# Symmetric exclusion as a model of non-elliptic dynamical random conductances

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**Abstract**

We consider a finite range symmetric exclusion process on the integer lattice in any dimension. We interpret it as a non-elliptic time-dependent random conductance model by setting conductances equal to one over the edges with end points occupied by particles of the exclusion process and to zero elsewhere. We prove a law of large number and a central limit theorem for the random walk driven by such a dynamical field of conductances by using the Kipnis-Varhadan martingale approximation. Unlike the tagged particle in the exclusion process, which is in some sense similar to this model, this random walk is diffusive even in the one-dimensional nearest-neighbor symmetric case.

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# 1 Introduction

## 1.1 Model and results

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Denote by $\xi = \{\xi(z); z \in \mathbb{Z}^d\}$ the elements of $\Omega$. For $\xi \in \Omega$ and $y, z \in \mathbb{Z}^d$, define $\xi^{y,z} \in \Omega$ as

$$\xi^{y,z}(x) = \begin{cases} \xi(z), & x = y \\ \xi(y), & x = z \\ \xi(x), & x \neq z, y, \end{cases}$$

that is, $\xi^{y,z}$ is obtained from $\xi$ by exchanging the occupation variables at $y$ and $z$. Fix $R \geq 1$. Consider the transition kernel $p(z, y)$ of a translation-invariant, symmetric, irreducible random walk with range size $R$, i. e., $p(0, y - z) = p(z, y) = p(y, z) > 0$ iff $|z - y|_1 \leq R$, and $\sum_{y \in \mathbb{Z}^d} p(0, y) = 1$. Due to translation invariance we will denote $p(x) := p(0, x)$.

Let $\{(\xi_t, X_t); t \geq 0\}$ be the Markov process on the state space $\Omega \times \mathbb{Z}^d$ with generator given by

$$\begin{aligned} Lf(\xi, x) = &\sum_{y,z \in \mathbb{Z}^d} p(z - y)\big[f(\xi^{y,z}, x) - f(\xi, x)\big] \\ &+ \sum_{y \in \mathbb{Z}^d} c_{x,y}(\xi)\big[f(\xi, y) - f(\xi, x)\big], \end{aligned} \tag{1.1}$$

for any local function $f : \Omega \times \mathbb{Z}^d \to \mathbb{R}$, with

$$c_{x,y}(\xi) = \begin{cases} \xi(x)\xi(y) & \text{if } |x - y|_1 \leq R, \\ 0 & \text{else.} \end{cases} \tag{1.2}$$

We interpret the dynamics of the process $\{(\xi_t, X_t); t \geq 0\}$ as follows. Checking the action of $L$ over functions $f$ which do not depend on $z$, we see that $\{\xi_t; t \geq 0\}$ has a Markovian evolution, which corresponds to the well known *symmetric exclusion process* on $\mathbb{Z}^d$, see e.g. [6]. Conditioned on a realization of $\{\xi_t; t \geq 0\}$, the process $\{X_t; t \geq 0\}$ is a continuous time random walk among the field of dynamical random conductances

$$\{c_{x,y}(\xi_t) = \xi_t(x)\xi_t(y)\mathbb{1}_{\{|x-y|_1 \leq R\}} \,|\, x, y \in \mathbb{Z}^d, t \geq 0\}. \tag{1.3}$$

Our main results are the following law of large numbers and functional central limit theorem for the random walk $X_t$.

**Theorem 1.1. (LLN)** *Assume that the exclusion process $\xi_t$ starts from the Bernoulli product measure $\nu_\rho$ of density $\rho \in [0,1]$. Then $X_t/t$ converges a.s. and in $L_1$ to $0$.*

**Theorem 1.2. (Annealed functional CLT)** *Under the assumptions of Theorem 1.1, the process $(\epsilon X_{t/\epsilon^2})$ converges in distribution, as $\epsilon$ goes to zero, to a non-degenerate Brownian motion with covariance $\sigma$ in the Skorohod topology.*

## 1.2 Motivation

Random walks in random media represents one of the main research area within the field of disordered system of particles. The aim is to understand the motion of a particle in a inhomogeneous medium. This is clearly interesting for applied purposes and has turned out to be a very challenging mathematical program. Lots of effort has been made in recent years in this direction. We refer to [7, 8] for recent overviews of rigorous results in this subject.

