THE GEOMETRIC INVARIANTS OF GROUP EXTENSIONS

NIC KOBAN AND PETER WONG

ABSTRACT. In this paper, we compute the $\Sigma^n(G)$ and $\Omega^n(G)$ invariants when $1 \to H \to G \to K \to 1$ is a short exact sequence of finitely generated groups with K finite. We also give sufficient conditions for G to have the R_∞ property in terms of $\Omega^n(H)$ and $\Omega^n(K)$ when either K is finite or the sequence splits. As an application, we construct a group $F \rtimes_\rho \mathbb{Z}_2$ where F is the R. Thompson's group F and show that $F \rtimes_\rho \mathbb{Z}_2$ has the R_∞ property while F is not characteristic.

1. Introduction

The Bieri-Neumann-Strebel-Renz invariants (and their homological analogs) $\Sigma^n(G)$ of a group have been useful in obtaining finiteness properties of subgroups of G with abelian quotients. Connections to other areas of mathematics have been made while the computation of these invariants remains difficult in general. In fact, the so-called direct product conjecture for $\Sigma^n(H \times K)$ has been shown to be false in general ([21] for the homotopical version and [22] for the homological, although in [2], the product conjecture for the homological version of the Σ-invariants is proven over a field). On the other hand, an analogous geometric invariant $\Omega^n(G)$ has been introduced and has proven somewhat easier to compute. For example, the product formula $\Omega^n(H \times K) = \Omega^n(H) \circledast \Omega^n(K)$, the spherical join of $\Omega^n(H)$ and $\Omega^n(K)$, holds. More recently, the product formula for Ω^n has been employed to yield new families of groups for which the R_∞ property holds [18]. The R_∞ property arises from the study of twisted conjugacy classes of elements of the fundamental group in topological fixed point theory (see § 6).

Motivated by [18] and the product formula for Ω^n [17], we conjectured similar formulas for Ω^1 for finite and split extensions in the unpublished manuscripts [19] and [20]. We have since found counterexamples to the formulas. Although these formulas are false in general, we have found use for the formulas which we discuss in § 6 of this paper.

The main objective of this paper is to use the Σ- and Ω-invariants to detect the R_∞ property for an extension $1 \to H \to G \to K \to 1$ where either K is finite or the sequence

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splits. As an application, we construct a finitely presented group with Ω^2 consisting of a single discrete point and hence with the R_∞ property whereas Ω^1 contains two antipodal points from which the R_∞ property cannot be detected. This is the first example known to the authors where $n = 2$ is needed to obtain this property where $n = 1$ fails to satisfy the conditions.

This paper is organized as follows. The invariants Ω^n were defined in [16] and are analogs of the Bieri-Neumann-Strebel-Renz invariants Σ^n defined in [4] for $n = 1$ and in [5] for $n \geq 2$. We recall these definitions in § 2.1 and § 2.2. In § 3, we describe the real vector space of characters, $\mathrm{Hom}(G, \mathbb{R})$, for a finite and split extension G in terms of $\mathrm{Hom}(H, \mathbb{R})$, $\mathrm{Hom}(K, \mathbb{R})$, and the action of K on H. In § 4, we prove the formula for $\Sigma^1(G)$ where G is a finite extension. In § 5, we investigate the conjectured Ω-formula for finite extensions, and give examples where this formula fails and conditions when the formula holds. In § 6, we give conditions using the Ω-invariant to detect the R_∞ property in finite and split extensions.

2. The Geometric Invariants Σ and Ω

Let G be a finitely generated group with generating set S. In this section, we define two invariants of G:

(1) the Bieri-Neumann-Strebel (or BNS) invariant Σ, and
(2) the invariant Ω.

2.1. **The BNS invariant Σ.** The set $\mathrm{Hom}(G, \mathbb{R})$ of homomorphisms from G to the additive group of reals is a real vector space with dimension equal to the $\mathbb{Z}$-rank of the abelianization of G, so $\mathrm{Hom}(G, \mathbb{R}) \cong \mathbb{R}^m$ for some m. Thus, there is a natural isomorphism between $\mathrm{Hom}(G, \mathbb{R})$ and the real vector space $G/G' \otimes_{\mathbb{Z}} \mathbb{R}$. The group G acts on $G/G' \otimes_{\mathbb{Z}} \mathbb{R}$ by left multiplication on the G/G' component, and this gives an action of G on $\mathrm{Hom}(G, \mathbb{R})$ (by translations).

Let Γ denote the Cayley graph of G with respect to a chosen generating set. Define $\hbar : \Gamma \to \mathbb{R}^m$ to be the abelianization map on the vertices and extend linearly on the edges. Denote by $\partial_\infty \mathbb{R}^m$ the boundary at infinity of $\mathbb{R}^m$ (ie. the set of geodesic rays in $\mathbb{R}^m$ initiating from the origin). This is isomorphic to the *character sphere of* G defined as the set of equivalence classes $S(G) := \{[\chi] | \chi \in \mathrm{Hom}(G, \mathbb{R}) - \{0\}\}$ where $\chi_1 \sim \chi_2$ if and only if $\chi_1 = r\chi_2$ for some $r > 0$. Let $e \in \partial_\infty \mathbb{R}^m$ and let γ be the geodesic ray defining e. We denote by H_e the half space perpendicular to γ that contains all of the image of γ. Denote by Γ_e the largest subgraph of Γ that is contained in $\hbar^{-1}(H_e)$. The direction $e \in \Sigma^1(G)$ if Γ_e is path connected.

We give an equivalent definition for $\Sigma^1(G)$ that we will use in this paper. It will also be useful to see the motivation for the definition of $\Omega^1(G)$. Let $e \in \partial_\infty \mathbb{R}^m$ and let γ be a geodesic ray defining e. For each $s \in \mathbb{R}$, let $H_{\gamma,s}$ be the closed half-space orthogonal to γ so that $H_{\gamma,s} \cap \gamma([0,\infty)) = \gamma([s,\infty))$. For each $s \in \mathbb{R}$, denote by $\Gamma_{\gamma,s}$ the largest subgraph of Γ contained in $\hbar^{-1}(H_{\gamma,s})$. The direction $e \in \Sigma^1(G)$ if and only if for every $s \geq 0$, there exists $\lambda = \lambda(s) \geq 0$ such that any two points $u, v \in \Gamma_{\gamma,s}$ can be joined by a path in $\Gamma_{\gamma,s-\lambda}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$.

