\bar{x})^{2r-1}\sum_{m=0}^{\infty}(\bar{a}+2\bar{x})^m.$$

Taking the degree-k part of (5.24) yields

$$c_k^G(\beta) = \sum_{j=0}^{k}\binom{2r-1}{j}(-2\bar{x})^j(\bar{a}+2\bar{x})^{k-j}$$

and so

$$c_k^G(\beta) = (\bar{a}+2\bar{x})c_{k-1}^G(\beta) + \binom{2r-1}{k}(-2\bar{x})^k.$$

In particular, since $\dim(\beta) = 2r-1$ and $\binom{2r-1}{2r} = 0$, we conclude

(5.25) $$c_{2r}^G(\beta) = (\bar{a}+2\bar{x})c_{2r-1}^G(\beta).$$

Also, for $k = 2r-1$,

(5.26) $$c_{2r-1}^G(\beta) = \sum_{j=0}^{2r-1}\binom{2r-1}{j}(-2x)^j(\bar{a}+2\bar{x})^{2r-1-j} = (\bar{a}+2\bar{x}-2\bar{x})^{2r-1} = \bar{a}^{2r-1}.$$

Putting (5.25) and (5.26) together yields the desired result.   □

Next recall that the cohomology Thom class and K-theory class are related by a formula involving the Chern character and the Todd class. Specifically, for an equivariant n-bundle ξ, we have

(5.27) $$\overline{U_\xi} = (-1)^n \bar{p}^*\big(\mathrm{Todd}_G(\xi)\big)\,\mathrm{ch}_G(U_\xi)$$

where

$$\mathrm{Todd}_G(L) := \frac{c_1^G(L)}{1 - e^{-c_1^G(L)}}$$

for an equivariant line bundle L, and the definition is extended to higher-rank bundles by means of the splitting principle [20, pages 13–14]. We have the following.

Lemma 5.11. $\bar{U} = -\bar{a}^{2r-1}$.

Proof. Applying (5.27) to the bundle τ^{2r-1} and using the multiplicativity of the Todd class yields

$$
\begin{aligned}
\bar{U} &= (-1)^{2r-1}\bar{p}^*\big(\mathrm{Todd}_G(\tau^{2r-1})\big)\,\mathrm{ch}_G(U) \\
&= \Big(-\bar{p}^*\big(\mathrm{Todd}_G(\tau)\big)\Big)^{2r-1}\big(1-e^{\bar{a}}\big)^{2r-1} \\
&= \left(\frac{2\bar{x}}{\big(1-e^{-2\bar{x}}\big)}\right)^{2r-1}\big(1-e^{\bar{a}}\big)^{2r-1} \\
&= \left(\frac{(-\bar{a})}{\big(1-e^{\bar{a}}\big)}\right)^{2r-1}\big(1-e^{\bar{a}}\big)^{2r-1} \\
&= (-1)^{2r-1}\bar{a}^{2r-1} \\
&= -\bar{a}^{2r-1}.
\end{aligned}
$$

where we have used Lemma 5.10 together with the fact that $(1-e^{\bar{a}})^{2r-1}$ is divisible by $\bar{a}^{2r-1}$.
$\square$

Using Lemmas 5.10 and 5.11, some algebraic manipulation (briefly sketched below) allows us to express $\mathrm{ch}_G(U) = (1-e^{\bar{a}})^{2r-1}$ in terms of $\bar{x}$ and $\bar{U}$. Given a variable y, note that the expression $g(y) := \left(\frac{e^y-1}{y}\right)^{2r-1}$ can be rewritten as a sum $g_1(y) + yg_2(y)$, where both g_1 and g_2 are even functions of y, as follows:

$$
(5.28) \qquad g(y) = e^{(2r-1)y/2}\left(\frac{\sinh(y/2)}{y/2}\right)^{2r-1} = g_1(y^2) + yg_2(y^2)
$$

where

$$
(5.29) \qquad g_1(y) = \cosh\big((2r-1)\sqrt{y}/2\big)\left(\frac{\sinh(\sqrt{y}/2)}{\sqrt{y}/2}\right)^{2r-1}
$$

and

$$
(5.30) \qquad g_2(y) = \frac{\sinh\big((2r-1)\sqrt{y}/2\big)}{\sqrt{y}}\left(\frac{\sinh(\sqrt{y}/2)}{\sqrt{y}/2}\right)^{2r-1}.
$$

