Singular parabolic p-Laplacian systems under non-smooth external forces. Regularity up to the boundary.

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Abstract

We study the regularity of the solutions to initial-boundary value problems for $N-$systems of the p-Laplacian type, in $n \geq 3$ space variables, with square-integrable external forces in the space-time cylinder. So, the ellipticity coefficient remains unbounded. The singular case $\mu = 0$ is covered.

Keywords: Initial-boundary value problems, p-Laplacian parabolic singular systems, regularity up to the boundary.

1 Introduction and main result.

In the sequel we consider the evolution problem

$$\text{(1.1)} \qquad \begin{cases} \partial_t u - \nabla \cdot \big((\mu + |\nabla u|)^{p-2} \nabla u \big) = f(t,x), \quad \text{in } (0,T) \times \Omega, \\ u = 0 \ \text{ on } (0,T) \times \partial\Omega, \\ u(0) = u_0 \ \text{ in } \Omega, \end{cases}$$

where $p \in (1,2]$, $T \in (0,\infty]$, and $\mu \geq 0$ are constants. Here u is an $N-$dimensional vector field, $N \geq 1$, defined in $Q_T \equiv (0,T) \times \Omega$ where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a regular, bounded open set. The main point here is that the external force f is only square-integrable in Q_T. This low integrability prevents boundedness of $|\nabla u(t,x)|$ (which holds, for instance, if $f \in L^q(Q_T)$, $q > n+2$). So, the ellipticity coefficient $((\mu + |\nabla u|)^{p-2}$ keeps unbounded. This obstacle is here by-passed, due to a simple, but fruitful, idea (which has a more wide range of application, as shown in a forthcoming work).

Let us illustrate the kind of results proved in the sequel, by the following example. Assume that p satisfies the condition (1.19), where K (see below) is a positive constant, independent of p. Then, the second order space derivatives satisfy

$$\text{(1.2)} \qquad D^2 u \in L^{2(p-1)}(0,T; L^{\widehat{q}}(\Omega)),$$

where $\widehat{q}$ is defined by (1.7).

The proof of our main result appeals to a regularity theorem, see the theorem 2.1 below, proved in reference [2] for the stationary problem

$$\text{(1.3)} \qquad \begin{cases} -\nabla \cdot \big((\mu + |\nabla u|)^{p-2} \nabla u \big) = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Actually, we need the above regularity result for the value $q = \widehat{q} < 2$, see (1.7) below. However, values smaller then 2 are out of the range considered in the statement of theorem 2.1, even though the proof applies to a range of values which includes $\widehat{q} < 2$. The check of this claim is straightforward. However, for the readers convenience, after the statement of the extension result to the $\widehat{q} < 2$ case (see proposition 2.1 below) we made a couple of comments plentifully sufficient to adapt the proof given in [2] to the $\widehat{q}$ case. To minimize the number of alterations, assume that

$$\begin{cases} \frac{2n}{n+2} < p \leq 2\,, & \text{if } n > 3\,, \\ \frac{5}{4} < p\,, & \text{if } n = 3\,. \end{cases} \tag{1.4}$$

In particular, the inclusion

$$L^2(\Omega) \subset W^{-1,\,p'}(\Omega) \tag{1.5}$$

holds. Actually, the assumption (1.4) is not strictly essential in the following.

We start by recalling that scalar multiplication of both sides of (1.1) by u, followed by classical manipulations, lead to the well known a priori estimate

$$\| u \|^2_{L^\infty(0,\,T;\,L^2(\Omega)\,)} + \| u \|^p_{L^p(0,\,T;\,W^{1,\,p}(\Omega)\,)}$$
$$\leq c\,\big(\,\| u_0 \|^2_{L^2(0,\,T;\,L^2(\Omega)\,)} + \| f \|^{p'}_{L^{p'}(0,\,T;\,W^{-1,\,p'}(\Omega)\,)}\big).$$

This estimate, useful in proving the existence of the weak solution, lead us to assume that

$$f \in L^{p'}(\,0,\,T;\,W^{-1,\,p'}(\Omega)\,)\,. \tag{1.6}$$

For the existence of the above weak solution we refer the reader to the Theorem 1.1, Chap. II, in [16].

