Uniform resolvent convergence for a strip with fast oscillating boundary

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Abstract

In a planar infinite strip with fast periodically oscillating boundary we consider an elliptic operator assuming that both the period and the amplitude of the oscillations are small. On the oscillating boundary we impose Dirichlet, Neumann or Robin boundary condition. In all cases we describe the homogenized operator, establish the uniform resolvent convergence of the perturbed resolvent to the homogenized one and prove the estimates for the rate of convergence. These results are obtained as the order of the amplitude of the oscillations is less, equal or greater than that of the period. It is shown that under the homogenization the type of the boundary condition can change.

Introduction

There are many papers devoted to the homogenization of the boundary value problems in the domains with fast oscillating boundary. The simplest example of such boundary is given by the graph of the function $x_2 = \eta(\varepsilon)b(x_1\varepsilon^{-1})$, where ε is a small positive parameter, $\eta(\varepsilon)$ is a positive function tending to zero as $\varepsilon \to +0$, and b is a smooth periodic function. The parameter ε describes the period of the boundary oscillations while $\eta(\varepsilon)$ is their amplitude.

Most of the papers on such topic are devoted to the case of bounded domains with fast oscillating boundary. Not trying to cite all papers in this field, we just mention [30, Ch. III, Sec. 4], [19], [20], [21], [3], [17], [18], [4], [1], [2], [23], [28], [27], [29], [16], [25], [26], see also the references therein. Main results concerned the identification of the homogenized problems and proving the convergence theorems for the solutions. The homogenized (limiting) problems were the boundary value problems for the same equations in the same domains but with the mollified boundary instead of the oscillating one. The type of the condition on the mollified boundary depended on the original boundary condition and the geometry of the oscillations. If the amplitude of the oscillations is of the same order as the period (i.e., in above example $\eta \sim \varepsilon$), then the homogenized boundary condition is of the same type as the original condition on the oscillating boundary. In the case of Robin or Neumann condition the homogenization gives rise to an additional term in the coefficient in the homogenized boundary condition; this term reflected the local geometry of the boundary oscillations. If the period of the boundary oscillations is smaller (in order) of the amplitude then the boundary is highly oscillating. To authors knowledge, such case was considered in [18] and [23].

In [23] the model was the spectral problem for the biharmonic operator with Dirichlet condition, while in [18] the Robin problem for the Poisson equation was studied. In the former case in particular it was shown that the homogenized boundary condition was the Dirichlet one while in the latter the authors discovered that in the case of highly oscillating boundary the homogenized boundary condition is also the Dirichlet while the perturbed problem involved the Robin condition.

Most of the results on the convergence of the solutions were established in the sense of the weak or strong resolvent convergence, and the resolvents were also treated in various possible norms. In some cases the estimates for the convergence rate were proven. It was also shown that constructing the next terms of the asymptotics for the perturbed solutions one get the estimates for the convergence rate or improves it [20], [21], [4], [29], [26], [29], [23], [27], [16]. In some cases even complete asymptotic expansions were constructed [2], [3], [22], [28].

One more type of the established results is the uniform resolvent convergence for the problems. Such convergence was established just for few models, see [30, Ch. III, Sec. 4], [29]. The estimates for the rates of convergence were also established. In both papers the amplitude and the period of oscillations were of the same order. At the same time, the uniform resolvent convergence for the models considered in the homogenization theory is a quite strong results. Moreover, recently the series of papers by M.Sh. Birman, T.A. Suslina and V.V. Zhikov, S.E. Pastukhova have stimulated the interest to this aspect, see [5], [6], [7], [8], [15], [31], [32], [35], [37], [38], [34], [36], the references therein and further papers by these authors. It was shown that the uniform resolvent convergence holds true for the elliptic operators with fast oscillating coefficients and even the estimates for the rates of convergence were obtained. Similar results were also established for some problems in bounded domains, see [37]. Similar results but for the boundary homogenization were established in [14], [11], [10], [12], [13]. Here the Laplacian in a planar straight infinite strip with frequently alternating boundary conditions was considered. Such boundary conditions were imposed by partitioning the boundary into small segments where Dirichlet and Robin conditions were imposed in turns. The homogenized problem involves one of the classical boundary conditions instead of the alternating ones. For all possible homogenized problems the uniform resolvent and the estimates for the rates of convergence were proven and asymptotics for the spectra were constructed.

In the present paper we also consider the boundary homogenization for the elliptic operators in unbounded domains but the perturbation is a fast oscillating boundary. As the domain we choose a planar straight infinite strip with a periodic fast oscillating boundary where a general self-adjoint second order elliptic operator is considered. The operator is regarded as an unbounded one in an appropriate L_2 space. On the oscillating boundary we impose Dirichlet, Neumann or Robin condition. Apart from a mathematical interest to this problem, as a physical motivation we can mention a model of a planar quantum or acoustic waveguide with a fast oscillating boundary.

Our main result is the form of the homogenized operator and the uniform resolvent convergence of the perturbed operator to the homogenized one. This convergence is established in the sense of the norm of the operator acting from L_2 into W_2^1. The estimates for the rate of convergence are provided. We show that in the case of the Dirichlet condition of the oscillating boundary the homogenized problem involves the same condition on the mollified boundary no matter what is the relation between the period and the amplitude of the oscillations. The Neumann or Robin condition on the oscillating boundary leads to the Robin condition with an additional term in the coefficient provided the amplitude is not greater the period (in order). If the amplitude is greater than the period, then the homogenization preserves the Neumann condition and transforms the Robin condition into the Dirichlet one. The last result is in a good accordance with the similar case treated in [18]. The difference is that in [18] the strong resolvent convergence was proven provided the coefficient in the Robin condition is positive, while we prove the uniform resolvent convergence provided the coefficient is non-negative and vanishes on the set of zero measure.

1 The problem and the main results

Let $x = (x_1, x_2)$ be the Cartesian coordinates in $\mathbb{R}^2$, ε be a small positive parameter, $\eta = \eta(\varepsilon)$ be an arbitrary non-negative 1-periodic function belonging to $C^2(\mathbb{R})$. We define two domains, cf. fig. 1,

$$\Omega_0 := \{x : 0 < x_2 < d\}, \quad \Omega_\varepsilon := \{x : \eta(\varepsilon) b(x_1 \varepsilon^{-1}) < x_2 < d\},$$

where $d > 0$ is a constant, and its boundaries are indicated as

$$\Gamma := \{x : x_2 = d\}, \quad \Gamma_0 := \{x : x_2 = 0\}, \quad \Gamma_\varepsilon := \{x : x_2 = \eta(\varepsilon) b(x_1 \varepsilon^{-1})\}.$$

By $A_{ij} = A_{ij}(x)$, $A_j = A_j(x)$, $A_0 = A_0(x)$, $i, j = 1, 2$, we denote the functions defined on Ω_0 and satisfying the belongings $A_{ij} \in W^2_\infty(\Omega_0)$, $A_j \in W^1_\infty(\Omega_0)$, $A_0 \in L_\infty(\Omega_0)$. The functions A_{ij}, A_j are assumed to be complex-valued, while A_0 is real-valued. In addition, the functions A_{ij} satisfy the ellipticity condition

$$A_{ij} = \overline{A}_{ji}, \quad \sum_{i,j=1}^{2} A_{ij} z_i \overline{z}_j \geqslant c_0(|z_1|^2 + |z_2|^2), \quad x \in \Omega_0, \quad z_j \in \mathbb{C}. \tag{1.1}$$

By $a = a(x)$ we denote a real function defined on $\{x : 0 < x_2 < \delta\}$ for some small fixed δ, and it is supposed that $a \in W^1_\infty(\{x : 0 < x_2 < d\})$.

The main object of our study is the operator

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} A_{ij} \frac{\partial}{\partial x_i} + \sum_{j=1}^{2} A_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j} \overline{A_j} + A_0 \quad \text{in} \quad L_2(\Omega_\varepsilon) \tag{1.2}$$

subject to the Dirichlet condition on Γ. On the other boundary we choose either Dirichlet condition

$$u = 0 \quad \text{on} \quad \Gamma_\varepsilon, \tag{1.3}$$

or Robin condition

$$\left(\frac{\partial}{\partial \nu^\varepsilon} + a \right) u = 0 \quad \text{on} \quad \Gamma_\varepsilon, \quad \frac{\partial}{\partial \nu^\varepsilon} = -\sum_{i,j=1}^{2} A_{ij} \nu_j^\varepsilon \frac{\partial}{\partial x_i} - \sum_{j=1}^{2} \overline{A}_j \nu_j^\varepsilon, \tag{1.4}$$

where $\nu^\varepsilon = (\nu_1^\varepsilon, \nu_2^\varepsilon)$ is the outward normal to Γ_ε. In the case of the Dirichlet condition on Γ_ε we denote this operator as $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$, while for the Robin condition it is $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$. The former includes also the case of the Neumann condition since the function a can be identically zero.

We introduce $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$ rigorously as the lower-semibounded self-adjoint operator in $L_2(\Omega_\varepsilon)$ associated with the closed symmetric lower-semibounded sesquilinear form

$$\begin{aligned} \mathfrak{h}^{\mathrm{D}}_{\varepsilon,\eta}(u, v) := &\sum_{i,j=1}^{2} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^{2} \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_\varepsilon)} \\ &+ \sum_{j=1}^{2} \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_\varepsilon)} + (A_0 u, v)_{L_2(\Omega_\varepsilon)} \end{aligned} \tag{1.5}$$

in $L_2(\Omega_\varepsilon)$ with the domain $\mathfrak{D}(\mathfrak{h}^{\mathrm{D}}_{\varepsilon,\eta}) := W^1_{2,0}(\Omega_\varepsilon, \partial\Omega_\varepsilon)$. Hereinafter $\mathfrak{D}(\cdot)$ is the domain of a form or an operator, and $W^j_{2,0}(\Omega, S)$ denotes the Sobolev space consisting of the functions in $W^j_2(\Omega)$ with zero trace on a curve S lying in a domain $\Omega \subset \mathbb{R}^2$. The operator $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$ is introduced in the same way via the sesquilinear form

$$\begin{aligned} \mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(u, v) := &\sum_{i,j=1}^{2} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^{2} \left(A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\Omega_\varepsilon)} \\ &+ \sum_{j=1}^{2} \left(u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\Omega_\varepsilon)} + (A_0 u, v)_{L_2(\Omega_\varepsilon)} + (au, v)_{L_2(\Gamma_\varepsilon)} \end{aligned} \tag{1.6}$$

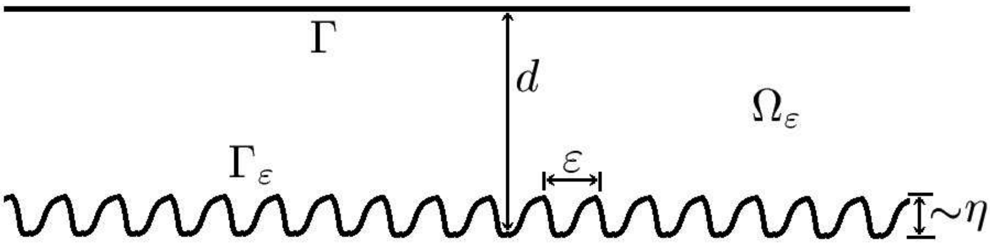

Figure 1: Domain with oscillating boundary

with the domain $\mathfrak{D}(\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}) := W^1_{2,0}(\Omega_\varepsilon,\Gamma)$.

