ON THE (STRICT) POSITIVITY OF SOLUTIONS OF THE STOCHASTIC HEAT EQUATION

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ABSTRACT. We give a new proof of the fact that the solutions of the stochastic heat equation, started with non-negative initial conditions, are strictly positive at positive times. The proof uses concentration of measure arguments for discrete directed polymers in Gaussian environments, originated in M. Talagrand's work on spin glasses and brought to directed polymers by Ph. Carmona and Y. Hu. We also get slightly improved bounds on the lower tail of the solutions of the stochastic heat equation started with a delta initial condition.

A very well known theorem proved by Mueller insures the strict positivity of the solution of the Stochastic Heat Equation (SHE) with non-negative initial data [9].

Mueller's theorem has gained new attention due to the links between the SHE and the Continuum Directed Polymer (CDP) [2], and, more generaly, with the KPZ equation (see the review [5]). In particular, it implies the positivity of the partition function of the CDP. This random measure on paths is defined by

$$\mu_{x,T}(X_{t_1} \in dx_1, \cdots, X_{t_k} \in dx_k) = \frac{1}{\mathcal{Z}(0,0;T,x)} \prod_{j=0}^{k} \mathcal{Z}(t_j, x_j; t_{j+1}, x_{j+1})\, \mathcal{Z}(t_k, x_k; T, x) dx_1 \cdots dx_k,$$

where $\mathcal{Z}(s,u;t,v)$ is obtained as the solution of

$$\begin{aligned} \partial_t \mathcal{Z}(s,u;\cdot,\cdot) &= \tfrac{1}{2}\Delta \mathcal{Z}(s,u;\cdot,\cdot) + \mathcal{Z}(s,u;\cdot,\cdot)\mathscr{W}, \\ \mathcal{Z}(s,u;s,\cdot) &= \delta_u(\cdot). \end{aligned}$$

The SHE arises as the limit of the renormalized partition function of discrete directed polymers [1] and the CDP as the weak limit of the discrete directed polymer path measure (see [4] for a general review on directed polymers).

A proof of the positivity of the solutions of the SHE contained inside the theory of directed polymers is hence desirable. This is the approach we will follow in this note. Our proof, together with providing a more straightforward argument, also improves existing bounds on the tails of the solution of the SHE. Our methods are strongly inspired by Talagrand's use of Gaussian concentration in spin glasses (see [3] where these ideas are applied to directed polymers in Gaussian environment).

1. RESULTS

In the following, unless stated otherwise, $\mathcal{Z}(t,x)$ is the continuous modification of the solution of the stochastic heat equation

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\Delta \mathcal{Z} + \mathcal{Z}\mathscr{W}, \tag{1}$$

$$\mathcal{Z}(0,x) = \delta_0(x), \tag{2}$$

where $\mathscr{W}$ is a space-time white noise.

Theorem 1. *a) There exists a locally bounded function $c(t,x) > 0$ such that*

$$\mathbb{P}\left[\mathcal{Z}(t,x) < c(t,x)\, e^{-u/c(t,x)}\right] \le e^{-u^2/2}, \tag{3}$$

hence, for all $p > 0$, there is a locally bounded function $\kappa_p(t,x)$ such that,

$$\mathbb{E}\mathcal{Z}(t,x)^{-p} \le \kappa_p(t,x)\, \exp\{\, p^2/\kappa_p(t,x)\}, \quad \forall\, t > 0,\ x \in \mathbb{R}. \tag{4}$$

b) We have

$$\mathbb{P}[\,\mathscr{Z}(t,x)=0,\ \textit{for some } t>0,\ x\in\mathbb{R}\,]=0. \tag{5}$$

Remark 1. A few remarks are in order:

(1) We note that, in [10], an estimate similar to (3) is proved, but the right hand side is $\exp\{-u^{3/2-\varepsilon}\}$. It is expected that the optimal bound is $\exp\{-u^3\}$. Our bound $\exp\{-u^2\}$ comes from Gaussian concentration arguments and is unlikely to be improved with the methods of this work.