Let us introduce the core exponent

$$\widehat{q} = \frac{2\,n\,(\,p-\,1\,)}{n-\,2\,(\,2-p\,)}\,. \tag{1.7}$$

As shown below, the central role of this exponent is due to the particular relation

$$r(\widehat{q}) = 2\,, \tag{1.8}$$

see (2.1). Our assumptions on p implies that $\widehat{q} \in (1,\,2]$, and also that the immersion

$$W^{2,\,\widehat{q}}(\Omega) \subset W^{1,\,p}(\Omega) \tag{1.9}$$

is compact.

Finally we recall the well known inequality

$$\|D^2\,v\|_q \leq C_2(q)\,\|\Delta v\|_q\,, \tag{1.10}$$

for $v \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$. Actually, there is a constant K, independent of q, such that

$$C_2(q) \leq K\,q\,, \tag{1.11}$$

at least for $q > \frac{2\,n}{n+2}$ (see [18]).

Our main result is the following.

Theorem 1.1. *Let* p *satisfy* (1.4), *and define* $\widehat{q}$ *by* (1.7). *Further, assume that*

$$(2-p)\,C_2(\widehat{q}) < 1\,, \tag{1.12}$$

where $C_2(\widehat{q})$ *is defined by* (1.10). *Let* $u_0 \in W_0^{1,\,p}(\Omega)$ *and assume that, for some* $T \in]\,0,\,+\infty\,]\,,$ f *satisfies* (1.6) *and*

$$f \in L^2(0,\,T;\,L^2(\Omega)\,)\,. \tag{1.13}$$

Then the weak solution u *of problem* (1.1) *enjoys the following properties:*

$$u \in L^\infty(0,\,T;\,W_0^{1,\,p}(\Omega)\,)\,, \tag{1.14}$$

$$\nabla\cdot\,\big(\,(\,\mu+|\,\nabla\,u|\,)^{p-2}\,\nabla\,u\,\big) \in L^2(0,\,T;\,L^2(\Omega)\,)\,, \tag{1.15}$$

$$\partial_t\,u \in L^2(0,\,T;\,L^2(\Omega)\,)\,, \tag{1.16}$$

and

$$u \in L^{2\,(p-1\,)}(0,\,T;\,W^{2,\,\widehat{q}}(\Omega)\,)\,. \tag{1.17}$$

Note that, if $n = 3$ and $p > \frac{3}{2}$, then

$$u \in L^{2\,(p-1\,)}(0,\;T\,;\,C^{\,0,\,\alpha}(\overline{\Omega})\,)\,,$$

where $\alpha = \frac{p-\frac{3}{2}}{p-1}$.

Remark 1.1. Due to (1.11), the condition (1.12) holds if

$$(2-\,p)\widehat{q} < \frac{1}{K}\,, \tag{1.18}$$

at least by assuming $p > 2 - \frac{2}{n}$ (as required by Yudovic assumption on n, actually, not strictly necessary). Further, straightforward calculations show that (1.18) holds if

$$2 - \frac{n}{2nK+\,2} < p \leq 2\,. \tag{1.19}$$

So, (1.19) by itself, is a sufficient condition to guarantee the results claimed in theorem 1.1. The main point is that this condition depends only on p, via Yudovic's constant K.