The main aim of the paper is to study the asymptotic behavior of the resolvents of $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$ and $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$ as $\varepsilon\to+0$. To formulate the main results we first introduce some additional operators.

By $\mathcal{H}^{\mathrm{D}}_0$ we denote the operator (1.2) in $L_2(\Omega)$ subject to the Dirichlet condition. We introduce it by analogue with $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$ as associated with the form

$$\begin{aligned}\mathfrak{h}^{\mathrm{D}}_0(u,v) := &\sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_i}\right)_{L_2(\Omega_0)}+\sum_{j=1}^{2}\left(A_j\frac{\partial u}{\partial x_j},v\right)_{L_2(\Omega_0)}\\ &+\sum_{j=1}^{2}\left(u,A_j\frac{\partial v}{\partial x_j}\right)_{L_2(\Omega_0)}+(A_0u,v)_{L_2(\Omega_0)}\end{aligned}\tag{1.7}$$

in $L_2(\Omega_0)$ with the domain $\mathfrak{D}(\mathfrak{h}^{\mathrm{D}}_0) := W^1_{2,0}(\Omega_0,\partial\Omega_0)$. The domain of the operator $\mathcal{H}^{\mathrm{D}}_0$ is $W^2_{2,0}(\Omega_0,\partial\Omega_0)$ that can be shown by analogy with [24, Ch. III, Sec. 7,8], [9, Lm. 2.2].

Our first main result describes the uniform resolvent convergence for $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$.

Theorem 1.1. *Let $f\in L_2(\Omega_0)$. For sufficiently small ε the estimate*

$$\|(\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}-\mathrm{i})^{-1}f-(\mathcal{H}^{\mathrm{D}}_0-\mathrm{i})^{-1}f\|_{W^1_2(\Omega_\varepsilon)}\leqslant C\eta^{1/2}(\varepsilon)\|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f.

The next four theorems describe the resolvent convergence of the operator $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$. We consider separately two cases,

$$\varepsilon^{-1}\eta(\varepsilon)\to\alpha=\mathrm{const}\geqslant 0,\quad \varepsilon\to+0,\tag{1.8}$$

$$\varepsilon^{-1}\eta(\varepsilon)\to+\infty,\qquad \varepsilon\to+0.\tag{1.9}$$

The first assumption means that the amplitude of the oscillation of the curve Γ_ε is of the same order (or smaller) as the period. The other assumption corresponds to the case when the amplitude is much greater than the period. In what follows the first case is referred to as the slowly oscillating boundary Γ_ε while the other describes highly oscillating boundary Γ_ε.

We begin with the slowly oscillating boundary. We denote

$$a_0(x_1) := a(x_1,0)\int_0^1\sqrt{1+\alpha^2\big(b'(t)\big)^2}\,dt.\tag{1.10}$$

Let $\mathcal{H}^{\mathrm{R},\alpha}_0$ be the self-adjoint operator in $L_2(\Omega_0)$ associated with the lower-semibounded sesquilinear symmetric form

$$\begin{aligned}\mathfrak{h}^{\mathrm{R},\alpha}_0(u,v) := &\sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial u}{\partial x_j},\frac{\partial v}{\partial x_i}\right)_{L_2(\Omega_0)}+\sum_{j=1}^{2}\left(A_j\frac{\partial u}{\partial x_j},v\right)_{L_2(\Omega_0)}\\ &+\sum_{j=1}^{2}\left(u,A_j\frac{\partial v}{\partial x_j}\right)_{L_2(\Omega_0)}+(A_0u,v)_{L_2(\Omega_0)}+(a_0u,v)_{L_2(\Gamma_0)}\end{aligned}\tag{1.11}$$

with the domain $\mathfrak{D}(\mathfrak{h}_0^{\mathrm{R},\alpha}) := W_{2,0}^1(\Omega_0,\Gamma)$. It can be shown by analogy with [24, Ch. III, Sec. 7,8], [9, Lm. 2.2] that the domain of $\mathcal{H}_0^{\mathrm{R},\alpha}$ consists of the functions $u \in W_{2,0}^2(\Omega_0,\Gamma_0)$ satisfying the Robin condition

$$\left(\frac{\partial}{\partial\nu^0} + a_0\right) u = 0 \quad \text{on} \quad \Gamma_0, \quad \frac{\partial}{\partial\nu^0} := \sum_{i=1}^{2} A_{i2}\frac{\partial}{\partial x_i} + \overline{A}_2. \tag{1.12}$$

Theorem 1.2. *Suppose (1.8) and let $f \in L_2(\Omega_\varepsilon)$. Then for sufficiently small ε the estimate*

$$\|(\mathcal{H}_{\varepsilon,\eta}^{\mathrm{R}} - \mathrm{i})^{-1}f - (\mathcal{H}_0^{\mathrm{R},\alpha} - \mathrm{i})^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \leqslant C(\eta^{1/2}(\varepsilon) + |\varepsilon^{-2}\eta^2(\varepsilon) - \alpha^2| + \alpha\varepsilon^{1/2})\|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f.

Now we consider the particular case of the Neumann condition on Γ_ε, i.e., $a = 0$. The operator $\mathcal{H}_{\varepsilon,\eta}^{\mathrm{R}}$ and the associated quadratic form $\mathfrak{h}_{\varepsilon,\eta}^{\mathrm{R}}$ are re-denoted in this case by $\mathcal{H}_{\varepsilon,\eta}^{\mathrm{N}}$ and $\mathfrak{h}_{\varepsilon,\eta}^{\mathrm{N}}$. By $\mathcal{H}_0^{\mathrm{N}}$ we denote the self-adjoint lower-semibounded operator in $L_2(\Omega_0)$ associated with the sesquilinear form $\mathfrak{h}_0^{\mathrm{N}}$ which is $\mathfrak{h}_0^{\mathrm{R},\alpha}$ taken for $a_0 \equiv 0$. Its domain is the set of the functions in $W_{2,0}^2(\Omega_0,\Gamma)$ satisfying the boundary condition (1.12) with $a_0 = 0$. The resolvent convergence in this case is given in

Theorem 1.3. *Let $f \in L_2(\Omega_\varepsilon)$. Suppose (1.8). Then for sufficiently small ε the estimate*

$$\|(\mathcal{H}_{\varepsilon,\eta}^{\mathrm{N}} - \mathrm{i})^{-1}f - (\mathcal{H}_0^{\mathrm{N}} - \mathrm{i})^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \leqslant C\eta^{1/2}(\varepsilon)\|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f.

Suppose (1.9). Then for sufficiently small ε the estimate

$$\|(\mathcal{H}_{\varepsilon,\eta}^{\mathrm{N}} - \mathrm{i})^{-1}f - (\mathcal{H}_0^{\mathrm{N}} - \mathrm{i})^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \leqslant C\eta^{1/4}(\varepsilon)\|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f.

We proceed to the case of the Robin condition on the highly oscillating boundary Γ_ε. Here the homogenized operator happens to be quite sensitive to the sign of a and zero level set of this function. In the paper we describe the resolvent convergence as a is non-negative. We first suppose that a is bounded from below by a positive constant. Surprisingly, but here the homogenized operator has the Dirichlet condition on Γ_0 as in Theorem 1.1.

Theorem 1.4. *Suppose (1.9),*

$$a(x) \geqslant c = \mathrm{const} > 0, \tag{1.13}$$

and that the function b is not identically constant. Let $f \in L_2(\Omega_0)$. Then for sufficiently small ε the estimate

$$\|(\mathcal{H}_{\varepsilon,\eta}^{\mathrm{R}} - \mathrm{i})^{-1}f - (\mathcal{H}_0^{\mathrm{D}} - \mathrm{i})^{-1}f\|_{W_2^1(\Omega_\varepsilon)} \leqslant C\big(\varepsilon^{1/4}\eta^{-1/4}(\varepsilon) + \eta^{1/4}(\varepsilon)\big)\|f\|_{L_2(\Omega_0)}$$

holds true, where C is a constant independent of ε and f.

In the next theorem we still suppose that a is non-negative but can have zeroes. An essential assumption that zero level set of a is of zero measure. We let $b_* := \max\limits_{[0,1]} b$.

Theorem 1.5. *Suppose (1.9),*

$$a \geqslant 0, \tag{1.14}$$

and that the function b is not identically constant. Assume also that for all sufficiently small δ the set $\{x : a(x) \leqslant \delta,\ 0 < x_2 < (b_* + 1)\eta\}$ is contained in an at most countable union of the rectangles $\{x : |x_1 - X_n| < \mu(\delta),\ 0 < x_2 < (b_* + 1)\eta\}$, where $\mu(\delta)$ is a some nonnegative function such that $\mu(\delta) \to +0$ as $\delta \to +0$, and the numbers X_n, $n \in \mathbb{Z}$, *are independent of* δ*, are taken in the ascending order, and satisfy uniform in* n *and* m *estimate*

$$|X_n - X_m| \geqslant c > 0, \quad n \neq m. \tag{1.15}$$

Let $f \in L_2(\Omega_0)$*. Then for sufficiently small* ε *the estimate*

$$\begin{aligned}\|(\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta} - \mathrm{i})^{-1} f - (\mathcal{H}^{\mathrm{D}}_0 - \mathrm{i})^{-1} f\|_{W_2^1(\Omega_\varepsilon)}& \\ \leqslant C\big(\mu^{1/2}(\delta)|\ln \mu(\delta)|^{1/2} + (\varepsilon^{1/4}\eta^{-1/4} + \eta^{1/4})\delta^{-1/4}\big)\|f\|_{L_2(\Omega_0)}&\end{aligned}$$

holds true, where C *is a constant independent of* ε *and* f*, and* $\delta = \delta(\varepsilon)$ *is any function tending to zero as* $\varepsilon \to +0$*.*

Let us discuss the main results. We first observe that under the hypotheses of all theorems we have the corresponding spectral convergence, namely, the convergence of the spectrum and the associated spectral projectors – see, for instance, [33, Thms. VIII.23, VIII.24]. We also stress that in all Theorems 1.1-1.5 the resolvent convergence is established in the sense of the uniform norm of bounded operator acting from $L_2(\Omega_0)$ into $W_2^1(\Omega_\varepsilon)$.

In the case of the Dirichlet condition on Γ_ε the homogenized operator has the same condition on Γ_0 no matter how the boundary Γ_ε oscillates, slowly or highly. The estimate for the rate of convergence is also universal being $\mathcal{O}(\eta^{1/2})$. Once we have Robin condition on Γ_ε, the situation is completely different. If the boundary oscillates slowly, the homogenized operator still has Robin condition on Γ_0, but the coefficient depends on the geometry of the original oscillations, cf. (1.10). The estimate for the rate of the resolvent convergence in this case involves additional term in comparison with the Dirichlet case, cf. Theorem 1.2.