(2) The AKQ theory described in Section 2.1 can be trivially extended to cover the case of the SHE

$$\partial_t \mathscr{Z} \quad = \quad \tfrac{1}{2}\Delta\mathscr{Z} + b\mathscr{Z} + \sigma\mathscr{W}\mathscr{Z},$$

for a bounded drift $b=b(t,x)$ and some nice $\sigma=\sigma(t,x)$. The drift can be handled using the comparison arguments of Proposition 1 (see [10], Proof of Theorem 2, where this argument is presented) and the arguments of our proof will also follow with minor modifications. More general positive initial conditions can be handled by integrating the solution of (1)-(2) against the initial condition and we would be able to recover the full strength of Mueller's theorem.

(3) We also note that Mueller's theorem is usually stated for $x\in[0,1]$ with Dirichlet boundary conditions. However the techniques in [9] and [10] carry on to our setting as well.

The proof of Theorem 1 using concentration of measure is given in Section 3. Section 2 provides useful preliminaries, while the technical estimates are deferred to the appendix.

2. Some preliminaries

2.1. **Directed polymers and the AKQ theory.** Let P be the law of the simple symmetric random walk S_t on $\mathbb{Z}$, let $\{\omega(i,x):\, i,x\}$ be a collection of real numbers (the environment) and let

$$Z_{N,x}(\beta) = P\left[e^{\beta\sum_{i=1}^N \omega(i,S_i)}|\, S_N = x\right]. \tag{6}$$

be the partition function of the directed polymers in environment ω at inverse temperature $\beta>0$. Note that $\log\mathbb{E}Z_N(\beta)=\frac{N}{2}\beta^2$ if the ω's are chosen to be independent standard normal random variables. Define

$$\mathscr{Z}_N(t,x) := e^{-\frac{1}{2}t\sqrt{N}}\, Z_{tN,x\sqrt{N}}(N^{-1/4}) = \frac{Z_{tN,x\sqrt{N}}(N^{-1/4})}{\mathbb{E}Z_{tN,x\sqrt{N}}(N^{-1/4})}. \tag{7}$$

Theorem 2. [1] *For each $t>$ and $x\in\mathbb{R}$, we have the convergence in law,*

$$\mathscr{Z}_N(t,x) \Rightarrow \mathscr{Z}(2t,x), \tag{8}$$

where $\mathscr{Z}$ is the solution of (1)-(2). Furthermore, the convergence holds at the process level in t and x.

This convergence has been extended in [8] to cover more general initial conditions. Consider a sequence of boundary conditions $\{\varphi_N(i):\, i\in\mathbb{Z}\}$ such that

$$\Phi_N(x) := N^{-1/4}\sum_{i=0}^{x\sqrt{N}} \varphi_N(i) \Rightarrow \varphi(x),$$

for some process $\varphi(\cdot)$. Define a new environment

$$\omega^{\varphi}(i,j) = \begin{cases} \omega(i,j) & \text{if } |j|<i \\ \\ \omega(i,\pm i) + \varphi_N(\pm i) + N^{1/4}\log 2 + \frac{1}{2}N^{-1/4} & \end{cases} \tag{9}$$

Assume that $\sup_x e^{-c|x|}\,\mathbb{E}[\exp\{2\Phi_N(x)\}] < +\infty$ for some $c>0$ (the reader familiar with the SHE will recognize such a natural technical condition). Let

$$\mathscr{Z}_N^{\varphi}(t,x) = e^{-\frac{1}{2}t\sqrt{N}} P\left[e^{N^{-1/4}\sum_{i=1}^{tN}\omega^{\varphi}(i,S_i)}\mathbf{1}_{\{S_N = x\sqrt{N}\}}\right] \tag{10}$$

Theorem 3. [8] *We have the convergence in law*

$$\mathcal{Z}_N^{\varphi}(t,x) \Rightarrow \mathcal{Z}^{\varphi}(2t,x), \tag{11}$$

as processes in t and x, where $\mathcal{Z}^{\varphi}$ solves the SHE

$$\partial_t \mathcal{Z} = \tfrac{1}{2}\Delta \mathcal{Z} + \mathcal{Z}\mathscr{W}, \tag{12}$$

$$\mathcal{Z}(0,x) = \exp\{\varphi(x)\}. \tag{13}$$

From this, it is easy to obtain comparison inequalities for the SHE. These are usually proved by discretizing the SHE and applying comparison arguments for diffusions. Our argument is more direct.