Sharp estimates for the norms of the left hand sides of the above equations, in terms of data norms and μ, follow immediately from the proofs. For a more detailed discussion see sections 3 and 4. We state here these estimates in the singular case $\mu = 0$. One has the following result:

Theorem 1.2. *Consider the singular parabolic problem*

$$\left\{\begin{array}{l} \partial_t\, u - \nabla\cdot\,(\,|\,\nabla\, u|^{p-2}\,\nabla\, u\,) = f(t,\,x)\,, \quad \textit{in } (0,\,T)\times\,\Omega\,, \\ u = 0 \quad \textit{on } (0,\,T)\times\,\partial\Omega\,, \\ u(0) = u_0 \quad \textit{in } \Omega\,. \end{array}\right. \tag{1.20}$$

Let the hypothesis assumed in Theorem 1.1, concerning p, $\widehat{q}$, *and* f, *hold. Then, one has*

$$\frac{2}{p}\,\|\,\nabla\, u\,\|^p_{L^\infty(0,\,T;\,L^p(\Omega)\,)} \leq \frac{2}{p}\,\|\,\nabla\, u_0\,\|^p_p + \|\,f\,\|^2_{L^2(0,\,T;\,L^2(\Omega)\,)}\,, \tag{1.21}$$

$$\|\,\nabla\cdot\,(\,|\,\nabla\, u\,|^{p-2}\,\nabla\, u\,)\,\|^2_{L^2(0,\,T;\,L^2(\Omega)\,)} \leq \frac{2}{p}\,\|\,\nabla\, u_0\,\|^p_p + \|\,f\,\|^2_{L^2(0,\,T;\,L^2(\Omega)\,)}\,, \tag{1.22}$$

$$\|\,\partial_t\, u\,\|^2_{L^2(0,\,T;\,L^2(\Omega)\,)} \leq \frac{2}{p}\,\|\,\nabla\, u_0\,\|^p_p + 2\,\|\,f\,\|^2_{L^2(0,\,T;\,L^2(\Omega)\,)}\,, \tag{1.23}$$

and

$$\begin{aligned} \|\,u\,\|^2_{L^{2(p-1)}(\,0,\,T;\,W^{2,\,\widehat{q}}(\Omega)\,)} &\leq C\,T^{\frac{2-p}{p-1}}\Big(\,\|\,\nabla\, u_0\,\|^p_p + \|\,f\,\|^2_{L^2(0,\,T;\,L^2(\Omega)\,)}\Big) \\ &+ C\,\Big(\,\|\,\nabla\, u_0\,\|^{\frac{p}{p-1}}_p + \|\,f\,\|^{\frac{2}{p-1}}_{L^2(0,\,T;\,L^2(\Omega)\,)}\Big)\,. \end{aligned} \tag{1.24}$$

It is worth noting that, for $p = 2$, the above estimates turn into the classical "heat equation" estimates. For instance, (1.24) reduces to

$$\|\,u\,\|^2_{L^2(\,0,\,T;\,W^{2,\,2}(\Omega)\,)} \leq C\,\big(\,\|\,\nabla\, u_0\,\|^2_2 + \|\,f\,\|^2_{L^2(0,\,T;\,L^2(\Omega)\,)}\,\big)\,. \tag{1.25}$$

For related results we refer, for instance, to the well-know monographs [5], [12], [16], and to references [3], [4], [5], [7], [8], [9], [10] [13], [14], [15].

In references [7], [8] (see [5] chapters IX, X) local Hölder continuity in $(0,\,T)\times\,\Omega$ of the space gradient of local weak solutions is proved. Regularity results, *up to the boundary*, are stated in the chapter X of [5] (see the Theorems 1.1 and 1.2 therein). In particular, in the Theorem 1.2, the Hölder continuity up to the parabolic boundary (where $u = 0$) of the spatial gradient of weak solutions u is proved. However, regularity results, up to the boundary, for the second order space derivatives in $L^q(\Omega)$ spaces, for solutions to the parabolic singular system (1.20), were not know in the literature. Actually, the two types of estimates are not comparable. It is worth noting that in the elliptic case, see [2], the $W^{2,\,q}(\Omega)$ estimates imply $C^{1,\,\alpha}(\overline{\Omega})$ regularity, since $q > n$ is admissible.