Applying this to our situation, we get $\mathrm{ch}_G(U) = -\bar{a}^{2r-1}g(\bar{a}) = -\bar{a}^{2r-1}\big(g_1(\bar{a}^2) + \bar{a}g_2(\bar{a}^2)\big)$. Also note that $\bar{x}^2 = \bar{t}$, as can be seen from Lemmas 3.8 and 3.5 and the definition of $\bar{x}$. This, together with Lemmas 5.10 and 5.11, gives

$$
\begin{aligned}
(5.31) \qquad \mathrm{ch}_G(U) &= -\bar{a}^{2r-1}\big(g_1\big((-2\bar{x})^2\big) + \bar{a}g_2\big((-2\bar{x})^2\big)\big) \\
&= -\bar{a}^{2r-1}\big(g_1(4\bar{t}) - 2\bar{x}g_2(4\bar{t})\big) = g_1(4\bar{t})\bar{U} - 2g_2(4\bar{t})\bar{x}\bar{U}.
\end{aligned}
$$

Since $\bar{x}^2 = \bar{t}$, we also get

$$
\begin{aligned}
\mathrm{ch}_G\big(xU\big) &= \big(\mathrm{ch}_G(x)\mathrm{ch}_G(U)\big) \\
&= \big(p(\bar{t})\bar{x} + q(\bar{t})\big)\big(g_1(4\bar{t})\big(\bar{U}\big) - 2g_2(4\bar{t})\big(\bar{x}\bar{U}\big)\big) \\
&= q(\bar{t})g_1(4\bar{t})\big(\bar{U}\big) - 2\bar{t}p(\bar{t})g_2(4\bar{t})\big(\bar{U}\big) \\
&\quad + p(\bar{t})g_1(4\bar{t})\bar{x}\bar{U} - 2q(\bar{t})g_2(4\bar{t})\bar{x}\bar{U} \\
&= \big(q(\bar{t})g_1(4\bar{t}) - 2\bar{t}p(\bar{t})g_2(4\bar{t})\big)\bar{U} + \big(p(\bar{t})g_1(4\bar{t}) - 2q(\bar{t})g_2(4\bar{t})\big)\bar{x}\bar{U}
\end{aligned}
$$

where $q(t) := \cosh(\sqrt{t})$ as in Equation (5.21).

Thus we conclude

$$
(5.32) \qquad \mathbf{M} = \begin{pmatrix} g_1(4\bar{t}) & -2g_2(4\bar{t}) \\ q(\bar{t})g_1(4\bar{t}) - 2\bar{t}p(\bar{t})g_2(4\bar{t}) & p(\bar{t})g_1(4\bar{t}) - 2q(\bar{t})g_2(4\bar{t}) \end{pmatrix}.
$$

5.4. Computation of matrix $\bar{Q}$ and conclusion of proof of Theorem 5.6.

In this section we prove that $\bar{Q}$ is the identity matrix, and show the determinants of $\mathrm{ch}_T(Q)$ (and hence Q) are 1. This allows us to conclude the proof of Proposition 5.9 and Theorem 5.6.

We begin with the following.

Lemma 5.12. *The matrix $\bar{Q}$ is diagonal with integer entries.*

Proof. The elements $\Phi_{2r}^*\kappa^*(\bar{U})$ and $\Phi_{2r}^*\kappa^*(\bar{x}\bar{U})$ are Weyl-invariant since τ is a G-bundle. We know that

$$
\mathrm{Ker}\, i^* \cap \{\text{symmetric polynomials}\} \cong \mathbb{Z} \qquad \text{in degree } 2(2r-1),
$$

generated by $\bar{K}_1$ and

$$
\mathrm{Ker}\, i^* \cap \{\text{symmetric polynomials}\} \cong \mathbb{Z} \oplus \mathbb{Z} \qquad \text{in degree } 4r,
$$