As the boundary Γ_ε oscillates highly, the resolvent convergence again changes. In the particular case of Neumann condition on Γ_ε the homogenized operator still has the Neumann condition on Γ_ε but the estimate for the rate of the resolvent convergence is of order $\mathcal{O}(\eta^{1/4})$, cf. Theorem 1.3. We note that this theorem also treat the case of the slowly oscillating boundary which is better than that in Theorem 1.2.

Once the coefficient a in the Robin condition is non-zero and the boundary oscillates highly, the situation changes dramatically in comparison to all previous cases. Namely, provided the function a is non-negative and does not vanishes identically on a set of non-zero measure, the homogenized operator has the Dirichlet condition on Γ_0, i.e., in the limit the type of the boundary condition changes. We stress that in all previous cases the type of the boundary condition was preserved under the homogenization. In the present case the estimate for the rate of the resolvent convergence is sensible to the presence of zeroes of a. If this function is lower-semibounded by a positive constant, the estimate for the rate is $\mathcal{O}(\varepsilon^{1/4}\eta^{-1/4} + \eta^{1/4})$, cf. Theorem 1.4. Once it has zeroes and they form a set of zero measure, then it becomes important how fast the function a vanishes in a vicinity of these zeroes. This is reflected by the function $\mu(\delta)$ introduced in Theorem 1.5. As we see, the estimate for the rate of convergence provided by this theorem is of order $\mathcal{O}\big(\mu^{1/2}|\ln \mu|^{1/2} + (\varepsilon^{1/4}\eta^{-1/4} + \eta^{1/4})\delta^{-1/4}\big)$, where $\mu = \mu(\delta(\varepsilon))$. Here one should choose δ so that $(\varepsilon^{1/4}\eta^{-1/4} + \eta^{1/4})\delta^{-1/4} \to +0$, $\delta \to +0$ as $\varepsilon \to +0$, and it is always possible. The optimal choice of δ is so that

$$\begin{aligned}&\mu^{1/2}|\ln \mu|^{1/2} \sim (\varepsilon^{1/4}\eta^{-1/4} + \eta^{1/4})\delta^{-1/4}, \\ &\delta^{1/2}\mu|\ln \mu| \sim \varepsilon^{1/2}\eta^{-1/2} + \eta^{1/2}.\end{aligned} \tag{1.16}$$

As we see, it depends on the structure of the function μ. The most typical case is $\mu(\delta) \sim \delta^{1/2}$, i.e., the function a vanishes by the quadratic law in a vicinity of its zeroes. In this case the condition (1.16) becomes

$$\delta|\ln \delta| \sim \varepsilon^{1/2}\eta^{-1/2} + \eta^{1/2},$$

which implies

$$\delta \sim (\varepsilon^{1/2}\eta^{-1/2} + \eta^{1/2})|\ln(\varepsilon^{1/2}\eta^{-1/2} + \eta^{1/2})|^{-1}.$$

Then the estimate for the resolvent convergence in Theorem 1.5 is of order $\mathcal{O}\big((\varepsilon^{1/8}\eta^{-1/8}+\eta^{1/8})|\ln(\varepsilon\eta^{-1}+\eta)|^{1/4}\big)$.

In conclusion we discuss the case of Robin condition on highly oscillating Γ_ε when the coefficient a does not satisfy the hypotheses of Theorems 1.4, 1.5. If it is still non-negative but vanishes on a set of non-zero measure, then we conjecture that the homogenized involves mixed Dirichlet and Neumann condition on Γ_0. Namely, if $a(x_1,0)\equiv 0$ on G_0^N and $a(x_1,0)>0$ on G_0^D, $\Gamma_0=\Gamma_0^N\cup\Gamma_0^D$, then it is natural to expect that the homogenized operator has Neumann condition on Γ_0^N and the Dirichlet one on Γ_0^D. This conjecture can be regarded as the mixture of the statements of Theorems 1.3 and 1.5. The main difficulty of proving this conjecture is that the domain of such homogenized operator is no longer a subset of $W_2^2(\Omega_0)$ because of the mixed boundary conditions. At the same time, this fact was essentially used in all our proofs. Even a more complicated situation occurs once a is negative or sign-indefinite. If a is negative at a set of non-zero measure, it can be shown that the bottom of the spectrum of the perturbed operator goes to $-\infty$ as $\varepsilon\to+0$. In such case one should study the resolvent convergence near this bottom, i.e., for the spectral parameter which goes to $-\infty$. This makes the issue quite troublesome. We stress that under the hypotheses of all Theorems 1.1-1.5 the bottom of the spectrum was lower-semibounded uniformly in ε.

2 Dirichlet condition

In this section we study the resolvent convergence of the operator $\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}$ and prove Theorem 1.1.

By $\chi_0=\chi_0(t)$ we denote an infinitely differentiable non-negative cut-off function with the values in $[0,1]$ vanishing as $t>1$ and being one as $t<0$. We also assume that the values of χ_0 are in $[0,1]$. We choose the function K as

$$K(x_2,\eta):=\chi_0\left(\frac{x_2-b_*\eta}{\eta}\right). \tag{2.1}$$

We observe that the function $(1-K)$ vanishes for $0<x_2<b_*\eta$ and is independent of x_1.

Given a function $f\in L_2(\Omega_0)$, we denote $u_\varepsilon:=(\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}-\mathrm{i})^{-1}f$, $u_0:=(\mathcal{H}^{\mathrm{D}}_0-\mathrm{i})^{-1}f$, $v_\varepsilon:=u_\varepsilon-(1-K)u_0$. In accordance with the definition of u_ε and u_0 these functions satisfy the integral identities

$$\mathfrak{h}^{\mathrm{D}}_{\varepsilon,\eta}(u_\varepsilon,\phi)+\mathrm{i}(u_\varepsilon,\phi)_{L_2(\Omega_\varepsilon)}=(f,\phi)_{L_2(\Omega_\varepsilon)} \tag{2.2}$$

for each $\phi\in W^1_{2,0}(\Omega_\varepsilon,\partial\Omega_\varepsilon)$, and

$$\mathfrak{h}^{\mathrm{D}}_0(u_0,\phi)+\mathrm{i}(u_0,\phi)_{L_2(\Omega_0)}=(f,\phi)_{L_2(\Omega_0)} \tag{2.3}$$

for each $\phi\in W^1_{2,0}(\Omega_0,\partial\Omega_0)$. It is clear that $(1-K)v_\varepsilon\in W^1_{2,0}(\Omega_\varepsilon,\partial\Omega_\varepsilon)$, $(1-K)u_0\in W^1_{2,0}(\Omega_\varepsilon,\partial\Omega_\varepsilon)$, and the extension of v_ε by zero in $\Omega_0\setminus\Omega_\varepsilon$ belongs to $W^1_{2,0}(\Omega_0,\partial\Omega_0)$. Bearing these facts in mind, as the test function in (2.2) we choose $\phi=v_\varepsilon$, and in (2.3) we let $\phi=(1-K)v_\varepsilon$ assuming that v_ε is extended by zero in $\Omega_0\setminus\Omega_\varepsilon$. It yields

$$\mathfrak{h}^{\mathrm{D}}_{\varepsilon,\eta}(u_\varepsilon,v_\varepsilon)+\mathrm{i}(u_\varepsilon,v_\varepsilon)=(f,v_\varepsilon)_{L_2(\Omega_\varepsilon)}, \tag{2.4}$$

$$\mathfrak{h}^{\mathrm{D}}_0(u_0,(1-K)v_\varepsilon)+\mathrm{i}(u_0,(1-K)v_\varepsilon)_{L_2(\Omega_\varepsilon)}=(f,(1-K)v_\varepsilon)_{L_2(\Omega_\varepsilon)}. \tag{2.5}$$

Employing (1.7), we rewrite the term $\mathfrak{h}_0^{\mathrm{D}}(u_0,(1-K)v_\varepsilon)$,

$$
\begin{aligned}
\mathfrak{h}_0^{\mathrm{D}}(u_0,&(1-K)v_\varepsilon)=\sum_{i,j=1}^{2}\left((1-K)A_{ij}\frac{\partial u_0}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}+\sum_{j=1}^{2}\left(A_j(1-K)\frac{\partial u_0}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}\\
&+\sum_{j=1}^{2}\left((1-K)u_0,A_j\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega_\varepsilon)}+(A_0(1-K)u_0,v_\varepsilon)_{L_2(\Omega_\varepsilon)}\\
&-\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}\\
=&\mathfrak{h}_{\varepsilon,\eta}^{\mathrm{D}}((1-K)u_0,v_\varepsilon)+\left(A_2u_0,v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}\\
&-\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}+\sum_{i=1}^{2}\left(A_{i2}u_0\frac{\partial K}{\partial x_2},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}.
\end{aligned}
\tag{2.6}
$$

It implies

$$
\begin{aligned}
\mathfrak{h}_{\varepsilon,\eta}^{\mathrm{D}}(v_\varepsilon,v_\varepsilon)+\mathrm{i}\|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2=&(f,Kv_\varepsilon)_{L_2(\Omega_\varepsilon)}\\
&+\left(A_2u_0,v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}\\
&-\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}+\sum_{i=1}^{2}\left(A_{i2}u_0\frac{\partial K}{\partial x_2},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}.
\end{aligned}
\tag{2.7}
$$

Our main idea of the proof of Theorem 1.1 is to estimate the right hand side of (2.7) with the introduced function K and get in this way an estimate for v_ε. In order to do it, we need some auxiliary statements.

We first observe obvious inequalities

$$
\|u_0\|_{W_2^1(\Omega_0)}\leqslant C\|f\|_{L_2(\Omega_0)},\quad \|u_\varepsilon\|_{W_2^1(\Omega_0)}\leqslant C\|f\|_{L_2(\Omega_0)}. \tag{2.8}
$$

Here and till the end of the section by C we denote inessential constants independent of ε, x, and f. Proceeding as in [24, Ch. III, Sec. 7,8] (see also [9, Lm. 2.2]), one can also check that

$$
\|u_0\|_{W_2^2(\Omega_0)}\leqslant C\|f\|_{L_2(\Omega_0)}. \tag{2.9}
$$

The next required statement is

Lemma 2.1. *Suppose $u\in W_{2,0}^2(\Omega_0,\Gamma_0)$, $v\in W_{2,0}^1(\Omega_\varepsilon,\Gamma_\varepsilon)$. Then for almost each $x_1\in\mathbb{R}$, $x_2\in[0,d]$ the estimates*

$$
\begin{aligned}
&|u(x)|^2\leqslant Cx_2^2\|u(x_1,\cdot)\|_{W_2^2(0,d)}^2,\\
&|\nabla u(x)|^2\leqslant C\|\nabla u(x_1,\cdot)\|_{W_2^1(0,d)}^2,\\
&|v(x)|^2\leqslant Cx_2\|v(x_1,\cdot)\|_{W_2^1(\eta b(x_1\varepsilon^{-1}),d)}^2
\end{aligned}
$$

hold true, where C are constants independent of x, ε, u, and v.