A classical related subject, in the case $N = 1$, are the Harnack's inequalities. See references and results in the recent monograph [6]. We learned in reference [5] that the first parabolic versions of Harnack's inequality are due to Hadamard [11] and Pini [17].

NOTATION: We follow the notation introduced in reference [2]. By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, m nonnegative integer and $p \in (1,+\infty)$, we denote the usual

Lebesgue and Sobolev spaces, with the standard norms $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$. We set $\|\cdot\| = \|\cdot\|_2$. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$, and by $W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, the strong dual of $W_0^{1,p}(\Omega)$ with norm $\|\cdot\|_{-1,p'}$.

The symbols c, c_1, c_2, etc., denote positive constants that may depend on μ; by capital letters, C, C_1, C_2, we denote positive constants independent of $\mu \geq 0$ (eventually, μ bounded from above). The same symbol c or C may denote different constants, even in the same equation. we set $\partial_t u = \frac{\partial u}{\partial t}$.

2 The stationary problem. Known results.

As already referred, a main ingredient used here to prove the core estimate (1.17) concerns the stationary problem (1.3). The following result was proved in reference [2], in collaboration with Francesca Crispo (we also recall the previous work [1], by the same authors).

Theorem 2.1. *Let $p \in (1,2]$ and $q \geq 2$, $q \neq n$, be given. Assume that $(2-p)\, C_2(q) < 1$, where $C_2(q)$ satisfies (1.10). Further, assume that $\mu \geq 0$. Let $f \in L^{r(q)}(\Omega)$, where*

$$r(q) = \begin{cases} \dfrac{nq}{n(p-1)+q(2-p)} & \text{if } q \in [2, n], \\ q & \text{if } q \geq n, \end{cases} \tag{2.1}$$

and let u be the unique weak solution of problem (1.3). Then u belongs to $W^{2,q}(\Omega)$. Moreover, the following estimate holds

$$\|u\|_{2,q} \leq C \left(\|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}} \right). \tag{2.2}$$

The reader directly interested in the above result is refereed to [2], where significance and range of application of the above statement are discussed.

In reference [2] the authors assume that $q \geq 2$ since they were mainly interested in maximal regularity. However results and proofs hold also for values $q \in (1, 2)$, at most under some small modification. Here we need the above result only for the particular value $q = \widehat{q} < 2$. For simplicity, we take into account only this value, and show the single points in the proof given in [2] where some small remark may be useful to adapt the proof to the value $\widehat{q}$.

Proposition 2.1. *Let be $\mu \geq 0$, and let p satisfy (1.12) and (1.4). Assume that $f \in L^2(\Omega)$. Then, the weak solution u to the problem (1.3) belongs to $W^{2,\widehat{q}}(\Omega)$. Moreover,*

$$\|u\|_{2,\widehat{q}} \leq C \left(\|f\|_{\widehat{q}} + \|f\|_2^{\frac{1}{p-1}} \right). \tag{2.3}$$

Proof. Clearly, we assume that the reader have in hands the proof of theorem 2.1 given in reference [2]. The unique real modification to be made in this proof, in order to adapt it to the present situation, is the following. In reference [2], at the end of section 3, the authors prove the following convergence

$$f\,(\mu + |\nabla v^m|)^{2-p} \to f\,(\mu + |\nabla v|)^{2-p} \tag{2.4}$$

in the $L^{\frac{q}{2}}$ norm. This is not suitable here, since $q = \widehat{q} < 2$. However, as remarked in [2], convergence in the distributional sense is obviously sufficient. As in [2], one has

$$|\,(\mu + |\nabla v^m|\,)^{2-p} - \,(\mu + |\nabla v|\,)^{2-p}\,| \leq \frac{2-p}{\mu^{p-1}}\,|\nabla v^m - \nabla v|\,.$$

In particular, it follows that (2.4) holds a.e. in Ω.