2.2. **The invariant** Ω**.** In the compactified space $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$, the compactified half-spaces play the role of neighborhoods of the point $e \in \partial_\infty \mathbb{R}^m$, but this gives an unsatisfactory topology to $\mathbb{R}^m \cup \partial_\infty \mathbb{R}^m$. From the point of view of topology, it is more natural to have a similar definition to $\Sigma^1(G)$ using "ordinary" neighborhoods of e. A basis for these neighborhoods consists of "truncated cones". For each $s \geq 0$, define the *truncated cone* $C_{\gamma,s} := Cone_\theta(\gamma) \cap H_{\gamma,s}$ where $Cone_\theta(\gamma)$ is the closed cone of angle θ and vertex $\gamma(0)$ and $\theta := \arctan(\frac{1}{s})$ if $s > 0$ and $\theta := \frac{\pi}{2}$ if $s = 0$. For each $s \geq 0$, denote by $\Delta_{\gamma,s}$ the largest subgraph of Γ contained in $\hbar^{-1}(C_{\gamma,s})$. We say that $e \in \Omega^1(G)$ if and only if there exists $s_0 \geq 0$ such that for each $s \geq s_0$, there exists $\lambda = \lambda(s) \geq 0$ such that any two points $u, v \in \Delta_{\gamma,s}$ can be joined by a path in $\Delta_{\gamma,s-\lambda}$ and $s - \lambda(s) \to \infty$ as $s \to \infty$.

When $n > 1$, we can make the following changes to the definitions to obtain $\Sigma^n(G)$ and $\Omega^n(G)$: replace G being finitely generated with type F_n, replace the Cayley graph Γ with an n-dimensional, $(n-1)$-connected CW-complex X on which G acts freely as a group of cell permuting homeomorphisms with $G \backslash X$ a finite complex, $\hbar$ is a G-map from X to $\mathrm{Hom}(G, \mathbb{R})$, and replace the path-connected property in the definition with the analogous $(n-1)$-connected property (see [5] for Σ and [16] for Ω). We should mention that $\Omega^n(G)$ is always a closed set while $\Sigma^n(G)$ is open.

The following theorem relates the invariants $\Sigma^n(G)$ and $\Omega^n(G)$.

Theorem 2.1. [16, Theorem 3.1] *Let $e \in \partial_\infty \mathbb{R}^m$. Then $e \in \Omega^n(G)$ if and only if $e' \in \Sigma^n(G)$ for every e' in an open $\frac{\pi}{2}$-neighborhood of e.*

Given $\Sigma^n(G)$, we can completely determine $\Omega^n(G)$: for each $e \in \partial_\infty \mathbb{R}^m$, $e \in \Omega^n(G)$ if and only if the open $\frac{\pi}{2}$-neighborhood of e is in $\Sigma^n(G)$. However, it is not the case that $\Omega^n(G)$ completely determines $\Sigma^n(G)$; examples of such groups are given in [16, § 1.3].

The following theorem completely describes $\Omega^n(H \times K)$ in terms of $\Omega^n(H)$ and $\Omega^n(K)$. This theorem will be useful in § 6.

Theorem 2.2. [17, Theorem 3.8] *$\Omega^n(H \times K) = \Omega^n(H) \circledast \Omega^n(K)$ where $\circledast$ represents the spherical join.*

3. The Space $\mathrm{Hom}(G,\mathbb{R})$

Consider a group extension G given by the following short exact sequence of groups

$$1 \to H \xrightarrow{i} G \xrightarrow{p} K \to 1.$$

Without loss of generality we may assume that the monomorphism i is an inclusion so that H is identified with its image in G as a normal subgroup. Let $\tilde{K}$ be a left transversal for H, i.e., $\tilde{K} = \nu(K)$ where $\nu : K \to G$ is an injective function so that $(p \circ \nu)(k) = k$ for all $k \in K$. In addition, we assume that $\nu(1_K) = 1_G$. There is an *"action"* of $\tilde{K}$ on $\mathrm{Hom}(H,\mathbb{R})$ given by

$$(\tilde{k} \cdot \phi)(h) = \phi(\tilde{k}h\tilde{k}^{-1})$$

for all $h \in H$ where $\phi \in \mathrm{Hom}(H,\mathbb{R})$.

Let

$$Fix\hat{\nu} = \{\phi \in \mathrm{Hom}(H,\mathbb{R}) | \tilde{k} \cdot \phi = \phi, \ \forall \tilde{k} \in \tilde{K} = \nu(K)\}.$$

Note that $Fix\hat{\nu}$ is a vector subspace of $\mathrm{Hom}(H,\mathbb{R})$.

Remark 3.1. *If $\nu_1, \nu_2 : K \to G$ are two left transversals for H then $Fix\hat{\nu}_1 = Fix\hat{\nu}_2$. To see this, let $\phi \in Fix\hat{\nu}_1$ and $\tilde{k}_2 \in \nu_2(K)$. Write $\tilde{k}_2 = \nu_2(k)$ for some $k \in K$. Since $(p \circ \nu_i)(k) = k$ for $i = 1, 2$, it follows that $\nu_2(k) = \nu_1(k)h'$ for some $h' \in H$. Now,*

$$\begin{aligned}
(\nu_2(k) \cdot \phi)(h) &= \phi(\nu_2(k)h\nu_2(k)^{-1}) \\
&= \phi(\nu_1(k)h'h(h')^{-1}\nu_1(k)^{-1}) \\
&= \phi(h'h(h')^{-1}) \quad \textit{since } \phi \in Fix\hat{\nu}_1 \\
&= \phi(h).
\end{aligned}$$

Thus, $\phi \in Fix\hat{\nu}_2$. A similar argument shows that if $\phi \in Fix\hat{\nu}_2$ then $\phi \in Fix\hat{\nu}_1$.

In the special case when $G = H \rtimes_\rho K$ is the semi-direct product given by an action $\rho : K \to \mathrm{Aut}(H)$, the canonical left transversal ν is the section given by $\nu(k) = (1, k)$. Then $Fix\hat{\nu} = \{\phi \in \mathrm{Hom}(H,\mathbb{R}) | \, \phi(\rho(k)(h)) = \phi(h) \text{ for all } h \in H, k \in K\}$. In this case, we also write $Fix\hat{\rho}$ for $Fix\hat{\nu}$ as the fixed subspace induced by the action ρ.