Proof. Since $u\in W_2^2(\Omega_0)$, $v\in W_2^1(\Omega_\varepsilon)$, for almost all $x_1\in\mathbb{R}$ we have $u(x_1,\cdot)\in W_2^2(0,d)$, $v\in W_2^1(0,d)$. We represent the function u as

$$
u(x_1,x_2)=\int\limits_0^{x_2}\frac{\partial u}{\partial x_2}(x_1,t)\,dt,
$$

and by Cauchy-Schwarz inequality we obtain

$$
|u(x_1,x_2)|^2\leqslant Cx_2\int\limits_0^{x_2}\left|\frac{\partial u}{\partial x_2}(x_1,t)\right|^2dt. \tag{2.10}
$$

Let $\chi_1=\chi_1(x_2)$ be an infinitely differentiable smooth function vanishing as $x_2>3d/4$ and equalling one as $x_2<d/2$. Then for $x_2\in[0,d/2]$ we have

$$\frac{\partial u}{\partial x_2}(x_1,x_2)=\int\limits_d^{x_2}\left(\frac{\partial}{\partial x_2}\chi_1\frac{\partial u}{\partial x_2}\right)(x_1,t)\,dt,$$

and thus

$$\left|\frac{\partial u}{\partial x_2}(x_1,x_2)\right|^2\leqslant C\int\limits_0^d\left(\left|\frac{\partial^2 u}{\partial x_2^2}(x_1,t)\right|^2+\left|\frac{\partial u}{\partial x_2}(x_1,t)\right|^2\right)\,dt.$$

Substituting this inequality into (2.10), we arrive at the first required estimate. To prove two others one should proceed as above starting with the representation

$$v(x_1,x_2)=\int\limits_0^{x_2}\frac{\partial v}{\partial x_2}(x_1,t)\,dt,$$

where v is assumed to be extended by zero outside Ω_ε, and the representation

$$\frac{\partial u}{\partial x_j}(x_1,x_2)=\int\limits_d^{x_2}\left(\frac{\partial}{\partial x_2}\chi_1\frac{\partial u}{\partial x_j}\right)(x_1,t)\,dt.$$

□

Now we proceed to the estimating the right hand side of (2.7). Denote $\Omega^\eta:=\Omega_\varepsilon\cap\{x: 0<x_2<(b_*+1)\eta\}$. Since the function K vanishes outside Ω^η and $|\nabla K|\leqslant C\eta^{-1}$, $0\leqslant K\leqslant 1$, it is easy to see that

$$\begin{aligned}&\Bigg|(f,Kv_\varepsilon)_{L_2(\Omega_\varepsilon)}+\left(A_2u_0,v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}\\&-\sum_{j=1}^2\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}+\sum_{i=1}^2\left(A_{i2}u_0\frac{\partial K}{\partial x_2},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}\Bigg|\\&\leqslant C\Big(\|f\|_{L_2(\Omega_\varepsilon)}\|v_\varepsilon\|_{L_2(\Omega^\eta)}+\eta^{-1}\|u_0\|_{W_2^1(\Omega^\eta)}\|v_\varepsilon\|_{L_2(\Omega^\eta)}\\&\quad+\eta^{-1}\|u_0\|_{L_2(\Omega^\eta)}\|\nabla v_\varepsilon\|_{L_2(\Omega^\varepsilon)}\Big).\end{aligned}\tag{2.11}$$

We estimate the terms in the right hand side by applying Lemma 2.1,

$$\begin{aligned}\|v_\varepsilon\|^2_{L_2(\Omega^\eta)}&\leqslant\int\limits_{\mathbb{R}}\|v_\varepsilon(x_1,\cdot)\|^2_{L_2(\eta b(x\varepsilon^{-1}),(b_*+1)\eta)}\,dx_1\\&\leqslant C\|v_\varepsilon\|^2_{W_2^1(\Omega_\varepsilon)}\int\limits_0^{(b_*+1)\eta}x_2\,dx_2\leqslant C\eta^2\|v_\varepsilon\|^2_{W_2^1(\Omega_\varepsilon)},\\\|u_0\|^2_{L_2(\Omega^\eta)}&\leqslant C\|u_0\|^2_{W_2^2(\Omega_\varepsilon)}\int\limits_0^{(b_*+1)\eta}x_2^2\,dx_2\leqslant C\eta^3\|u_0\|^2_{W_2^2(\Omega_\varepsilon)},\\\|\nabla u_0\|^2_{L_2(\Omega^\eta)}&\leqslant C\|u_0\|^2_{W_2^2(\Omega_\varepsilon)}\int\limits_0^{(b_*+1)\eta}dx_2\leqslant C\eta\|u_0\|^2_{W_2^2(\Omega_\varepsilon)}.\end{aligned}\tag{2.12}$$

We substitute the obtained estimates and (2.9) into (2.11),

$$\begin{aligned}&\Bigg|(f,Kv_\varepsilon)_{L_2(\Omega_\varepsilon)}+\left(A_2u_0,v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}\\&-\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}+\sum_{i=1}^{2}\left(A_{i2}u_0\frac{\partial K}{\partial x_2},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}\Bigg|\\&\leqslant C\eta^{1/2}(\varepsilon)\|f\|_{L_2(\Omega_\varepsilon)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.\end{aligned}$$

We substitute the last obtained estimate and (2.9) into the right hand side of (2.7) and arrive at the final estimate for v_ε,

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\leqslant C\eta^{1/2}(\varepsilon)\|f\|_{L_2(\Omega_\varepsilon)}.$$

Using (2.9), by analogy with (2.12) one can check easily that

$$\|Ku_0\|_{W_2^1(\Omega_0)}\leqslant C\eta^{1/2}\|u_0\|_{W_2^2(\Omega_0)}\leqslant C\eta^{1/2}\|f\|_{L_2(\Omega_0)}. \tag{2.13}$$

The statement of Theorem 1.1 follows from two last estimates and the definition of v_ε.

3 Robin condition and slowly oscillating boundary and Neumann condition

In this section we study the resolvent convergence for the operator $\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}$ and prove Theorems 1.2, 1.3. Throughout the proofs by C we indicate various inessential constants independent of ε, x, and f.

Proof of Theorem 1.2. Denote $u_\varepsilon:=(\mathcal{H}^{\mathrm{R}}_{\varepsilon,\eta}-\mathrm{i})^{-1}f$, $u_0:=(\mathcal{H}^{\mathrm{R},\alpha}_0-\mathrm{i})^{-1}f$. First we write the integral identities for the functions u_ε and u_0,

$$\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(u_\varepsilon,\phi)-\mathrm{i}(u_\varepsilon,\phi)_{L_2(\Omega_\varepsilon)}=(f,\phi)_{L_2(\Omega_\varepsilon)}\qquad\text{for all}\quad \phi\in W^1_{2,0}(\Omega_\varepsilon,\Gamma), \tag{3.1}$$

$$\mathfrak{h}^{\mathrm{R},\alpha}_0(u_0,\phi)-\mathrm{i}(u_0,\phi)_{L_2(\Omega_0)}=(f,\phi)_{L_2(\Omega_0)}\qquad\text{for all}\quad \phi\in W^1_{2,0}(\Omega_0,\Gamma). \tag{3.2}$$

We extend the function u_ε in $\Omega_0\setminus\Omega_\varepsilon$ by the rule

$$u_\varepsilon(x)=u_\varepsilon\big(x_1,2\eta(\varepsilon)b(x_1\varepsilon^{-1})-x_2\big),\quad x_2<\eta(\varepsilon)b(x_1\varepsilon^{-1}). \tag{3.3}$$

Bearing in mind (1.8), it is easy to show that after this extension the function u_ε belongs to $W_2^1(\Omega_0)$ and the estimates

$$\|\nabla u_\varepsilon\|_{L_2(\Omega_0\setminus\Omega_\varepsilon)}\leqslant C\|\nabla u_\varepsilon\|_{L_2(\Omega_\varepsilon)}. \tag{3.4}$$

The next lemma is an analogue of Lemma 2.1 and is proven in the similar way.

Lemma 3.1. *Let $u\in W_2^2(\Omega_\varepsilon)$, $v\in W_2^1(\Omega_\varepsilon)$. Then for almost all $x_1\in\mathbb{R}$, $x_2\in[0,d/2]$ the estimates*

$$\begin{aligned}&|u(x)|+|\nabla u(x)|\leqslant C\|u(x_1,\cdot)\|_{W_2^2(\eta b(x_1\varepsilon^{-1}),d)},\\&|v(x)|\leqslant C\|u(x_1,\cdot)\|_{W_2^1(\eta b(x_1\varepsilon^{-1}),d)}\end{aligned}$$

hold true, where the constant C are independent of x, ε, u, and v.

The next lemma gives an apriori estimate for the form $\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}$.

Lemma 3.2. *For any $u\in W^1_{2,0}(\Omega_\varepsilon,\Gamma)$ the estimate*

$$\|u\|^2_{W_2^1(\Omega_\varepsilon)}\leqslant\big|\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(u,u)-\mathrm{i}\|u\|^2_{L_2(\Omega_\varepsilon)}\big|$$

hold true, where the constant C is independent of ε and u.

Proof. For $x\in\Gamma_\varepsilon$ we have

$$|u(x)|^2 \leqslant \int\limits_{\eta b(x_1\varepsilon^{-1})}^{d} \frac{\partial |u|^2}{\partial x_2}(x_1,t)\,dt \leqslant \delta\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} + C(\delta)\|u\|^2_{L_2(\Omega_\varepsilon)}, \tag{3.5}$$

where the constant δ can be chosen arbitrarily small. Hence, due to (1.8), for an appropriate choice of δ

$$\begin{aligned} |(au,u)_{L_2(\Gamma_\varepsilon)}| &= \Big|\int\limits_{\Gamma_\varepsilon} a(x)|u(x)|^2\sqrt{1+\varepsilon^{-2}\eta^2\big(b'(x_1\varepsilon^{-1})\big)^2}\,dx_1\Big| \\ &\leqslant \frac{c_0}{4}\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} + C\|u\|^2_{L_2(\Omega_\varepsilon)}. \end{aligned} \tag{3.6}$$

It is also clear that

$$\left|\sum_{j=1}^{2}\left(A_j\frac{\partial u}{\partial x_j},u\right)_{L_2(\Omega_\varepsilon)} + \sum_{j=1}^{2}\left(u,A_j\frac{\partial u}{\partial x_j}\right)_{L_2(\Omega_\varepsilon)}\right| \leqslant \frac{c_0}{4}\|\nabla u\|^2_{L_2(\Omega_\varepsilon)} + C\|u\|^2_{L_2(\Omega_\varepsilon)}. \tag{3.7}$$

Last two estimates and the definition (1.6) of $\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}$ imply the desired estimate. □