Proposition 1. *Let $\mathcal{Z}_0^{(1)}(x) \le \mathcal{Z}_0^{(2)}(x)$, $\forall\, x \in \mathbb{R}$ and $b^{(1)}(t,x) \le b^{(2)}(t,x)$, $\forall\,(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ and*

$$\partial_t \mathcal{Z}^{(i)} = \tfrac{1}{2}\Delta \mathcal{Z}^{(i)} + b^{(i)}\mathcal{Z}^{(i)} + \mathcal{Z}^{(i)}\mathscr{W}, \tag{14}$$

$$\mathcal{Z}^{(i)}(0,x) = \mathcal{Z}_0^{(i)}(x), \tag{15}$$

$i = 1, 2$. Then, $P[\mathcal{Z}^{(1)}(t,x) \le \mathcal{Z}^{(2)}(t,x),\ \forall\, t \ge 0,\ x \in \mathbb{R}] = 1$.

Proof. From the initial conditions above, we can easily built suitable boundary conditions for directed polymers. Also the drift term $b^{(i)}$ can be obtained by an easy modification of the environment. This can be done in such a way that the comparison inequalities hold path by path at the discrete level, and take the limit. □

2.2. **Gaussian concentration.** We borrow the following from [11] (Lemma 2.2.11). Let $d(\cdot,\cdot)$ denote the usual euclidean distance.

Theorem 4 (Talagrand)**.** *Let ω be an $\mathbb{R}^m$-valued Gaussian vector with covariance matrix I. Then, for any measurable set $A \subset \mathbb{R}^m$, if $\mathbb{P}[\,\omega \in A\,] \ge c > 0$, then, for any $u > 0$,*

$$\mathbb{P}\left[d(\omega, A) > u + \sqrt{2\log(1/c)}\right] \le e^{-\frac{u^2}{2}}. \tag{16}$$

The distance appears naturally when we compare the partition function over different environments. Define the partition function and the polymer measure in a fixed environment ω by

$$Z_N(\omega,\beta) = E\left[e^{\beta\sum_{t=1}^N \omega(t,S_t)}\right], \tag{17}$$

$$\langle F(S)\rangle_{N,\omega} = \frac{1}{Z_N(\omega,\beta)} E\left[F(S)\, e^{\beta\sum_{t=1}^N \omega(t,S(t))}\right], \tag{18}$$

where we just disregard the dependence on the end-point of the paths. Denote the expected value over two independent copies of the polymer over the same environment by $\langle\cdot\rangle_{N,\omega}^{(2)}$ and, for two paths S_1 and S_2, let $L_N(S_1,S_2) = \sum_{t=1}^N \mathbf{1}_{S_1(t)=S_2(t)}$ be the overlap.

Lemma 1. *Let ω and ω' be two environments and let $d_N(\omega,\omega')$ denote the distance between these environments in the box $\{(t,x) :\ 0 \le t \le N, |x| \le t\}$. Then,*

$$\log Z_N(\omega',\beta,) \ge \log Z_N(\omega,\beta) - \beta\, d_N(\omega,\omega')\sqrt{\langle L_N(S_1,S_2)\rangle_{N,\omega}^{(2)}}. \tag{19}$$