Following [15], we will use the presentations for H and K to derive a presentation for G. Let $H \cong \langle A|R\rangle$ and $K \cong \langle B|S\rangle$. Since every word in S is equivalent to the identity, $\nu(S) \subseteq ker(p) = H$. Thus, every word $\nu(S)$ is equivalent to some word w_s in A. Denote by $X := \{\nu(s)w_s^{-1} | s \in S$ and w_s is the equivalent word to $\nu(s)$ in $A\}$. For each $\tilde{b} \in \nu(B)$ and each $a \in A$, there is a word $w_{a,b} \in H$ such that $\tilde{b}a\tilde{b}^{-1} = w_{a,b}$. Let $Y := \{\tilde{b}a\tilde{b}^{-1}w_{a,b}^{-1} | \tilde{b} \in \nu(B); a \in A\}$. Then $G \cong \langle A \cup \nu(B) | R \cup X \cup Y\rangle$.

Proposition 3.1. *Let G be a finite extension given by the short exact sequence of groups*

$$1 \to H \xrightarrow{i} G \xrightarrow{p} K \to 1$$

where K is finite and let $\nu : K \to G$ be a left transversal for H such that $\nu(1_K) = 1_G$. Then $\mathrm{Hom}(G, \mathbb{R}) \cong Fix\hat{\nu}$.

Proof. Suppose $\phi \in Fix\hat{\nu}$. Thus, ϕ is defined on A, so we need only define ϕ on B so that it satisfies the relations X and Y of G to get a homomorphism from G to the reals. Since K is finite, for each $b \in B$, there is a relation in S of the form b^m for some integer $m \geq 1$. Thus, there is a word w in A such that $\tilde{b}^m w^{-1} = 1$, so $\phi(\tilde{b}) = \frac{\phi(w)}{m}$. Since $\phi \in Fix\hat{\nu}$, the relations in Y are satisfied, and obviously the relations in X are satisfied. Let $\hat{\phi}$ be this extension of ϕ. Define $T : Fix\hat{\nu} \to \mathrm{Hom}(G)$ by $T(\phi) = \hat{\phi}$.

If $\phi \in \mathrm{Hom}(G, \mathbb{R})$ then define $Q : \mathrm{Hom}(G, \mathbb{R}) \to Fix\hat{\nu}$ by $Q(\phi) = \phi \circ i$. A priori, $Q(\phi) \in \mathrm{Hom}(H, \mathbb{R})$. To see that the image actually lies in $Fix\hat{\nu}$, we note that

$$\begin{aligned}
\tilde{k} \cdot (\phi \circ i))(h) &= (\phi \circ i)(\tilde{k} h \tilde{k}^{-1}) \\
&= \phi(\tilde{k} h \tilde{k}^{-1}) \\
&= \phi(\tilde{k})\phi(h)\phi(\tilde{k})^{-1} \qquad [\text{since } \phi \in \mathrm{Hom}(G, \mathbb{R})] \\
&= \phi(h).
\end{aligned}$$

It follows that $(\phi \circ i) \in Fix\hat{\nu}$. It is easy to see that $Q \circ T$ and $T \circ Q$ yield identity maps and thus the assertion follows. $\square$

We now give the analogous proposition in the split extension case.

Proposition 3.2. *Let G be a split extension given by the short exact sequence of groups*

$$1 \to H \xrightarrow{i} G \xrightarrow{p} K \to 1$$

and let $\nu : K \to G$ be a left transversal for K such that $\nu(1_K) = 1_G$. Then $\mathrm{Hom}(G, \mathbb{R}) \cong Fix\hat{\rho} \times \mathrm{Hom}(K, \mathbb{R})$.

Proof. Define $\Phi : \mathrm{Hom}(G, \mathbb{R}) \to Fix\hat{\rho} \times \mathrm{Hom}(K, \mathbb{R})$ by $\phi \mapsto (\phi \circ i, \phi \circ \sigma)$. To show that Φ is well-defined, we first show that $\phi \circ i \in Fix\hat{\rho}$. Let $h \in H$ and $k \in K$, so

$$(k \cdot \phi \circ i)(h) = \phi \circ i(\tilde{k} h \tilde{k}^{-1}) = \phi(\tilde{k} h \tilde{k}^{-1}) = \phi(\tilde{k}) + \phi(h) - \phi(\tilde{k}) = \phi \circ i(h).$$

Define $\Psi : Fix\hat{\rho} \times \mathrm{Hom}(K, \mathbb{R}) \to \mathrm{Hom}(G, \mathbb{R})$ by $(\alpha, \beta) \mapsto (\hat{\alpha} + \beta \circ \pi)$ where $\hat{\alpha}(g) = \hat{\alpha}(h\tilde{k}) := \alpha(h)$ and $\tilde{k} = \sigma \circ \pi(g)$. Since $ker(\pi) = H$, $(\beta \circ \pi)(h) = 0$ for all $h \in H$. To show that $\hat{\alpha}$ is a homomorphism, we note that since $\alpha \in Fix\hat{\rho}$, we have $\alpha(\tilde{k} h \tilde{k}^{-1}) = \alpha(h)$. Therefore,

$$\hat{\alpha}(g_1 g_2) = \hat{\alpha}(h_1 \tilde{k}_1 h_2 \tilde{k}_2) = \hat{\alpha}(h_1(\tilde{k}_1 h_2 \tilde{k}_1^{-1})\tilde{k}_1 \tilde{k}_2) = \alpha(h_1)\alpha(\tilde{k}_1 h_2 \tilde{k}_1^{-1}) = \alpha(h_1)\alpha(h_2) = \hat{\alpha}(g_1)\hat{\alpha}(g_2).$$

To see that Φ and Ψ are inverses, we have

$$\Phi \circ \Psi(\alpha, \beta)(h,k) = ((\hat{\alpha} + \beta \circ \pi) \circ i, (\hat{\alpha} + \beta \circ \pi) \circ \sigma)(h,k) = (\hat{\alpha}(h) + \beta \circ \pi(h), \beta \circ \pi(\sigma(k))) = (\alpha(h), \beta(k))$$

and

$$\Psi \circ \Phi(\phi)(g) = (\widehat{\phi \circ i} + \phi \circ \sigma \circ \pi)(g) = \widehat{\phi \circ i}(h\tilde{k}) + \phi(\sigma(\pi(h\tilde{k}))) = \phi(h) + \phi(\tilde{k}) = \phi(g).$$

□

4. The Σ-invariant for finite extensions

In this section, we prove for a finite extension $1 \to H \to G \to K \to 1$ that $\Sigma^1(G) = \partial_\infty Fix\hat{\nu} \cap \Sigma^1(H)$. We should mention that a more general result for finite index subgroups was given in [21].