Applying Lemma 3.1 with $v=u_\varepsilon$, we get

$$\|u_\varepsilon\|_{L_2(\Omega_0\setminus\Omega_\varepsilon)} \leqslant C\eta^{1/2}\left\|\frac{\partial u_\varepsilon}{\partial x_2}\right\|_{L_2(\Omega_\varepsilon)}. \tag{3.8}$$

We let $\phi=v_\varepsilon:=u_\varepsilon-u_0$ in (3.1), (3.2) and take the difference of these identities

$$\begin{aligned} \mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(v_\varepsilon,v_\varepsilon) - \mathrm{i}\|v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} = (f,v_\varepsilon)_{L_2(\Omega_\varepsilon)} &+ \sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial u_0}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} \\ &+ \sum_{j=1}^{2}\left(A_j\frac{\partial u_\varepsilon}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} + \sum_{j=1}^{2}\left(u_\varepsilon,A_j\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} \\ &+ (A_0u_0,v_\varepsilon)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} - \mathrm{i}(u_0,v_\varepsilon)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} \\ &- (au_0,v_\varepsilon)_{L_2(\Gamma_\varepsilon)} + (a_0u_0,v_\varepsilon)_{L_2(\Gamma_0)}. \end{aligned} \tag{3.9}$$

Proceeding as [24, Ch. III, Sec. 7,8], [9, Lm. 2.2], one can estimate u_0,

$$\|u_0\|_{W_2^2(\Omega_0)} \leqslant C\|f\|_{L_2(\Omega_0)}. \tag{3.10}$$

Employing this identity and Lemma 3.1 with $u=u_0$, we obtain

$$\|u_0\|_{W_2^1(\Omega_0\setminus\Omega_\varepsilon)} \leqslant C\eta^{1/2}\|f\|_{L_2(\Omega_0)}. \tag{3.11}$$

The inequalities (3.8), (3.10), (3.11), and Lemma 3.2 with $u=u_\varepsilon$ allow us to estimate the most part of the terms in the right hand side in (3.9),

$$\begin{aligned} &\Bigg|(f,v_\varepsilon)_{L_2(\Omega_\varepsilon)} + \sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial u_0}{\partial x_j},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} + \sum_{j=1}^{2}\left(A_j\frac{\partial u_\varepsilon}{\partial x_j},v_\varepsilon\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} \\ &+ \sum_{j=1}^{2}\left(u_\varepsilon,A_j\frac{\partial v_\varepsilon}{\partial x_j}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} + (A_0u_0,v_\varepsilon)_{L_2(\Omega_0\setminus\Omega_\varepsilon)} - \mathrm{i}(u_0,v_\varepsilon)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}\Bigg| \\ &\leqslant C\eta^{1/2}(\varepsilon)\|f\|_{L_2(\Omega_0)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}. \end{aligned} \tag{3.12}$$

It remains to estimate last two terms in the right hand side of (3.9). We denote

$$\begin{aligned} &\widetilde{a}(x_1,\varepsilon) := a\big(x_1,\eta(\varepsilon)b(x_1\varepsilon^{-1})\big)\sqrt{1+\varepsilon^{-2}\eta^2(\varepsilon)\big(b'(x_1\varepsilon^{-1})\big)^2}, \\ &\widehat{a}(x_1,\varepsilon) := a(x_1,0)\sqrt{1+\alpha^2\big(b'(x_1\varepsilon^{-1})\big)^2}, \end{aligned}$$

and have

$$(a_0u_0,v_\varepsilon)_{L_2(\Gamma_0)}-(au_0,v_\varepsilon)_{L_2(\Gamma_\varepsilon)}=\int\limits_{\mathbb{R}} dx_1\Big(a_0(x_1)u_0(x_1,0)\overline{v_\varepsilon(x_1,0)}$$

$$-\widetilde{a}(x_1,\varepsilon)u_0(x_1,\eta b(x_1\varepsilon^{-1}))\overline{v_\varepsilon(x_1,\eta b(x_1\varepsilon^{-1}))}\Big)$$

$$=\int\limits_{\mathbb{R}}\widetilde{a}(x_1,\varepsilon)\Big(u_0(x_1,0)\overline{v_\varepsilon(x_1,0)}-u_0(x_1,\eta b(x_1\varepsilon^{-1}))\overline{v_\varepsilon(x_1,\eta b(x_1\varepsilon^{-1}))}\Big)\,dx_1$$

$$+\int\limits_{\mathbb{R}}\big(\widehat{a}(x_1,\varepsilon)-\widetilde{a}(x_1,\varepsilon)\big)u_0(x_1,0)\overline{v_\varepsilon(x_1,0)}\,dx_1$$

$$+\int\limits_{\mathbb{R}}\big(a_0(x_1)-\widehat{a}(x_1,\varepsilon)\big)u_0(x_1,0)\overline{v_\varepsilon(x_1,0)}\,dx_1. \tag{3.13}$$

We estimate the first term by (3.8), (3.10), (3.11), and Lemma 3.2 with $u=u_\varepsilon$

$$\Bigg|\int\limits_{\mathbb{R}}\widetilde{a}(x_1,\varepsilon)\Big(u_0(x_1,0)\overline{v_\varepsilon(x_1,0)}-u_0(x_1,\eta b(x_1\varepsilon^{-1}))\overline{v_\varepsilon(x_1,\eta b(x_1\varepsilon^{-1}))}\Big)\,dx_1\Bigg|$$

$$\leqslant C\int\limits_{\mathbb{R}} dx_1\int\limits_0^{\eta b(x_1\varepsilon^{-1})}\left|\frac{\partial}{\partial x_2}u_0\overline{v_\varepsilon}\right|dx_2\leqslant C\|u_0\|_{W_2^1(\Omega_0\setminus\Omega_\varepsilon)}\|v_\varepsilon\|_{W_2^1(\Omega_0\setminus\Omega_\varepsilon)} \tag{3.14}$$

$$\leqslant C\eta^{1/2}\|f\|_{L_2(\Omega_0)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.$$

Employing (3.4), Lemma 3.1, and an obvious inequality

$$\sup_{x_1\in\mathbb{R}}\left|\widetilde{a}_1(x_1,\varepsilon)-a(x_1,0)\sqrt{1+\varepsilon^{-2}\eta^2(\varepsilon)\big(b'(x_1\varepsilon^{-1})\big)^2}\right|$$

$$\leqslant\sup_{x_1\in\mathbb{R}}\int\limits_0^{\eta b(x_1\varepsilon^{-1})}\left|\frac{\partial a}{\partial x_2}(x_1,t)\right|dt\leqslant C\eta(\varepsilon),$$

we obtain

$$\Bigg|\int\limits_{\mathbb{R}}\big(\widehat{a}(x_1,\varepsilon)-\widetilde{a}(x_1,\varepsilon)\big)u_0(x_1,0)\overline{v_\varepsilon(x_1,0)}\Bigg|\,dx_1$$

$$\leqslant C\big(|\varepsilon^{-2}\eta^2-\alpha^2|+\sup_{x_1\in\mathbb{R}}|\widetilde{a}_1(x_1,\varepsilon)-\widehat{a}(x_1,\varepsilon)|\big)\|f\|_{L_2(\Omega_0)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \tag{3.15}$$

$$\leqslant C\big(|\varepsilon^{-2}\eta^2-\alpha^2|+\eta\big)\|f\|_{L_2(\Omega_0)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.$$

Denote

$$q_0(z):=\sqrt{1+\alpha^2\big(b'(z)\big)^2}-\int\limits_0^1\sqrt{1+\alpha^2\big(b'(t)\big)^2}\,dt.$$

We observe that

$$\int\limits_0^1 q_0(z)\,dz=0,\quad \widehat{a}(x_1,\varepsilon)-a_0(x_1)=a(x_1,0)q_0(x_1\varepsilon^{-1}).$$

Employing this fact, the inequalities (3.4), (3.10), (3.5), and proceeding completely in the same way as in the proof of Lemma 2.9 in [30, Ch. II, Sec. 2.3], we show that

$$\Bigg|\int\limits_{\mathbb{R}}\big(a_0(x_1)-\widehat{a}(x_1,\varepsilon)\big)u_0(x_1,0)\overline{v_\varepsilon(x_1,0)}\,dx_1\Bigg|\leqslant CB\varepsilon^{1/2}\|f\|_{L_2(\Omega_0)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.$$

Due to the obtained inequality and (3.12), (3.13), (3.14), (3.15) we can estimate the right hand side of (3.9) by the quantity

$$C(\eta^{1/2}(\varepsilon)+|\varepsilon^{-2}\eta^2(\varepsilon)-\alpha^2|+\alpha\varepsilon^{1/2})\|f\|_{L_2(\Omega_0)}\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}.$$

Then we employ (1.1) and Lemma 3.2 with $u=u_\varepsilon$ it completes the proof. □

Proof of Theorem 1.3. The statement of the theorem in the case of slowly oscillating boundary follows directly from (3.9) with $a=a_0=0$, (3.12), and Lemma 3.2. The rest of the proof is devoted to the case of highly oscillating boundary.

Denote $u_\varepsilon:=(\mathcal{H}^{\mathrm{N}}_{\varepsilon,\eta}-\mathrm{i})^{-1}f$, $u_0:=(\mathcal{H}^{\mathrm{N}}_0-\mathrm{i})^{-1}f$, $v_\varepsilon:=u_\varepsilon-u_0$. It follows from the integral identities for u_ε and u_0 that

$$\begin{aligned}&\mathfrak{h}^{\mathrm{N}}_{\varepsilon,\eta}(u_\varepsilon,v_\varepsilon)-\mathrm{i}(u_\varepsilon,v_\varepsilon)_{L_2(\Omega_\varepsilon)}=(f,v_\varepsilon)_{L_2(\Omega_\varepsilon)},\\ &\mathfrak{h}^{\mathrm{N}}_0(u_0,u_0)-\mathrm{i}\|u_0\|^2_{L_2(\Omega_0)}=(f,u_0)_{L_2(\Omega_0)}.\end{aligned}\tag{3.16}$$

Since $\mathfrak{D}(\mathcal{H}^{\mathrm{N}}_0)\subset W_2^2(\Omega_0)$, the function u_0 solves the boundary value problem

$$\begin{gathered}-\sum_{i,j=1}^{2}\frac{\partial u_0}{\partial x_j}A_{ij}\frac{\partial}{\partial x_i}+\sum_{j=1}^{2}\left(A_j\frac{\partial u_0}{\partial x_j}-\frac{\partial}{\partial x_j}\overline{A_j}u_0\right)+A_0u_0-\mathrm{i}u_0=f\quad\text{in}\quad\Omega_0,\\ \frac{\partial u_0}{\partial\nu^0}=0\quad\text{on}\quad\Omega_0.\end{gathered}\tag{3.17}$$