Proof. We start computing

$$Z_N(\omega',\beta) = E\left[e^{\beta\sum_{t=1}^N \omega(t,S_t)} e^{\beta\sum_{t=1}^N(\omega'(t,S_t)-\omega(t,S_t))}\right] \tag{20}$$

$$= Z_N(\omega,\beta)\langle e^{\beta\sum_{t=1}^N(\omega'(t,S_t)-\omega(t,S_t))}\rangle_{N,\omega} \tag{21}$$

$$\ge Z_N(\omega,\beta) e^{\beta\langle\sum_{t=1}^N(\omega'(t,S_t)-\omega(t,S_t))\rangle_{N,\omega}}, \tag{22}$$

by Jensen's inequality. Now,

$$\left|\langle\sum_{t=1}^{N}\left(\omega'(t,S_t)-\omega(t,S_t)\right)\rangle_{N,\omega}\right| = \left|\sum_{t=1}^{N}\sum_{x}\left(\omega'(t,x)-\omega(t,x)\right)\ \langle\mathbf{1}_{S_t=x}\rangle_{N,\omega}\right| \tag{23}$$

$$\leq \left(\sum_{x}\sum_{t=1}^{N}\left(\omega'(t,x)-\omega(t,x)\right)^2\right)^{1/2}\left(\sum_{x}\sum_{t=1}^{N}\langle\mathbf{1}_{S_t=x}\rangle_{N,\omega}^2\right)^{1/2} \tag{24}$$

$$= d_N(\omega,\omega')\sqrt{\langle L_N(S_1,S_2)\rangle_{N,\omega}^{(2)}} \tag{25}$$

□

3. Proof of Theorem 1

In the following, we will omit the dependence on the end-point from the notation when it is equal to 0. Let E_2 denote the expected value with respect to two independent walks. Define the event

$$A=\left\{\omega:\ Z_N(\omega,\beta)\geq\frac{1}{2}\mathbb{E}Z_N(\beta),\ \langle L_N(S_1,S_2)\rangle_{N,\omega}^{(2)}\leq\frac{C}{4}\sqrt{N}\right\}, \tag{26}$$

Lemma 2. *Take $\beta=N^{-1/4}$. For $C>0$ large enough, there exists $\delta>0$ such that $\mathbb{P}[A]\geq\delta$, $\forall\, N\geq 1$.*

Proof. The key to prove this fact is the estimate (47) proved in Section 4. Let $H_N(S_1,S_2)=\sum_{t=1}^{N}\omega(t,S_1(t))+\omega(t,S_2(t))$.

$$\mathbb{P}[A] = \mathbb{P}\left\{Z_N(\beta)\geq\frac{1}{2}\mathbb{E}Z_N(\beta),\ E_2\left[L_N(S_1,S_2)\exp\{\beta H_N(S_1,S_2)\}\right]\leq\frac{C}{4}\sqrt{N}\left(\mathbb{E}Z_N(\beta)\right)^2\right\} \tag{27}$$

$$\geq \mathbb{P}\left\{Z_N(\beta)\geq\frac{1}{2}\mathbb{E}Z_N(\beta)\right\} \tag{28}$$

$$-\,\mathbb{P}\left\{E_2\left[L_N(S_1,S_2)\exp\{\beta H_N(S_1,S_2)\}\right]>\frac{C}{4}\sqrt{N}\left(\mathbb{E}Z_N(\beta)\right)^2\right\} \tag{29}$$

We treat the first summand: by Paley-Zygmund's inequality (see for example [11], Proposition 2.2.3),

$$\mathbb{P}\left\{Z_N(\beta)\geq\frac{1}{2}\mathbb{E}Z_N(\beta)\right\}\geq\frac{1}{4}\frac{\left(\mathbb{E}Z_N(\beta)\right)^2}{\mathbb{E}Z_N(\beta)^2}=\frac{1}{4}\frac{1}{\mathbb{E}\mathcal{Z}_N^2}, \tag{30}$$

if we take $\beta=N^{-1/4}$. Now, by an application of Fubini's theorem together with $\log\mathbb{E}e^{\beta\omega}=\beta^2/2$ if ω is standard normal, we have $\mathbb{E}\mathcal{Z}_N^2=E_2[\exp N^{-1/2}L_N(S_1,S_2)]$. The estimate (46) then provides a constant $0<L<+\infty$ such that

$$\mathbb{E}\mathcal{Z}_N^2\leq L,\quad \forall\, N\geq 1. \tag{31}$$

This gives

$$\mathbb{P}\left\{Z_N(\beta)\geq\frac{1}{2}\mathbb{E}Z_N(\beta)\right\}\geq\frac{1}{4L},\quad\forall\,N\geq 1\quad\text{when}\quad\beta=N^{-1/4}. \tag{32}$$