One of the easiest models of random walk in random media is represented by a random walk among (time-independent) random conductances. This model turned out to be relatively simple due the reversibility properties of the walker. In fact, the behavior of such random walks has been recently analyzed and understood in quite great generality. See, e. g. ,[3] for a recent overview and references therein. When considering a field of dynamical random conductances, the mentioned reversibility of the random walk is lost, and other types of techniques are needed. In the recent paper [1], annealed and quenched invariance principles for a random walk in a field of time-dependent random conductances have been derived by assuming fast enough space-time mixing conditions and uniform ellipticity for the field. In particular, the uniform ellipticity, which guarantees heat kernel estimates, is a crucial assumption in their approach even for the annealed statement(ellipticity plays a fundamental role also in the analysis of other random walks in random environments). The model we consider represents a first solvable example of non-elliptic time-dependent random conductances. Moreover it strengthens the connection between particle systems theory and the theory of random walks in random media. To overcome the loss of ellipticity we use the "good" properties of the symmetric exclusion in equilibrium.

The proof of our results rely on the martingale approximation method developed by Kipnis and Varhadan [5] for additive functionals of reversible Markov processes. In the original paper [5], the authors apply their method to study a tagged particle in the exclusion process. Indeed, this latter has some similarities with our model, and our proof is essentially an adaptation of their proof. Unlike the tagged particle behavior, our random walk is always diffusive even in the one dimensional nearest-neighbors case.

## 2 Proofs of the LLN and of the invariance principle

### 2.1 The environment from the position of the walker

Consider the process $\{\eta_t; t \geq 0\}$ with values in $\Omega$, defined by $\eta_t = \tau_{X_t}\xi_t$, where $\tau_y$ denotes the shift operator on $\Omega$ (i. e. $\eta_t(z) = \xi_t(z + X_t)$). The process $\{\eta_t; t \geq 0\}$ is usually called the *environment seen by the random walk*. For $\eta \in \Omega$, the process $\{\eta_t; t \geq 0\}$ is also Markovian with generator:

$$\begin{aligned}\mathcal{L}_{ew}f(\eta) &= \sum_{z,y} p(z-y)\big[f(\eta^{y,z}) - f(\eta)\big] + \sum_{y} c_{0,y}(\eta)\big[f(\tau_y\eta) - f(\eta)\big] \\ &=: \mathcal{L}_{se}f(\eta) + \mathcal{L}_{rc}f(\eta).\end{aligned} \tag{2.1}$$

for any local function $f : \Omega \to \mathbb{R}$. The choice of the subindexes in the generators above is just for notational convenience: "$ew$", "$se$" and "$rc$", stand for, Environment from the point of view of the Walker, Symmetric Exclusion and Random Conductances, respectively.

For any function $f, g : \Omega \to \mathbb{R}$, we denote the inner product in $L_2 = L_2(\nu_\rho)$ by

$$\langle f, g\rangle_{\nu_\rho} := \int_\Omega d\nu_\rho f(\eta)g(\eta),$$

where $\nu_\rho$ is the Bernoulli product measure of density $\rho \in [0, 1]$. In particular, it is well known that the family $\{\nu_\rho : \rho \in (0, 1)\}$ fully characterizes the set of extremal invariant measures for the symmetric exclusion process, and $\mathcal{L}_{se}$ is self-adjoint in $L_2$ (see e.g. [6]). The next lemma shows that the same hold for the environment as seen by the walker. Before proving it, we define the Dirichlet forms associated to the generators involved in (2.1) as

$$D_a(f) := \langle f, -\mathcal{L}_a f\rangle_{\nu_\rho} \text{ with } a \in \{ew, se, rc\}, \tag{2.2}$$

for $f$ in $L_2$. It follows by a standard computation (cf. [4], Prop. 10.1 P.343) that

$$\begin{aligned} D_{ew}(f) =& D_{se}(f) + D_{rc}(f) = \frac{1}{2}\sum_{z,y}\int d\nu_\rho\, p(z-y)\big[f(\eta^{y,z}) - f(\eta)\big]^2 \\ &+ \frac{1}{2}\sum_{y}\int d\nu_\rho\, c_{0,y}(\eta)\big[f(\tau_y\eta) - f(\eta)\big]^2. \end{aligned} \tag{2.3}$$

**Lemma 2.1.** *The process $\eta_t$ is reversible and ergodic with respect to the the Bernoulli product measure $\nu_\rho$.*

*Proof.* We first show that $\mathcal{L}_{ew}$ is self-adjoint in $L_2$, namely ,$\langle f, \mathcal{L}_{ew} g\rangle_{\nu_\rho} = \langle \mathcal{L}_{ew} f, g\rangle_{\nu_\rho}$, with $f, g$ arbitrary functions.