We multiply the equation by $\overline{u}_\varepsilon$ and integrate by parts once over Ω_ε,

$$\mathfrak{h}^{\mathrm{N}}_{\varepsilon,\eta}(u_0,u_\varepsilon)-\mathrm{i}(u_0,u_\varepsilon)_{L_2(\Omega_\varepsilon)}+\left(\frac{\partial u_0}{\partial\nu^\varepsilon},u_\varepsilon\right)_{L_2(\Gamma_\varepsilon)}=(f,u_\varepsilon)_{L_2(\Omega_\varepsilon)},$$

where, we remind, the derivative $\frac{\partial}{\partial\nu^\varepsilon}$ was defined in (1.4). We deduct the last identity from the sum of the identities (3.16),

$$\begin{aligned}\mathfrak{h}^{\mathrm{N}}_{\varepsilon,\eta}(v_\varepsilon,v_\varepsilon)-\mathrm{i}\|v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)}=&(f,u_0)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}+\left(\frac{\partial u_0}{\partial\nu^\varepsilon},u_\varepsilon\right)_{L_2(\Gamma_\varepsilon)}\\ &-\sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial u_0}{\partial x_j},\frac{\partial u_0}{\partial x_i}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}-\sum_{j=1}^{2}\left(A_j\frac{\partial u_0}{\partial x_j},u_0\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}\\ &-\sum_{j=1}^{2}\left(u_0,A_j\frac{\partial u_0}{\partial x_j}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}-(A_0u_0,u_0)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}+\mathrm{i}\|u_0\|^2_{L_2(\Omega_0\setminus\Omega_\varepsilon)}.\end{aligned}\tag{3.18}$$

Since the operator $\mathcal{H}^{\mathrm{N}}_0$ is a particular case of $\mathcal{H}^{\mathrm{R},\alpha}_0$, the estimates (3.10) and Lemma 3.2 hold true. Applying them, we obtain

$$\begin{aligned}\Bigg|&(f,u_0)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}-\sum_{i,j=1}^{2}\left(A_{ij}\frac{\partial u_0}{\partial x_j},\frac{\partial u_0}{\partial x_i}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}-\sum_{j=1}^{2}\left(A_j\frac{\partial u_0}{\partial x_j},u_0\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}\\ &-\sum_{j=1}^{2}\left(u_0,A_j\frac{\partial u_0}{\partial x_j}\right)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}-(A_0u_0,u_0)_{L_2(\Omega_0\setminus\Omega_\varepsilon)}+\mathrm{i}\|u_0\|^2_{L_2(\Omega_0\setminus\Omega_\varepsilon)}\Bigg|\leqslant C\eta^{1/2}\|f\|^2_{L_2(\Omega_0)}.\end{aligned}\tag{3.19}$$

It remains to estimate the boundary term over Γ_ε in the right hand side of (3.18).

Since

$$\nu^\varepsilon=\frac{1}{\sqrt{1+\varepsilon^{-2}\eta^2(\varepsilon)\left(b'\left(\frac{x_1}{\varepsilon}\right)\right)^2}}\left(-\varepsilon^{-1}\eta(\varepsilon)b'\left(\frac{x_1}{\varepsilon}\right),1\right),$$

then

$$\left(\frac{\partial u_0}{\partial\nu^\varepsilon},u_\varepsilon\right)_{L_2(\Gamma_\varepsilon)}=\int\limits_{\mathbb{R}}\left(\varepsilon^{-1}\eta(\varepsilon)b'\left(\frac{x_1}{\varepsilon}\right)w_1^\varepsilon(x)-w_2^\varepsilon(x)\right)\overline{u}_\varepsilon(x)\Big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}dx_1,\tag{3.20}$$

where

$$w_j^\varepsilon := \sum_{i=1}^{2} A_{ij}\frac{\partial u_0}{\partial x_i} + A_j u_0.$$

Denote

$$w_3^\varepsilon(x) := \int\limits_{b_*\eta}^{x_2} w_1^\varepsilon(x_1,t)\overline{u}_\varepsilon(x_1,t)\,dt,$$

where, we remind, $b_* := \max\limits_{[0,1]} b$. The identity

$$\varepsilon^{-1}\eta w_1^\varepsilon\left(x_1,\eta b\left(\frac{x_1}{\varepsilon}\right)\right)b'\left(\frac{x_1}{\varepsilon}\right) = \frac{d}{dx_1}w_3^\varepsilon\left(x_1,\eta b\left(\frac{x_1}{\varepsilon}\right)\right) - \frac{\partial w_3^\varepsilon}{\partial x_1}\left(x_1,\eta b\left(\frac{x_1}{\varepsilon}\right)\right),$$

implies

$$\left|\varepsilon^{-1}\eta\int\limits_{\mathbb{R}} b'\left(\frac{x_1}{\varepsilon}\right) w_1^\varepsilon\overline{u}_\varepsilon\Big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}dx_1\right| = \left|\int\limits_{\mathbb{R}}\frac{\partial w_3^\varepsilon}{\partial x_1}\Big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}dx_1\right| = \left|\int\limits_{\mathbb{R}}dx_1\int\limits_{\eta b_*}^{\eta b\left(\frac{x_1}{\varepsilon}\right)}\frac{\partial w_1^\varepsilon}{\partial x_1}(x)\,dx_2\right|$$
$$\leqslant C\left(\|u_0\|_{W_2^2(\Omega_\varepsilon)}\|u_\varepsilon\|_{L_2(\Omega_\varepsilon\cap\{x:x_2<\eta b_*\})} + \|u_0\|_{W_2^1(\Omega_\varepsilon)\cap\{x:x_2<\eta b_*\}}\|u_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\right).$$

In the same way how (3.11) was proven we show that

$$\|u_\varepsilon\|_{L_2(\Omega_\varepsilon\cap\{x:x_2<\eta b_*\})} + \|u_0\|_{W_2^1(\Omega_\varepsilon)\cap\{x:x_2<\eta b_*\}} \leqslant C\eta^{1/2}(\varepsilon)\|f\|_{L_2(\Omega_\varepsilon)}^2.$$

Thus,

$$\left|\varepsilon^{-1}\eta\int\limits_{\mathbb{R}} w_1^\varepsilon\overline{u}_\varepsilon\Big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}dx_1\right| \leqslant C\eta^{1/2}(\varepsilon)\|f\|_{L_2(\Omega_\varepsilon)}^2. \tag{3.21}$$

We estimate the second part of the right hand side of (3.20) as follows,

$$\left|\int\limits_{\mathbb{R}} w_2^\varepsilon(x)\overline{u}_\varepsilon(x)\Big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}dx_1\right| \leqslant \left\|w_2^\varepsilon\big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}\right\|_{L_2(\mathbb{R})}\left\|u_\varepsilon\big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}\right\|_{L_2(\mathbb{R})}.$$

In view of the boundary condition for u_0 in (3.17), the function w_2^ε vanishes at $x_2=0$. Since it also belongs to $W_2^1(\Omega_0)$, by analogy with Lemma 2.1 one can prove easily that

$$\left|w_2^\varepsilon\big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}\right|^2 \leqslant C\eta\|u_0(x_1,\cdot)\|_{L_2(0,d)}^2 \quad \text{for almost all } x_1\in\mathbb{R},$$
$$\left\|w_2^\varepsilon\big|_{x_2=\eta b\left(\frac{x_1}{\varepsilon}\right)}\right\|_{L_2(\mathbb{R})} \leqslant C\eta^{1/2}\|u_0\|_{W_2^2(\Omega_0)}.$$

The last estimate, (3.10), (3.20), (3.21), and Lemma 3.2 with $u=u_\varepsilon$ yield

$$\left|\left(\frac{\partial u_0}{\partial\nu^\varepsilon},u_\varepsilon\right)_{L_2(\Gamma_\varepsilon)}\right| \leqslant C\eta^{1/2}\|f\|_{L_2(\Omega_\varepsilon)}^2.$$

Together with (3.18), (3.19) it implies

$$\left|\mathfrak{h}_{\varepsilon,\eta}^{\mathrm{N}}[v_\varepsilon] - \mathrm{i}\|v_\varepsilon\|_{L_2(\Omega_\varepsilon)}^2\right| \leqslant C\eta^{1/2}\|f\|_{L_2(\Omega_\varepsilon)}^2,$$

and therefore

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leqslant C\eta^{1/4}(\varepsilon)\|f\|_{L_2(\Omega_\varepsilon)}.$$

The proof is complete. □

4 Robin condition for fast oscillating boundary

In this section we prove Theorems 1.4, 1.5. Throughout the proofs we indicate by C various inessential constants independent of ε, x, and f. Given a function $f\in L_2(\Omega_0)$, we let

$$u_\varepsilon:=(\mathcal{H}^{\mathrm{D}}_{\varepsilon,\eta}-\mathrm{i})^{-1}f,\quad u_0:=(\mathcal{H}^{\mathrm{D}}_{0}-\mathrm{i})^{-1}f,\quad v_\varepsilon:=u_\varepsilon-(1-K)u_0, \tag{4.1}$$

where the function K is introduced by (2.1). We remind that the function $1-K$ vanishes for $x_2<b_*\eta$.

Proof of Theorem 1.4. In the proof we employ some ideas used in the proof of Theorem 1.1. The first of them is the analogue of the identity (2.7).

We write the integral identity for u_ε choosing v_ε as the test function,

$$\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(u_\varepsilon,v_\varepsilon)-\mathrm{i}(u_\varepsilon,v_\varepsilon)_{L_2(\Omega_\varepsilon)}=(f,v_\varepsilon)_{L_2(\Omega_\varepsilon)}, \tag{4.2}$$

and that for u_0 with the test function $(1-K)v_\varepsilon$ extended by zero in $\Omega_0\setminus\Omega_\varepsilon$,

$$\mathfrak{h}^{\mathrm{D}}_{0}\big(u_0,(1-K)v_\varepsilon\big)-\mathrm{i}(u_0,(1-K)v_\varepsilon)_{L_2(\Omega_\varepsilon)}=(f,(1-K)v_\varepsilon)_{L_2(\Omega_\varepsilon)}. \tag{4.3}$$

We observe that

$$(au,(1-K)v)_{L_2(\Gamma_\varepsilon)}=0$$

for all $u,v\in W_2^1(\Omega_\varepsilon)$. Bearing this fact in mind, we reproduce the arguments used in obtaining (2.6) and check easily that

$$\begin{aligned}\mathfrak{h}^{\mathrm{D}}_{0}\big(u_0,(1-K)v_\varepsilon\big)=&\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}\big((1-K)u_0,v_\varepsilon\big)+\left(A_2u_0,v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}\\&-\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}+\sum_{i=1}^{2}\left(A_{i2}u_0\frac{\partial K}{\partial x_2},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}.\end{aligned}$$

We substitute this identity into (4.3) and deduct the result from (4.2),

$$\begin{aligned}\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(v_\varepsilon,v_\varepsilon)-\mathrm{i}\|v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)}=&(f,Kv_\varepsilon)_{L_2(\Omega_\varepsilon)}+\left(A_2u_0,v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}\\&-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}-\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}\\&+\sum_{i=1}^{2}\left(A_{i2}u_0\frac{\partial K}{\partial x_2},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}.\end{aligned} \tag{4.4}$$