For the second summand above, using Chebyshev followed by Fubini

$$\mathbb{P}\left\{E_2\left[L_N(S_1,S_2)\exp\{N^{-1/4}H_N(S_1,S_2)\}\right]>\frac{C}{4}\sqrt{N}\left(\mathbb{E}Z_N(\beta)\right)^2\right\} \tag{33}$$

$$\leq\frac{4}{C\sqrt{N}\left(\mathbb{E}Z_N(\beta)\right)^2}\mathbb{E}E_2\left[L_N(S_1,S_2)\exp\{N^{-1/4}H_N(S_1,S_2)\}\right] \tag{34}$$

$$=\frac{4}{C\sqrt{N}}E_2\left[L_N(S_1,S_2)\exp\{N^{-1/2}L_N(S_1,S_2)\}\right] \tag{35}$$

$$\leq\frac{4K}{C}, \tag{36}$$

thanks to (47), where we also used

$$\mathbb{E}E_2\left[L_N(S_1,S_2)\exp\{N^{-1/4}H_N(S_1,S_2)\}\right] \;=\; (\mathbb{E}Z_N(\beta))^2\; E_2\left[L_N(S_1,S_2)\exp\{N^{-1/2}L_N(S_1,S_2)\}\right].$$

Overall, we have $\mathbb{P}[A] \geq \frac{1}{4L} - \frac{4K}{C} =: \delta$, which is positive provided we choose C large enough. □

Proof of Theorem 1- a). Recall the distance $d_N(\cdot,\cdot)$ from Lemma 1. By Lemma 2 and Talagrand's theorem,

$$\mathbb{P}\left[\omega :\, d_N(\omega, A) > u + C'\right] \leq e^{-u^2/2}, \tag{37}$$

for all $u > 0$ and some explicit constant $0 < C' < +\infty$ depending on C, K and L. In particular, for any $\omega' \in A$, if ω is any environment, by Lemma 1,

$$\begin{aligned}
\log Z_N(\omega,\beta) \;&\geq\; \log \mathbb{E}Z_N(\beta) - \log 2 - \beta\, d_N(\omega,\omega')\sqrt{\langle L_N(S_1,S_2)\rangle_{N,\omega'}} && (38)\\
&\geq\; \log \mathbb{E}Z_N(\beta) - \log 2 - \beta N^{1/4}\sqrt{C} d_N(\omega,\omega'), && (39)\\
&\geq\; \log \mathbb{E}Z_N(\beta) - \log 2 - C'' d_N(\omega,\omega'), && (40)
\end{aligned}$$

for some $0 < C'' < +\infty$, if $\beta = N^{-1/4}$. Hence, for $c_2 = C''$ and $c_1 = \log 2 + C'C''$,

$$\mathbb{P}\left[\log Z_N(\omega,\beta) \leq \log \mathbb{E}Z_N(\beta) - c_1 u - c_2\right] \;\leq\; \mathbb{P}\left[d_N(\omega, A) \geq u\right] \;\leq\; e^{-u^2/2}. \tag{41}$$

This proves the following intermediate result: remember $\mathcal{Z}_N = Z_N/\mathbb{E}Z_N$, hence for all $u > 0$, $N \geq 1$,

$$\mathbb{P}\left[\mathcal{Z}_N < C_2 e^{-c_1 u}\right] \leq e^{-u^2/2}. \tag{42}$$

with $C_2 = e^{-c_2}$. Using that $\mathcal{Z}_N \to \mathcal{Z}(1,0)$ in law, we get

$$\mathbb{P}\left[\mathcal{Z}(1,0) < C_2 e^{-c_1 u}\right] \leq e^{-u^2/2}. \tag{43}$$

for all $u > 0$. This proves Theorem 1-*a*) when the end-point is taken to be 0. Finally, observe that, if we take the end-point of the path to be $x\sqrt{N}$ and the length of the polymer to be tN, the proof is unchanged, and the estimates of Section 4 imply that the constants C' and C'' above are uniformly bounded for (t,x) in a compact. □