Theorem 4.1. [21, Theorem 9.3] *Suppose that $H \leq G$ is a subgroup of finite index, and that χ restricts to a non-zero homomorphism of H. Then $[\chi|_H] \in \Sigma^n(H)$ if and only if $[\chi] \in \Sigma^n(G)$.*

We give a geometric proof of Theorem 4.3 for $n = 1$. The authors would like to thank Dessislava Kochloukova for pointing us to the result in [21].

Proposition 4.2. *Given an extension $1 \to H \to G \to K \to 1$ where K is finite, if H is finitely generated, then $\Sigma^1(G) \subseteq \Sigma^1(H)$.*

Proof. Since H is finitely generated and K is finite, G is finitely generated. First, we note that since $\mathrm{Hom}(G,\mathbb{R}) = Fix\hat{\nu}$, we have $\mathrm{Hom}(G,\mathbb{R}) \subseteq \mathrm{Hom}(H,\mathbb{R})$ as a vector subspace. Let Γ_H and Γ_G be the Cayley graphs of H and of G with respect to the generating sets A and $A \cup \nu(B)$ respectively. For any height function $\hbar : \Gamma_G \to \mathrm{Hom}(G,\mathbb{R})$, we have an induced height function $\bar{\hbar} : \Gamma_H \to \mathrm{Hom}(H,\mathbb{R})$ such that the following diagram commutes:

$$\begin{tikzcd} \Gamma_H \arrow[r, "\bar{\hbar}"] \arrow[d, "\hat{i}"'] & \mathrm{Hom}(H,\mathbb{R}) \\ \Gamma_G \arrow[r, "\hbar"] & \mathrm{Hom}(G,\mathbb{R}) \arrow[u, "\hat{Q}"'] \end{tikzcd}$$

where $\hat{Q}$ is the induced inclusion $\mathrm{Hom}(G,\mathbb{R}) \subseteq \mathrm{Hom}(H,\mathbb{R})$ from the proof of Proposition3.1, and $\hat{i}$ is induced by the inclusion $i : A \subseteq A \cup \tilde{B}$. In fact, $\hat{i}$ is the inclusion of Γ_H as the subgraph induced by the vertices $\{h\nu(1_K) | h \in H\}$ in Γ_G.

Let $e \in \Sigma^1(G)$. We will show that $e \in \Sigma^1(H)$. First, take a half space $\bar{H}_{e,s}$ for e in $\mathrm{Hom}(H,\mathbb{R})$. Since $\mathrm{Hom}(G,\mathbb{R}) \subseteq \mathrm{Hom}(H,\mathbb{R})$, there is a unique half space $H_{e,s}$ in $\mathrm{Hom}(G,\mathbb{R})$ such that $H_{e,s} = \bar{H}_{e,s} \cap \mathrm{Hom}(G,\mathbb{R})$. Take x and y to be vertices in $\bar{\hbar}^{-1}(\bar{H}_{e,s})$. Since $\bar{\hbar} = \hat{Q} \circ \hbar \circ \hat{i}$, it follows that $\bar{\hbar}^{-1}(\bar{H}_{e,s}) \subseteq \hbar^{-1}(H_{e,s})$ and thus $x, y \in \hbar^{-1}(H_{e,s})$, that is, there exist $g_1, g_2, ..., g_m \in (A \cup \tilde{B})$ such that $y = xg_1g_2...g_m$ and for each $1 \leq i \leq m$, $\hbar(xg_1 \ldots g_i) \subset H_{e,s}$. Let $\mu := \min\{\hbar(\tilde{k}|\tilde{k} \in \nu(K)\}$. We can rewrite the path from y to x as $h_1k_1h_2k_2 \ldots h_{p-1}k_{p-1}h_p$ where each $h_j \in H$ and each $k_j \in \nu(K)$ with the possibility that h_1 and h_p are the identity element. Since H is normal, we have for each $1 \leq i \leq p-1$, $k_ih_{i+1} = \bar{h}_{i+1}k_i$, and $\hbar(x\bar{h}_1 \ldots \bar{h}_{i+1}) \subset H_{e,s-\mu}$. Therefore, $y = x\bar{h}_1 \ldots \bar{h}_pk_1 \ldots k_{p-1}$, and since $x, y, (\bar{h}_1 \ldots \bar{h}_p) \in H$, we have that $k_1 \ldots k_{p-1}$ is trivial (otherwise it would be a non-trivial element of H). Thus, $y = x\bar{h}_1 \ldots \bar{h}_p$ and for each $1 \leq i \leq p$, $\hbar(x\bar{h}_1 \ldots \bar{h}_i) \subset H_{e,s-\mu}$. Hence x and y are connected in $\bar{\hbar}^{-1}(\bar{H}_{e,s})$ or $e \in \Sigma^1(H)$. □

Remark 4.1. *Note that the inclusion* $\mathrm{Hom}(G, \mathbb{R}) \subseteq \mathrm{Hom}(H, \mathbb{R})$ *of Proposition 4.2 can be strict. For instance, take* $H = \mathbb{Z}^2$ *and* G *to be the fundamental group of the Klein bottle with* $K = \mathbb{Z}_2$*. Here,* $\mathrm{rk}_{\mathbb{Z}}(G) = 1 < 2 = \mathrm{rk}_{\mathbb{Z}}(\mathbb{Z}^2)$*. Moreover, Proposition 4.2 is false if* K *is not finite, e.g., take* $G = \mathbb{Z}^2$ *and* $H = \mathbb{Z}$*.*

Theorem 4.3. *Let* G *be a finite extension given by the short exact sequence of groups*

$$1 \to H \xrightarrow{i} G \xrightarrow{p} K \to 1$$

where K *is finite,* H *is finitely generated, and let* $\nu : K \to G$ *be a left transversal for* K *such that* $\nu(1_K) = 1_G$*. Then,* $\Sigma^1(G) = \partial_\infty Fix\hat{\nu} \cap \Sigma^1(H)$*.*

Proof. By Prop. 4.2, $\Sigma^1(G) \subseteq \Sigma^1(H)$. Since K is finite, we have $\mathrm{Hom}(G, \mathbb{R}) = Fix\hat{\rho}$ by Prop. 3.1. Thus, $\Sigma^1(G) \subseteq \partial_\infty \mathrm{Hom}(G) = \partial_\infty Fix\hat{\rho}$, and we have $\Sigma^1(G) \subseteq \partial_\infty Fix\hat{\rho} \cap \Sigma^1(H)$.