By translation invariance, we have

$$\begin{aligned} \langle f, \mathcal{L}_{rc} g\rangle_{\nu_\rho} &= \sum_y \int d\nu_\rho\, f(\eta)\,[g(\tau_y\eta) - g(\eta)]\, c_{0,y}(\eta) \\ &= \sum_y \Big(\int d\nu_\rho\, f(\tau_{-y}\eta) g(\eta) c_{0,-y}(\eta) - \int d\nu_\rho\, f(\eta) g(\eta) c_{0,y}(\eta)\Big) \\ &= \sum_y \Big(\int d\nu_\rho\, f(\tau_{y}\eta) g(\eta) c_{0,y}(\eta) - \int d\nu_\rho\, f(\eta) g(\eta) c_{0,y}(\eta)\Big) = \langle \mathcal{L}_{rc} f, g\rangle_{\nu_\rho}. \end{aligned} \tag{2.4}$$

Together with the fact that $\mathcal{L}_{se}$ is also self-adjoint, we get

$$\langle f, \mathcal{L}_{ew} g\rangle_{\nu_\rho} = \langle f, \mathcal{L}_{se} g\rangle_{\nu_\rho} + \langle f, \mathcal{L}_{rc} g\rangle_{\nu_\rho} = \langle \mathcal{L}_{se} f, g\rangle_{\nu_\rho} + \langle \mathcal{L}_{rc} f, g\rangle_{\nu_\rho} = \langle \mathcal{L}_{ew} f, g\rangle_{\nu_\rho}.$$

It remains to show the ergodicity. Following the argument in [5], we show that any harmonic function $h$ such that $\mathcal{L}_{ew} h = 0$ is $\nu_\rho$-a. s. constant.

Indeed $\mathcal{L}_{ew} h = 0$ implies that $D_{se}(h) = -D_{rc}(h)$. Since the Dirichlet forms are non-negative, then $D_{se}(h) = 0 = D_{rc}(h)$, but $\mathcal{L}_{se}$ is reversible and ergodic, hence $h$ must be $\nu_\rho$-a. s. constant. ∎

## 2.2 Proof of Theorem 1.1

We now express the position of the RW $X_t$ in terms of the process $\eta_t$. For $y \in \mathbb{Z}^d$, let $J_t^y$ denote the number of spatial shifts in direction $y$ of the process $\eta_t$ up to time $t$. Then

$$X_t = \sum_y y J_t^y. \tag{2.5}$$

By compensating the process $J_t^y$ by its intensity $\int_0^t c_{0,y}(\eta_s) ds$, it is standard to check that

$$M_t^y := J_t^y - \int_0^t ds\, c_{0,y}(\eta_s) \quad \text{and} \quad (M_t^y)^2 - \int_0^t ds\, c_{0,y}(\eta_s) \tag{2.6}$$

are martingales with stationary increments vanishing at $t = 0$.

Next, define

$$M_t := \sum_y y M_t^y \quad \text{and} \quad \phi(\eta_s) := \sum_y y c_{0,y}(\eta_s), \tag{2.7}$$

by combining (2.5) and (2.6), we obtain

$$X_t = M_t + \int_0^t ds\, \phi(\eta_s), \tag{2.8}$$

from which we easily obtain the law of large numbers in Theorem 1.1. Indeed, due to Lemma 2.1, the representation in (2.8) express $X_t$ as a sum of a zero-mean martingale with stationary and ergodic increments $M_t$, plus the term $\int_0^t ds\, \phi(\eta_s)$, which by the ergodic theorem, when divided by $t$, it converges to its average

$$\mathbb{E}_{\nu_\rho}[\phi(\eta)] = \sum_{|y|_1 \le R} y \int d\nu_\rho\, \eta(0)\eta(y) = \rho^2 \sum_{|y|_1 \le R} y = 0.$$