On the other hand, $(\mu + |\nabla v^m|)^{2-p}$ is bounded in L^t, where $t := \frac{\widehat{q}^*}{2-p} > \frac{n}{2}$. So, it follows from Lemma 1.3, in Chap.1, [16] that

$$(\mu + |\nabla v^m|)^{2-p} \,\to\, f\,(\mu + |\nabla v|)^{2-p}\,,$$

weakly in L^t. Moreover, (1.4) implies that $(\frac{n}{2})' \leq 2$. Consequently, $f \in L^{(\frac{n}{2})'}$. It readily follows that (2.4) holds in the distributional sense.

Just for the reader's convenience we add two (we believe dispensable) remarks:

i) The set

$$\mathbb{K} = \{v \in W^{2,\,\widehat{q}}(\Omega) \,:\, \|\Delta\, v\|_{\widehat{q}} \leq R\,,\, v = 0 \;\text{ on } \partial\Omega\}\,,$$

introduced in [2], section 3, is still contained in $W_0^{1,\,p}(\Omega)$, since $p > \frac{2\,n}{n+2}$.

ii) As at the very beginning of section 4 in [2], one still has here $p < q^*$. So, as in [2], u^μ converges to u in $W^{1,p}(\Omega)$. □

3 A more general setting.

Besides proving the theorem 1.1, we also want to show that the statement may be extended to other systems of equations of the form

$$\left\{\begin{array}{l} \partial_t\, u - \,\nabla\cdot\, S(\,\nabla\, u\,) \,=\, f(t,\,x) \;\text{ in } (0,\,T\,)\times\,\Omega\,,\\ u \,=\, 0 \;\text{ on } (0,\,T)\times\,\partial\Omega\,,\\ u(0) \,=\, u_0 \;\text{ in } \Omega\,, \end{array}\right. \tag{3.1}$$

where $S(\cdot)$ is given by

$$S(\,\nabla\, u\,) := B(\,|\nabla u|\,)\,\nabla\, u\,. \tag{3.2}$$

Note that the $N-$dimensional vector field vector $\nabla\cdot\, S(\,\nabla\, u\,)$ has components given by

$$(\,\nabla\cdot\, S(\,\nabla\, u\,)\,)_j = \sum_i \partial_i\big(\,B(\,|\nabla u|\,)\,\partial_i\, u_j\,\big)\,. \tag{3.3}$$

In the sequel we show that the above extension is possible, provided that proposition 2.1 applies to the corresponding stationary problem

$$\left\{\begin{array}{l} -\,\nabla\cdot\,\big(\,B(\,|\nabla u|\,)\,\nabla\, u\,\big) = \,f \;\text{ in } \Omega\,,\\ u \,=\, 0 \;\text{ on } \partial\Omega\,. \end{array}\right. \tag{3.4}$$

This last possibility was, rightly, claimed in reference [2]. So, in this section, we consider the system (3.1), and *assume* that a suitable extension of proposition 2.1 to the system (3.4) holds. Further, we introduce the following notation, suited to treat the general situation.

We set, for $y > 0$,

$$A(y) := B(\sqrt{y}\,), \tag{3.5}$$

and define, for $y \geq 0$,

$$G(y) := \int A(y)\, dy\,. \tag{3.6}$$

Furthermore, we assume that there are positive constants c_0 and c_1 such that

$$c_0\, y^p - c_1 \leq G(y^2) \leq \widetilde{c}_0\, y^p + \widetilde{c}_1\,, \tag{3.7}$$

for $y \geq 0$.

In this section we show the following result.