Proof of Theorem 1- b). We will use the following standard estimate:

$$\mathbb{E}|\mathcal{Z}(t,x) - \mathcal{Z}(s,y)|^p \leq C\left(|x-y|^{p/2} + |t-s|^{p/4}\right), \tag{44}$$

for any $p > 1$. See for example [7], where these estimates are used to show the existence of a continuous extension of the solution of the stochastic heat equation.

As $\mathcal{Z}$ is continuous, the only possible singularities of $\mathcal{Z}^{-1}$ correspond to zeros of $\mathcal{Z}$. We will show that $\mathcal{Z}(\cdot,\cdot)^{-1}$ has a continuous modification as well. We estimate

$$\begin{aligned}
\mathbb{E}|\mathcal{Z}(t,x)^{-1} - \mathcal{Z}(s,y)^{-1}|^M \;&=\; \mathbb{E}\left|\frac{\mathcal{Z}(t,x) - \mathcal{Z}(s,y)}{\mathcal{Z}(t,x)\mathcal{Z}(s,y)}\right|^M\\
&\leq\; \mathbb{E}\left[|\mathcal{Z}(t,x) - \mathcal{Z}(s,y)|^{2M}\right]^{1/2} \mathbb{E}\left[\mathcal{Z}(t,x)^{-4M}\right]^{1/4} \mathbb{E}\left[\mathcal{Z}(s,y)^{-4M}\right]^{1/4}.
\end{aligned}$$

By (4), the moments of order $-4M$ are locally bounded. Together with (44), we conclude that, for each compact $K \subset (0,+\infty)\times\mathbb{R}$, there is a constant $C_K < +\infty$, such that

$$\sup_{(t,x),(s,y)\in K} \mathbb{E}|\mathcal{Z}(t,x)^{-1} - \mathcal{Z}(s,y)^{-1}|^M < C_K\left(|x-y|^{M/2} + |t-s|^{M/4}\right). \tag{45}$$

Hence, by Kolmogorov criterion, $\{\mathcal{Z}(t,x)^{-1} : (t,x) \in K\}$ has a continuous modification $\mathcal{Y}(\cdot,\cdot)$, and hence stays bounded. It follows that $\mathcal{Y}^{-1}$ cannot assume the value 0 in K. This proves (5). □

4. Appendix: Overlap Estimates

The goal of this section is to prove the needed overlap estimates. These estimates are very familiar in the context of homogeneous pinning models. Recall that $L_N(S_1,S_2)=\sum_{i=1}^N \mathbf{1}_{S_i^{(1)}=S_i^{(2)}}$.

Lemma 3. *There is a locally bounded function $\kappa(t,x)\in(0,+\infty)$ such that*

$$\sup_{N\geq 1} E_2\left[e^{N^{-1/2}L_{tN}(S_1,S_2)}|S_{tN}^{(1)}=S_{tN}^{(2)}=x\sqrt{N}\right] \;\leq\; \kappa(t,x), \tag{46}$$

$$\sup_{N\geq 1}\frac{1}{\sqrt{N}}E_2\left[L_{tN}(S_1,S_2)e^{N^{-1/2}L_{tN}(S_1,S_2)}|S_{tN}^{(1)}=S_{tN}^{(2)}=x\sqrt{N}\right] \;\leq\; \kappa(t,x). \tag{47}$$