### 2.3 Proof of Theorem 1.2

Next, we want to prove a functional CLT for the process $X_t$. To this aim we will use again the representation in (2.8) and the well known Kipnis-Varhadan method [5] for additive functional of reversible Markov processes. Indeed, $\int_0^t ds\, \phi(\eta_s)$ in (2.8) is an additive functional of the reversible process $\eta_t$. To recall briefly the Kipnis-Varadhan method, we first introduce the Sobolev spaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ associated to a generator $\mathcal{L}$. Let $\mathcal{D}(\mathcal{L})$ be the domain of this generator. Consider in $\mathcal{D}(\mathcal{L})$, the equivalence relation $\sim_1$ defined as $f \sim_1 g$ if $\|f - g\|_1 = 0$, where $\|\cdot\|_1$ is the semi-norm given by

$$\|f\|_1^2 := \langle f, -\mathcal{L}f \rangle_{\nu_\rho}. \tag{2.9}$$

Define the space $\mathcal{H}_1$ as the completion of the normed space $(\mathcal{D}(\mathcal{L})|_{\sim_1}, \|\cdot\|_1)$. It can be check that $\mathcal{H}_1$ is a Hilbert space with inner product $\langle f, g \rangle_1 := \langle f, -\mathcal{L}g \rangle_{\nu_\rho}$. Next, for $f \in L_2$, let

$$\|f\|_{-1} := \sup \left\{ \frac{\langle f, g \rangle_{\nu_\rho}}{\|g\|_1} : g \in L_2, \|g\|_1 \neq 0 \right\}. \tag{2.10}$$

Consider $\mathcal{G}_{-1} := \{f \in L_2 : \|f\|_{-1} < \infty\}$. As for the $\|\cdot\|_1$ norm, define the equivalence relation $\sim_{-1}$, and let $\mathcal{H}_{-1}$ be the completion of the normed space $(\mathcal{G}_{-1}|_{\sim_1}, \|\cdot\|_1)$. $\mathcal{H}_{-1}$ is the dual of $\mathcal{H}_1$ and it is also a Hilbert space. Theorem 1.8 in [5] states that, if $\mathcal{L}$ is self-adjoint and $\phi \in \mathcal{H}_{-1}$ (which we prove in the next lemma), then there exists a square integrable martingale $\tilde{M}_t$ and an error term $E_t$ such that

$$\int_0^t ds\, \phi(\eta_s) = \tilde{M}_t + E_t, \tag{2.11}$$

and $|E_t|/\sqrt{t}$ converges to zero in $L_2$.

In particular, denoting by $\cdot$ the standard inner product and considering a vector $l$ in $\mathbb{R}^d$, the martingale $\tilde{M}_t \cdot l$ in (2.11) is obtained as the limit as $\lambda \to 0$ of the martingale

$$\tilde{M}_t(\lambda, l) := f_\lambda(\eta_t) - f_\lambda(\eta_0) - \int_0^t ds\, \mathcal{L} f_\lambda(\eta_s), \tag{2.12}$$

where $f_\lambda$ is the solution of the resolvent equation

$$(\lambda I - \mathcal{L}) f_\lambda = \phi \cdot l. \tag{2.13}$$

Moreover

$$\mathbb{E}_{\nu_\rho}[\tilde{M}_1(\lambda, l)^2] = \|f_\lambda\|_1^2. \tag{2.14}$$

We are now ready to show the crucial estimate which, in view of what we said, by Theorem 1.8 in [5], implies the decomposition in (2.11).

**Lemma 2.2.** *There exists a constant $K > 0$ such that, for any function $f \in \mathcal{D}(\mathcal{L}_{ew})$ and $l$ in $\mathbb{R}^d$,*

$$|\langle \phi \cdot l, f \rangle_{\nu_\rho}| \leq K\, D_{ew}(f)^{1/2}. \tag{2.15}$$

*Proof.* Recall (2.7) and estimate

$$\begin{aligned}
\left|\langle \phi \cdot l, f \rangle_{\nu_\rho}\right| &= \left| \int d\nu_\rho \sum_y (y \cdot l) c_{0,y}(\eta) f(\eta) \right| = \left| \frac{1}{2} \int d\nu_\rho \sum_y (y \cdot l) \left[c_{0,y}(\eta) - c_{0,-y}(\eta)\right] f(\eta) \right| \\
&= \left| \frac{1}{2} \int d\nu_\rho \sum_y (y \cdot l) c_{0,y}(\eta) \left[f(\tau_y \eta) - f(\eta)\right] \right| \\
&\leq \frac{1}{2} \left( \sum_y (y \cdot l)^2 c_{0,y}(\eta) \right)^{1/2} \left( \int d\nu_\rho \sum_y c_{0,y}(\eta) \left[f(\tau_y \eta) - f(\eta)\right]^2 \right)^{1/2} \\
&\leq K\, D_{rc}(f)^{1/2} \leq K\, D_{ew}(f)^{1/2},
\end{aligned} \tag{2.16}$$

where we have used translation invariance, $c_{0,y}(\eta)^2 = c_{0,y}(\eta)$, Cauchy-Schwartz, the finite range assumption on $p(\cdot)$, and the representation of the Dirichlet forms in (2.3), respectively. ∎