Proposition 3.1. *Assume that the solutions to the stationary problem* (3.4) *enjoy the regularity result stated in proposition 2.1, and that* (3.7) *holds. Further, assume that* $f \in L^2(0,\, T;\, L^2(\Omega)\,)$, *for some* $T \in]0, +\infty\,]$. *Then*

(3.8)
$$c_0 \,\|\, u \,\|^p_{L^\infty(0,\, T;\, W^{1,\, p}_0(\Omega)\,)} \leq \widetilde{c}_0 \,\|\, u_0 \,\|^p_{W^{1,\, p}_0(\Omega)} + \|\, f \,\|^2_{L^2(0,\, T;\, L^2(\Omega)\,)} + (c_1 + \widetilde{c}_1)\,|\,\Omega\,|\,,$$

(3.9)
$$\|\, \nabla \cdot S(\,\nabla\, u\,) \,\|^2_{L^2(0,\, T;\, L^2(\Omega)\,)} \leq \widetilde{c}_0 \,\|\, u_0 \,\|^p_{W^{1,\, p}_0(\Omega)} + \|\, f \,\|^2_{L^2(0,\, T;\, L^2(\Omega)\,)} + (c_1 + \widetilde{c}_1)\,|\,\Omega\,|\,,$$

(3.10)
$$\|\, \partial_t\, u \,\|^2_{L^2(0,\, T;\, L^2(\Omega)\,)} \leq \widetilde{c}_0 \,\|\, u_0 \,\|^p_{W^{1,\, p}_0(\Omega)} + 2\,\|\, f \,\|^2_{L^2(0,\, T;\, L^2(\Omega)\,)} + (c_1 + \widetilde{c}_1)\,|\,\Omega\,|\,,$$

and

$$\|\, u \,\|_{L^{2(p-1)}_T(W^{2,\,\hat{q}}(\Omega)\,)} \leq C\Big(T^{\frac{1}{2(p-1)}} + \widetilde{c}_0^{\,\frac{1}{2(p-1)}} \,\|\, u \,\|^{\frac{p}{2(p-1)}}_{W^{1,\, p}_0(\Omega)} + \|\, f \,\|^{\frac{1}{(p-1)}}_{L^2_T(L^2(\Omega))} + \big(\,(c_1 + \widetilde{c}_1)\,|\,\Omega\,|\,\big)^{\frac{1}{2(p-1)}}\Big)\,. \tag{3.11}$$

Proof. By scalar multiplication of booth sides of the first equation (3.1) by $-\,\nabla \cdot \big(\, B(|\,\nabla\, u|)\,\nabla\, u\,\big)$, followed by integration in Ω, one gets

$$\begin{aligned} &\frac{1}{2}\int_\Omega B(|\,\nabla\, u|)\,\partial_t\,|\,\nabla\, u|^2\, dx + \int_\Omega |\,\nabla \cdot\, S(\,\nabla\, u\,)\,|^2\, dx \,= \\ &-\int_\Omega f \cdot (\,\nabla \cdot\, S(\,\nabla\, u\,)\,)\, dx\,. \end{aligned} \tag{3.12}$$

Recall (3.3). We have appealed to an integration by parts and to the fact that $\partial_t\, u = 0$ on $\partial\Omega$. Next, we write the equation (3.12) in the form

$$\begin{aligned} &\frac{1}{2}\,\frac{d}{d\,t}\int_\Omega G(|\,\nabla\, u|^2)\, dx + \int_\Omega |\,\nabla \cdot\, S(\,\nabla\, u\,)\,|^2\, dx \,= \\ &-\int_\Omega f \cdot (\,\nabla \cdot\, S(\,\nabla\, u\,)\,)\, dx\,, \end{aligned} \tag{3.13}$$

Note that, if $f = 0$, the quantity

$$\int_\Omega G(|\nabla u(t)|^2)\,dx$$

is decreasing with respect to time. For instance, in the singular case (1.20), the norm $\|\nabla u(t)\|_p$ decreases with time.