Suppose $e \in \partial_\infty Fix\hat{\rho} \cap \Sigma^1(H)$ and suppose γ is a geodesic ray defining e. To show that $e \in \Sigma^1(G)$, let $s \in \mathbb{R}$, and let $H_{\gamma,s}$ be the corresponding half space of e in $\mathrm{Hom}(G, \mathbb{R}) = Fix\hat{\rho} \subseteq \mathrm{Hom}(H, \mathbb{R})$. Let $\pi : G \to G/G' \cong \mathbb{Z}^m$ be the natural projection epimorphism. Let Γ_G be the Cayley graph of G with respect to the generating set $A \cup \nu(B)$ from the presentations $H = \langle A|R \rangle$ and $K = \langle B|S \rangle$. Define $\hbar : \Gamma_G \to \mathrm{Hom}(G, \mathbb{R}) \cong \mathbb{R}^m$ by: $\hbar(g) = \pi(g)$ for all vertices $g \in \Gamma_G$, and extend linearly on edges. Choose two points $x, y \in \hbar^{-1}(H_{\gamma,s})$. Since G is a finite extension, x and y can be uniquely written as $x = h_1\tilde{k}_1$ and $y = h_2\tilde{k}_2$ for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Let $\lambda_1 := \min\{d(\hbar(\tilde{k}), H_{\gamma,0})|\tilde{k} \in \nu(K)\}$ where $d(\hbar(\tilde{k}), H_{\gamma,0})$ is the distance between the point $\hbar(\tilde{k})$ and the half space $H_{\gamma,0}$. Since $x, y \in \hbar^{-1}(H_{\gamma,s})$, we have $h_1, h_2 \in \hbar^{-1}(H_{\gamma,s-\lambda_1})$. Since $e \in \Sigma^1(H)$, there exists $\lambda_2 \geq 0$ such that there is a path w in $(\Gamma_H)_{\gamma,s-\lambda_1-\lambda_2} \subseteq (\Gamma_G)_{\gamma,s-\lambda_1-\lambda_2}$ from h_1 to h_2. Thus, $\tilde{k}_1^{-1}w\tilde{k}_2$ is a path in $(\Gamma_G)_{\gamma-\lambda_1-\lambda_2}$ from x to y. Since K is finite, $s - \lambda_1 - \lambda_2 \to \infty$ as $s \to \infty$. □

Remark 4.2. *It should be noted that due to Theorem 4.1 and Proposition 3.1 the result in Theorem 4.3 holds for all* $n \geq 1$*. We will use this fact in later constructions in this paper.*

Corollary 4.4. *For the semi-direct product* $H \rtimes_\rho K$ *of a finitely generated group* H *and a finite group* K*, we have* $\Sigma^1(H \rtimes_\rho K) = \Sigma^1(H) \cap \partial_\infty Fix\hat{\rho}$*.*

Example 4.3. *Consider the fundamental group of the Klein Bottle with the standard presentation*

$$G = \langle \alpha, \beta | \alpha\beta\alpha\beta^{-1} = 1 \rangle.$$

The subgroup generated by α *and* β^2 *is isomorphic to* $\mathbb{Z}^2$ *since* $\alpha\beta^2 = \alpha\beta(\alpha\beta\alpha) = (\alpha\beta\alpha)(\beta\alpha) = \beta(\beta(\alpha) = \beta^2\alpha$*. This subgroup is the fundamental group of the 2-torus as a double cover of the Klein Bottle. Moreover,* G *admits the following short (non-split) exact sequence*

$$0 \to \langle \alpha, \beta^2 \rangle \to G \xrightarrow{p} \mathbb{Z}_2 = \langle \bar{\beta} | \bar{\beta}^2 = \bar{1} \rangle \to 0$$

where the projection p *sends* α *to* $\bar{1}$ *and* β *to* $\bar{\beta}$*.It follows from Theorem 4.3 that* $\Sigma^1(G) = \Sigma^1(\mathbb{Z}^2) \cap \partial_\infty Fix\hat{\nu}$ *where* $\nu : \mathbb{Z}_2 \to G$ *is given by* $\bar{\beta} \mapsto \beta$*. Thus,*

$$\begin{aligned}\Sigma^1(G) &= \mathbb{S}^1 \cap \partial_\infty Fix\hat{\nu} \\ &= \mathbb{S}^1 \cap \partial_\infty \{\phi \in \mathrm{Hom}(\mathbb{Z}^2) | \beta \cdot \phi = \phi\} \\ &= \mathbb{S}^1 \cap \partial_\infty \{\phi \in \mathrm{Hom}(\mathbb{Z}^2) | \phi(\beta h \beta^{-1}) = \phi(h), \forall h \in H\}.\end{aligned}$$

Note that $H = \langle \alpha, \beta^2 \rangle$ *and* $\phi(\beta\alpha\beta^{-1}) = \phi(\alpha)$ *implies that* $\phi(\alpha^{-1}) = \phi(\alpha)$ *which in turn implies that* $\phi(\alpha) = 0$*. This implies that* $Fix\hat{\nu} = \mathbb{R}$ *and so* $\Sigma^1(G) = \{\pm\infty\}$*. Since* $\mathrm{Hom}(G, \mathbb{R}) = Fix\hat{\nu} = \mathbb{R}$ *is one dimensional, we have* $\Omega^1(G) = \Sigma^1(G) = \{\pm\infty\}$*.*

Example 4.4. *Consider the infinite dihedral group* D_∞*. It is known that* $D_\infty \cong \mathbb{Z}_2 * \mathbb{Z}_2$*, the free product of* $\mathbb{Z}_2$ *with* $\mathbb{Z}_2$*. Moreover, it is also isomorphic to* $\mathbb{Z} \rtimes_\rho \mathbb{Z}_2$ *where the action* ρ *is the non-trivial one. It is easy to see that* $Fix\hat{\rho}$ *is the origin* $\{0\}$ *so that* $\partial_\infty Fix\hat{\rho} = \emptyset$*. It follows from Theorem 4.3 that* $\Sigma^1(D_\infty) = \emptyset = \Omega^1(D_\infty)$*. On the other hand,* D_∞ *admits the following non-split extension*

$$1 \to (\mathbb{Z}_2 * \mathbb{Z}_2)' \to \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 0$$

where Z' *denotes the commutator subgroup of a group* Z*. Furthermore,* $(\mathbb{Z}_2 * \mathbb{Z}_2)'$ *is isomorphic to the infinite cyclic group* $\mathbb{Z}$ *(see e.g. Exercise 10 on p.134 of* [15]*). Again similar arguments show that, using Theorem 4.3, that* $\Sigma^1(D_\infty) = \emptyset = \Omega^1(D_\infty)$*.*

5. The Ω-invariant for finite extensions

The authors originally conjectured that $\Omega^1(G) = \Omega^1(H) \cap \partial_\infty Fix\hat{\nu}$. This however turned out not to be true as the following examples show.