In view of the assumption (1.13) the boundary term in the definition of the form $\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}$ can be estimated as

$$\begin{aligned}(au,u)_{L_2(\Gamma_\varepsilon)}&\geqslant c_0\|u\|^2_{L_2(\Gamma_\varepsilon)}\\&=c_1\int\limits_{\mathbb{R}}\Big|u\big(x_1,\eta b(x_1\varepsilon^{-1})\big)\Big|^2\sqrt{1+\varepsilon^{-2}\eta^2\big(b'(x_1\varepsilon^{-1})\big)^2}\,dx_1\\&\geqslant\frac{c_1\eta}{\varepsilon}\int\limits_{\mathbb{R}}\Big|u\big(x_1,\eta b(x_1\varepsilon^{-1})\big)\Big|^2\big|b'(x_1\varepsilon^{-1})\big|\,dx_1\end{aligned} \tag{4.5}$$

for any $u\in W_2^1(\Omega_\varepsilon)$. Together with (1.1), (3.7) it implies

$$\begin{aligned}\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(u,u)\geqslant&C_1\|\nabla u\|^2_{L_2(\Omega_\varepsilon)}+C_2\|u\|^2_{L_2(\Omega_\varepsilon)}\\&+\frac{c_1\eta}{\varepsilon}\int\limits_{\mathbb{R}}\Big|u\big(x_1,\eta b(x_1\varepsilon^{-1})\big)\Big|^2\big|b'(x_1\varepsilon^{-1})\big|\,dx_1,\end{aligned} \tag{4.6}$$

where the constants C_1, C_2 are independent of ε and u, and $C_1 > 0$. The last inequality with $u = u_\varepsilon$ and the definition of K yield the apriori estimate for v_ε and u_ε,

$$\|u_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leqslant C\|f\|_{L_2(\Omega_\varepsilon)}, \tag{4.7}$$

$$\begin{aligned}\int\limits_{\mathbb{R}} \Big|v_\varepsilon\big(x_1, \eta b(x_1\varepsilon^{-1})\big)\Big|^2 \big|b'(x_1\varepsilon^{-1})\big|\,dx_1 = \int\limits_{\mathbb{R}} \Big|u_\varepsilon\big(x_1, \eta b(x_1\varepsilon^{-1})\big)\Big|^2 \big|b'(x_1\varepsilon^{-1})\big|\,dx_1 \\ \leqslant C\varepsilon\eta^{-1}(\varepsilon)\|f\|^2_{L_2(\Omega_\varepsilon)}.\end{aligned} \tag{4.8}$$

The next lemma is one of the main ingredients in the proof.

Lemma 4.1. *The estimate*

$$\|v_\varepsilon\|_{L_2(\widetilde{\Omega}^\eta)} \leqslant C\Big(\varepsilon^{1/2}\|f\|_{L_2(\Omega_\varepsilon)} + (\varepsilon+\eta)\|\nabla v_\varepsilon\|_{L_2(\Omega_\varepsilon)}\Big),$$

holds true, where $\widetilde{\Omega}^\eta := \Omega \cap \{x : b_*\eta < x_2 < (b_*+1)\eta\}$.

Proof. For almost all $x_1 \in \mathbb{R}$ and $x_2 \in (b_*\eta, (b_*+1)\eta)$ the function v_ε satisfies the representation

$$v_\varepsilon(x) = v_\varepsilon\big(x_1, \eta b(x_1\varepsilon^{-1})\big) + \int\limits_{\eta b(x_1\varepsilon^{-1})}^{x_2} \frac{\partial v_\varepsilon}{\partial x_2}(x_1, t)\,dt,$$

which implies

$$|v_\varepsilon(x)|^2 \leqslant 2|v_\varepsilon\big(x_1, \eta b(x_1\varepsilon^{-1})\big)|^2 + 2\big(x_2 - \eta b(x_1\varepsilon^{-1})\big) \int\limits_{\eta b(x_1\varepsilon^{-1})}^{(b_*+1)\eta} \left|\frac{\partial v_\varepsilon}{\partial x_2}(x_1, t)\right|^2 dt.$$

We multiply this inequality by $|b'(x_1\varepsilon^{-1})|$ and integrate it over $\widetilde{\Omega}^\eta$,

$$\int\limits_{\widetilde{\Omega}^\eta} |v_\varepsilon(x)|^2 |b'(x_1\varepsilon^{-1})|\,dx \leqslant C\eta \int\limits_{\mathbb{R}} |v_\varepsilon\big(x_1, \eta b(x_1\varepsilon^{-1})\big)|^2 |b'(x_1\varepsilon^{-1})|\,dx_1 + C\eta^2\|\nabla v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)}.$$

By (4.8) it yields

$$\int\limits_{\widetilde{\Omega}^\eta} |v_\varepsilon(x)|^2 |b'(x_1\varepsilon^{-1})|\,dx \leqslant C\left(\varepsilon\|f\|^2_{L_2(\Omega_\varepsilon)} + \eta^2\|\nabla v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)}\right). \tag{4.9}$$

Denote

$$\widetilde{b}(t) := \int\limits_0^t |b'(z)|\,dz - t\int\limits_0^1 |b'(z)|\,dz.$$

This function is obviously continuous and 1-periodic. It also satisfies the identity

$$\widetilde{b}'(t) := |b'(t)| - \int\limits_0^1 |b'(z)|\,dz.$$

Hence,

$$\begin{aligned}\int\limits_{\widetilde{\Omega}^\eta} |v_\varepsilon(x)|^2 |b'(x_1\varepsilon^{-1})|\,dx &= \int\limits_0^1 |b'(t)|\,dt \int\limits_{\widetilde{\Omega}^\eta} |v_\varepsilon(x)|^2\,dx + \varepsilon\int\limits_{\widetilde{\Omega}^\eta} |v_\varepsilon(x)|^2 \frac{\partial}{\partial x_1}\widetilde{b}(x_1\varepsilon^{-1})\,dx \\ &= \int\limits_0^1 |b'(t)|\,dt \int\limits_{\widetilde{\Omega}^\eta} |v_\varepsilon(x)|^2\,dx - \varepsilon\int\limits_{\widetilde{\Omega}^\eta} \widetilde{b}(x_1\varepsilon^{-1})\frac{\partial}{\partial x_1}|v_\varepsilon(x)|^2\,dx.\end{aligned} \tag{4.10}$$

The function $\widetilde{b}$ is bounded and thus

$$\|v_\varepsilon\|^2_{L_2(\widetilde{\Omega}^\eta)}\int\limits_0^1 |b'(t)|\,dt\leqslant \int\limits_{\widetilde{\Omega}^\eta}|v_\varepsilon(x)|^2|b'(x_1\varepsilon^{-1})|\,dx + C\varepsilon\|v_\varepsilon\|_{L_2(\widetilde{\Omega}^\eta)}\left\|\frac{\partial v_\varepsilon}{\partial x_1}\right\|_{L_2(\widetilde{\Omega}^\eta)}.$$

Since the function b' is not identically zero, the last inequality implies

$$\|v_\varepsilon\|^2_{L_2(\widetilde{\Omega}^\eta)}\leqslant C\int\limits_{\widetilde{\Omega}^\eta}|v_\varepsilon(x)|^2|b'(x_1\varepsilon^{-1})|\,dx + C\varepsilon^2\left\|\frac{\partial v_\varepsilon}{\partial x_1}\right\|_{L_2(\Omega_\varepsilon)}.$$

Together with (4.9) it leads us to the desired estimate for $\|v_\varepsilon\|_{L_2(\widetilde{\Omega}^\eta)}$. □

Let us estimate the right hand side of (4.4). By the definition of K, (2.9), and Lemma 2.1 we have

$$\begin{aligned}
&\bigg|(f,Kv_\varepsilon)_{L_2(\Omega_\varepsilon)}+\left(A_2u_0,v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}\\
&-\left(u_0,A_2\frac{\partial K}{\partial x_2}v_\varepsilon\right)_{L_2(\Omega_\varepsilon)}+\sum_{i=1}^2\left(A_{i2}u_0\frac{\partial K}{\partial x_2},\frac{\partial v_\varepsilon}{\partial x_i}\right)_{L_2(\Omega_\varepsilon)}\bigg|\\
&\leqslant C\Big(\|f\|_{L_2(\Omega^\eta)}\|v_\varepsilon\|_{L_2(\Omega^\eta)}+\eta^{-1}\|u_0\|_{L_2(\widetilde{\Omega}^\eta)}\|v_\varepsilon\|_{W_2^1(\widetilde{\Omega}^\eta)}\Big)\\
&\leqslant C\eta^{1/2}\|f\|^2_{L_2(\Omega_0)},
\end{aligned}\tag{4.11}$$

where, we remind, $\Omega^\eta=\Omega_\varepsilon\cap\{x:0<x_2<(b_*+1)\eta\}$. Lemma 4.1 implies the estimate for the remaining term in the right hand side of (4.4),

$$\left|\sum_{j=1}^2\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}\right|\leqslant C\varepsilon^{1/2}\eta^{-1/2}(\varepsilon)\|f\|^2_{L_2(\Omega_0)}.\tag{4.12}$$

We substitute the obtained estimates into (4.4) and get

$$\left|\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}(v_\varepsilon,v_\varepsilon)-\mathrm{i}\|v_\varepsilon\|^2_{L_2(\Omega_\varepsilon)}\right|\leqslant C\big(\varepsilon^{1/2}\eta^{-1/2}+\eta^{1/2}\big)\|f\|^2_{L_2(\Omega_0)}.$$

This estimate and (4.5), (4.6) with $u=v_\varepsilon$ imply

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\leqslant C\big(\varepsilon^{1/4}\eta^{-1/4}+\eta^{1/4}\big)\|f\|_{L_2(\Omega_0)}.$$

It remains to employ the definition of v_ε and (2.13) complete the proof. □

Proof of Theorem 1.5. The proof of this theorem follows the same lines as that of the previous theorem with one substantial modification due to the replacement of the assumption (1.13) by (1.14). As in the previous proof, we define the function u_0, u_ε, v_ε by (4.1) and obtain the identity (4.4) for v_ε. The estimate (4.11) is still valid since it is independent on the assumptions for a, while (4.12) is no longer valid. And the aforementioned modification in the proof is exactly a new estimate substituting (4.12). Let us obtain it.