generated by $\bar{K}_2$ and $\bar{b}\bar{K}_1$. The intersections of the left hand sides of the above two equations with the Weyl invariants are generated by $\bar{K}_1$ and $\bar{K}_2$ respectively. Recall that $\bar{K}_1$ and $\bar{K}_2$ are homogeneous. Thus $\Phi_{2r}{}^*\kappa^*(\bar{U}) = \lambda_1\bar{K}_1$ and $\Phi_{2r}{}^*\kappa^*(\bar{x}\bar{U}) = \lambda_2\bar{K}_2$ for some integers λ_1 and λ_2. $\qquad \square$

Let U be the Thom class of τ^{2r-1} in ordinary cohomology. We next show that the diagonal matrix entries λ_1 and λ_2 (from the proof of the lemma above) are both 1, by doing a computation in ordinary cohomology.

Lemma 5.13. *The constants λ_1 and λ_2 appearing in the proof of Lemma 5.12 are both equal to 1. In particular, $\bar{Q}$ is the 2×2 identity matrix.*

Proof. Let $\mathsf{K}_1, \mathsf{K}_2, \mathsf{x}, \mathsf{L}_j$ and s_j be the ordinary cohomology verions of $\bar{K}_1, \bar{K}_2, \bar{x}, \bar{L}_j$ and $\bar{s}_j$. Using that $\mathsf{L}_1^2 = 0$ in the exterior algebra $H^*\big((\mathbb{P}^1)^{2r}\big)$ we get

$$
\mathsf{x}\mathsf{K}_1 = -\mathsf{L}_1\mathsf{s}_{2r-1} = -\mathsf{s}_{2r} = \mathsf{K}_2,
$$

and it follows that $\lambda_1 = \lambda_2$. From Section 2 (2.4), we know that $\kappa^*(\mathsf{U})$ generates $H^*(F_{2r})$ in degree $4r-2$. (This is by the Thom isomorphism in ordinary cohomology.) Since $\Phi_{2r}{}^*\kappa^*(\mathsf{U})$ lies in $\mathrm{Ker}\, i^* \cap \{\text{symmetric polynomials}\}$, it is a multiple of the ordinary cohomology version of the generator K_1. After dualizing, up to sign, the statement in the Lemma becomes equivalent to the statement that $\kappa_*\Phi_{2r*}$ is onto in homology in degree $4r - 2$. Since κ_* is an isomorphism in these degrees, we are asking whether Φ_{2r*} is onto on $H_{4r-2}(\ )$. Since $F_{2r} \hookrightarrow \Omega G$ is an isomorphism in these degrees, we may instead consider the composition

$(\mathbb{P}^1)^{2r} \xrightarrow{\Phi_{2r}} F_{2r} \hookrightarrow \Omega G$. By construction, the composition $(\mathbb{P}^1)^{2r} \xrightarrow{\Phi_{2r}} F_{2r} \hookrightarrow \Omega G$ is the composition

$$(\mathbb{P}^1)^{2r} \xrightarrow{i^{2r}} (\Omega G)^{2r} \xrightarrow{\text{mult}} \Omega G$$

where $i : S^2 \cong \mathbb{P}^1 \to \Omega G \cong \Omega S^3$ is the standard inclusion. Therefore Φ_{2r}^* can be computed from knowledge of i_* and mult_*. The latter is the Pontrjagin multiplication in the Hopf algebra $H_*(\Omega S^3)$, which is the polynomial algebra on the image, y, under i_* of a generator of $H_2(S^2)$. Since y^{2r-1} is a group generator of this polynomial algebra in degree $4r-2$ the map Φ_{2r*} is surjective in degree $4r-2$. That is, $\lambda_1 = \lambda_2 = \pm 1$.

The sign is not critical to us — it does not even affect the determinant — so we omit the details, but finer analysis shows that $\lambda_1 = \lambda_2 = 1$, as claimed. $\qquad\square$

We are now in a position to prove Proposition 5.9.