Example 5.1. *Recall that the R. Thompson's group* F *can be given the following presentation*

$$F = \langle x_0, x_1, x_2, ... | x_k^{-1} x_n x_k = x_{n+1}, k < n \rangle.$$

The elements x_0 and x_1 correspond to the following piecewise linear homeomorphisms of the unit interval:

$$x_0(t) = \begin{cases} \frac{t}{2}, & 0 \le t \le \frac{1}{2} \\ t - \frac{1}{4}, & \frac{1}{2} \le t \le \frac{3}{4} \\ 2t-1, & \frac{3}{4} \le t \le 1 \end{cases} \tag{1}$$

and

$$x_1(t) = \begin{cases} t, & 0 \le t \le \frac{1}{2} \\ \frac{t}{2} + \frac{1}{4}, & \frac{1}{2} \le t \le \frac{3}{4} \\ t - \frac{1}{8}, & \frac{3}{4} \le t \le \frac{7}{8} \\ 2t-1, & \frac{7}{8} \le t \le 1 \end{cases} \tag{2}$$

The 180 degree rotation of the square $[0,1] \times [0,1]$ centered at the point $(\frac{1}{2}, \frac{1}{2})$ induces an order 2 automorphism ρ of the group F. It is easy to see that F can be generated by x_0 and x_1. This automorphism ρ is given by $\rho(x_0) = x_0^{-1}$ and $\rho(x_1) = x_0 x_1 x_0^{-2}$. Using the automorphism ρ of F, we form the semi-direct product $G = F \rtimes_\rho \mathbb{Z}_2$. The vector space Hom$(G, \mathbb{R}) = \{\chi | \chi(x_0) = 0\} \cong \mathbb{R}^1$, and we use Theorem 4.3 to show that $\Sigma^1(G) = \{\pm\infty\}$. To see this, first recall that $\Sigma^1(F)^c = \{[\chi_1], [\chi_2]\}$ where $\chi_1(x_0) = 1$, $\chi_1(x_1) = 0$, and $\chi_2(x_0) = \chi_2(x_1) = -1$. Since neither of these points are in $\partial Fix\hat{\rho}$, by Theorem 4.3, $\Sigma^1(G) = \Sigma^1(F) \cap \partial Fix\hat{\rho} = \{\pm\infty\}$. Since Hom$(G, \mathbb{R})$ is one-dimensional, we have that $\Omega^1(G) = \Sigma^1(G)$. Using the $\pi/2$-neighborhood result of Theorem 2.1, it follows that $\Omega^1(F)$ is a single arc which contains the north pole $+\infty$ but not the south pole $-\infty$. Thus $\Omega^1(F) \cap \partial Fix\hat{\rho} = \{+\infty\}$. Thus, $\Omega^1(G) \neq \Omega^1(H) \cap \partial_\infty Fix\hat{\nu}$ in general.

Example 5.2. *Let $H \cong \langle a, b | b^{-1}ab = a^2 \rangle \times \langle c, d | d^{-1}cd = c^2 \rangle \times \langle x, y \rangle$ (so H is the product of two Baumslag-Solitar groups and the free group on two generators), and define the action ρ of $\mathbb{Z}_2 \cong \langle t | t^2 = 1 \rangle$ on H by $t \cdot a = c, t \cdot b = d, t \cdot c = a, t \cdot d = b, t \cdot x = y$, and $t \cdot y = x$. Let $G \cong H \rtimes_\rho \mathbb{Z}_2$. The vector space Hom$(H, \mathbb{R}) \cong \mathbb{R}^4$ as any homomorphism must send a and c to zero, and $Fix\hat{\rho} = \{\phi | \phi(b) = \phi(d); \phi(x) = \phi(y)\} \cong \mathbb{R}^2$. The complement of $\Sigma^1(H)$ is the set $\{[\chi] | \chi(b) = \chi(d) = 0\} \cup \{[\chi] | \chi(x) = \chi(y) = \chi(b) = 0; \chi(d) = -1\} \cup \{[\chi] | \chi(x) = \chi(y) = \chi(d) = 0; \chi(b) = -1\}$. Thus, by Theorem 4.1, the complement of $\Sigma^1(G)$ is the two-point set $\{[\chi] | \chi(b) = \chi(d) = 0\}$. By Theorem 2.1, $\Omega^1(G)$ is the two-point set $\{[\chi] | \chi(x) = \chi(y) = 0\}$. However, by Theorem 2.2, $\Omega^1(H) = \{[\chi] | \chi(x) = \chi(y) = 0; \chi(b) > 0; \chi(d) > 0\}$, so $\Omega^1(H) \cap \partial_\infty Fix\hat{\rho}$ is the one-point set $\{[\chi] | \chi(x) = \chi(y) = 0; \chi(b) = \chi(d) = 1\}$.*

We do have the following containments.

Proposition 5.1. *$\Omega^n(H) \cap \partial_\infty Fix\hat{\nu} \subseteq \Omega^n(G) \subseteq \Sigma^n(H) \cap \partial_\infty Fix\hat{\nu}$.*

Proof. To see that $\Omega^n(G) \supseteq \Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$, if $[\chi] \in \Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$, then the open $\frac{\pi}{2}$-neighborhood of $[\chi]$ in $\partial_\infty \mathrm{Hom}(H, \mathbb{R})$, denoted $N^H_{\pi/2}([\chi])$, is contained in $\Sigma^n(H)$. Thus, $N^G_{\pi/2}([\chi]) = N^H_{\pi/2}([\chi]) \cap \partial_\infty Fix\hat{\nu} \subseteq \Sigma^n(H) \cap \partial_\infty Fix\hat{\nu} = \Sigma^n(G)$ which implies $[\chi] \in \Omega^n(G)$. Further, by Theorem 4.1 and remark 4.2, $\Omega^n(G) \subseteq \Sigma^n(G) = \Sigma^n(H) \cap \partial_\infty Fix\hat{\nu}$ which finishes the proof. ☐

It should be noted that these containments can be strict. Example 5.1 shows that the first containment can be strict, and example 5.2 shows the second containment can be strict.