From (3.13), it follows that

$$\frac{d}{dt}\int_\Omega G(|\nabla u|^2)\,dx + \int_\Omega \left|\nabla\cdot\left(A(|\nabla u|^2)\nabla u\right)\right|^2 dx \le \int_\Omega |f|^2\,dx\,. \tag{3.14}$$

By integration with respect to t, one gets

$$\int_\Omega G(|\nabla u(t)|^2)dx + \int_0^t \left\|\nabla\cdot S(\nabla u(s))\right\|_2^2 ds \le \int_\Omega G(|\nabla u_0|^2)dx + \int_0^t \|f(s)\|_2^2\,ds\,. \tag{3.15}$$

From (3.15) and (3.7), by appealing to well know manipulations, one proves (3.8), (3.9), and (3.10). Note that (3.10) follows immediately from the identity

$$\partial_t u = \nabla\cdot S(\nabla u) + f(t,x)\,.$$

In the above estimates $W_0^{1,p}(\Omega)$ is endowed with the norm $\|\nabla u\|_{L^p(\Omega)}$.

Finally, we prove (3.11). By assumption, the solutions to the stationary problem (3.4) enjoy the regularity results stated in proposition 2.1. So, it follows from equations (3.1), (2.3), (3.9), and (3.10) that

$$u \in L^{2(p-1)}(0,T;W^{2,\widehat{q}}(\Omega))\,. \tag{3.16}$$

More precisely, by appealing to (2.3), one gets, for a.a. $t\in(0,T)$,

$$\|u(t)\|_{2,\widehat{q}}^{2(p-1)} \le C\left(\|\partial_t u - f\|_2^{2(p-1)} + \|\partial_t u - f\|_2^2\right).$$

Note that, for $p = 2$, we get the classical estimate.

Since $2(p-1)\le 2$, we may replace the above estimate simply by

$$\|u(t)\|_{2,\widehat{q}}^{2(p-1)} \le C\left(1 + \|\partial_t u - f\|_2^2\right).$$

So, with obvious notation,

$$\|u\|_{L_T^{2(p-1)}(W^{2,\widehat{q}})} \le C\,T^{\frac{1}{2(p-1)}} + C\,\|\partial_t u - f\|_{L_T^2(L^2)}^{\frac{1}{p-1}}\,. \tag{3.17}$$

Hence, by the assumptions on f together with (1.16), one gets (3.11). This completes the proof of proposition 3.1.

□

Note that proposition 3.1 is only a *partial extension* of theorem 1.1 to more general systems of the form (3.1) since, in this last case the proposition 2.1 was not proved. However, the corresponding extension should be routine.

4 Proof of Theorem 1.1.

To prove the theorem 1.1, we simply assume that B is given by

$$B(|\nabla u|) = (\mu + |\nabla u|)^{p-2}. \tag{4.1}$$

It remains to show that, in the case of equation (1.1), an estimate like (3.7) holds.

Lemma 4.1. *Set, for* $y \geq 0$,

$$A(y) = (\mu + y^{\frac{1}{2}})^{p-2}. \tag{4.2}$$

It follows that

$$G(y^2) = \frac{2}{p}(\mu + y)^p - \frac{2\mu}{p-1}(\mu + y)^{p-1}. \tag{4.3}$$

in particular

$$\frac{1}{p}(\mu + y)^p - C_1\mu^2 \leq G(y^2) \leq \frac{2}{p}(\mu + y)^p \leq \frac{2^p}{p}(y^p + \mu^p), \tag{4.4}$$

where $C_1 = \frac{2^p}{p(p-1)}$.

The second inequality (4.4) follows by setting $z = \mu + y$,where $z \geq 0$, and by writing

$$\frac{2}{p}z^p - \frac{2\mu}{p-1}z^{p-1} = \frac{1}{p}z^p + \left(\frac{1}{p}z^p - \frac{2\mu}{p-1}z^{p-1}\right).$$

The minimum of the function between open brackets is attained for $z = 2\mu$.

It readily follows that the estimates (3.8), (3.9), (3.10), and (3.11) hold by setting, for instance,

$$c_0 = \frac{1}{p}, \quad c_1 = C_1\mu^2, \quad \widetilde{c}_0 = \widetilde{c}_1 = \frac{2^p}{p}.$$

In the singular case (1.20) one has $G(y^2) = \frac{2}{p}y^p$. Straightforward manipulations lead to the estimates claimed in the theorem 1.2.

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