Proceeding as in (4.9), (4.10), we obtain

$$\begin{aligned}
&\int\limits_0^1|b'(t)|\,dt\sum_{j=1}^2\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}=\sum_{j=1}^2\left(A_{2j}\frac{\partial u_0}{\partial x_j},|b'|\frac{\partial K}{\partial x_2}\overline{v}_\varepsilon\right)_{L_2(\widetilde{\Omega}^\eta)}\\
&-\varepsilon\sum_{j=1}^2\int\limits_{\Omega_\varepsilon}|\widetilde{b}|\frac{\partial K}{\partial x_2}\frac{\partial}{\partial x_1}A_{2j}\frac{\partial u_0}{\partial x_j}\overline{v}_\varepsilon\,dx=-\sum_{j=1}^2\left(A_{2j}\frac{\partial u_0}{\partial x_j},|b'|\overline{v}_\varepsilon\right)_{L_2(\Gamma^\eta)}\\
&-\sum_{j=1}^2\int\limits_{\widetilde{\Omega}^\eta}K|b'|\frac{\partial}{\partial x_2}A_{2j}\frac{\partial u_0}{\partial x_j}\overline{v}_\varepsilon\,dx-\varepsilon\sum_{j=1}^2\int\limits_{\Omega_\varepsilon}|\widetilde{b}|\frac{\partial K}{\partial x_2}\frac{\partial}{\partial x_1}A_{2j}\frac{\partial u_0}{\partial x_j}\overline{v}_\varepsilon\,dx,
\end{aligned}$$

where $b' = b'(x_1\varepsilon^{-1})$, $\widetilde{b} = \widetilde{b}(x_1\varepsilon^{-1})$, $\Gamma^\eta := \{x : x_2 = b_*\eta\}$. In view of Lemma 2.1 and the definition of K it yields

$$\begin{aligned}\left|\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j}, v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}\right| \leqslant & C\|f\|_{L_2(\Omega_0)}\||b'|^{1/2}v_\varepsilon\|_{L_2(\Gamma^\eta)} \\ & + C(\varepsilon\eta^{-1/2}+\eta^{1/2})\|f\|^2_{L_2(\Omega_0)}.\end{aligned} \tag{4.13}$$

It remains to estimate $\||b'|^{1/2}v_\varepsilon\|_{L_2(\Gamma^\eta)}$.

Proceeding as in (4.5), (4.8), and in the proof of Lemma 4.1, one can show easily that

$$\int\limits_{\Gamma^\eta} a|b'||v_\varepsilon|^2\,dx_1 \leqslant C(\varepsilon\eta^{-1}+\eta)\|f\|^2_{L_2(\Omega_\varepsilon)}. \tag{4.14}$$

Given any $\delta > 0$, we split the set Γ^η into two parts, $\Gamma^\eta = \Gamma^\eta_\delta \cup \Gamma^{\eta,\delta}$,

$$\Gamma^\eta_\delta := \{x : a(x_1, b_*\eta) > \delta, x_2 = b_*\eta\}, \quad \Gamma^{\eta,\delta} := \{x : a(x_1, b_*\eta) \leqslant \delta, x_2 = b_*\eta\}.$$

In view of (4.14) it is clear that

$$\begin{aligned}\||b'|^{1/2}v_\varepsilon\|_{L_2(\Gamma^\eta)} &\leqslant \||b'|^{1/2}v_\varepsilon\|_{L_2(\Gamma^{\eta,\delta})} + \delta^{-1/2}\|a^{1/2}|b'|^{1/2}v_\varepsilon\|_{L_2(\Gamma^\eta_\delta)} \\ &\leqslant C\|v_\varepsilon\|_{L_2(\Gamma^{\eta,\delta})} + C(\varepsilon^{1/2}\eta^{-1/2}+\eta^{1/2})\delta^{-1/2}\|f\|_{L_2(\Omega_0)}.\end{aligned} \tag{4.15}$$

The next auxiliary lemma will allow us to estimate $\|v_\varepsilon\|_{L_2(\Gamma^{\eta,\delta})}$.

Lemma 4.2. *Let $v \in W^1_2(\square)$, where $\square := \{x : d_- < x_1 < d_+, 0 < x_2 < d\}$, $d_+ > d_-$, $d_\pm$ are some constants. Denote $\Xi_\mu := \{x : |x_1 - d_0| < \mu, x_2 = 0\}$ and suppose that there exists a positive constant c independent of μ such that $d_0 - d_- \geqslant c$, $d_+ - d_0 \geqslant c$. Then for sufficiently small μ there exists a positive constant C independent of μ and v but dependent on c such that the inequality*

$$\|v\|_{L_2(\Xi_\mu)} \leqslant C\mu|\ln\mu|\|v\|_{W^1_2(\square)}$$

holds true.

Proof. We expand v in a Fourier series

$$v(x) = \sum_{m,n=0}^{\infty} c_{mn}\cos\frac{\pi n}{d_+ - d_-}(x_1 - d_-)\cos\frac{\pi m}{d}x_2 \tag{4.16}$$

converging in $W^1_2(\square)$. In view of the Parseval identity one has

$$\sum_{m,n=0}^{\infty} |c_{mn}|^2(m^2+n^2) \leqslant C\|v\|^2_{W^1_2(\square)}. \tag{4.17}$$

Due to the embedding of $W^1_2(\square)$ into $L_2(\xi_\mu)$ we can employ (4.16) to calculate $\|v\|_{L_2(\Xi_\mu)}$,

$$\begin{aligned}\|v\|^2_{L_2(\Xi_\mu)} = &\sum_{m,n,p,q=0}^{\infty} c_{mn}\overline{c_{pq}}\int\limits_{d_0-\mu}^{d_0+\mu}\cos\frac{\pi n}{d_+ - d_-}(x_1 - d_-)\cos\frac{\pi p}{d_+ - d_-}(x_1 - d_-)\,dx_1 \\ =&\frac{d_+ - d_-}{\pi}\sum_{m,n,p,q=0}^{\infty} c_{mn}\overline{c_{pq}}\left(\sin\frac{\pi(n+p)\mu}{d_+ - d_-}\frac{\cos\pi(n+p)}{n+p}\right. \\ &\left.-\sin\frac{\pi(n-p)\mu}{d_+ - d_-}\frac{\cos\pi(n-p)}{n-p}\right),\end{aligned}$$

where $\sin\frac{\pi(n-p)\mu}{d_+-d_-}/(n-p)$ is to be replaced by $\pi\mu/d$ as $n=p$. We employ the estimate

$$\sin^2 t\leqslant \frac{t^2}{1+t^2},\quad t\geqslant 0,$$

and by (4.17) we obtain

$$\begin{aligned}\|v\|^2_{L_2(\Xi_\mu)}&\leqslant C\mu^2\sum_{m,n,p,q=0}^{\infty}\frac{1}{(m^2+n^2)(p^2+q^2)}\left(\frac{1}{1+\mu^2(n+p)^2}+\frac{1}{1+\mu^2(n-p)^2}\right)\\ &\leqslant C\mu^2\sum_{n,p=0}^{\infty}\left(\frac{1}{1+\mu^2(n+p)^2}+\frac{1}{1+\mu^2(n-p)^2}\right)\int\limits_1^{+\infty}\frac{dz}{z^2+n^2}\int\limits_1^{+\infty}\frac{dz}{z^2+p^2}\\ &\leqslant C\mu^2\sum_{n,p=0}^{\infty}\frac{1}{1+nq}\left(\frac{1}{1+\mu^2(n+p)^2}+\frac{1}{1+\mu^2(n-p)^2}\right).\end{aligned}$$

In the last sum we extract the terms for $(n,p)=(0,0)$, $(n,p)=(0,1)$, and $(n,p)=(1,0)$. Then we replace the remaining summation by the integration and estimate in this way the sum by a two-dimensional integral,

$$\|v\|^2_{L_2(\Xi_\mu)}\leqslant 3\mu^2+C\mu^2\int\limits_{\substack{z_1^2+z_2^2>3\\ z_1,z_2>0}}\left(\frac{1}{1+\mu^2(z_1+z_2)^2}+\frac{1}{1+\mu^2(z_1-z_2)^2}\right)\frac{dz_1\,dz_2}{1+z_1z_2}.$$

Passing to the polar coordinates (r,θ) associated with (z_1,z_2), we get

$$\begin{aligned}&\int\limits_{\substack{z_1^2+z_2^2>3\\ z_1,z_2>0}}\left(\frac{1}{1+\mu^2(z_1+z_2)^2}+\frac{1}{1+\mu^2(z_1-z_2)^2}\right)\frac{dz_1\,dz_2}{1+z_1z_2}\\ &\qquad\leqslant 2\int\limits_{\sqrt{3}}^{+\infty}\int\limits_0^{\pi/2}\left(\frac{1}{1+\mu^2r^2(1+\sin 2\theta)}+\frac{1}{1+\mu^2r^2(1-\sin 2\theta)}\right)\frac{r\,dr\,d\theta}{1+r^2\sin 2\theta}\\ &\qquad\leqslant C\int\limits_3^{+\infty}\left(\frac{\ln\tau}{\tau(1+\mu^2\tau)}+\frac{\mu^2}{(1+\mu^2\tau)^{3/2}}\right)d\tau\\ &\qquad=C\int\limits_{3\mu^2}^{+\infty}\left(\frac{\ln\tau-2\ln\mu}{\tau(1+\tau)}+\frac{1}{(1+\tau)^{3/2}}\right)d\tau\leqslant C\ln^2\mu.\end{aligned}$$

Two last formulas proves the desired estimate for $\|v\|_{L_2(\Xi_\mu)}$. $\square$

We apply the proven lemma with $v=v_\varepsilon$, $d_-=X_n-c/2$, $d_+=X_n+c/2$, $d_0=X_n$ and sum the obtained inequalities over $n\in\mathbb{Z}$. It gives the estimate for $\|v_\varepsilon\|_{L_2(\Gamma^{\eta,\delta})}$,

$$\|v_\varepsilon\|_{L_2(\Gamma^{\eta,\delta})}\leqslant C\mu|\ln\mu|\|f\|_{L_2(\Omega_\varepsilon)}.$$

This estimate and (4.13), (4.15) yield

$$\left|\sum_{j=1}^{2}\left(A_{2j}\frac{\partial u_0}{\partial x_j},v_\varepsilon\frac{\partial K}{\partial x_2}\right)_{L_2(\Omega_\varepsilon)}\right|\leqslant C\big(\mu(\delta)|\ln\mu(\delta)|+(\varepsilon^{1/2}\eta^{-1/2}+\eta^{1/2})\delta^{-1/2}\big)\|f\|^2_{L_2(\Omega_0)},\tag{4.18}$$

where δ is assumed to tends to zero as $\varepsilon\to+0$. Together with (4.11), (4.4), (1.14), and the definition (1.6) of $\mathfrak{h}^{\mathrm{R}}_{\varepsilon,\eta}$ it implies the final estimate for v_ε,

$$\|v_\varepsilon\|_{W_2^1(\Omega_\varepsilon)}\leqslant C\Big(\mu^{1/2}(\delta)|\ln\mu(\delta)|^{1/2}+(\varepsilon^{1/4}\eta^{-1/4}+\eta^{1/4})\delta^{-1/4}\Big)\|f\|_{L_2(\Omega_0)}.$$

Together with (2.13) it completes the proof. $\square$

Acknowledgments

The authors thank G.A. Chechkin and T.A. Mel'nyk for useful remarks.

D.B. was supported by RFBR and Federal Targeted Program "Scientific and pedagogical staff of innovative Russia for 2009-2013" (contract 02.740.11.0612) and by the Project "Metodi asintotici per lo studio di alcuni funzionali e alcuni tipi di equazioni alle derivate parziali", GNAMPA, 2011.

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