Proposition 5.1 leads to the following sufficient conditions to obtain equality.

Theorem 5.2. *$\Omega^n(G) = \Omega^n(H) \cap \partial_\infty Fix\widehat{\nu}$ if*

(1) *$Hom(H, \mathbb{R}) \cong \mathbb{R}^1$,*
(2) *$\Sigma^n(H) = S(H)$ which is the character sphere of $Hom(H, \mathbb{R})$, or*
(3) *$\Sigma^n(H) = \emptyset$.*

Proof. Each condition implies that $\Sigma^n(H) = \Omega^n(H)$ which gives equality for the left and right ends of the above subset inclusion. ☐

Remark 5.3. *It is worth noting that many groups H satisfy the conditions in Theorem 5.2 such as free groups, free abelian groups, nilpotent groups, polycyclic groups, and the Baumslag-Solitar groups $BS(1, m)$.*

Example 5.4. *The above conditions are not necessary. Revisiting example 5.1, it was shown in [3] that $\Sigma^2(F)$ contains the larger arc but not the smaller arc from $\Sigma^1(F)$. Therefore, by Theorem 4.1 and remark 4.2, we have $\Sigma^2(F \rtimes_\rho \mathbb{Z}_2) = \Sigma^2(F) \cap \partial_\infty Fix\hat{\rho} = \{+\infty\}$. Therefore, $\Omega^2(F \rtimes_\rho \mathbb{Z}_2) = \{+\infty\} = \Omega^2(F) \cap \partial_\infty Fix\hat{\rho}$, but Thompson's group F does not satisfy any of the conditions of Theorem 5.2.*

6. Twisted Conjugacy and the Ω-invariant of extenstions

6.1. **Twisted conjugacy.** Following [23], a group G is said to have the property R_∞ if $R(\varphi) = \infty$ for all $\varphi \in \mathrm{Aut}(G)$ where $R(\varphi)$ denotes the cardinality of the set of φ-twisted conjugacy classes of elements of G (i. e. the number of orbits of the left action of G on G via $g \cdot h \mapsto gh\varphi(g)^{-1}$). For instance, $R(1_G)$ is the number of ordinary conjugacy classes of elements of G. It has been shown in [18] that G has property R_∞ if $\Omega^n(G)$ consists of a single discrete point. However, for such a group G with $\#\Omega^1(G) = 1$, Theorem 2.1 implies that $\Sigma^1(G) \neq -\Sigma^1(G)$ so in particular, G cannot be the fundamental group of a closed 3-manifold (See [4, Cor. F]). The only known examples of groups G with $\#\Omega^n(G) = 1$ are of the form $BS(1, n) \times W$ where $n \geq 2$ and $\#\Omega^n(W) = 0$.

The basic algebraic techniques used in the present paper for showing $R(\varphi) = \infty$ is the relationship among the Reidemeister numbers of group homomorphisms of a short exact sequence. In general, given a commutative diagram of groups and homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\eta} & B \\ \psi \big\downarrow & & \big\downarrow \varphi \\ A & \xrightarrow{\eta} & B \end{array}$$

the homomorphism η induces a function $\hat{\eta} : \mathcal{R}(\psi) \to \mathcal{R}(\varphi)$ where $\mathcal{R}(\alpha)$ denotes the set of α-twisted conjugacy classes. For our purposes, we are only concerned with automorphisms. For more general results, see [12] and [24]. We will use the following lemma; for a proof, see [18].

Lemma 6.1. *Consider the following commutative diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \\ & & \varphi' \big\downarrow & & \varphi \big\downarrow & & \overline{\varphi} \big\downarrow & & \\ 1 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \end{array}$$

where the rows are short exact sequences of groups and the vertical arrows are group automorphisms.

(1) *If $R(\overline{\varphi}) = \infty$ then $R(\varphi) = \infty$.*
(2) *If $|Fix\overline{\varphi}| < \infty$ and $R(\varphi') = \infty$ then $R(\varphi) = \infty$.*

6.2. **Using Ω for finite extensions.**

Theorem 6.2. *Let G be a finite extension given by the short exact sequence of groups*

$$1 \to H \xrightarrow{i} G \xrightarrow{p} K \to 1$$

where K is finite, H is finitely generated, and let $\nu : K \to G$ be a left transversal for K such that $\nu(1_K) = 1_G$. Let $\varphi \in Aut(G)$ such that H is invariant under φ. If $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$ has exactly one rational point, then $R(\varphi) = \infty$. In particular, if H is characteristic in G and $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$ has exactly one rational point, then G has the R_∞ property.

Proof. Suppose $\varphi \in Aut(G)$ with $\varphi(H) = H$, and suppose $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu} = \{[\chi]\}$. Let $N = ker(\chi)$ and $V := \mathrm{Hom}(G/N, \mathbb{R})$. Since $[\chi]$ is rational, G/N has rank 1, so V is 1-dimensional. Define $\tilde{\varphi} : \mathrm{Hom}(G, \mathbb{R}) \to \mathrm{Hom}(G, \mathbb{R})$ by $\tilde{\varphi}(\alpha) = \alpha \circ \varphi$. Since $\varphi(H) = H$ and both $\Omega^n(H)$ and $\partial_\infty Fix\hat{\nu}$ are invariant under automorphisms, $[\tilde{\varphi}(\chi)] \in \Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$, so $[\tilde{\varphi}(\chi)] = [\chi]$. Thus, $\chi \circ \varphi = c\chi$ for some $c \in \mathbb{Z}$, so $\varphi(N) \subseteq N$ and N is invariant under φ.