Proof. As the estimates will be clearly uniform for $0<t\leq T$, we specify to $t=1$. We will first show that we can reduce to consider the overlap up to time $n/2$: abbreviate $L_m=L_m(S_1,S_2)$. Then,

$$E\left[L_N e^{\beta L_N}|S_N^{(i)}=x\sqrt{N},\, i=1,2\right] \;\leq\; 4E\left[L_{N/2}e^{2\beta L_{N/2}}|S_N^{(i)}=x\sqrt{N},\, i=1,2\right]E\left[e^{\beta L_{N/2}}|S_N^{(i)}=x\sqrt{N},\, i=1,2\right],$$

$$E\left[e^{\beta L_N}|S_N^{(i)}=x\sqrt{N},\, i=1,2\right] \;\leq\; 2E\left[e^{2\beta L_{N/2}}|S_N^{(i)}=x\sqrt{N},\, i=1,2\right]E\left[e^{\beta L_{N/2}}|S_N^{(i)}=x\sqrt{N},\, i=1,2\right].$$

This shows that it is enough to prove (46) and (47) for the overlap up to time $N/2$. Now, we can reduce to consider the overlap of two unconditioned random walks with a suitable drift. Define,

$$P_{x,N}[S_0=0]=1, \qquad P_{x,N}[S_{i+1}=a\pm 1|S_i=a]=\frac{1}{2}\left(1\pm\frac{x}{\sqrt{N}}\right). \tag{48}$$

An application of the local limit theorem shows that $S_{[0,N/2]}$ under $P[\cdot|S_n=x\sqrt{N}]$ and $S_{[0,N/2]}$ under $P_{x,N}[\cdot]$ are absolutely continuous. Furthermore, the Radon-Nykodim derivative is locally bounded in x, uniformly in N. It is therefore enough to prove (46) and (47) with respect to $P_{x,N}[\cdot]$. Let $Y_i=S_i^{(1)}-S_i^{(2)}$ where $S^{(i)}$ are two independent walks with law $P_{x,N}$. Then, $Y_0=0$ and

$$P_{x,N}[Y_{i+1}-Y_i=0]=\frac{1}{2}\left(1+\frac{x^2}{N}\right), \quad P_{x,N}[Y_{i+1}-Y_i=\pm 2] \;=\; \frac{1}{4}\left(1-\frac{x^2}{N}\right). \tag{49}$$

The problem is now reduced to estimate the local time at 0 for the walk Y, which is a homogeneous pinning problem. Accordingly, we introduce some notions and results from [6]. Let

$$z_N(x,\beta)=E_{x,N}\left[e^{\beta\sum_{i=1}^N \mathbf{1}_{Y_i=0}}\right]. \tag{50}$$

From [6] (1.6) and the proof of [6] (2.12), it follows that, for any compact K, there are two finite constants c_1, $c_2>0$ such that

$$z_N(x,\beta)\leq c_1\, e^{c_2\beta^2 N}, \quad \forall\, x\in K. \tag{51}$$

Taking $\beta=N^{-1/2}$ yields (46). For (47), all we need is a bound on the derivative of $z_N(x,\beta)$ with respect to β. Notice that $g(u)=z_N(x,u)$ is an increasing and convex function with $g(0)=1$ and

$$g'(u)=E_2\left[L_N e^{uL_N}\right], \tag{52}$$

where $L_N=\sum_{i=1}^N \mathbf{1}_{Y_i=0}$. By convexity,

$$1+ug'(u)\leq g(u)+ug'(u)\leq g(2u) \tag{53}$$

and consequently,

$$\tfrac{1}{2}g'(u)\leq\frac{g(2u)-1}{2u}. \tag{54}$$

Together with (51),

$$\tfrac{1}{2}\partial_u z_N(x,u)\leq\frac{g(2u)-1}{2u}\leq\frac{c_1 e^{4c_2Nu^2}-1}{2u}\leq 4c_3Nue^{4c_2Nu^2}, \tag{55}$$

with $c_3=c_1c_2$. The last inequality follows from the convexity of $\exp CNu^2$. Remember these estimates are uniform for $x\in K$. Taking $u=N^{-1/2}$ in the string of inequalities above ends the proof of (47). □

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