In view of the discussion above, from (2.8) and (2.11), we have that

$$X_t = M_t + \tilde{M}_t + o(\sqrt{t}). \tag{2.17}$$

Since the sum of two martingales is again a martingale, the functional CLT for $X_t$ follows immediately from the standard functional CLT for martingales provided that we prove the non-degeneracy of the covariance matrix of the martingale given by $M_t + \tilde{M}_t$. Roughly speaking, we have to prove that $M_t$ and $\tilde{M}_t$ do not cancel each other. This is the content of the next proposition which concludes the proof of Theorem 1.2.

**Proposition 2.3.** *The sum of the two martingales $M_t + \tilde{M}_t$ is a non-degenerate martingale.*

*Proof.* For $z, y \in \mathbb{Z}^d$ with $p(z-y) > 0$, let $I_t^{y,z}$ denote the total number of jumps of particles from $y$ to $x$ up to time $t$. Similarly to (2.6), by compensating the process $I_t^{y,z}$ by its intensity, it is standard to check that

$$N_t^{y,z} := I_t^{y,z} - p(z-y)t \quad \text{and} \tag{2.18}$$

$$(N_t^{y,z})^2 - p(z-y)t \tag{2.19}$$

are martingales.

In particular, the martingales $\{M_t^y | \, y \in \mathbb{Z}^d\}$ (recall (2.6)) and $\{N_t^{y,z} | \, y, z \in \mathbb{Z}^d, p(z-y) > 0\}$ are jump processes which do not have common jumps. Therefore they are orthogonal, namely, the product of two such martingales is still a martingale.

On the other hand, we can check that the martingale in (2.12) can be expressed as

$$\begin{aligned} \tilde{M}_t(\lambda, l) &= \sum_{y,z} \int_0^t dN_s^{y,z} \left[ f_\lambda(\eta_s^{y,z}) - f_\lambda(\eta_s) \right] \\ &\quad + \sum_y \int_0^t dM_s^y \left[ f_\lambda(\tau_y \eta_s) - f_\lambda(\eta_s) \right]. \end{aligned} \tag{2.20}$$

Since $M_t, \tilde{M}_t$ are mean-zero square integrable martingales with stationary increments, to prove that $M_t + \tilde{M}_t$ is a non-degenerate martingale, we show that for any vector $l \in \mathbb{R}^d$,

$$\mathbb{E}_{\nu_\rho} \left[ (M_1 \cdot l + \tilde{M}_1 \cdot l)^2 \right] > 0. \tag{2.21}$$

By using (2.20), the orthogonality and the form of the quadratic variations of $M_t^y$ and $N_t^{y,z}$(see (2.6) and (2.19)), and (2.3), we have that

$$\begin{aligned} \mathbb{E}_{\nu_\rho} \left[ (M_1 \cdot l + \tilde{M}_1 \cdot l)^2 \right] &= \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho} \left[ \left( M_1 \cdot l + \tilde{M}_1(\lambda, l) \right)^2 \right] \\ &= \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho} \left[ \left( \int_0^1 \sum_{y,z} \left[ f_\lambda(\eta_s^{y,z}) - f_\lambda(\eta_s) \right] dN_s^{y,z} \right)^2 \right] \\ &+ \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho} \left[ \left( \int_0^1 \sum_y \left\{ (y \cdot l) + \left[ f_\lambda(\tau_y \eta_s) - f_\lambda(\eta_s) \right] \right\} dM_s^y \right)^2 \right] \\ &= \lim_{\lambda \to 0} 2 D_{se}(f_\lambda) \\ &+ \lim_{\lambda \to 0} \mathbb{E}_{\nu_\rho} \left[ \sum_y c_{0,y}(\eta) \left\{ (y \cdot l) + \left[ f_\lambda(\tau_y \eta) - f_\lambda(\eta) \right] \right\}^2 \right]. \end{aligned} \tag{2.22}$$

Hence, to conclude (2.21), we argue a follows. Assume that there exists a constant $K > 0$ such that

$$|\langle \phi \cdot l, f_\lambda \rangle_{\nu_\rho}| \leq K \, D_{se}(f_\lambda)^{1/2}, \tag{2.23}$$

then

$$D_{ew}(f_\lambda) \leq |\langle \phi \cdot l, f_\lambda \rangle_{\nu_\rho}| \leq K \, D_{se}(f_\lambda)^{1/2}, \tag{2.24}$$

where the first inequality follows by $D_{ew}(f_\lambda) \leq D_{ew}(f_\lambda) + \lambda |\langle f_\lambda, f_\lambda \rangle_{\nu_\rho}| = |\langle \phi \cdot l, f_\lambda \rangle_{\nu_\rho}|$.