Proof of Proposition 5.9. We begin by computing some determinants. From (5.32) we have

$$\det \mathbf{M} = p(\bar{t})\big(g_1(4\bar{t})\big)^2 - 2q(\bar{t})g_1(4\bar{t})g_2(4\bar{t}) + 2q(\bar{t})g_1(4\bar{t})g_2(4\bar{t}) - 4\bar{t}p(\bar{t})\big(g_2(4\bar{t})\big)^2$$
$$= p(\bar{t})\Big(\big(g_1(4\bar{t})\big)^2 - 4\bar{t}\big(g_2(4\bar{t})\big)^2\Big).$$

From the definitions (5.29) and (5.30) of g_1 and g_2, it follows that $\big(g_1(4\bar{t})\big)^2 - 4\bar{t}\big(g_2(4\bar{t})\big)^2 = \big(p(4\bar{t}/4)\big)^{2(2r-1)} = (p(\bar{t})^{2(2r-1)}$ so we conclude

$$(5.33) \qquad\qquad \det \mathbf{M} = p(\bar{t})\big(p(\bar{t})\big)^{2(2r-1)} = \big(p(\bar{t})\big)^{4r-1}.$$

For the matrix $\mathbf{N}$, the description (5.22) straightforwardly yields

$$(5.34) \qquad\qquad \det \mathbf{N} = \big(p(\bar{t})\big)^{2r-1}\big(p(\bar{t})\big)^{2r} = \big(p(t)\big)^{4r-1}.$$

We know from (5.12) that

$$\det \mathbf{N} \det \text{ch}_T(Q) = \det \bar{Q} \det \mathbf{M}$$

in the (torsion-free) module $H_T^\pi(\text{pt}; \mathbb{Q})$. The above computations of $\det \mathbf{M}$, $\det \mathbf{N}$, and $\bar{Q}$, together with Lemma 5.13, immediately yields

$$p(\bar{t})^{4r-1}\det\big(\text{ch}_T(Q)\big) = p(\bar{t})^{4r-1}$$

from which it follows that $\det\big(\text{ch}_T(Q)\big) = 1$. Since ch_T is an injective homomorphism we conclude $\det Q = 1$, hence in particular Q is invertible. This in turn implies that the images of U and xU under $\Phi_{2r}^* \circ \kappa^*$ form an $R(T)$-module basis for $\text{Ker}\, i^* \cap \left(K_T\big((\mathbb{P}^1)^{2r}\big)\right)^{S_{2r}}$. The statements of the proposition now follow. $\qquad\square$

Next we turn to the proof of Theorem 5.6. We will use the following statement which can be proven using the monomial basis for symmetric polynomials (see e.g. the classic text by MacDonald [21]).

Lemma 5.14. *Let R be a ring and let $p(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ be a symmetric polynomial which is linear in the variable x_j for all j. Then $p(x_1, \ldots, x_n)$ is an R-linear combination of the elementary symmetric polynomials*

$$s_0(x_1, \ldots, x_n), \ldots, s_n(x_1, \ldots, x_n).$$

$\qquad\square$

Proof of Theorem 5.6. Consider the base case $r = 0$. In this case the claim of the theorem is immediate since $F_0 = \text{pt}$. Now suppose by induction that the claim of the theorem holds for $r - 1$. We wish to prove the claim holds for r.

Due to the quadratic relation $L_j^2 = vL_j - 1$ in Theorem 3.13, any symmetric polynomial in the L_j in $K_T^*\big((\mathbb{P}^1)^{2r}\big)$ can be expressed as a symmetric polynomial which is linear in each L_j. By Lemma 5.14 it also follows that it can be written as an $R(T)$-linear combination of $s_0, \ldots, s_{2r}$. It follows that

$$\big(K_T^*((\mathbb{P}^1)^{2r})\big)^{S_{2r}} = \text{the } K_T^*(\text{pt})\text{-subalgebra of } K_T^*((\mathbb{P}^1)^{2r}) \text{ generated by } s_1, \ldots, s_{2r}$$
$$= \text{the } K_T^*(\text{pt})\text{-submodule of } K_T^*((\mathbb{P}^1)^{2r}) \text{ generated by } s_0, \ldots, s_{2r}.$$

Lemma 5.7 shows

$$\Phi_{2r}^*\big(K_T^*(F_{2r})\big) \subseteq (K_T^*((\mathbb{P}^1)^{2r}))^{S_{2r}},$$

and by Lemma 5.8, the claim proven in Proposition 5.9 suffices to show the containment in the other direction. Since Φ_{2r}^* is injective by Proposition 5.1, the claim of the theorem now follows.