The automorphism φ induces the map $\bar{\varphi} : G/N \to G/N$ defined by $\bar{\varphi}(gN) = \varphi(g)N$ and the map $\hat{\varphi} : V \to V$ defined by $\hat{\varphi}(\alpha)(gN) = \alpha(\varphi(g)N)$. We will show $\bar{\varphi} = id$. Since φ is invertible and N is invariant under φ, we have that $\hat{\varphi}$ is invertible, and $\{[\bar{\chi}]\}$ is a basis for V where $\bar{\chi}: G/N \to \mathbb{R}$ is induced by χ. Since $\hat{\varphi}$ is invertible, $c = \pm 1$, but since $-[\chi] \notin \Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$, $c = 1$. Thus, $\hat{\varphi}(\bar{\chi}) = \bar{\chi}$ which implies that $\bar{\chi}(gN) = \bar{\chi}(\varphi(g)N)$, and so $g^{-1}\varphi(g) \in N$. Therefore, $\varphi(g)N = gN$ which implies $\bar{\varphi}(gN) = \varphi(g)N = gN$. The free abelian group G/N has rank 1, so $(G/N)/\{\text{torsion}\} \cong \mathbb{Z}$, and $\bar{\varphi}$ is the identity on G/N, so it is also induces the identity on $(G/N)/\{\text{torsion}\}$. It is clear that $R(1_{\mathbb{Z}}) = \infty$. It follows from Lemma 6.1 that $R(\bar{\varphi}) = \infty$, and hence, $R(\varphi) = \infty$. In particular, if H is characteristic in G, then G has property R_∞. □

Remark 6.1. *It is known that if Ω^n of a group is finite, then it contains either* $0, 1,$ *or* 2 *points (in the case with two points, the points are antipodal). Although $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$ is not equal to $\Omega^n(G)$ in general, the set $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$ has the same property that if it is finite, then it contains either* $0, 1,$ *or* 2 *(antipodal) points. This is due to the fact that if $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$ contains two non-antipodal points, then by Theorem 2.1, the arc joining those points will also be in $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$.*

Example 6.2. *Revisiting example 5.2, we showed that $\Omega^1(G)$ was two antipodal points while $\Omega^1(H) \cap \partial_\infty Fix\hat{\rho}$ contains exactly one rational point. The automorphism $\varphi : G \to G$ defined by $\varphi(a,1) = (a,1)$, $\varphi(b,1) = (b,1)$, $\varphi(c,1) = (c,1)$, $\varphi(d,1) = (d,1)$, $\varphi(1,t) = (1,t)$, $\varphi(x,1) = (y,t)$, and $\varphi(y,1) = (x,t)$ is an order two automorphism of G that is not H-invariant, so H is not characteristic in G. However, there are automorphisms of G that are H-invariant (for example, send $a \mapsto c$, $b \mapsto d$, $c \mapsto a$, $d \mapsto b$, $x \mapsto y$, $y \mapsto x$, and $t \mapsto t$), and by Theorem 6.2, these automorphisms ϕ have $R(\phi) = \infty$. This information cannot be obtained from either* [11] *(since $\Sigma^1(G)^c$ is two antipodal points) or* [18] *(since $\Omega^1(G)$ is two antipodal points).*

Example 6.3. *Revisiting example 5.4, since $\Omega^2(F) \cap \partial_\infty Fix\hat{\rho}$ contains exactly one point, any automorphism φ of this group that leaves F invariant would have $R(\varphi) = \infty$. Note that the map given by $(x_0,1) \mapsto (x_0,1); (x_1,1) \to (x_1,t); (1,t) \mapsto (x_0,t)$ defines an automorphism and it does not preserve F so that F is not characteristic in G. The fact that F is not characteristic in G is the reason that the R_∞ property of G does not follow from case (2) of Lemma 6.1 from the fact that F has property R_∞* [7]*. However, by* [18, Theorem 4.3]*, since $\Omega^2(F \rtimes_\rho \mathbb{Z})$ contains exactly one point, the group $F \rtimes_\rho \mathbb{Z}$ has the R_∞ property.*

6.3. **Using Ω for split extensions.**

Theorem 6.3. *Let G be a split extension given by the short exact sequence of groups*

$$1 \to H \xrightarrow{i} G \xrightarrow{p} K \to 1$$

H and K are finitely generated, and let $\nu : K \to G$ be a left transversal for K such that $\nu(1_K) = 1_G$. Let $\varphi \in Aut(G)$ such that H is invariant under φ. If $(\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}) \circledast \Omega^n(K)$ has exactly one point, then $R(\varphi) = \infty$. In particular, if H is characteristic in G and $(\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}) \circledast \Omega^n(K)$ has exactly one rational point, then G has the R_∞ property.

Proof. For $(\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}) \circledast \Omega^n(K)$ to have exactly one rational point, either $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$ contains exactly one rational point or $\Omega^n(K)$ contains exactly one rational point. In the case where $\Omega^n(H) \cap \partial_\infty Fix\hat{\nu}$ contains exactly one rational point, the proof follows the proof of Theorem 6.2.

In the case where $\Omega^n(K)$ contains exactly one rational point, it follows from [18] that K has property R_∞. Since φ is H-invariant, φ induces an automorphism $\bar{\varphi}$ on K. Therefore, $R(\bar{\varphi}) = \infty$. By Lemma 6.1, we have $R(\varphi) = \infty$.

In particular, if H is characteristic in G, then G has property R_∞. ☐

Just as in remark 6.1, if $(\Omega^1(H) \cap \partial_\infty Fix\hat{\nu}) \circledast \Omega^1(K)$ is finite, then it will either contain $0, 1$, or 2 (antipodal) points.

It is fair to wonder about the conjecture that $\Omega^1(G) = (\Omega^1(H) \cap \partial_\infty Fix\hat{\nu}) \circledast \Omega^1(K)$. The authors know of examples where the containment $(\Omega^1(H) \cap \partial_\infty Fix\hat{\nu}) \circledast \Omega^1(K) \subseteq \Omega^1(G)$ is false. Example 5.1 shows the reverse containment $\Omega^1(G) \subseteq (\Omega^1(H) \cap \partial_\infty Fix\hat{\nu}) \circledast \Omega^1(K)$ is also false. Are there sufficient and/or necessary conditions where the conjecture does hold?

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Nic Koban
Department of Mathematics
University of Maine Farmington
Farmington, ME 04938
USA
nicholas.koban@maine.edu

Peter Wong
Department of Mathematics
Bates College
Lewiston, ME 04240
USA
pwong@bates.edu

Department of Mathematics, University of Maine Farmington, Farmington, ME 04938, USA

E-mail address: nicholas.koban@maine.edu

Department of Mathematics, Bates College, Lewiston, ME 04240, USA

E-mail address: pwong@bates.edu