In view of (2.24), if $D_{ew}(f_\lambda)$ stays positive in the limit as $\lambda \to 0$, the same holds for $D_{se}(f_\lambda)$ and the variance is positive. On the other hand, if $D_{ew}(f_\lambda)$ vanishes, then (recall (2.14)), $\mathbb{E}_{\nu_\rho}[\tilde{M}_1(\lambda, l)^2] = D_{ew}(f_\lambda) \to 0$ and the limit variance is just $\mathbb{E}_{\nu_\rho} \left[ (M_1 \cdot l)^2 \right] > 0$.

It remains to show the claim in (2.23). Indeed, for an arbitrary $f$, we can estimate

$$\begin{aligned}|\langle \phi\cdot l, f\rangle_{\nu_\rho}| &= \left|\frac{1}{2}\int d\nu_\rho \sum_y (y\cdot l)\,[c_{0,y}(\eta)-c_{0,-y}(\eta)]\,f(\eta)\right|\\ &= \left|\frac{1}{2}\sum_{|y|_1\le R}\int d\nu_\rho (y\cdot l)\eta(0)\,[\eta(y)-\eta(-y)]\,f(\eta)\right| \qquad (2.25)\\ &\le \frac{1}{2}\sum_{|y|_1\le R}|y\cdot l|\left|\int d\nu_\rho\,[\eta(y)-\eta(-y)]\,f(\eta)\right|.\end{aligned}$$

Note that due to the irreducibility of $p(\cdot)$, for any $y\in\mathbb{Z}^d$ with $|y|_1\le R$, we can write

$$\eta(y)-\eta(-y)=\sum_{i=1}^{n}[\eta(z_i)-\eta(z_{i-1})]$$

for some sequence $(z_0=y, z_1,\dots,z_n=-y)$, with $p(z_i-z_{i-1})>0$ for $i=1,\dots,n$. Moreover

$$\begin{aligned}&\left|\int d\nu_\rho\,[\eta(z_i)-\eta(z_{i-1})]\,f(\eta)\right| = \left|\int d\nu_\rho\,\eta(z_{i-1})\,[f(\eta^{z_{i-1},z_i})-f(\eta)]\right|\\ &\le \rho\left(\int d\nu_\rho\,[f(\eta^{z_{i-1},z_i})-f(\eta)]^2\right)^{1/2} \le p(z_i-z_{i-1})^{-1/2}D_{se}(f)^{1/2}. \qquad (2.26)\end{aligned}$$

Combining (2.25) and (2.26), we obtain (2.23) which concludes the proof. ∎

## 2.4 Concluding remarks

**Remark 2.4. (On the tagged particle in symmetric exclusion)** *In the original paper by Kipnis-Varhadan, the authors used their general theorem to show the diffusivity of a tagged particle in the symmetric exclusion process in any dimension. An exceptional case is when the symmetric exclusion is nearest-neighbor and one-dimensional, which has been shown to be sub-diffusive [2] due to the "traffic jam" created by the other particles in the system. In particular, in this latter context, the analogous two martingales involved in* (2.17) *do annihilate each other. In fact, the crucial estimate in* (2.23) *does not hold.*

**Remark 2.5. (Particle systems as non-elliptic dynamical random conductances)** *The model we introduced is an example of time-dependent random conductances, non-elliptic from below, but bounded from above since $c_{\{x,y\}}(t)\in\{0,1\}$. In a similar fashion, we can interpret more general particle systems as models of non-elliptic dynamical random conductances, even unbounded from above. This can be done by considering a particle system $\xi_t\in\mathbb{N}^{\mathbb{Z}^d}$ and again setting $c_{x,y}(t)=\xi_t(x)\xi_t(y)$ (e.g. a Poissoinian field of independent random walks), provided that the particle system has "well behaving" space-time correlations and good spectral properties. Furthermore, in principle, Theorem 1.2 can be pushed to obtain the analogous quenched statement. We plan to address these natural generalizations in future work.*

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