$\square$

We also record the G-equivariant version.

Theorem 5.15. *Let $G = SU(2)$. Let $\Phi_{2r} : (\mathbb{P}^1)^{2r} \to F_{2r}$ be the map given in Definition 3.2. Then*

$$K_G^*(F_{2r}) \cong \Phi_{2r}^*\big(K_G^*(F_{2r})\big) = \{\text{symmetric polynomials in } K_G^*\big((\mathbb{P}^1)^{2r}\big)\}$$
$$= \text{the } K_G^*(\text{pt})\text{-subalgebra of } K_G^*\big((\mathbb{P}^1)^{2r}\big) \text{ generated by } s_1, \ldots, s_{2r}$$
$$= \text{the } K_G^*(\text{pt})\text{-submodule of } K_G^*\big((\mathbb{P}^1)^{2r}\big) \text{ generated by } s_0, \ldots, s_{2r}.$$

Proof. Since Φ_{2r}^* is an algebra injection, the theorem is equivalent to the statement that $\Phi_{2r}^*\big(K_G^*(F_{2r})\big)$ is the subalgebra of $K_G^*\big((\mathbb{P}^1)^n\big)$ generated by $s_0, s_1, \ldots, s_{2r}$. We know that $K_G^*(F_{2r}) \cong \big(K_T^*(F_{2r})\big)^W$ from [10]. In Section 2.4 we showed that

$$K_G^*\big((\mathbb{P}^1)^n\big) \cong \Big(K_T^*\big((\mathbb{P}^1)^n\big)\Big)^W.$$

Therefore, by taking Weyl-invariants, we conclude that $\Phi_{2r}^*\big(K_G^*(F_{2r})\big)$ is the submodule of $K_G^*\big((\mathbb{P}^1)^{2r}\big)$ generated by $s_0, \ldots, s_{2r}$. The other statements follow. $\square$

6. Proof of the main theorem

The following is now an immediate consequence of the previous results.

Theorem 6.1. *Let $G = SU(2)$ and let ΩG be the space of (continuous) based loops in G, equipped with the natural G-action by pointwise conjugation. Then*

$$K_G^*(\Omega G) = \varprojlim_r \Big(K_G^*\big((\mathbb{P}^1)^{2r}\big)\Big)^{S_{2r}} = \varprojlim_r \{\text{symmetric polynomials in } K_G^*\big((\mathbb{P}^1)^{2r}\big)\}$$

where $K_G^*\big((\mathbb{P}^1)^{2r}\big) \cong R(G)[L_1, \ldots, L_{2r}]/I$, and I is the ideal generated by $\{L_j^2 - vL_j + 1\}_{j=1}^n$. Here $R(G)$ is the representation ring of G, v is the standard representation of $G = SU(2)$ on $\mathbb{C}^2$

and L_j is the pullback of either the canonical line bundle over the jth factor of $(\mathbb{P}^1)^{2r}$ or its inverse, depending on j (see Definition 3.12). The maps in this inverse system are given by

$$i^*(s_j) = \begin{cases} s'_0 & \text{if } j = 0; \\ s'_1 + vs'_0 & \text{if } j = 1; \\ s'_j + vs'_{j-1} + s'_{j-2} & \text{if } 1 < j \leq 2r - 2; \\ vs'_{2r-2} + s'_{2r-3} & \text{if } j = 2r - 1; \\ s'_{2r-2} & \text{if } j = 2r, \end{cases}$$

where s_j and s'_j are respectively the jth elementary symmetric polynomials in $\{L_1, \ldots, L_{2r}\}$ and $\{L'_1, \ldots, L'_{2r-1}\}$.

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